State-of-Affairs Semantics for Positive Free Logic

Hans-Peter Leeb

Abstract. In the following the details of a state-of-affairs semantics for positive free logic are worked out, based on the models of common inner domain-outer domain semantics. Lambert's PFL system is proven to be weakly adequate (i.e. sound and complete) with respect to that semantics by demonstrating that the concept of logical truth definable therein coincides with that one of common truth-value semantics for PFL. Furthermore, this state-of-affairs semantics resists the challenges stemming from the slingshot argument since logically equivalent statements do not always have the same extension according to it. Finally, it is argued that in such a semantics all statements of a certain language for PFL are state-of-affairs-related extensional as well as *salva extensione* extensional, even though their *salva veritate* extensionality fails.

1. Introduction

In modern philosophy of language it is to a large extent undisputed that singular terms have individuals as extensions and *n*-place general terms, or predicates, sets of ordered *n*-tuples of such individuals. However, in the early phase of modern philosophy of language, the question "What has to be taken as the *extensions* of statements?" was answered quite differently.[1] Whereas Frege argued that the extensions of statements are *truth-values*, Carnap argued that they are *states of affairs*. Nowadays, states of affairs are widely held to be a rather unfortunate choice for the extensions of statements, due mostly to the influence of the slingshot argument.[2] According to this argument, it can be proven that all true statements have the same extension (as well as all false ones); so, as long as a semantics proceeds from the assumptions of this argument, all true and false statements taken together can have at most two different entities as extensions. These might be, but do not have to be, the two truth-values True and False, since the assumptions of this argument only determine the number, but not the kind of the extensions of statements.[3] The upshot of all this for the assumption that states of affairs are such extensions is that all true and false statements taken together can have at most two different states of affairs as extensions. This unpleasant consequence of the slingshot argument, however, conflicts with the pre-semantic intuition that, generally, different true statements talk about different things (and analogously for the false ones).

[1] See Frege 1892 and Carnap 1942. Further, in this paper, a *singular term* is a linguistic expression serving to refer to exactly one individual. An *n-place general term* is a linguistic expression true of all ordered *n*-tuples of individuals from a given class of individuals; it is *simple* iff it does not contain any connectives and bound individual variables. Finally, a *statement* is a linguistic expression that can be true or false.

[2] See Gödel 1944, p.450 f. and Church 1943, p.299 f.

[3] See Leeb 2004, p.175 and p.156.

What are the principal assumptions of this argument? The first assumption is that of the principle of compositionality, understood as a substitution thesis, according to which co-extensional expressions can always be substituted for each other in a statement without changing its extension. Further, it is assumed that logically true statements always have the same extension.

With respect to these assumptions, the following points merit consideration:

(i) The substitution thesis underdetermines any answer to the question what the extensions of statements are. According to this thesis, every entity is admissible as the extension of a statement as long as it does not change when co-extensional expressions are substituted for each other. Hence, in classical first-order logic, truth-values are admissible as such extensions, and in free logic – as will be demonstrated later on – states of affairs, understood in a certain way, are also admissible.

(ii) A state-of-affairs semantics for PFL can be developed in such a way that logically equivalent statements do not always have the same extension, as will be shown. That way one of the two principal assumptions of the slingshot argument is refuted allowing one to retain the substitution thesis as a principle that guides the search for the extensions of statements.

(iii) The substitution thesis is closely connected to the concept of *extensionality* (in the sense of substitutivity of co-extensional expressions). Accordingly, a statement is extensional in the *neutral* meaning of 'extensional', namely, in the sense of *salva extensione* substitutivity (in short: *SE-extensional*) iff co-extensional expressions can always be substituted for each other in this statement without changing its extension.[4]

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[4] In the concept of SE-extensionality it is not fixed what entities the extensions of statements are. By contrast, in the different notions of *non-neutral* extensionality this is specified by an *additional* assumption, e.g. in the concept of *salva veritate* extensionality not only the notion of SE-extensionality is involved, but it is also assumed that truth-values are the extensions of statements.

Lambert, however, has argued that in free logic, one-place general terms that are co-extensional cannot always be substituted for each other in simple statements containing empty singular terms without changing their truth-value as extension.[5] Thus, in free logic such simple statements turn out to be non-extensional in a *non-neutral* meaning of 'extensional', namely, in the sense of *salva veritate* non-substitutivity (i.e. they turn out to be, in short, *non-SV-extensional*). This raises the question whether their non-SV-extensionality also involves their non-SE-extensionality: Can co-extensional expressions always be substituted for each other in such simple statements without changing their extension, even though this cannot be done without changing their truth-value as extension?

I argue that SE-extensionality of such simple statements with empty singular terms can be ensured if one replaces the assumption that truth-values are the extensions of statements with the following:

S The extension of a statement is composed of the extensions of its singular as well as general terms and sub-statements, and the logical form (in Kaplan's sense[6]) of this statement determines how that extension is composed (I call such complexes *abstract states of affairs*).

Here the logical form of a statement depends on how its truth-value is determined. Hence, in so far as SE-extensionality is a desirable feature of scientific language, truth-values cannot be looked on in free logic as a reliable choice for the extensions of statements. Actually, such states of affairs seem to be preferable – were it not for the slingshot argument to discredit such a choice.

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[5] For his non-extensionality argument see Lambert 2003, p.96 f. and 1991, p.278 f. Further, a statement is *simple* iff it does not contain connectives and quantifiers; and a singular term is *empty* iff it does not refer to an existent, e.g. 'Vulcan'.

[6] See Kaplan 1970, p.283 and Lambert 2003, p.104.

In what follows the details of a state-of-affairs semantics are worked out. First, I shall prove (i) the adequacy of the PFL system[7] with respect to this semantics. Two further goals are then pursued: (ii) I argue that in this semantics logically equivalent statements do not always have the same extension (thus refusing one of the principal assumptions of the slingshot argument); and (iii) I demonstrate that therein all statements of a certain language for PFL are SE-extensional (as a result of which SE-extensionality of simple statements with empty singular terms is ensured as well).

2. Syntax of *L*

Since the syntax I have choosen to achieve my goals is somewhat unusual, the alphabet and formation rules of a language *L* for Lambert's PFL system will be given.

The *alphabet* of *L* shall contain as *descriptive* signs infinitely many singular terms a, $a_1,\ldots, a_n,\ldots$, and for each n (with $n = 1, 2, \ldots$) infinitely many simple n-place general terms. As *logical* signs it shall contain the connectives $\neg$, $\wedge$, the all-quantifier $\forall$, infinitely many individual variables $x, x_1,\ldots, x_n,\ldots, y, y_1,\ldots, y_n,\ldots$, the existence predicate E!, the identity sign =, the term abstractor Δ (read: 'thing such that'), and the auxiliary signs (,).

The *formation rules* of *L* are as follows:

(1) If F^n is a simple n-place general term and $y, y_1,\ldots, y_n$ are individual variables, then $F^n y_1\ldots y_n$, $\mathrm{E!}y$ and $y_1 = y_2$ are formulas.

(2) If $O_{y_1\ldots y_n}$ is a homogenous formula[8] with $y_1,\ldots, y_n$ as the only individual variables, but without quantifiers, singular terms,

[7] For the PFL system see Lambert 1997, p.39 f., p.65 f., and p.114 f.

[8] A *homogenous* formula is one whose simple general terms all have the same number of places for individual expressions ('E!' is here added to the one-place and '=' to the two-place simple general terms).

and Δ-operators, then $\Delta y_1 \ldots \Delta y_n O_{y_1 \ldots y_n}$ is a complex *n*-place general term.

(3) If G^n is a complex *n*-place general term and $t_1, \ldots, t_n$ are singular terms or individual variables, then $G^n t_1 \ldots t_n$ is a formula.[9]

(4) If *A*, *B* are formulas, then $\neg A$ and $(A \wedge B)$ are formulas.

(5) If *A* is a formula in prenex normal form[10] (with *m* complex general terms $G_1, \ldots, G_m$ not necessarily distinct from each other) and x_i is a free individual variable contained in *A* (with i, m = 1, 2,…), then $\forall x_i A[G_1, \ldots, G_m]$ is a formula.

(6) Nothing else is a complex *n*-place general term or formula.

In the following the *n*-place general term $\Delta y_1 \ldots \Delta y_n O_{y_1 \ldots y_n}$ is abbreviated by '$G^n_{\Delta y}$' and $\forall x_i A[G_1, \ldots, G_m]$ by '$\forall x_i A[G_{\bar{m}}]$'. Further connectives and the existence-quantifier may be introduced as usual, according to demand. The *statements* of *L* are the closed formulas of *L*.

The following is an *abbreviating convention*[11]:

$$\Delta^- \qquad C[O_{y_1 \ldots y_n}(t_1/y_1, \ldots, t_n/y_n)] :\leftrightarrow C[\Delta y_1 \ldots \Delta y_n O_{y_1 \ldots y_n} t_1 \ldots t_n]$$

Here *C* is a context and $O_{y_1 \ldots y_n}(t_1/y_1, \ldots, t_n/y_n)$ is the result of simul-

[9] The formulas $G^n t_1 \ldots t_n$ are the *complex predications* of *L*. A *predication* is a formula in which an *n*-place general term and *n* singular terms and/or individual variables are joined by the copula; it is *complex* iff its general term is complex (i.e. not simple); and it is *closed* iff it contains no free individual variables.

[10] A formula in *prenex normal form* is one whose quantifiers, if any, are all put in front of it. As is well known, every formula with quantifiers can be transformed into a deductively equivalent one in prenex normal form (see Kalish/Montague/Mar 1980, p.225 and p.427 ff.). Hence, for the purpose of this paper the axioms of PFL can be transformed into their deductively equivalent prenex normal forms.

[11] This convention restores the familiar picture of a first-order language whose formulas only contain simple general terms. Complex general terms are here introduced into *L* merely because of a certain application I have in mind: A state-of-affairs semantics for such a language, based on single domain models for free logic, will be developed in another paper to discuss Lambert's non-extensionality argument.

taneously substituting all occurrences of $y_1,\ldots, y_n$ in $O_{y_1\ldots y_n}$ by $t_1,\ldots, t_n$.

3. Truth-value semantics for *L*

In the following I briefly summarize common truth-value semantics for *L* to gain a basis for the development of its state-of-affairs semantics.

3.1. Models and assignments

An *inner domain-outer domain model* (in short: IODM) *M* is an ordered triple $\langle D_I, D_O, f \rangle$ such that the following conditions are fulfilled:

(1) D_I and D_O are sets (possibly empty);
intuitively, D_I (the *inner domain*) is a set of existents and D_O (the *outer domain*) a set of non-existents

(2) $D_I \cap D_O = \varnothing$

(3) $D_I \cup D_O \neq \varnothing$
f is a total function (interpretation function) such that the following conditions are fulfilled:

(4) for all singular terms *a*: $f(a) \in D_I \cup D_O$

(5) for all simple *n*-place general terms F^n: $f(F^n)$ is a set of ordered *n*-tuples of elements of $D_I \cup D_O$

(6) for every $d \in D_I \cup D_O$ there is a singular term *a* such that: $f(a) = d$

(7) there is exactly one *D* (*domain*) such that: $D = D_I \cup D_O$.

Further, an *assignment of term-operands* is a function *g* that assigns to every homogenous formula $O_{y_1\ldots y_n}$ with $y_1,\ldots, y_n$ as the only individual variables, but without quantifiers, singular terms, and Δ-operators exactly one set of ordered *n*-tuples of elements of *D* (these formulas are called *term-operands* and are *assigned* such sets).

If $M = \langle D_I, D_O, f \rangle$ is an IODM and $O'_{y_1 \ldots y_n}$ as well as $O''_{y_1 \ldots y_n}$ are term-operands, then:

(1) $g(F^n y_1 \ldots y_n) = f(F^n)$
(2) $g(\mathrm{E}!y) = D_I$
(3) $g(y_1 = y_2) = \{\langle d_1, d_2 \rangle \in D^2 \mid d_1 = d_2\}$[12]
(4) $g(\neg O'_{y_1 \ldots y_n}) = D^n \setminus g(O'_{y_1 \ldots y_n})$[13]
(5) $g(O'_{y_1 \ldots y_n} \wedge O''_{y_1 \ldots y_n}) = g(O'_{y_1 \ldots y_n}) \cap g(O''_{y_1 \ldots y_n})$

3.2. Semantic concepts$_1$

According to common truth-value semantics, a truth concept for L can be introduced as follows (read: '$\models_{1\, M, g} S$' as 'a statement S is *true*$_1$ in an IODM M under an assignment of term-operands g'):

(1) $\models_{1\, M, g} G^n_{\Delta y}\, a_1 \ldots a_n \Leftrightarrow \langle f(a_1), \ldots, f(a_n) \rangle \in g(O_{y_1 \ldots y_n})$
(2) $\models_{1\, M, g} \neg A \Leftrightarrow$ not $\models_{1\, M, g} A$
(3) $\models_{1\, M, g} A \wedge B \Leftrightarrow\ \models_{1\, M, g} A$ and $\models_{1\, M, g} B$
(4) $\models_{1\, M, g} \forall x_i A[G_{\bar{m}}] \Leftrightarrow$ for all singular terms a_i
(if $f(a_i) \in D_I$, then $\models_{1\, M, g} A[G_{\bar{m}}](a_i / x_i)$)[14]

Df.LT$_1$ A statement S is *logically true*$_1$ (in short: $\models_1 S$) :$\Leftrightarrow$
for all IODM M and assignments of term-operands g:
$\models_{1\, M, g} S$

Lambert's PFL system is *weakly adequate* with respect to common truth-value semantics in the sense that for all statements S of L the following holds: $\models_1 S \Leftrightarrow S$ is a theorem of PFL.[15]

[12] $D^2 = D \times D = \{\langle d_1, d_2 \rangle \mid d_1 \in D \;\&\; d_2 \in D\}$.

[13] D^n is the n-times Cartesian product of D with itself.

[14] In the framework of such a truth-value semantics one can define the extension of a statement S in M under g as follows: $ext_{M, g}(S) = \mathrm{T(rue)}$:$\Leftrightarrow\ \models_{1\, M, g} S$, and $ext_{M, g}(S) = \mathrm{F(alse)}$:$\Leftrightarrow$ not $\models_{1\, M, g} S$.

[15] See Lambert 1997, p.65 f. and p.114 f. and Leblanc/Thomason 1968.

4. State-of-affairs semantics for *L*

The state-of-affairs semantics for *L* I propose here is based on the IODM and assignments of term-operands as presented in §3. The idea is that statements are either true or false because they have states of affairs as extensions that either obtain or do not obtain; for the obtaining or not obtaining of a state of affairs is it what makes a statement true or false.[16] Thus, in such a semantics a statement is true (respectively, false) if it has a state of affairs as extension that obtains (respectively, does not obtain). Intuitively a state of affairs obtains if in a model (world) things happen to be that way, but not another way.

4.1. The set of states of affairs

I shall now say a word or two about the complexes that are (according to S) composed of the extensions of the sub-expressions of a statement. The extensions of *closed predications* are taken to be composed of three basic elements, namely, the set-theoretical element relation (correlating to the copula), an ordered *n*-tuple of individuals, and a set of ordered *n*-tuples of such individuals. For example the state of affairs such that Mercury is a thing such that it rotates, is an ordered triple composed of the element relation, the individual Mercury, and the set of the rotating individuals. According to whether the empty singular term 'Vulcan' refers to a non-existent or to nothing at all, there are in principal two possibilities to compose the state of affairs such that Vulcan is a thing such that it rotates:

(i) If 'Vulcan' refers to a non-existent, then let that complex be composed of the element relation, the non-existent Vulcan, and the set of the rotating individuals.

(ii) If 'Vulcan' does not refer to anything at all, then an individual is just missing and that complex may be composed only of the

[16] See van Fraassen 1969, p.479 f.

element relation and the set of the rotating individuals. Such a complex I would like to call ‘degenerate’ since one can neither sensibly say that it obtains nor that it does not obtain.

Whereas state-of-affairs semantics as modelled on supervaluation semantics would have to proceed, at least in the first phase of evaluation, from alternative (ii), state-of-affairs semantics as modelled on inner domain-outer domain semantics might rest on alternative (i).

Further, the extensions of *all-quantified statements* in prenex normal form are complexes composed of sets of ordered *n*-tuples of individuals and so-called *logical attributes* of such sets. An example of a logical attribute is the all-existing-things-are-such-that-they-lie-therein-property that can be defined as follows:

R_1^1 A set X has the all-existing-things-are-such-that-they-lie-therein-property (in short: R_1^1-property) $\Leftrightarrow$
for all singular terms a (if the extension of a is an existent, then this extension lies in X).

Actually, those logical attributes are defined inductively only later on in this paper, they are nevertheless mentioned here in advance merely for systematic exposition. The idea is to capture the whole logical structure of an all-quantified statement $\forall x_i A[G_{\bar{m}}]$ in a single logical attribute that is then claimed to hold of the extensions of the m general terms contained in such an all-quantified statement. Finally, the set of states of affairs is supposed to be closed under the operations of negation and conjunction.

4.2. Definition of (not) obtaining

In the following I shall define what it is for a state of affairs to obtain in an IODM under an assignment of term-operands. Next, it will be defined what state of affairs a statement has as extension in an IODM under such an assignment. Both inductive definitions are put forward successively because of the linear order of written English. However, the concepts of the obtaining of a state of affairs and of the extension of an expression are, in fact, defined simultaneously in so far as, e.g., the extensions of the all-quantified statements with one quantifier depend on the obtaining of the extensions (i.e. state of affairs) of certain quantifier-free statements.[17] Finally, I will define what it actually is for a statement to be true in an IODM under an assignment of term-operands, on the basis of which further semantic concepts can be introduced.

If $M = \langle D_I, D_O, f\rangle$ is an IODM, g an assignment of term-operands, $d_1,\ldots, d_n$ are individuals, X^n, $X_1,\ldots, X_m$ are sets[18], R^m_k is the k^{th} m-place logical attribute[19] (with k, m, n = 1, 2,...), and s_1, s_2 are states of affairs, then:

(1) The state of affairs such that: $\langle d_1,\ldots, d_n\rangle$ lies in X^n,
i.e. $\langle \in, \langle d_1,\ldots, d_n\rangle, X^n\rangle$,
obtains (respectively, does not obtain) in $M = \langle D_I, D_O, f\rangle$ under g $\Leftrightarrow$
$\langle d_1,\ldots, d_n\rangle \in X^n$ (respectively, $\langle d_1,\ldots, d_n\rangle \notin X^n$).

(2) The state of affairs such that: $X_1,\ldots, X_m$ have the R^m_k-attribute, i.e. $\langle R^m_k, X_1,\ldots, X_m\rangle$,
obtains (respectively, does not obtain) in $M = \langle D_I, D_O, f\rangle$ under g $\Leftrightarrow$
$X_1,\ldots, X_m$ have (respectively, have not) the R^m_k-attribute in $M = \langle D_I, D_O, f\rangle$ under g.[20]

(3) The state of affairs such that: s_1 is not the case,
i.e. $\langle \text{not}, s_1\rangle$,
obtains (respectively, does not obtain) in $M = \langle D_I, D_O, f\rangle$ under g $\Leftrightarrow$
s_1 does not obtain (respectively, obtains) in $M = \langle D_I, D_O, f\rangle$ under g.

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[17] See p.14.

[18] X^n is the n-times Cartesian product of X with itself.

[19] The logical attributes will be defined in §4.4.

[20] What the latter means will be defined on p.16.

(4) The state of affairs such that: s_1 and s_2 are the case,
i.e. $\langle$and, s_1, $s_2\rangle$,
obtains (respectively, does not obtain) in $M = \langle D_I, D_O, f\rangle$ under $g \Leftrightarrow$
s_1 and s_2 both obtain (respectively, at least one of s_1 and s_2 does not obtain) in $M = \langle D_I, D_O, f\rangle$ under g.

4.3. Definition of an extension function

In the following I shall define what state of affairs a statement has as extension in an IODM under an assignment of term-operands. In doing so the following kinds of expressions in primitive notation are matched with *extensions in M under g*: singular as well as complex *n*-place general terms and statements in prenex normal form. By contrast, simple general terms, individual variables, and open formulas are not matched with such extensions (nevertheless, the first are interpreted by the interpretation functions and the latter by the assignments of term-operands, at least in so far as they are homogenous):

(1) $ext_{M,\,g}(a) = f(a)$
(2) $ext_{M,\,g}(G^n_{\Delta y}) = g(O_{y_1 \ldots y_n})$
(3) $ext_{M,\,g}(G^n_{\Delta y}\, a_1 \ldots a_n) =$
$\langle \in, \langle ext_{M,\,g}(a_1), \ldots, ext_{M,\,g}(a_n)\rangle, ext_{M,\,g}(G^n_{\Delta y})\rangle$
(4) $ext_{M,\,g}(\neg A) = \langle \text{not}, ext_{M,\,g}(A)\rangle$
(5) $ext_{M,\,g}(A \wedge B) = \langle \text{and}, ext_{M,\,g}(A), ext_{M,\,g}(B)\rangle$
(6) $ext_{M,\,g}(\forall x_i A[G_{\bar{m}}]) = \langle R^m_k, ext_{M,\,g}(G_1), \ldots, ext_{M,\,g}(G_m)\rangle$

4.4. All-quantified statements and logical attributes

In the following the extension of $\forall x_i A[G_{\bar{m}}]$ in M under g, i.e. the state of affairs

(1) $\langle R^m_k, ext_{M,\,g}(G_1), \ldots, ext_{M,\,g}(G_m)\rangle$,

is defined inductively. For this purpose I suggest a procedure whereby one can construct for every all-quantified *statement* $\forall x_i A[G_{\bar{m}}]$ the corresponding logical attribute (with i, k, m = 1, 2, …). It is here assumed that this all-quantified statement is in primitive notation (hence, in prenex normal form) and, further, that it contains m complex general terms (not necessarily distinct from each other). Since

(2) $\forall x_i A[G_{\bar{m}}]$

is in prenex normal form, (2) can be described as follows:

(3) $\forall x_i (\neg)\forall x_{i-1}(\neg)\ldots(\neg)\forall x_1 A^{-}[G_{\bar{m}}]$

The negation signs enclosed in brackets shall indicate zero or one occurrence[21] thereof and, further,

(4) $A^{-}[G_{\bar{m}}]$

is the remaining open formula without quantifiers (with at least x_i as free individual variable and at most i different individual variables). Replace now simultaneously the individual variables x_1, x_2,…, x_i, distinctly from each other and contained in the open formula $A^{-}[G_{\bar{m}}]$, by the singular terms a_1, a_2,…, a_i, also distinctly from each other. The latter all have to be different from singular terms possibly occurring in $A^{-}[G_{\bar{m}}]$. That is construct the quantifier-free *statement*

(5) $A^{-}[G_{\bar{m}}](a_1/x_1, a_2/x_2,\ldots, a_i/x_i) = A°$

such that these requirements are fulfilled. The statement $A°$ then can have only one of the following forms:

[21] *Simplifying convention*: If the number of negation signs occurring between two all-quantifiers is even, then they are reduced to 0, and if it is odd, then to 1.

(a) A° has the form $G^n_{\Delta y}\, a_1 \ldots a_n$ (i.e. is a closed predication), or

(b) A° has the form $\neg B$ (i.e. is a negation statement), or

(c) A° has the form $B \wedge C$ (i.e. is a conjunction statement).

Then, due to §4.3.(1)–(5), $ext_{M,\,g}\,(A^\circ)$ is already defined as follows:

ad (a) If A° has the form $G^n_{\Delta y}\, a_1 \ldots a_n$, then it holds because of clause §4.3.(3) and the fact that $G^n_{\Delta y}$ contains no quantifiers that:
$ext_{M,\,g}\,(A^\circ) = \langle \in, \langle ext_{M,\,g}\,(a_1), \ldots, ext_{M,\,g}\,(a_n) \rangle, ext_{M,\,g}\,(G^n_{\Delta y}) \rangle$.

ad (b) If A° has the form $\neg B$, then it holds because of clause §4.3.(4) and the fact that A° contains no quantifiers that:
$ext_{M,\,g}\,(A^\circ) = \langle \text{not}, ext_{M,\,g}\,(B) \rangle$.

ad (c) If A° has the form $B \wedge C$, then it holds because of clause §4.3.(5) and once again the fact that A° contains no quantifiers that:
$ext_{M,\,g}\,(A^\circ) = \langle \text{and}, ext_{M,\,g}\,(B), ext_{M,\,g}\,(C) \rangle$.

The simplest logical attributes are those which are contained in the extensions of all-quantified statements with only one quantifier. They can be defined by means of the obtaining of the extensions of the *quantifier-free* statements $A^{-}[G_{\bar{m}}](a_1/x_1)$ already previously defined, as follows:

R^m_j The sets $ext_{M,\,g}\,(G_1), \ldots, ext_{M,\,g}\,(G_m)$ have the R^m_j-attribute in $M = \langle D_I, D_O, f \rangle$ under $g \Leftrightarrow$
for all singular terms a_1 ($ext_{M,\,g}\,(a_1) \in D_I \Rightarrow$
$ext_{M,\,g}\,(A^{-}[G_{\bar{m}}](a_1/x_1))$ obtains in $M = \langle D_I, D_O, f \rangle$ under g)

Accordingly, the extensions of the all-quantified statements with only one quantifier are:

(6) If $\forall x_i A[G_{\bar{m}}]$ has the form $\forall x_1(A^{-}[G_{\bar{m}}](a_1/x_1)(x_1/a_1))$[22], then $ext_{M,\,g}(\forall x_i A[G_{\bar{m}}]) = \langle R^m_j,\ ext_{M,\,g}(G_1),\ldots,\ ext_{M,\,g}(G_m)\rangle$ (with $j = 1, 2,\ldots$)

An *example* might be helpful to fix ideas. Because of the definitions of (i) the obtaining of a state of affairs as well as of (ii) an extension function and, further, (iii) clause $\mathrm{R}^{\mathrm{m}}_{\mathrm{j}}$ the following holds: for all singular terms a_2,

(7) $ext_{M,\,g}(\forall x_1(G_1x_1 \wedge G_2a_2))$ obtains in M under $g \Leftrightarrow_{(ii)}$
$\langle R^2_1, ext_{M,\,g}(G_1), ext_{M,\,g}(G_2)\rangle$ obtains in M under $g \Leftrightarrow_{(i)}$
the sets $ext_{M,\,g}(G_1)$ and $ext_{M,\,g}(G_2)$ have the R^2_1-attribute in M under $g \Leftrightarrow_{(iii)}$
for all singular terms a_1 ($ext_{M,\,g}(a_1) \in D_I \Rightarrow$
$ext_{M,\,g}(G_1a_1 \wedge G_2a_2)$ obtains in M under g) $\Leftrightarrow_{(i),(ii)}$
for all singular terms a_1 ($ext_{M,\,g}(a_1) \in D_I \Rightarrow$
$ext_{M,\,g}(a_1) \in ext_{M,\,g}(G_1)$ and $ext_{M,\,g}(a_2) \in ext_{M,\,g}(G_2)$)

Due to (7) the following holds: for all singular terms a_2,

(8) $ext_{M,\,g}(\forall x_1(G_1x_1 \wedge G_2a_2))$ obtains in M under $g \Leftrightarrow$
for all singular terms a_1 ($ext_{M,\,g}(a_1) \in D_I \Rightarrow$
$ext_{M,\,g}(a_1) \in ext_{M,\,g}(G_1)$ and $ext_{M,\,g}(a_2) \in ext_{M,\,g}(G_2)$)

Finally, in the following *construction-schema* for the k^{th} m-place logical attribute R^m_k the object-linguistic statement

(9) $\forall x_i(\neg)\forall x_{i-1}(\neg)\ldots(\neg)\forall x_1$
$(A^{-}[G_{\bar{m}}](a_1/x_1,\ldots, a_i/x_i)(x_1/a_1,\ldots, x_i/a_i))$

is translated by the meta-linguistic expression

[22] The reasons for this peculiar notation will get obviously when it comes to the proof of the theorem of adequacy.

(10) for all singular terms a_i ($ext_{M,g}(a_i) \in D_I \Rightarrow$
(not)[23]
$ext_{M,g}(\forall x_{i-1}(\neg)\ldots(\neg)\forall x_1$
$(A^{-}[G_{\bar{m}}](a_1/x_1,\ldots, a_i/x_i)(x_1/a_1,\ldots, x_{i-1}/a_{i-1})))$
obtains in M under g)

Thus:

R^m_k The sets $ext_{M,g}(G_1),\ldots, ext_{M,g}(G_m)$ have the R^m_k-attribute in $M = \langle D_I, D_O, f\rangle$ under $g \Leftrightarrow$
for all singular terms a_i ($ext_{M,g}(a_i) \in D_I \Rightarrow$
(not)
$ext_{M,g}(\forall x_{i-1}(\neg)\ldots(\neg)\forall x_1$
$(A^{-}[G_{\bar{m}}](a_1/x_1,\ldots, a_i/x_i)(x_1/a_1,\ldots, x_{i-1}/a_{i-1})))$
obtains in M under g)

Accordingly, the extension of an all-quantified statement with i quantifiers is:

(11) If $\forall x_i A[G_{\bar{m}}]$ has the form
$\forall x_i(\neg)\forall x_{i-1}(\neg)\ldots(\neg)\forall x_1$
$(A^{-}[G_{\bar{m}}](a_1/x_1,\ldots, a_i/x_i)(x_1/a_1,\ldots, x_i/a_i))$,
then $ext_{M,g}(\forall x_i A[G_{\bar{m}}]) = \langle R^m_k, ext_{M,g}(G_1),\ldots, ext_{M,g}(G_m)\rangle$

Once again an *example* will help to fix these ideas. Because of the definitions of (i) the obtaining of a state of affairs as well as of (ii) an extension function and, further, (iii) clause R^m_k the following holds:

(12) $ext_{M,g}(\forall x_2\forall x_1(G_1x_1 \wedge G_2x_2))$ obtains in M under $g \Leftrightarrow_{(ii)}$
$\langle R^2_2, ext_{M,g}(G_1), ext_{M,g}(G_2)\rangle$ obtains in M under $g \Leftrightarrow_{(i)}$
the sets $ext_{M,g}(G_1)$ and $ext_{M,g}(G_2)$ have the R^2_2-attribute in M under $g \Leftrightarrow_{(iii)}$

[23] Here occurrences of 'not' put into brackets indicate zero or one occurrence thereof, according to whether the object-linguistic all-quantified statement contains at the corresponding positions zero or one occurrence of the negation sign.

for all singular terms a_2 ($ext_{M,g}(a_2) \in D_I \Rightarrow$
$ext_{M,g}(\forall x_1(G_1 x_1 \wedge G_2 a_2))$ obtains in M under g) $\Leftrightarrow_{(8)}$
for all singular terms a_2 ($ext_{M,g}(a_2) \in D_I \Rightarrow$
for all singular terms a_1 ($ext_{M,g}(a_1) \in D_I \Rightarrow$
$ext_{M,g}(a_1) \in ext_{M,g}(G_1)$ and $ext_{M,g}(a_2) \in ext_{M,g}(G_2)$))

Moreover, in accordance with a *set-theoretical* view, logical attributes might be looked on as certain second-order sets. Hence, one might understand here, e.g. the two one-place logical attributes R^1_1 and R^1_2 that are contained in the extensions

(13) $\langle R^1_1, ext_{M,g}(G^1_{\Delta y})\rangle$

and

(14) $\langle R^1_2, ext_{M,g}(G^1_{\Delta y})\rangle$

of the two all-quantified statements

(15) $\forall x_1 G^1_{\Delta y} x_1$

and

(16) $\forall x_1 \neg G^1_{\Delta y} x_1$

having the form $\forall x_1(A^{-}[G^{-}_1](a_1/x_1)(x_1/a_1))$ as follows:

R^1_1 the set $ext_{M,g}(G^1_{\Delta y})$ has the R^1_1-attribute
in $M = \langle D_I, D_O, f\rangle$ under $g \Leftrightarrow$
for all singular terms a_1 ($ext_{M,g}(a_1) \in D_I \Rightarrow$
$ext_{M,g}(G^1_{\Delta y} a_1)$ obtains in $M = \langle D_I, D_O, f\rangle$ under g),
i.e. iff for all singular terms a_1 ($ext_{M,g}(a_1) \in D_I \Rightarrow$
$ext_{M,g}(a_1) \in ext_{M,g}(G^1_{\Delta y})$)

and

R^1_2 the set $ext_{M,g}(G^1_{\Delta y})$ has the R^1_2-attribute
in $M = \langle D_I, D_O, f\rangle$ under $g \Leftrightarrow$

for all singular terms a_1 ($ext_{M,g}(a_1) \in D_I \Rightarrow$
$ext_{M,g}(\neg G^1_{\Delta y}\, a_1)$ obtains in $M = \langle D_I, D_O, f\rangle$ under g),
i.e. iff for all singular terms a_1 ($ext_{M,g}(a_1) \in D_I \Rightarrow$
$ext_{M,g}(a_1) \notin ext_{M,g}(G^1_{\Delta y})$)

Hence, the logical attributes R^1_1 and R^1_2 can be understood as certain sets of sub-sets X of the domain D as follows:

(17) $R^1_1 = \{X \mid$ for all singular terms a_1
$(ext_{M,g}(a_1) \in D_I \Rightarrow ext_{M,g}(a_1) \in X)\} = \{X \mid X \supseteq D_I\}$

and

(18) $R^1_2 = \{X \mid$ for all singular terms a_1
$(ext_{M,g}(a_1) \in D_I \Rightarrow ext_{M,g}(a_1) \notin X)\} = \{X \mid X \subseteq D_O\}$

Accordingly, the following then holds:

(19) $ext_{M,g}(G^1_{\Delta y})$ has the R^1_1-attribut in $M = \langle D_I, D_O, f\rangle$ under $g \Leftrightarrow$
$ext_{M,g}(G^1_{\Delta y}) \in R^1_1$ (i.e. $ext_{M,g}(G^1_{\Delta y}) \supseteq D_I$)

(20) $ext_{M,g}(G^1_{\Delta y})$ has the R^1_2-attribut in $M = \langle D_I, D_O, f\rangle$ under $g \Leftrightarrow$
$ext_{M,g}(G^1_{\Delta y}) \in R^1_2$ (i.e. $ext_{M,g}(G^1_{\Delta y}) \subseteq D_O$)[24]

This view of logical attributes as certain second-order sets can easily be generalized for the k^{th} m-place logical attribute R^m_k. Consider for this purpose an object-linguistic all-quantified statement

(21) $\forall x_i\, A[G_{\bar{m}}]$

having the form

(22) $\forall x_i (\neg)\forall x_{i-1}(\neg)\ldots(\neg)\forall x_1$
$(A^{-}[G_{\bar{m}}](a_1/x_1,\ldots, a_i/x_i)(x_1/a_1,\ldots, x_i/a_i))$

[24] Thus, all one-place logical attributes of the extensions of complex one-place general terms can be reduced to these two second-order sets R^1_1 and R^1_2.

This all-quantified statement can be translated into our metalanguage as follows:[25]

(23) for all singular terms a_i ($ext_{M,\,g}(a_i) \in D_I \Rightarrow$
(not) for all singular terms a_{i-1} ($ext_{M,\,g}(a_{i-1}) \in D_I \Rightarrow$
(not) … (not)
for all singular terms a_1 ($ext_{M,\,g}(a_1) \in D_I \Rightarrow$
$ext_{M,\,g}(A^{-}[G_{\bar{m}}](a_1/x_1,\ldots, a_i/x_i))$ obtains in
$M = \langle D_I, D_O, f\rangle$ under g)…))

The quantifier-free (object-linguistic) statement

(24) $A^{-}[G_{\bar{m}}](a_1/x_1,\ldots, a_i/x_i)$

contained in (23) gets now transformed by replacing every predication occurring in (24) having the form

(25) $G^{n}_{\Delta y}\, a_1 \ldots a_n$

as well as every negated predication occurring in (24) having the form

(26) $\neg G^{n}_{\Delta y}\, a_1 \ldots a_n$

by expressions having the following forms:

(27) $\langle ext_{M,\,g}(a_1),\ldots, ext_{M,\,g}(a_n)\rangle \in X^n$

(28) $\langle ext_{M,\,g}(a_1),\ldots, ext_{M,\,g}(a_n)\rangle \notin X^n$

assuming that $X^n \subseteq D^n$.[26] I designate the result of this replacement by:

(29) $A^{-}[G_{\bar{m}}](a_1/x_1,\ldots, a_i/x_i)\ /\ X_1,\ldots, X_m$

---

[25] The following translation is equivalent to that one on p.15.

[26] The following common convention shall hold that for all $d \in D$: $\langle d\rangle = d$.

assuming that for all l with $1 \leq l \leq m$ the following shall hold:

(30) $X_l \subseteq D^n \Leftrightarrow ext_{M,\,g}(G_l) \subseteq D^n$

One can then construct the k^{th} m-place logical attribute as a second-order set as follows:

(31) $R^m_k = \{\langle X_1,\ldots, X_m\rangle \mid$ for all singular terms a_i $(ext_{M,\,g}(a_i) \in D_I$
$\Rightarrow$ (not) for all singular terms a_{i-1} $(ext_{M,\,g}(a_{i-1}) \in D_I \Rightarrow$
(not) … (not)
for all singular terms a_1 $(ext_{M,\,g}(a_1) \in D_I \Rightarrow$
$A^{-}[G_{\bar{m}}](a_1/x_1,\ldots, a_i/x_i)$ / $X_1,\ldots, X_m)\ldots))\}$

In this way the formulation 'the sets $ext_{M,\,g}(G_1),\ldots,$ $ext_{M,\,g}(G_m)$ have the R^m_k-attribute in $M = \langle D_I, D_O, f\,\rangle$ under g' can be understood to be a *second-order* predication as follows:

(32) The sets $ext_{M,\,g}(G_1),\ldots,$ $ext_{M,\,g}(G_m)$ have the R^m_k-attribute in $M = \langle D_I, D_O, f\,\rangle$ under $g \Leftrightarrow$
$\langle ext_{M,\,g}(G_1),\ldots, ext_{M,\,g}(G_m)\rangle \in R^m_k$

Hence, our examples (19) and (20) turn out to be special cases of this general construction of logical attributes.

The following three *criteria of identity* for logical attributes that apply to object-linguistic all-quantified statements in primitive notation will aid in the examination of identity in a given case:

ICA_1 If two meta-linguistic translations of two object-linguistic all-quantified statements with i quantifiers and m complex general terms differ only in two singular terms s_1 and s_2 for *identical* states of affairs from each other, then the two attributes defined by those translations are identical.

ICA_2 If two meta-linguistic translations of two object-linguistic all-quantified statements with i quantifiers and m complex general terms are *deductively equivalent*, then the two attributes defined by those translations are identical.

ICA_3 If two object-linguistic all-quantified statements with *i* quantifiers and *m* complex general terms have the same *syntactical structure* (i.e. the same succession of quantifiers, connectives, individual variables, singular terms, complex general terms, and brackets), then the two attributes defined by their meta-linguistic translations are identical.

4.5. Semantic concepts$_2$

In the framework of this state-of-affairs semantics a concept of truth and further semantic notions can be defined as follows:

$Df.T_2$ A statement S is *true*$_2$ in an IODM M under an assignment of term-operands g (in short: $\models_{2\,M,\,g} S$) $:\Leftrightarrow$
there is a state of affairs s such that:
$ext_{M\,,\,g}(S) = s$ and s obtains in M under g.

$Df.LT_2$ A statement S is *logically true*$_2$ (in short: $\models_2 S$) $:\Leftrightarrow$
for all IODM M and assignments of term-operands g:
$\models_{2\,M,\,g} S$.

$Df.LE_2$ Two statements S_1, S_2 are *logically equivalent*$_2$ $:\Leftrightarrow$
$\models_2 S_1 \leftrightarrow S_2$[27]

4.6. Theorem of adequacy

I shall now prove some metalogical results concerning adequacy and extensionality.

T1 Every system of positive free logic that is weakly adequate with respect to common inner domain-outer domain truth-value semantics is one that is also *weakly adequate* with respect to the state-of-affairs semantics under consideration (e.g. the PFL system).

[27] D1 $(A \leftrightarrow B) :\leftrightarrow (\neg(A \wedge \neg B) \wedge \neg(B \wedge \neg A))$

Proof
It holds for all statements S of L: $\models_1 S \Leftrightarrow \models_2 S$. In order to prove this, it has to be demonstrated that for all IODM M, assignments of term-operands g, and statements S of L holds:

(1) $\models_{1\,M,\,g} S \Leftrightarrow \models_{2\,M,\,g} S$

The latter holds since one can establish the following for all IODM M, assignments of term-operands g, and statements S of L:

(2) $ext_{M,\,g}(S)$ obtains in M under $g \Leftrightarrow \models_{1\,M,\,g} S$

For it follows from (2) that

(3) $\models_{1\,M,\,g} S \Rightarrow ext_{M,\,g}(S)$ obtains in M under g

Since further the following holds:

(4) $ext_{M,\,g}(S)$ obtains in M under $g \Rightarrow \models_{2\,M,\,g} S$

it follows from (3) and (4):

(5) $\models_{1\,M,\,g} S \Rightarrow \models_{2\,M,\,g} S$

Moreover, it follows from (2):

(6) $ext_{M,\,g}(S)$ obtains in M under $g \Rightarrow \models_{1\,M,\,g} S$

As further the following holds:

(7) $\models_{2\,M,\,g} S \Rightarrow ext_{M,\,g}(S)$ obtains in M under g

it follows from (7) and (6):

(8) $\models_{2\,M,\,g} S \Rightarrow \models_{1\,M,\,g} S$

Therefore, (1) holds if (2) holds (for all M, g, S).

In the following the assertion (2) will be proven by induction on the construction of a formula:

(a) Predications

(9) $ext_{M,g}(G^n_{\Delta y}\, a_1 \ldots a_n)$ obtains in M under $g \Leftrightarrow$
$(\langle ext_{M,g}(a_1), \ldots, ext_{M,g}(a_n)\rangle =)\ \langle f(a_1), \ldots, f(a_n)\rangle \in g(O_{y_1 \ldots y_n})$
$(= ext_{M,g}(G^n_{\Delta y})) \Leftrightarrow$
$\models_{1\, M,g} G^n_{\Delta y}\, a_1 \ldots a_n$

(b) Negation statements

The induction-hypothesis here is:

IH $ext_{M,g}(A)$ obtains in M under $g \Leftrightarrow \models_{1\, M,g} A$

(10) $ext_{M,g}(\neg A)$ obtains in M under $g \Leftrightarrow$
$ext_{M,g}(A)$ does not obtain in M under $g \Leftrightarrow_{\text{IH}}$
not $\models_{1\, M,g} A \Leftrightarrow$
$\models_{1\, M,g} \neg A$

(c) Conjunction statements

The induction-hypothesis here is:

IH $ext_{M,g}(A)$ and $ext_{M,g}(B)$ obtain in M under $g \Leftrightarrow$
$\models_{1\, M,g} A$ and $\models_{1\, M,g} B$

(11) $ext_{M,g}(A \wedge B)$ obtains in M under $g \Leftrightarrow$
$ext_{M,g}(A)$ and $ext_{M,g}(B)$ obtain in M under $g \Leftrightarrow_{\text{IH}}$
$\models_{1\, M,g} A$ and $\models_{1\, M,g} B \Leftrightarrow$
$\models_{1\, M,g} A \wedge B$

(d) All-quantified statements I

If $\forall x_i A[G_{\bar{m}}]$ has the form

(12) $\forall x_1(A^{-}[G_{\bar{m}}](a_1/x_1)(x_1/a_1))$,

then the following holds because of (9)–(11): for all singular terms a_1,

(13) $ext_{M,g}(A^{-}[G_{\bar{m}}](a_1/x_1))$ obtains in M under $g \Leftrightarrow$
$\models_{1\, M,g} A^{-}[G_{\bar{m}}](a_1/x_1)$

Then the following holds:

(14) $ext_{M,\,g}\,(\forall x_i\, A[G_{\bar{m}}])$ obtains in M under $g \Leftrightarrow$
$\langle R^m_j,\, ext_{M,\,g}\,(G_1), \ldots,\, ext_{M,\,g}\,(G_m)\rangle$ obtains in M under $g \Leftrightarrow$
$ext_{M,\,g}\,(G_1), \ldots,\, ext_{M,\,g}\,(G_m)$ have the R^m_j-attribute
in M under $g \Leftrightarrow$
for all singular terms a_1 ($ext_{M,\,g}\,(a_1) \in D_I \Rightarrow$
$ext_{M,\,g}\,(A^{-}[G_{\bar{m}}](a_1/x_1))$ obtains in M under g) $\Leftrightarrow_{(13)}$
for all singular terms a_1 ($ext_{M,\,g}\,(a_1) \in D_I \Rightarrow$
$\models_{1\,M,\,g} A^{-}[G_{\bar{m}}](a_1/x_1)) \Leftrightarrow$
$\models_{1\,M,\,g} \forall x_i\, A[G_{\bar{m}}]$

(e) All-quantified statements II
If $\forall x_i\, A[G_{\bar{m}}]$ has the form

(15) $\forall x_i\,(\neg)\forall x_{i-1}(\neg)\ldots(\neg)\forall x_1$
$(A^{-}[G_{\bar{m}}](a_1/x_1, \ldots,\, a_i\,/x_i)(x_1/a_1, \ldots,\, x_i\,/a_i))$,

then the induction-hypothesis here is: for all singular terms a_i,

IH $ext_{M,\,g}\,(\forall x_{i-1}(\neg)\ldots(\neg)\forall x_1$
$(A^{-}[G_{\bar{m}}](a_1/x_1, \ldots,\, a_i\,/x_i)(x_1/a_1, \ldots,\, x_{i-1}/a_{i-1})))$
obtains in M under $g \Leftrightarrow$
$\models_{1\,M,\,g} \forall x_{i-1}(\neg)\ldots(\neg)\forall x_1$
$(A^{-}[G_{\bar{m}}](a_1/x_1, \ldots,\, a_i\,/x_i)(x_1/a_1, \ldots,\, x_{i-1}/a_{i-1}))$

Then the following holds:

(16) $ext_{M,\,g}\,(\forall x_i\, A[G_{\bar{m}}])$ obtains in M under $g \Leftrightarrow$
$\langle R^m_k,\, ext_{M,\,g}\,(G_1), \ldots,\, ext_{M,\,g}\,(G_m)\rangle$ obtains in M under $g \Leftrightarrow$
$ext_{M,\,g}\,(G_1), \ldots,\, ext_{M,\,g}\,(G_m)$ have the R^m_k-attribute
in M under $g \Leftrightarrow$
for all singular terms a_i ($ext_{M,\,g}\,(a_i) \in D_I \Rightarrow$
(not)
$ext_{M,\,g}\,(\forall x_{i-1}(\neg)\ldots(\neg)\forall x_1$
$(A^{-}[G_{\bar{m}}](a_1/x_1, \ldots,\, a_i\,/x_i)(x_1/a_1, \ldots,\, x_{i-1}/a_{i-1})))$

obtains in M under g) $\Leftrightarrow_{\text{IH}}$
for all singular terms a_i ($ext_{M,g}(a_i) \in D_I \Rightarrow$
(not)
$\models_{1\,M,g} \forall x_{i-1}(\neg)\ldots(\neg)\forall x_1$
$(A^{-}[G_{\bar{m}}](a_1/x_1,\ldots, a_i/x_i)(x_1/a_1,\ldots, x_{i-1}/a_{i-1}))) \Leftrightarrow$
for all singular terms a_i ($ext_{M,g}(a_i) \in D_I \Rightarrow$
$\models_{1\,M,g} (\neg)\forall x_{i-1}(\neg)\ldots(\neg)\forall x_1$
$(A^{-}[G_{\bar{m}}](a_1/x_1,\ldots, a_i/x_i)(x_1/a_1,\ldots, x_{i-1}/a_{i-1}))) \Leftrightarrow$
$\models_{1\,M,g} \forall x_i(\neg)\forall x_{i-1}(\neg)\ldots(\neg)\forall x_1$
$(A^{-}[G_{\bar{m}}](a_1/x_1,\ldots, a_i/x_i)(x_1/a_1,\ldots, x_i/a_i)) \Leftrightarrow$
$\models_{1\,M,g} \forall x_i A[G_{\bar{m}}]$

Thanks to the cases (a)–(e) it holds for all IODM M, assignments of term-operands g, and statements S of L:

(17) $ext_{M,g}(S)$ obtains in M under $g \Leftrightarrow \models_{1\,M,g} S$

Thus, it follows because of (2)–(8) that: $\models_1 S \Leftrightarrow \models_2 S$ (for all S of L). Since it already holds that: $\models_1 S \Leftrightarrow S$ is a theorem of PFL (for all S of L), it follows that: $\models_2 S \Leftrightarrow S$ is a theorem of PFL (for all S of L). Therefore, T1 holds.□[28]

4.7. The slingshot argument

The present state-of-affairs semantics resists the challenges of the *slingshot argument* since in this semantics the principle does not hold that logically equivalent$_2$ statements are always strongly (respectively, weakly) co-extensional. Strong co-extensionality can be defined as follows:

[28] The semantics on hand can easily be adapted to a state-of-affairs semantics with respect to which classical first-order logic is weakly adequate by carrying out the appropriate changes in the model definition and dropping the existence premises from this proof. After these changes a result analogously to the theorem of extensionality of §4.8 can also be achieved for classical first-order logic.

Df.CE An expression A_1 is *strongly co-extensional* in an IODM M under an assignment of term-operands g to an expression A_2
(in short: $A_1 \nabla_{ext_{M,g}} A_2$) :⇔
$(\vee e_1)(ext_{M,g}(A_1) = e_1)$ & $(\vee e_2)(ext_{M,g}(A_2) = e_2)$ &
$(\wedge e_1, e_2)(ext_{M,g}(A_1) = e_1$ & $ext_{M,g}(A_2) = e_2 \Rightarrow e_1 = e_2)$

For weak co-extensionality the condition is dropped that both expressions have an extension. Both relations collapse in the framework of the present semantics since therein the extension function $ext_{M,g}$ is a total one. Hence, these two relations are equivalence relations. Accordingly, it holds with respect to *co-extensionality* $\equiv_{ext_{M,g}}$ for all expressions A_1 and A_2 of L:

T2 $A_1 \equiv_{ext_{M,g}} A_2 \Leftrightarrow ext_{M,g}(A_1) = ext_{M,g}(A_2)$

Moreover, in the state-of-affairs semantics under consideration it can be illustrated by means of many examples that logically equivalent$_2$ statements do not always have the same extension, e.g. by way of the difference between *inner* and *outer negation*:

(i) $ext_{M,g}(\Delta yF^1y)$ is the set of all $d \in D$ that lie in $f(F^1)$; thus, $ext_{M,g}((\Delta yF^1y)a)$ is the state of affairs such that the individual $f(a)$ lies in $f(F^1)$.

(ii) $ext_{M,g}(\Delta y\neg F^1y)$ is the set of all $d \in D$ that lie in the *difference* set $D \setminus f(F^1)$; thus, $ext_{M,g}((\Delta y\neg F^1y)a)$ is the state of affairs such that the individual $f(a)$ lies in the difference set $D \setminus f(F^1)$. It is for this reason that the statements $(\Delta yF^1y)a$ and $(\Delta y\neg F^1y)a$ are *affirmative* ones.

(iii) The outer negation $\neg(\Delta yF^1y)a$ has the state of affairs as extension such that the individual $f(a)$ does *not* lie in $f(F^1)$.

(iv) By contrast, the outer negation $\neg(\Delta y\neg F^1y)a$ has the state of affairs as extension such that the individual $f(a)$ does *not* lie in the *difference* set $D \setminus f(F^1)$. Accordingly, the statements $\neg(\Delta yF^1y)a$ and $\neg(\Delta y\neg F^1y)a$ are *negative* ones.

Since it holds for all $d \in D$: $d \in f(F^1) \Leftrightarrow d \notin D \setminus f(F^1)$, the states of affairs as extensions of the statements in (i) and (iv) obtain in the same IODM under the same assignments of term-operands. Consequently, they are logically equivalent$_2$, although they are different from each other according to the criterion of identity for *n*-tuples. (Things are analogously for the statements in (ii) and (iii)).

4.8. Extensionality

Quite generally, an extension function is a function *ext* from a set of expressions that are capable of having extensions to a set of entities that are admissible as the extensions of such expressions. An expression capable of having an extension is a linguistic expression that is well formed and independent (the latter in the sense that it can have an extension just standing alone). Further, every entity is admissible as the extension of such an expression that is always preserved when in this expression co-extensional expressions are substituted for each other, e.g. appropriate set-theoretical entities that do not change when co-extensional expressions are substituted. In *neutral extensionality* one abstracts from every specific commitment to the kind of extensions of statements (e.g. from the assumptions that they are truth-values or states of affairs etc.) and, nevertheless, retains *salva extensione* substitutivity as a constituting feature of extensionality. According to my analysis of the notions of neutral as well as non-neutral extensionality that I can only briefly sketch here, the following theorem holds:[29]

T3 A statement S_1 is *SE-extensional* $\Leftrightarrow$
for all IODM *M*, assignments of term-operands *g*, statements S_2, and expressions A_1 and A_2
$(Res(S_2, S_1(A_1//A_2))\ \&\ A_1 \equiv_{ext_{M,g}} A_2 \Rightarrow S_1 \equiv_{ext_{M,g}} S_2)$[30]

[29] See Leeb 2004, p.104–108 for a fuller account.

[30] That is a statement S_1 is SE-extensional iff in S_1 expressions that are co-extensional in *M* under *g* can always be substituted for each other without chang-

Here the set-theoretical entities that function as placeholders for the extensions of statements are not yet interpreted philosophically at all; it has not been said at all whether they are supposed to be abstract states of affairs or anything else. As a further result of this analysis, the following theorem also holds:

T4 A statement S_1 is *non-SV-extensional* $\Leftrightarrow$
S_1 is non-SE-extensional or
truth-values are not the extensions of statements.

Thus, non-SV-extensionality can have different sources: (i) co-extensional expressions cannot always be substituted for each other without changing the extension, or (ii) the set-theoretical placeholders for the extensions of statements are not understood as truth-values.

Moreover, a state-of-affairs-related concept of extensionality can be defined as follows:[31]

Df.SS A statement S_1 is *SS-extensional* $:\Leftrightarrow$
S_1 is SE-extensional &
states of affairs (according to S) are the extensions of statements.

In contrast to T3, here it is fixed how the set-theoretical placeholders for the extensions of statements have to be understood, namely, as states of affairs (according to S).

I shall now demonstrate that all statements of the language *L* are SS-extensional and thus SE-extensional. The propositional-logical cases are fairly simple. However, for the predicate-logical cases some of the above-mentioned criteria of identity for logical attributes are useful, as will become obviously later on. For this purpose

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ing the extension (for all IODM *M* and assignments of term-operands *g*). The expression '$Res(S_2, S_1(A_1//A_2))$' expresses that S_2 is a result of substituting one or more occurrences of A_1 in S_1 by A_2.

[31] See Leeb 2004, p.57 for a fuller account.

I shall select ICA_1, although one could take ICA_3. Then the following *theorem of extensionality* can be proven:

T5 In the state-of-affairs semantics under consideration all statements of *L* are SS-extensional and thus SE-extensional, even though their SV-extensionality fails.

Proof

Because of the assumption that states of affairs (according to S) are extensions of statements and due to theorem T4, all statements of *L* are non-SV-extensional.

In the following it is shown by induction on the construction of a formula that in every statement S_1 of *L* expressions that are co-extensional in *M* under *g* can always be substituted for each other without changing the state of affairs as extension (i.e. that every such statement is SS-extensional in *M* under *g*). As the idea of proving this is similar in the several cases, I summarize them into two groups.

(a)–(c) Predications, negation and conjunction statements

The induction-hypotheses for negation and conjunction statements are:

IH1 In the negated sub-statement of a negation statement expressions that are co-extensional in *M* under *g* can always be substituted for each other without changing the state of affairs as extension.

IH2 In the two sub-statements of a conjunction statement expressions that are co-extensional in *M* under *g* can always be substituted for each other without changing the state of affairs as extension.

Consider further the statements

(1) $G^n_{\Delta y} a_1 \ldots a_n$

(2) $\neg A$

(3) $A_1 \wedge A_2$

and the states of affairs that (1)–(3) have as extensions in an IODM M under g:

(4) $s_1 = \langle \in, \langle ext_{M,\,g}(a_1),\ldots, ext_{M,\,g}(a_n)\rangle, ext_{M,\,g}(G^n_{\Delta y})\rangle$

(5) $s_3 = \langle \text{not}, ext_{M,\,g}(A)\rangle$

(6) $s_5 = \langle \text{and}, ext_{M,\,g}(A_1), ext_{M,\,g}(A_2)\rangle$

Replace now some sub-expressions of (1) as well as of A in (2) and of A_1 and A_2 in (3) by expressions that are co-extensional in M under g. Further, I designate the results of these substitutions by the following expressions:

(7) $H^n_{\Delta y} b_1 \ldots b_n$

(8) $\neg B$

(9) $B_1 \wedge B_2$

Consider, moreover, the states of affairs that (7)–(9) have as extensions in M under g:

(10) $s_2 = \langle \in, \langle ext_{M,\,g}(b_1),\ldots, ext_{M,\,g}(b_n)\rangle, ext_{M,\,g}(H^n_{\Delta y})\rangle$

(11) $s_4 = \langle \text{not}, ext_{M,\,g}(B)\rangle$

(12) $s_6 = \langle \text{and}, ext_{M,\,g}(B_1), ext_{M,\,g}(B_2)\rangle$

Since in the present semantics the relation of co-extensionality $\equiv_{ext_{M,\,g}}$ is an equivalence relation and due to T2, IH1, IH2, and the co-extensionality assumptions above, the following holds:

(13) $ext_{M,\,g}(G^n_{\Delta y}) = ext_{M,\,g}(H^n_{\Delta y})$ &
$ext_{M,\,g}(a_1) = ext_{M,\,g}(b_1)$ & … & $ext_{M,\,g}(a_n) = ext_{M,\,g}(b_n)$

(14) $ext_{M,\,g}(A) = ext_{M,\,g}(B)$

(15) $ext_{M,\,g}(A_1) = ext_{M,\,g}(B_1)$ & $ext_{M,\,g}(A_2) = ext_{M,\,g}(B_2)$

Accordingly, the states of affairs s_1, s_2 as well as s_3, s_4 and s_5, s_6 are composed of identical elements. As further (1) and (7) are predications as well as (2) and (8) are negation statements and (3) and (9) conjunction statements, the truth-values of these pairs of statements are determined in the same way (i.e. they have in pairs the same logical form). Therefore, those identical elements in the states of affairs s_1, s_2 as well as s_3, s_4 and s_5, s_6 are composed (according to S) in an identical way. Thus, those ordered *n*-tuples are composed of the same elements that are ordered in the same way. Consequently, they are identical in view of the criterion of identity for such ordered *n*-tuples:

(16) $s_1 = s_2$

(17) $s_3 = s_4$

(18) $s_5 = s_6$

Hence, because of T2, the following holds:

(19) $G^n_{\Delta y}\, a_1 \ldots a_n \equiv_{ext_{M,\,g}} H^n_{\Delta y}\, b_1 \ldots b_n$

(20) $\neg A \equiv_{ext_{M,\,g}} \neg B$

(21) $A_1 \wedge A_2 \equiv_{ext_{M,\,g}} B_1 \wedge B_2$

Therefore, because of the cases (a)–(c) in predications, negation and conjunction statements expressions that are co-extensional in *M* under *g* can always be substituted for each other without changing the state of affairs as extension.

(d)–(e) All-quantified statements I and II

The induction-hypothesis for all-quantified statements II is:

IH In all-quantified statements II with i–1 quantifiers and m general terms expressions that are co-extensional in M under g can always be substituted for each other without changing the state of affairs as extension.

$\forall x_i A[G_{\bar{m}}]$ is either an all-quantified statement I, i.e. has the form:

(22) $\forall x_1(A^{-}[G_{\bar{m}}](a_1/x_1)(x_1/a_1))$,

or an all-quantified statement II, i.e. has the form:

(23) $\forall x_i (\neg)\forall x_{i-1}(\neg)\ldots(\neg)\forall x_1$
$(A^{-}[G_{\bar{m}}](a_1/x_1,\ldots, a_i/x_i)(x_1/a_1,\ldots, x_i/a_i))$

Consider further the states of affairs that (22) and (23), respectively, have as extensions in an IODM M under g:

(24) $s_7 = \langle R^m_j, ext_{M,g}(G_1),\ldots, ext_{M,g}(G_m)\rangle$

(25) $s_9 = \langle R^m_k, ext_{M,g}(G_1),\ldots, ext_{M,g}(G_m)\rangle$

Further, the following holds:

(26) the sets $ext_{M,g}(G_1),\ldots, ext_{M,g}(G_m)$ have the R^m_j-attribute in M under $g \Leftrightarrow$
for all singular terms a_1 ($ext_{M,g}(a_1) \in D_I \Rightarrow$
$ext_{M,g}(A^{-}[G_{\bar{m}}](a_1/x_1))$ obtains in M under g)

and

(27) the sets $ext_{M,g}(G_1),\ldots, ext_{M,g}(G_m)$ have the R^m_k-attribute
in $M = \langle D_I, D_O, f\rangle$ under $g \Leftrightarrow$
for all singular terms a_i ($ext_{M,g}(a_i) \in D_I \Rightarrow$
(not)
$ext_{M,g}(\forall x_{i-1}(\neg)\ldots(\neg)\forall x_1$
$(A^{-}[G_{\bar{m}}](a_1/x_1,\ldots, a_i/x_i)(x_1/a_1,\ldots, x_{i-1}/a_{i-1})))$
obtains in M under g)

Replace now some sub-expressions of (22) and (23), respectively, by expressions that are co-extensional in M under g. Further, I designate the results $\forall x_i B[H_{\bar{m}}]$ of these substitutions by the following expressions, namely, if $\forall x_i B[H_{\bar{m}}]$ is an all-quantified statement I, by:

(28) $\forall x_1(B^{-}[H_{\bar{m}}](a_1/x_1)(x_1/a_1))$,

and, if $\forall x_i B[H_{\bar{m}}]$ is an all-quantified statement II, by:

(29) $\forall x_i (\neg)\forall x_{i-1}(\neg)\ldots(\neg)\forall x_1$
$(B^{-}[H_{\bar{m}}](a_1/x_1,\ldots, a_i/x_i)(x_1/a_1,\ldots, x_i/a_i))$

Consider, moreover, the states of affairs that (28) and (29), respectively, have as extensions in M under g:

(30) $s_8 = \langle R^{m}_{j}{}', ext_{M,g}(H_1),\ldots, ext_{M,g}(H_m)\rangle$

(31) $s_{10} = \langle R^{m}_{k}{}', ext_{M,g}(H_1),\ldots, ext_{M,g}(H_m)\rangle$

Further, the following holds:

(32) the sets $ext_{M,g}(H_1),\ldots, ext_{M,g}(H_m)$ have the $R^{m}_{j}{}'$-attribute in M under $g \Leftrightarrow$
for all singular terms a_1 ($ext_{M,g}(a_1) \in D_I \Rightarrow$
$ext_{M,g}(B^{-}[H_{\bar{m}}](a_1/x_1))$ obtains in M under g)

and

(33) the sets $ext_{M,g}(H_1),\ldots, ext_{M,g}(H_m)$ have the $R^{m}_{k}{}'$-attribute in M under $g \Leftrightarrow$
for all singular terms a_i ($ext_{M,g}(a_i) \in D_I \Rightarrow$
(not)
$ext_{M,g}(\forall x_{i-1}(\neg)\ldots(\neg)\forall x_1$
$(B^{-}[H_{\bar{m}}](a_1/x_1,\ldots, a_i/x_i)(x_1/a_1,\ldots, x_{i-1}/a_{i-1})))$
obtains in under g)

Because of T2 it follows from the co-extensionality assumptions above that:

(34) $ext_{M,\,g}(G_1) = ext_{M,\,g}(H_1)$ & … & $ext_{M,\,g}(G_m) = ext_{M,\,g}(H_m)$

Since, if $\forall x_i\, B[H_{\bar{m}}]$ is an all-quantified statement I, the *quantifier-free* statements

(35) $A^{-}[G_{\bar{m}}](a_1/x_1)$

and

(36) $B^{-}[H_{\bar{m}}](a_1/x_1)$

differ from each other only with respect to co-extensional expressions, the following holds because of the cases (a)–(c): for all singular terms a_1,

(37) $ext_{M,\,g}(A^{-}[G_{\bar{m}}](a_1/x_1)) = ext_{M,\,g}(B^{-}[H_{\bar{m}}](a_1/x_1))$

Moreover, if $\forall x_i\, B[H_{\bar{m}}]$ is an all-quantified statement II, it follows because of T2 and IH from the co-extensionality assumptions above that: for all singular terms a_i,

(38) $ext_{M,\,g}(\forall x_{i-1}(\neg)\ldots(\neg)\forall x_1$
$(A^{-}[G_{\bar{m}}](a_1/x_1,\ldots,\, a_i/x_i)(x_1/a_1,\ldots,\, x_{i-1}/a_{i-1}))) =$
$ext_{M,\,g}(\forall x_{i-1}(\neg)\ldots(\neg)\forall x_1$
$(B^{-}[H_{\bar{m}}](a_1/x_1,\ldots,\, a_i/x_i)(x_1/a_1,\ldots,\, x_{i-1}/a_{i-1})))$

That is in both cases the states of affairs in (37) and (38), respectively, turn out to be identical. Hence, it follows from (26), (32) and (37) because of the criterion of identity ICA_1 that:

(39) $R^{m}_{j} = R^{m}_{j}{'}$

Moreover, it follows from (27), (33) and (38) because of ICA_1 that:

(40) $R^{m}_{k} = R^{m}_{k}{'}$

Because of (39) and (34) – as well as (40) and (34) – the states of affairs s_7 and s_8 – as well as s_9 and s_{10} – are composed of identical elements. As (22) and (28) are all-quantified statements I with one quantifier – as well as (23) and (29) are all-quantified statements II with *i* quantifiers – (22) and (28) – as well as (23) and (29) – have the same logical form. Thus, it follows because of the criterion of identity for ordered *n*-tuples that:

(41) $s_7 = s_8$

(42) $s_9 = s_{10}$

Therefore, it follows because of T2 that:

(43) $\forall x_i A[G_{\bar{m}}] \equiv_{ext_{M,g}} \forall x_i B[H_{\bar{m}}]$

Because of the cases (a)–(e) in all statements S_1 of L, expressions that are co-extensional in M under g can always be substituted for each other without changing the state of affairs as extension (for all IODM M and assignments of term-operands g). Therefore, all statements S_1 of L are SS-extensional and thus due to Df.SS SE-extensional, too.□

Finally, I would like to point out some relevant *consequences* of the theorems T3–T5. One is that there are several sources not only for non-SV-extensionality, as already mentioned above, but also for SE-extensionality: a statement can be SE-extensional thanks to its SS-extensionality or its SV-extensionality. Further, its non-SV-extensionality need not exclude its SE-extensionality as long as one can prove it to be SS-extensional. Hence, positive free logic might be in view of T4 and Lambert’s non-extensionality argument non-SV-extensional, but, what I wanted to demonstrate in this paper, it is nevertheless SE-extensional. Though, of course, it remains the task to demonstrate the latter also within the framework of a single domain based state-of-affairs semantics for free logic that I shall propose in a future paper.

Acknowledgments

I am indebted to *FWF*, the Austrian research fund, for funding a project on the philosophical foundations of free logic in the course of which this article was written and to Christopher Lay, UCI, for proof-reading the English passages of an earlier version.

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