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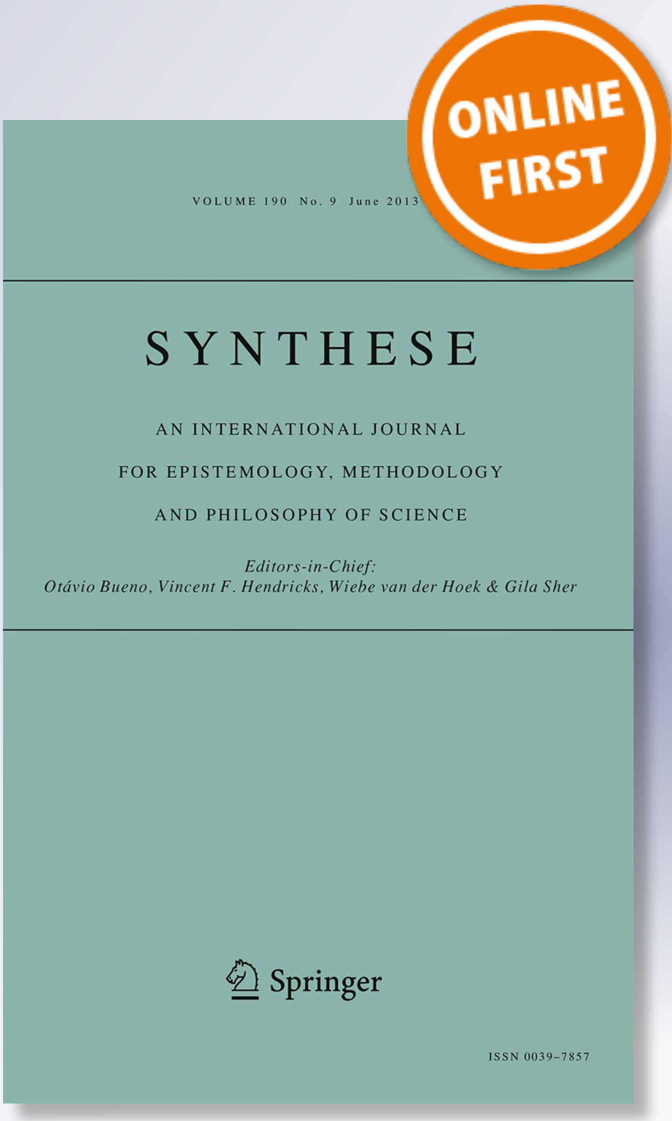

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Why pure mathematical truths are metaphysically necessary: a set-theoretic explanation

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Abstract
Pure mathematical truths are commonly thought to be metaphysically necessary. Assuming the truth of pure mathematics as currently pursued, and presupposing that set theory serves as a foundation of pure mathematics, this article aims to provide a metaphysical explanation of *why* pure mathematics is metaphysically necessary.

Keywords Pure mathematics · Metaphysical necessity · Explanation · Set theory

1 Introduction

Pure mathematical truths, such as '0 < 1', Fermat's Last Theorem, the Four-Colour-Theorem, the Fundamental Theorem of the Calculus, the Fundamental Theorem of Algebra, or the Well-Ordering Theorem are commonly thought to be metaphysically necessary. It is less common to explain why that is so. If traditional logicism had been right, pure mathematical truths would have turned out logically true and *therefore* necessary; unfortunately, it did not work out. In what follows, I will develop a different explanation for the metaphysical necessity of pure mathematics.

When doing so, I will make two basic presuppositions that will show up in Sect. 2 as the initial two premises. First, I will take for granted that our current standard mathematical theories—arithmetic, graph theory, the calculus, algebra, ZFC set theory, and the like—are *true*. (If one did not commit to the truth of pure mathematics, the project of trying to explain its metaphysical *necessity* would be a non-starter.) For instance, according to ZFC, $\{\emptyset\}$ exists, that is, there is a set that includes the empty set as its only member, which is why I will presuppose '$\{\emptyset\}$ exists' to be true, and the same holds for all other theorems of standard pure mathematics as accepted at this point. My argument will not extend to statements of *applied*—as opposed to *pure*—mathematics

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in science, engineering, and the like (at least not without further work). Furthermore, I will not deal at all with the much more ambitious project of explaining or justifying the *truth* of pure mathematics as accepted at this point. Instead, I will merely argue in Sect. 2 that *assuming* a pure mathematical sentence A is true—in the current sense of 'mathematical', that is, arithmetical or graph-theoretic or…—*then* A is metaphysically necessary, and I will simply presuppose that all theorems derivable in our standard mathematical theories belong to these truths. Section 3 will complement this result by showing that the argument from Sect. 2 does not just derive but even *explain* the metaphysical necessity of these sentences.

Secondly, I will presuppose that first-order ZFC set theory functions as a *foundation* of ordinary pure mathematics as presently pursued: every mathematical statement A in our current mathematical languages can be reformulated as a statement B in the first-order language of ZFC, such that A is necessarily equivalent to B, and if A is true, the truth of B explains the truth of A. While clearly controversial philosophically, at least some form of set-theoretic foundationalism is widely presupposed in present-day mathematics where mathematical statements about natural numbers, graphs, real numbers, complex numbers, and ordinal numbers are usually regarded as translatable into set-theoretic statements about certain kinds of pure sets, where mathematical proofs are usually considered as convertible into set-theoretic proofs, and where set-theoretic definitions and proofs are meant to explain mathematical concepts and truths on the foundational level (see Mayberry 2000, p. 3).[1] My argument will only concern non-controversial statements in the standard areas of pure mathematics, for which set-theoretic foundations are broadly accepted.

Along the way, I will also make clear that at least parts of my argumentative strategy could be recycled in the service of alternative explanations for the metaphysical necessity of pure mathematics that would not rely on set-theoretic foundationalism. But my main goal will be to argue that, whatever one may think about set-theoretic foundationalism in general, it *does* possess the attractive feature of contributing to the explanation of the metaphysical necessity of pure mathematics as we know it. If, and when, an opponent of set-theoretic foundationalism aims to explain the metaphysical necessity of pure mathematics in some different manner, the set-theoretic metaphysical explanation to be developed below should serve, at the very least, as non-negligible competitor and foil.

2 The argument

As promised in the introductory section, the first premise of my (as we shall see) explanatory argument expresses the commitment to the truth of pure mathematics as accepted right now:

P1 All theorems of our present standard purely mathematical theories are true.

[1] Friedman (2000) positions the foundations of mathematics "in between mathematics and philosophy". In our second premise, this combination of disciplines will show up as a combination of claims concerning set-theoretic expressibility (as confirmed by mathematical practice) and metaphysical necessity and explanation (as studied by philosophers).

The second premise pins down set-theoretic foundationalism in the form explained before:

P2 Every statement A in the language of pure mathematics, as presently practiced, can be reformulated as a statement B in the first-order language of pure set-theory, such that their material equivalence is metaphysically necessary (in short, $L(A \leftrightarrow B)$), and if A is true, then the truth of B explains the truth of A. ('L' is the sentential operator for metaphysical necessity.)

I will not go into the details of how exactly the required set-theoretic foundations are provided. Following typical textbooks in set theory, one option would be: by replacing definiendum by definiens in a set-theoretic definition. For instance, by applying the set-theoretic definition of natural numbers as Von Neumann ordinals, the arithmetical statement '$0 < 1$' becomes the set-theoretic statement '$\emptyset \in \{\emptyset\}$', and P2 would maintain: $L(0 < 1 \leftrightarrow \emptyset \in \{\emptyset\})$ and, since '$0 < 1$' is true, the truth of '$\emptyset \in \{\emptyset\}$' explains the truth of '$0 < 1$'. Reformulating, in similar set-theoretic terms, Fermat's Last Theorem, the Four-Colour-Theorem (with graphs defined as set-theoretic pairs of a set of vertices and a set of edges), the Fundamental Theorem of the Calculus (with real numbers defined as Dedekind cuts, that is, as special sets), the Fundamental Theorem of Algebra (with complex numbers defined as set-theoretic pairs of real numbers), and so on, would only be more cumbersome but not different in principle.

Alternatively, a set-theoretic foundationalist might be a set-theoretic structuralist moved by Benacerraf's (1965) objections to identifying the natural numbers with, e.g., the Von Neumann ordinals (rather than, say, Zermelo ordinals): if so, she might prefer set-theoretic reformulations by which, e.g., an arithmetical statement such as '$0<1$' is translated into a set-theoretic statement that quantifies universally over all set-theoretic systems satisfying the Dedekind–Peano axioms, and which claims the following about all these systems: the initial "zero" element in the system stands in the "less-than" relation of the system to the "successor" in the system of the "zero" of the system. (I use scare quotes, since, e.g., the "zero" in such a system could really be any set whatsoever.) Similarly, statements in the language of the calculus might be taken to translate into set-theoretic statements that quantify universally over all set-theoretic systems satisfying Dedekind's axioms for real numbers, statements of Euclidean geometry might be taken to translate into set-theoretic statements quantifying universally over all set-theoretic systems satisfying Hilbert's axioms for Euclidean geometry, and so forth. See Shapiro (1997) and what is called "universalist structuralism" in Reck and Price (2000) for more details.

The fact that many present-day mathematicians adhere (if often tacitly) to a version of P2 conveys a non-trivial insight into pure mathematics as developed up to this point of time: it took mathematicians and philosophers a good part of the nineteenth and twentieth century to determine the set-theoretic foundations of pure mathematics, and initially there was no guarantee at all that the project would succeed—but it did: standard pure mathematics as we know it *can* be reformulated within ZFC set theory. Of course, the language of pure mathematics may well progress in a way that will eventually invalidate P2. But for the time being, if one is inclined towards set-theoretic foundationalism at all, P2 should be safe at least so far as the generally acknowledged parts of *present-day* pure mathematics are concerned.

The third premise concerns the *modal* properties of set-theoretic membership:

P3 Set membership is rigid: $\forall x \forall y(x \in y \rightarrow L(x \in y))$, and $\forall x \forall y(x \notin y \rightarrow L(x \notin y))$. (Here, the universal quantifiers are meant to range over all and only pure sets, that is, members of the cumulative hierarchy of sets that is based solely on the empty set Ø.)

On the one hand, P3 is grounded in the fact that sets are extensional combinatorial collections the identity conditions of which are given by reference to their members and to their members only: for by the Axiom of Extensionality, two sets are identical just in case they have precisely the same members. It is fair to say that if anything captures the nature of sets, this is: a set is what it is (rather than something else) in virtue of its members—because of the members it has, and because it does not have any other members. For that reason, it is highly plausible that if $x \in y$, then this is essential to y (for otherwise it would something else), and if $x \notin y$, this is essential to y, too (or it would be something else again).[2] Finally, essentiality implies metaphysical necessity (cf. Fine 1994a, b).

Since 'x' and 'y' range just over pure sets, there should not be any worry either that '$L(x \in y)$' might still fail to be implied by '$x \in y$' in view of x or y failing to exist at some possible world: for instance, by P1, Ø exists at the actual world, and if a pure set such as Ø exists at the actual world, it is hard to see how it could fail to exist at some other world. Things would be different for *impure* sets of sets of… *empirical* atoms or urelements, such as e.g., the singleton set {Socrates} that includes Socrates as its only member, since many metaphysicians would want to argue that Socrates might not have existed, which is why {Socrates} might not have existed either. But what metaphysical law could prohibit the application of the *set-of* operation to *nothing at all* in some possible world when the application of the *set-of* operation to *nothing at all* does yield an entity in the actual world? Similarly, if {Ø} exists at the actual world, which is indeed the case by P1, it should do so at any other one; and so forth, for all other pure sets such as {{Ø}}, {Ø, {Ø}},… resulting from iterated applications of the set-theoretic power set operation and union operation to Ø. The same may be said in support of $x \notin y$ implying $L(x \notin y)$. Hence, P3 should be fine. (Compare, e.g., Fine 1981; Linnebo 2016.)

By P2, statements about natural numbers or graphs or… can be reformulated as set-theoretic claims. By P3, set-theoretic membership is rigid. As we are going to see, given the rest of our premises, all statements of pure mathematics can be shown to inherit their "own" rigidity properties from the rigidity of set-theoretical membership. If, alternatively, one were to think of natural numbers as objects *sui generis*, the metaphysical necessity of arithmetical truths might perhaps still be defended along these lines: by first arguing for the rigidity of all basic properties and relations for natural numbers, such as the rigidity of the successor relation for natural numbers[3] (that is, $\forall x \forall y(S(y, x) \rightarrow L(S(y, x)))$ and $\forall x \forall y({\sim}S(y, x) \rightarrow L({\sim}S(y, x)))$, with quantifiers ranging over the natural numbers), and then applying an argument analogous to the one below just to arithmetical claims. Similarly, if unlabelled graphs were regarded as

[2] Recently, Korbmacher (2016) has defended a general explication of essential properties as properties grounded in identity.

[3] I am grateful to Ed Zalta for suggesting this in personal communication.

objects *sui generis*, their edge-relations might be argued to be rigid, and the metaphysical necessity of graph-theoretic truths could again be demonstrated by an argument analogous to the one below. Perhaps, for *any* domain of current pure mathematics taken as *sui generis*, an analogous argument could be used to demonstrate the metaphysical necessity of its truths—I will leave that open here. The advantage of P2 is to allow us to replace that piecemeal argumentative strategy by just one argument in which pure set theory is used as a foundation for pure mathematics *as a whole*.

Let me now turn to the fourth premise. At least at first glance, one of its conjuncts might seem more controversial than the previous premises:

P4 (i) All theorems of the system K of modal logic formulated for the modal first-order language with the predicates '$\in$' and '=' and with the operator 'L' are true (and the underlying non-modal logic contains the rule of universal instantiation with free variables). (ii) Identity is rigid: $\forall x \forall y(x = y \rightarrow L(x = y))$, and $\forall x \forall y(x \neq y \rightarrow L(x \neq y))$. (iii) All instances of the (universal quantifier version of the) Barcan formula scheme '$\forall x\, L(A) \rightarrow L(\forall x\, A)$' are true. (In all three cases, the respective universal quantifiers range over all and only pure sets.)

The seemingly more controversial component is (iii), whereas (i) and (ii) are pretty much standard and should be unproblematic at least for pure sets x and y. Linsky and Zalta (1994) and Williamson (1998) mounted some influential arguments in favour of the *logical truth* of the Barcan formula, but I will not need to rely on their arguments. In the present context, I only presuppose the Barcan formula to be *true* when '$\forall x$' ranges over *the universe of pure sets*. If formulated in the language of possible worlds again: no "new" members of the cumulative hierarchy should ever "emerge" in the transition from one metaphysically possible world to another, as would have to be the case in order for an instance of '$\forall x\, L(A) \rightarrow L(\forall x\, A)$' to be false. Analogously to what was said in support of P3, if a pure set exists at a metaphysically possible world, it should exist at any other world, including the actual one. Once again, this would not be plausible at all for impure sets, but as far as quantification over pure sets is concerned, every instance of the Barcan formula scheme should be true.[4]

Now we can draw a first intermediate conclusion based just on P3 and P4:

C1 The first-order language of pure set theory is rigid: Every statement of the form '$B \rightarrow L(B)$' with whatever antecedent (B) within the first-order language of pure set theory is derivable. And: Every statement of the form '$\sim B \rightarrow L(\sim B)$' with whatever negative antecedent ($\sim B$) within the language of first-order pure set theory is derivable.

For one can show that all rigidity formulas mentioned in C1 can be derived by classical logic from the principles stated in P3 together with the principles mentioned in P4. The proof proceeds by induction over the syntactic complexity of arbitrary open or closed formulas B in the first-order language of pure set theory. (I will suppress quotation marks or other devices of reference to formulas whenever feasible.)

[4] Linnebo (2013) rejects the Barcan formula in his potentialist conception of the hierarchy of sets, but that is in a context in which the corresponding necessity operator does not express *metaphysical* necessity (see Linnebo 2013, p. 207).

The induction basis is: for all atomic formulas F—which have complexity 0, and which must involve either '∈' or '='—it holds that both $F \rightarrow L(F)$ and $\sim F \rightarrow L(\sim F)$ are derivable. This follows directly using the principles in P3 and (ii) of P4.

The inductive hypothesis states: for all formulas F, G up to a certain complexity, it holds that $F \rightarrow L(F), \sim F \rightarrow L(\sim F)$, $G \rightarrow L(G), \sim G \rightarrow L(\sim G)$ are derivable from the relevant principles. The inductive step extends this to the derivability of the corresponding rigidity claims for all formulas of the next greater degree of complexity:

- For negation formulas, one concludes from $F \rightarrow L(F)$ and $\sim F \rightarrow L(\sim F)$ that $\sim F \rightarrow L(\sim F)$ and $\sim\sim F \rightarrow L(\sim\sim F)$. In the latter case, this is by classical logic, which yields $\sim\sim F \rightarrow F$, furthermore by the inductive hypothesis $F \rightarrow L(F)$, and finally by (i) of P4, which yields $L(F) \rightarrow L(\sim\sim F)$. Combining the three of them leads to the intended conclusion.
- For conjunction formulas, one derives from $F \rightarrow L(F), \sim F \rightarrow L(\sim F)$, $G \rightarrow L(G), \sim G \rightarrow L(\sim G)$ that $F \wedge G \rightarrow L(F \wedge G)$ and $\sim(F \wedge G) \rightarrow L(\sim(F \wedge G))$. Both derivations rely on classical logic and (i) of P4. E.g., the main steps in the second case are: $\sim(F \wedge G) \rightarrow \sim F \vee \sim G, \sim F \vee \sim G \rightarrow L(\sim F) \vee L(\sim G), L(\sim F) \vee L(\sim G) \rightarrow L(\sim F \vee \sim G), L(\sim F \vee \sim G) \rightarrow L(\sim(F \wedge G))$. (For disjunctions one proceeds analogously.)
- For universally quantified formulas, one uses the assumption that $F \rightarrow L(F)$ and $\sim F \rightarrow L(\sim F)$ are derivable from the principles stated in P3 together with the principles mentioned in P4 to show that also $\forall x\, F \rightarrow L(\forall x\, F)$ and $\sim\forall x\, F \rightarrow L(\sim\forall x\, F)$ are derivable from these principles. The proof exploits classical logic and (i) and (iii) of P4: in the first case, one concludes from the derivability of $F \rightarrow L(F)$ the derivability of $\forall x\, (F \rightarrow L(F))$, from this the derivability of $\forall x\, F \rightarrow \forall x\, L(F)$, which one combines with the Barcan formula $\forall x\, L(F) \rightarrow L(\forall x\, F)$. In the second case, one infers from the derivability of $\sim F \rightarrow L(\sim F)$ the derivability of $\forall x\, (\sim F \rightarrow L(\sim F))$, thus the derivability of $\exists x \sim F \rightarrow \exists x\, L(\sim F)$ and therefore of $\sim\forall x\, F \rightarrow L(\exists x \sim F)$. (Other than exchanging the antecedent by a logically equivalent one, the last step uses: $\sim F \rightarrow \exists x \sim F$ by logic, therefore $L(\sim F \rightarrow \exists x \sim F)$ and hence $L(\sim F) \rightarrow L(\exists x \sim F)$ by (i) of P3, and so $\forall x\, (L(\sim F) \rightarrow L(\exists x \sim F))$ and thus $\exists x\, L(\sim F) \rightarrow L(\exists x \sim F)$ by logic again.) Finally, from $\sim\forall x\, F \rightarrow L(\exists x \sim F)$ one can infer $\sim\forall x\, F \rightarrow L(\sim\forall x\, F)$, by (i) of P4. (For existential formulas, the proof is analogous.)

By induction, C1 follows. For instance, one can show in this way: $\emptyset \in \{\emptyset\} \rightarrow L(\emptyset \in \{\emptyset\})$ (and also $\emptyset \notin \{\emptyset\} \rightarrow L\, (\emptyset \notin \{\emptyset\})$). With respect to more complex set-theoretic statements B, the proof of C1 demonstrates how one may first derive rigidity for the atomic parts of B, from which one derives rigidity for larger parts of B, until eventually rigidity applies to all of B, in which case one may conclude $B \rightarrow L(B)$ (and similarly $\sim B \rightarrow L(\sim B)$).

This leads to the ultimate conclusion of my argument, which is based on all premises taken together:

C2 Every true statement within the language of pure mathematics, as presently practiced, is metaphysically necessary. In particular, all theorems of standard theories of pure mathematics, as currently accepted, are metaphysically necessary.

For consider an arbitrary statement A in the language of pure mathematics as used right now. Assume A to be true. By P2, A can be reformulated as a statement B in the first-order language of pure set theory, such that: $L(A \leftrightarrow B)$. (The additional explanation part of P2 will only become important in Sect. 3.) By (i) of P4, this entails $L(B) \rightarrow L(A)$. By C1 we know that $B \rightarrow L(B)$, from which we can conclude: $B \rightarrow L(A)$. Since A is true (by assumption), B is true as well, by P2; thus, we also have: B. Therefore: $L(A)$. Thus, finishing the initial conditional proof: $A \rightarrow L(A)$. In other words: if a statement A in the language of pure mathematics is true, it is metaphysically necessary. Finally, by P1, all theorems A of our present standard purely mathematical theories are true. Hence all such statements A are also metaphysically necessary.

3 The argument from Sect. 2 is explanatory

The argument from the last section lends strong support to C2, which is certainly worth noting. E.g., one can derive from this that for every A in the language of pure mathematics (using $A \vee {\sim} A$): $L(A) \vee L({\sim} A)$.

But does the argument also amount to an *explanation* for the metaphysical necessity of pure mathematics? After all, not every sound argument has explanatory power. Does the argument also manage to convey *why* mathematical truths are necessary? In order to see the answer is affirmative, let us reconsider the essential steps in the argument from before. We started with an arbitrary true statement A in the language of pure mathematics, and we argued it to be metaphysically necessary. Why is A metaphysically necessary? *Because*, first, A has a set-theoretic reformulation B, such that B explains A by set-theoretic foundationalism, and B's necessity implies A's necessity, that is, $L(B) \rightarrow L(A)$ (see P2). And, second, *because* B is indeed metaphysically necessary. (Taking all of this together, $L(B)$ should thus count as explaining $L(A)$). Why is B itself metaphysically necessary? It is so *because* B is true, and the conditional $B \rightarrow L(B)$ (in C1) is true and explanatory. Why is the conditional $B \rightarrow L(B)$ true and explanatory? *Because* the truth of B implies the metaphysical necessity of B by the rigidity of both membership and identity (that is, $F \rightarrow L(F)$ and ${\sim} F \rightarrow L({\sim} F)$ for membership and identity claims F, see P3 and ii of P4), and by the application of certain (see P4) truth-preserving and explanatory inference steps. Why are membership and identity rigid? *Because* of the nature of sets, the nature of entities in general, and the concepts expressed by '$\in$' and '='. Why are the relevant inference steps explanatory? *Because* they are of two especially salient kinds: either they go along with the determination of the truth conditions of more complex statements from those of less complex ones, as in the step from ${\sim} F \vee {\sim} G$ to the more complex $L({\sim} F) \vee L({\sim} G)$, or in the inference from $F \rightarrow L(F)$ via the more complex $\forall x\, (F \rightarrow L(F))$ to $\forall x\, F \rightarrow \forall x\, L(F)$, or when $L({\sim}{\sim} F)$ is inferred from the less complex $L(F)$. (Schnieder's 2011 logic of 'because' is based on precisely that idea, though in a non-modal context: e.g., in Schnieder's logic, assuming F is true, one can indeed derive '${\sim}{\sim} F$ because F'; and the like.) Or the relevant inference steps are explanatorily distinguished instances of substitutions of formulas by logically equivalent ones, whether inside or outside of an L context: e.g., inferring ${\sim} F \vee {\sim} G$ from ${\sim}(F \wedge G)$, or $L({\sim}(F \wedge G))$ from $L({\sim} F \vee {\sim} G)$, or ${\sim}\forall x\, F \rightarrow \ldots$ from $\exists x {\sim} F \rightarrow \ldots$ The logical equivalences in question are not just valid in classical logic but also according to first-degree entailment (Anderson and Belnap 1975), analytic entailment (Angell 2002), and their truth-maker semantics (Fine 2016). For that reason, the corresponding substitutions should not just preserve truth but even explanatory power (see Fine 2012). Summing up: the rigidity of membership (P3) and identity (P4, ii) do not just *entail* the necessity of pure mathematical truths (via intermediate steps afforded by P2 and P4, i, iii), they also *explain* it. By P1, this also applies to the theorems of our present standard theories of pure mathematics.

Therefore, the argument from above is indeed an explanation for why pure mathematics is metaphysically necessary. At least it seems to be as good an explanation as permitted by our current understanding of metaphysical explanation (or lack thereof[5]).

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[5] If there were anything like a generally accepted and fully worked out "logic of (metaphysical) explanation" for languages with quantification, one could—and should—try to formalize the argument of this final section within it. Alas, to the best of my knowledge, no such logical system is available at this point.