GALOIS PARSIMONY

GREGORY LIM

ABSTRACT. In philosophy, Occam's razor is a principle stating that "simpler" explanations ought to be preferred over "less simple" ones. In mathematics, a sentential Galois connection is a Galois connection between a space of sentences and a space of models in a first-order logic. In this paper, we connect Occam's razor to sentential Galois connections.

Historically, Occam's razor has been stated as, "entia non sunt multiplicanda praeter necessitatem," which roughly translates to "entities must not be multiplied beyond necessity."[1] The razor is occasionally described as a principle of parsimony. It has been used to guide human reasoning in many areas, serving as a general "heuristic" or "rule" for selecting between competing explanations. In physics, for instance, one often has to deal with competing hypotheses. Assuming the hypotheses *can* be tested experimentally, Occam's razor could motivate prioritising the "simplest" one. One might ask, however, why simplicity or parsimony should be a reliable guide to truth. We propose that an examination of first-order logic can offer insight.

Let $\mathcal{L}$ be a first-order language of logic, and $\mathbb{S}$ be a space of sentences in $\mathcal{L}$, and $\mathbb{M}$ be a space of models in $\mathcal{L}$.

Definition 1 (Subject of Set of Sentences)**.** Let $\Phi \subseteq \mathbb{S}$.

$$\mathrm{subj}(\Phi) = \{m \in \mathbb{M} : \forall \varphi \in \Phi(m \vDash \varphi)\}$$

Definition 2 (Theory of Set of Models)**.** Let $M \subseteq \mathbb{M}$.

$$\mathrm{th}(M) = \{\varphi \in \mathbb{S} : \forall m \in M(m \vDash \varphi)\}$$

For a set of sentences Φ, the subject of Φ is the set of models satisfying every sentence in Φ. Similarly, for a set of models M, the theory of M is the set of sentences satisfied by every model in M.

Theorem 1 (Sentential Galois Connection)**.** *Let* $\Phi \subseteq \mathbb{S}$*, and* $M \subseteq \mathbb{M}$*.*

$$\Phi \subseteq \mathrm{th}(M) \iff M \subseteq \mathrm{subj}(\Phi)$$

Proof. Let $\Phi \subseteq \mathbb{S}$, and $M \subseteq \mathbb{M}$. In the forward direction, if Φ is a subset of $\mathrm{th}(M)$, then Φ is a set of sentences satisfied by every model in M. By definition, $\mathrm{subj}(\Phi)$ is the set of models satisfying every sentence in Φ. Since every model in M satisfies every sentence in Φ, it follows that M is a subset of $\mathrm{subj}(\Phi)$. Similarly, in the backward direction, if M is a subset of $\mathrm{subj}(\Phi)$, then every model in M satisfies every sentence in Φ. By definition, $\mathrm{th}(M)$ is the set of sentences satisfied by every

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model in M. Since every sentence in Φ is satisfied by every model in M, it follows that Φ is a subset of $\mathrm{th}(M)$. $\square$

To make intuitive sense of Theorem 1, one can consider how subsets of the space of sentences relate to subsets of the space of models. For any set of sentences $\Phi \subseteq \mathbb{S}$, one can consider its subject $\mathrm{subj}(\Phi) \subseteq \mathbb{M}$. Increasing the number of sentences in Φ will either maintain or decrease the number of models in $\mathrm{subj}(\Phi)$, since a conjunction with more sentences will either be satisfied by the same set of models, or a smaller set of models. Intuitively, adding to one's set of requirements on models can only ever keep or shrink the set of models satisfying what one requires. Similarly, for any set of models $M \subseteq \mathbb{M}$, one can consider its theory $\mathrm{th}(M) \subseteq \mathbb{S}$. Increasing the number of models in M will either maintain or decrease the number of sentences in $\mathrm{th}(M)$, since a larger set of models will either satisfy the same set of sentences, or a smaller set of sentences. Intuitively, adding to one's set of models can only ever keep or shrink one's set of requirements on models.

A hypothesis can be defined as a "testable" set of assumptions. In a "physical" setting, what we defined as *subjects* and *theories* could broadly correspond to *hypotheses* and *universes*, respectively. In particular, a set of sentences can be seen as a set of assumptions on universes, and a set of models can be seen as a set of universes. Thus, if H is said to be a hypothesis, then subj(H) is said to be the set of universes satisfying H.

Suppose one has a finite hypothesis H whose subject is a finite set of universes U. One may have the intuition that expanding H can only ever keep or shrink U. This can be seen as an instance of the backward direction of Theorem 1. In this way, a "simplicity" in hypothesis could give rise to a "generality" in explanatory power.

References

[1] Jonathan Schaffer, *What not to multiply without necessity*, Australasian Journal of Philosophy **93** (2015), no. 4, 644–664, DOI 10.1080/00048402.2014.992447. `https://doi.org/10.1080/00048402.2014.992447`.