POSSIBLE WORLDS SEMANTICS AND THE LIAR

Reflections on a Problem Posed by Kaplan

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Introduction

In this paper I discuss a paradox, due to David Kaplan, that in his view threatens the use of possible worlds semantics as a model-theoretic framework for intensional logic.[1] Kaplan's paradox starts out from an intuitively reasonable principle that I refer to as the *Principle of Plenitude*. From this principle he derives a contradiction in what he calls *Naive Possible World Theory*. Kaplan's metatheoretic argument can be restated in the modal object language as an intensional version of the *Liar paradox*.

To dissolve the paradox, Kaplan favors a ramified theory of propositions, along the lines of Russell's ramified theory of types. According to this approach: (i) the propositions are divided into orders 0, 1,...; (ii) a propositional quantifier never ranges over the totality of all propositions but only over the propositions of some given order; and (iii) a proposition that involves quantification over the propositions of a given order will itself be of higher order than the propositions in the domain of quantification. Accordingly, a sentence can never express a proposition that is itself within the range of one of the propositional quantifiers in the sentence. This approach avoids the intensional Liar paradox, but its implementation involves severe complications both in the syntax and in the model theory of intensional logic.[2]

Here I shall attempt an alternative, less drastic, modification of the standard possible worlds methodology than the one favored by Kaplan. The idea is to regard sentences that involve propositional quantifiers, like the sentence

All propositions contemplated by Epimenides are false, $\quad (\Lambda)$

as being, in a sense, indexical: one and the same sentence can express different propositions when used in different possible worlds. The context of use fixes the range of the propositional quantifiers in the sentence and thereby the proposition expressed by the sentence. Using this approach, I try to show that the intensional Liar paradox can be defused and no longer poses a threat to possible worlds semantics. This means that the Principle of Plenitude can consistently be upheld in the modified semantics.

1. BACKGROUND: INTENSIONAL LOGIC AND PARADOX

General intensional logic — the logical study of so-called oblique or intensional constructions — is an area that is thriving on paradoxes and puzzles but is also haunted by them. There are, of course, the puzzles of intensionality that started the whole enterprise and constitute its raison d'être: Frege's puzzle about the information value of true identity statements, Russell's puzzle concerning the author of Waverley, puzzles by Quine and others about quantifying in and *de re* constructions, Kripke's puzzle about belief, and so on. It is probably safe to say that even today there is no consensus among logicians concerning the correct treatment and diagnosis of these puzzles.

The various approaches to intensional logic that were proposed to take care of the initial puzzles soon turned out to create paradoxes and foundational difficulties of their own. These approaches can be divided into direct-discourse and indirect discourse ones. According to the *direct discourse treatments*, oblique constructions in ordinary language (like "John knows *that* A", "It is necessary *that* A") are represented in the regimented languages of logic by direct discourse analogues ("John knows $\boldsymbol{A}$", "$\boldsymbol{A}$ is a necessary truth"). Necessity, for instance, is represented, not by a sentential *operator* that takes a sentence (or a formula) A and yields a new sentence (or formula) $\Box$A, but by a *predicate* Nec which when appended to a *name* $\boldsymbol{A}$ yields a sentence Nec($\boldsymbol{A}$). Direct-discourse approaches come in two varieties, according to whether the name $\boldsymbol{A}$ is taken to designate the expression A itself or the sense (or intension) associated with A. The first, the *syntactic approach*, has been advocated by Quine and the second, the *intensional direct-discourse approach*, was adopted by Church in his logic of sense and denotation.[3]

The direct-discourse approaches view the intensionality of natural language, i.e., its failure of extensionality, as a surface phenomenon: the underlying logic and its semantics is taken to be fully extensional. The indirect-discourse approaches, on the other hand, work with logical languages where oblique constructions are represented by intensional operators. The direct-discourse approaches are, however, threatened by (self-referential) paradoxes. Montague (1963) showed that the syntactic treatment of necessity as a predicate of sentences in the object language leads to inconsistency. Kaplan and Montague (1960) contains a similar result for knowledge ("the Knower paradox"). These results generalize Tarski's theorem on the undefinability of the truth-predicate in a sufficiently rich object language.

Also the intensional direct-discourse approach is threatened by self-referential paradoxes. Church's logic of sense and denotation is not directly concerned with linguistic expressions and their senses and denotations, but rather with the language-independent concept relation Δ between concepts (Fregean senses) and the entities they determine (denotations). The more finely the concepts are individuated, the more closely will the abstract theory of concepts and their objects resemble the more concrete theory of linguistic expressions and their denotations, with Church's concept relation playing a role similar to the one played by the denotation predicate of semantics. Consequently, antinomies analogous to the semantic antinomies may arise for formulations of the logic of sense and denotation along the lines of Church's so-called Alternatives (0) or (1), where logically equivalent expressions are allowed to have distinct senses. Indeed, as Myhill (1958) points out Church's Alternative (0) is threatened by the antinomy described by Russell in *The Principles of Mathematics* (Appendix B, p. 527) — the so-called *Russell-Myhill Paradox*. Myhill argues as follows:

> [...] the system Church designed to embody Alternative 0 is formally inconsistent! For if f and g are two distinct sets of propositions, the proposition that every proposition belongs to f is distinct, in virtue of the "principle of maximum distinction" characteristic of Alternative 0, from the proposition that every proposition belongs to g. There is thus established a one-to-one mapping of sets of propositions into propositions, in violation of Cantor's theorem. There is no difficulty in formalizing this argument as a derivation from Church's axioms.

The remedy proposed by Church (1974) and C. Anthony Anderson (1980) is to split up the concept relation $\Delta(x, y)$ holding between a concept x and an object y when the former determines the latter, into an infinite sequence Δ^m of relations satisfying the condition:

$$\forall x \forall y[\Delta^m(x, y) \rightarrow \Delta^{m+1}(x, y)],$$

thereby creating a *ramified* intensional type theory rather than a simple one. Anderson (1980) presented axioms for such a system and showed them to be consistent by constructing a model. The resulting system, however, is extremely complicated. As indicated, the direct-discourse approach has had to face immense difficulties and the systems that have been created have been so complicated that to most researchers the approach has hardly seemed worth pursuing. In comparison the indirect-discourse approach of modal logic and Montague's intensional logic has been greatly successful. Oblique constructions of natural language are here represented by intensional operators and the formal language is provided with a model-theoretic semantics of the possible worlds variety. Although there are problems with this approach, for example the problem of logical omniscience — and difficulties of an interpretational nature concerning the concept of a possible world — it seems that at least the self-referential paradoxes have been avoided. But is this really so? In this paper I am going to discuss an argument due to David Kaplan according to which possible worlds semantics may, after all, also be threatened by the Liar paradox and it relatives. If Kaplan is right, the indirect-discourse approach and possible world-semantics are faced with problems analogous to those facing the direct-discourse approach.

2. KAPLAN'S PROBLEM

There is, according to Kaplan, a serious foundational problem in what might be called *naive possible world theory* — a problem that is analogous to the paradoxes of naive set theory and that threatens the use of the theory as a basis for the model-theory of intensional languages. Kaplan's argument starts out from the following observation: Consider some thinker and some time t. Then the following principle appears to be logically consistent:

> For any proposition p, it is possible that the thinker entertains the proposition p at time t and that p is the only proposition that he entertains at time t. (P)

It may be that the principle (P) is ruled out for metaphysical reasons: perhaps, there must be, for any thinker, propositions that are not possible contents of thought for that individual. However, Kaplan argues, logic alone should not exclude the situation envisaged in (P). In other word, intensional logic should not rule out the existence of a property Q of propositions satisfying the principle:

$$\forall p \, \Diamond(Qp \wedge \forall q(Qq \rightarrow q = p)), \qquad \text{(P)}$$

where the quantifiers are thought of as ranging over the collection of (absolutely) all propositions.[4] I shall refer to this principle as Kaplan's *principle of plenitude*.

Now, Kaplan's argument shows that the principle of plenitude is incompatible with assumptions commonly made in possible worlds semantics. Here is how the argument goes:

(i) There is a set W of possible worlds and a set $Prop$ of propositions.

(ii) There is, for every subset X of W, a corresponding proposition $[X]$ which is true with respect to any possible world w just in case w belongs to X.

(iii) Different sets of possible worlds correspond to different propositions. So $card(Prop) \geqslant card(\wp(W)) > card(W)$.

(iv) By the principle of plenitude, there is a property Q of propositions such that: for each proposition p, it is possible that p should have had the property Q and that p should have been the only proposition having that property.

(v) Hence, for each proposition p, there is a possible world w_p in which it is true that p has the property Q and that p is the only proposition having the property Q.

(vi) Clearly, if p and q are distinct propositions, then the possible worlds w_p and w_q must also be distinct. That is, the function mapping p to w_p is one-to-one.

(vii) From (vi) follows that $card(W) \geqslant card(Prop)$, contrary to (iii).

So this is Kaplan's paradox: According to possible worlds semantics, there are at least as many propositions as there are sets of possible worlds. So, by Cantor's theorem, there are more propositions than there are worlds. On the other hand, an adequate intensional logic should be compatible with the principle of plenitude. But according to this principle, there are at least as many possible worlds as there are propositions. Hence, possible worlds semantics cannot meet the demands of intensional logic.

It should perhaps be pointed out that Kaplan's problem has nothing to do with the assumption, standardly made within possible worlds semantics, that propositions with the same intension, i.e., that are true in exactly the same worlds, are identical. This so-called *principle of intensionality* was not assumed above and does not play any role in Kaplan's argument. In other words, the argument is relevant also for conceptions of propositions that take them to be more fine-grained than Carnapian intensions (i.e., sets of possible worlds).

3. THE PRINCIPLE OF PLENITUDE AND THE LIAR

One obvious reaction to Kaplan's problem is to question the assumption that for every set of possible worlds there exists a proposition having that set as its intension. In his *On the Plurality of Worlds* (1966, section 2, pp. 104–108), David Lewis suggests this as the solution to the paradox: not all propositions are possible contents of thought for a given agent, therefore the class of propositions to which the principle of plenitude is applicable cannot correspond to the entire power-set of the set of worlds.

This does not seem to be a fully adequate reaction to Kaplan's problem. First of all, as Kaplan stresses, even if, for deep metaphysical reasons, the set of all possible contents of thought cannot exhaust the power-set of W, it does not seem reasonable to exclude the possibility that it does when trying to devise a model theory for intensional languages. But there is an even stronger reason why the above reaction is inadequate: even if we give up the assumption that there are propositions corresponding to all sets of worlds, paradox still threatens. As Kaplan (1994) himself points out, the principle (P) gives rise to an intensional version of the *Liar Paradox*. Consider, namely, the sentence:

$$\text{No proposition having the property } Q \text{ is true.} \qquad (\Lambda)$$

We make the intuitively reasonable assumption that there is a proposition, let us call it λ, that the sentence Λ expresses. Then,

(i) It is necessary that: λ is true iff no proposition having the property Q is true.

However, according to the principle of plenitude:

(ii) It is possible that: λ has the property Q and λ is the only proposition having the property Q.

From (i) and (ii) we get:

(iii) It is possible that: λ is the one and only proposition having property Q and (λ is true iff no proposition having the property Q is true).

Finally, (iii) yields:

(iv) It is possible that: λ is true iff λ is not true,

which is an obvious contradiction.

In order to formalize the above argument, we introduce a suitable formal language L. The *symbols* of L are: (i) a denumerable sequence of propositional variables $p_1, p_2, \ldots$; (ii) the Boolean connectives $\bot$ and $\rightarrow$; (iii) the identity connective $=$; (iv) the universal quantifier symbol $\forall$; (v) the logical propositional operator $\Box$ (the necessity symbol); (vi) a non-logical propositional operator Q; (vii) parentheses: "(" and ")". We use the letters "p", "q" and "r" to range over propositional variables in L. The symbols "$\neg$", "$\wedge$", "$\vee$", "$\leftrightarrow$", "$\exists$", "$\Diamond$", are introduced as abbreviations in the usual way.

The set of *formulas* of L is the smallest set Φ that is closed under the conditions:

(i) all propositional variables of L belong to Φ;

(ii) $\bot \in \Phi$;

(iii) if $A, B \in \Phi$, then $(A \rightarrow B)$ and $(A = B) \in \Phi$;

(iv) if $A \in \Phi$, then $\Box A \in \Phi$ and $QA \in \Phi$;

(v) for any propositional variable p and $A \in \Phi$, $\forall p\ A \in \Phi$.

The notions of a variable occurring *free* in a formula is defined in the usual way. A *sentence* of L is a formula in which no variables occur free.

We now turn to the task of formalizing the above argument as a natural deduction proof in L. We symbolize the Liar sentence as $\forall p(Qp \rightarrow \neg p)$, or in primitive notation, $\forall p(Qp \rightarrow (p \rightarrow \bot))$. Below we use "$\Lambda$" as an abbreviation for this sentence.

We start the argument by assuming that (1) there is a proposition expressed by Λ; and (2) the Principle of Plenitude holds:

{1}	(1)	$\exists r(r = \Lambda)$	Premise
{2}	(2)	$\forall p\, \Diamond \exists q(q = p \wedge Qq \wedge \forall r(Qr \rightarrow r = q))$	Premise

Next, we go through the Liar argument itself:

{3}	(3)	$\exists q(q = \Lambda \wedge Qq \wedge$ $\wedge \forall r(Qr \rightarrow r = q))$	Premise
{3}	(4)	$Q\Lambda$	3 by predicate logic
{3}	(5)	$\forall p(Qp \rightarrow p = \Lambda)$	3 by predicate logic
{6}	(6)	Λ (i.e., $\forall p(Qp \rightarrow (p \rightarrow \bot))$	Premise
{3, 6}	(7)	$\Lambda \rightarrow \bot$	4, 6 by predicate logic
{3, 6}	(8)	$\bot$	6, 7 modus ponens ($\rightarrow E$)

{3}	(9)	$\Lambda \rightarrow\perp$	6–8 by $\rightarrow I$
{3}	(10)	$Q\Lambda \rightarrow (\Lambda \rightarrow\perp)$	9 using $\rightarrow I$
{3}	(11)	Λ	5,10 by predicate logic
{3}	(12)	$\perp$	9, 11 modus ponens
{}	(13)	$\exists q(q = \Lambda \wedge Qq\wedge$ $\wedge\forall r(Qr \rightarrow r = q)) \rightarrow\perp$	3, 12 $\rightarrow I$

That is, from the assumption:

> The one and only proposition having the property Q is the proposition that all propositions having the property Q are false;

follows logically the necessarily false proposition $\perp$.

So far we have only used nonmodal predicate logic (as a matter of fact, we have only used the very weak system of *minimal logic* (cf. Prawitz, 1965). Assuming that the modal logic for $\Box$ is normal, we have the inference rule:

> If $A \rightarrow B$ is provable, then $\Diamond A \rightarrow \Diamond B$ is also provable.

By means of this rule, we infer from (13):

{} (14) $\Diamond\exists q(q = \Lambda \wedge Qq \wedge \forall r(Qr \rightarrow r = q)) \rightarrow \Diamond \perp$.

But, (1) and (2) yield:

{1, 2} (15) $\Diamond\exists q(q = \Lambda \wedge Qq \wedge \forall r(Qr \rightarrow r = q))$.

So, finally we have:

{1, 2} (16) $\Diamond \perp$,

contrary to the assumption that the modal logic is normal.

In light of this argument it seems that we have to give up either the principle of plenitude or the intuitively reasonable assumption that the Liar sentence Λ expresses an (actually existing) proposition. The argument does not explicitly involve possible worlds or possible worlds semantics. Instead it uses simple and seemingly natural principles of modal logic. At first sight, the argument casts considerable doubt on the principle of plenitude. As a matter of fact, it can easily be formalized within Montague's intensional logic IL or Gallin's higher-order modal logic (cf. Montague 1970, 1974 and Gallin, 1975). Within Montague's intensional logic, it is provable that every sentence A expresses a proposition. Hence, we can formally prove within IL that the principle of plenitude is false.

But, as the saying goes, "one person's modus ponens is another person's modus tollens". So, this result can as well be taken as an indication that systems like Montague's IL are unreasonably strong. As a matter of fact, we think that this is the right moral to draw from Kaplan's argument. The argument is not so much an objection to possible worlds semantics, as a warning against an uncritical application of the possible worlds methodology that might lead logicians to construct systems of intensional logic in which intuitively consistent principles, like the principle of plenitude, are provably inconsistent.

4. THE INTENSIONAL LIAR AS A DIAGONAL ARGUMENT

Suppose that we distinguish between the set $Prop$ of all propositions and the subset U of $Prop$ over which the propositional quantifiers of L range. We speak of U as the domain of quantification. The Liar proposition for U is the proposition λ_U expressed by the sentence:

$$\forall p(Qp \rightarrow \neg p), \qquad (\Lambda)$$

when we take the universal quantifier in Λ to range over U. We can now view the Liar argument as a proof of the following:

Theorem. *If the domain of quantification U satisfies the principle of plenitude, then the Liar proposition for U is not a member of U.*

Proof. Suppose that U satisfies plenitude, i.e.,

1 for every $p \in U$, there exists a $w \in W$ such that Qp is true in w and for every $q \in U$, if Qq is true in w, then $q = p$.

For any world $w \in W$, we have that:

2 λ_U is true in w iff for every $p \in U$, if Qp is true in w, then p is not true in w.

Suppose now that $\lambda_U \in U$. Then, by (1),

3 there exists $w \in W$ such that $Q\lambda_U$ is true in w and for every $q \in U$, if Qq is true in w, then $q = \lambda_U$.

However, by the usual Liar argument, we get for the world w that exists according to (3) that:

4 λ_U is true in w iff λ_U is not true in w. Hence, by reductio, $\lambda_U \notin U$.

□

Corollary. *If the domain of quantification U satisfies the principle of plenitude, then it cannot be the whole set Prop.*

5. A RESOLUTION OF KAPLAN'S PARADOX SUGGESTED BY THE DIAGONAL ARGUMENT

Viewing the Liar paradox as a diagonal argument suggests a way out of Kaplan's problem. In the diagonal argument above, we interpreted the Liar sentence Λ relative to a parameter, namely, the domain U of propositions over which we quantify. The idea is that the sentence Λ will express different propositions for different values of the parameter U.

Suppose now that this parameter is fixed by context, i.e., for every context of use c, there is a domain U_c of all the propositions that are *available* in c. Propositions that lie outside of U_c, we think of as *unavailable* in c: they cannot be quantified over from the perspective of c. When from the perspective c, we speak of "All propositions" we always refer to the totality U_c of all propositions that are available in c. This is assumed to be the case also when we speak of counterfactual situations (possible worlds). Hence, the domain of quantification is assumed to be rigidly fixed by context. On this approach, the Liar sentence:

All propositions contemplated by Epimenides are false, (Λ)

will in general express different propositions relative to different contexts of use c.

For any context of use c, Λ expresses relative to c the proposition λ_c such that for any world w,

> λ_c is true at w iff for all propositions p in U_c, if p is contemplated by Epimenides in w, then p is false (not true) in w.

Given that the proposition expressed by the Liar sentence has been determined as λ_c by the actual context c, we can evaluate the proposition λ_c at various points of evaluation w. Letting the possible worlds serve the double roles of *contexts of use* as well as *points of evaluation*, we can in particular evaluate the proposition λ_c at the point c itself (the actual world). We write $\models_{u,v} A$, for a formula A taken in the context u being true at the point of evaluation v. In the next section, we shall give a rigorous definition of this notion. Let us for the time being leave it at the intuitive level.

Returning now to the Liar, we ask ourselves whether it is possible for λ_c to be the one and only proposition contemplated by Epimenides in the actual world c. Let us say that a sentence A is *true in a context* c (in symbols, $\models_c A$) iff $\models_{c,c} A$, i.e., iff A taken in the context c is true at c. A is false in the context c iff $\neg A$ is true in the context c. Thus, A is true in the context c just in case the proposition expressed by A in c

is true at c. In particular, the Liar sentence is true in the context c iff the proposition λ_c is true at c. Hence, we have:

> $\models_c \Lambda$ iff for all propositions p in U_c, if p is contemplated by Epimenides in c, then p is not true in c.

Suppose now that the following sentence is true in c,

$$\exists p(Qp \wedge \forall q(Qq \rightarrow q = p) \wedge p = \Lambda), \tag{1}$$

where we interpret "Qp" as "Epimenides contemplates the proposition p".

Then it is true (in the actual world) that:

> The proposition that all propositions contemplated by Epimenides are false is the one and only (available) proposition contemplated by Epimenides.

But this means that the proposition λ_c is in U_c and is the unique proposition contemplated by Epimenides in c. By the diagonal argument, again, we get:

> $\models_c \Lambda$ iff it is not the case that $\models_c \Lambda$.

We conclude that the sentence (1) cannot be true in the actual world. In other words, the negation of (1):

$$\neg\exists p(Qp \wedge \wedge q(Qq \rightarrow q = p) \wedge p = \Lambda), \tag{2}$$

is *logically true* in the sense of expressing a true proposition in every context (in every model). Regardless of which world is the actual one, (2) expresses a true proposition. Hence, we can say that the *sentence* (2) is *a priori* true.

(2) leaves the possibility that λ_c is the one and only proposition p in *Prop* that Epimenides contemplates in the actual world c, *provided that* λ_c is not a member of U_c. In other words, it is still possible that the proposition expressed by

$$\forall p(Qp \rightarrow \neg p) \tag{Λ}$$

is the one and only proposition contemplated by Epimenides in the actual world. This possibility obtains, as long as this proposition is not within the range assigned to the quantifier in Λ in the actual world.

As Myhill (1979, p. 83) points out, the intensional Liar that we have considered, should be distinguished from those so-called semantic paradoxes that concern linguistic entities like sentences and predicates rather than propositions and concepts:

> Actually what are called semantic paradoxes are of two kinds, exemplified by the Epimenides and the heterological paradox, respectively. The first kind does not involve any mention of expressions, but only of propositions; it does not depend on syntax, but (in the case of Epimenides) on the pragmatic relation of assertion. The other kind includes, besides the heterological paradox, the paradox of the least integer not nameable in fewer than nineteen syllables; of the least indefinable ordinal; and of Richard's. It *also* includes "This sentence is false" which can be rigorously presented using Gödel's diagonalisation technique, and which is concerned with falsehoods of *sentences* not *propositions* (and therefore is distinct from the Epimenides with which it is frequently confused).

I am inclined to look upon the intensional Liar ("Epimenides") on the analogy of the pseudo-paradox of the Barber.[5] We escape the Barber by denying the existence of any such person (i.e., a village barber that shaves all and only those inhabitants of his village that do not shave themselves). The assumption that there is such a person is simply logically inconsistent. Similarly, it seems that we can avoid the intensional Liar by viewing it as a *reductio ad absurdum* of the assumption that there is a person who actually contemplates (or asserts) one and only one proposition in the domain U_c of available propositions, namely, the proposition that all propositions (in U_c) that he contemplates are false. On this view, the assumption of the intensional Liar:

$$\exists p(Qp \wedge \forall q(Qq \rightarrow q = p) \wedge p = \Lambda) \qquad (1)$$

is simply inconsistent.

Suppose now that one and only one sentence is engraved on Epimenides' tomb, namely:

> All propositions expressed by sentences engraved on Epimenides' tomb are false;[6] (Λ)

and let λ_c be the proposition that is actually expressed by Λ. Then, it seems reasonable to assume that λ_c is the only proposition satisfying the condition of being expressed by a sentence engraved on Epimenides' tomb. By the diagonal argument however, we can conclude that:

$$\lambda_c \notin U_c,$$

that is, the proposition expressed by Λ in the actual world cannot be in the domain of propositions over which the propositional quantifier in Λ ranges, when Λ is interpreted in the actual world. In other words, if the sentence on Epimenides' tomb expresses a (unique) proposition at all, we can be sure that this proposition is not within the range of the propositional quantifiers that occurs in it. Therefore, (2) is not true and the Liar paradox is avoided.

But where does this leave Kaplan's Principle of Plenitude? According to standard modal logic, the logical falsity of (2) implies the logical truth of

$$\neg\Diamond\exists p(Qp \wedge \forall q(Qq \rightarrow q = p) \wedge p = \Lambda), \tag{2}$$

which apparently contradicts the Principle of Plenitude:

$$\forall r \Diamond p(Qp \wedge \forall q(Qq \rightarrow q = p) \wedge p = r).$$

Does this mean that the sentence (2) is necessarily false? Couldn't there have been a counterfactual situation w in which λ_c is the one and only proposition that is contemplated by Epimenides? According to the approach envisaged here, such a situation is indeed possible. Consider, for example, a world w whose set of available propositions U_w is a proper superset of the set U_c of actually available propositions. Then, λ_c could very well be the one and only proposition that Epimenides contemplated in w. Thus, we could have:

$$\models_{c,w} \exists p(Qp \wedge \forall q(Qq \rightarrow q = p) \wedge p = \Lambda),$$

from which we could infer:

$$\models_{c,c} \Diamond\exists p(Qp \wedge \forall q(Qq \rightarrow q = p) \wedge p = \Lambda).$$

The explanation for this possibility is that λ_c is not a Liar proposition with respect to the situation w: it is not necessarily equivalent to the proposition λ_w saying that for all propositions in U_w, if Epimenides contemplates p, then p is false. It would indeed have been impossible for λ_w — the proposition expressed by Λ relative to w — to be the one and only proposition contemplated by Epimenides in w. In order to express this impossibility, we need to extend the language L with the *index operator* † having the evaluation clause:

$$\models_{c,u} \dagger A \text{ iff } \models_{u,u} A.$$

Thus, the index operator takes the current point of evaluation and makes it the point of reference.[7] In terms of †, we can now express:

$$\neg\Diamond\dagger\exists p(Qp \wedge \forall q(Qq \rightarrow q = p) \wedge p = \Lambda).$$

This says that there cannot be a world w such that $\lambda_w \in U_w$ and λ_w is the one and only proposition contemplated by Epimenides in w.

It should be clear from the above discussion that given this account, there is nothing that stops the Principle of Plenitude:

$$\forall p \Diamond\exists q(q = p \wedge Qq \wedge \forall r(Qr \rightarrow r = q)).$$

from holding in the actual world.

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Appendix: The Formal Semantics

For the sake of completeness, we write up the formal semantics for L (with the added operator †) that was outlined informally in the previous section.

We define a *frame* to be a sequence:

$$F = \langle W, U \rangle,$$

satisfying the conditions:

(i) W is a non-empty set (referred to as the set of *possible worlds*).

(ii) U is a function which assigns to each $w \in W$ a subset U_w of $\wp(W)$.

Subsets of W are called *propositions*. For each $w \in W$, we say that U_w is the set of propositions that are *available* in w.

A *model* for the language L is a sequence $M = \langle W, U, V \rangle$, where

(i) $F = \langle W, U \rangle$ is a frame; and

(ii) V is a function that assigns to the non-logical operator symbol Q, an operation on propositions $V(Q) : \wp(W) \to \wp(W)$.

$M = \langle W, U, V \rangle$ is said to be a model based on $F = \langle W, U \rangle$. An *assignment* (in M) is a function g which assigns a proposition $g(p) \in \wp(W)$ to every propositional variable p in L.

Given a model $M = \langle W, U, V \rangle$ and a possible world $w \in W$ (serving as context), we define for every formula A of L and every assignment g, the proposition $|A[g]|_w$ expressed by A at w, with respect to M, g, as follows:

(i) if p is a propositional variable, then $|p[g]|_w = g(p)$;

(ii) $|Q(A)[g]|_w = V(Q)(|A[g]|_w)$;

(iii) $|(A = B)[g]|_w = W$, if $|A[g]|_w = |B[g]|_w$; and $|(A = B)[g]|_w = \emptyset$, otherwise;

(iv) $| \perp [g]|_w = \emptyset$;

(v) $|(A \to B)[g]|_w = (W - |A[g]|_w) \cup |B[g]|_w$;

(vi) $|\forall pA(p)[g]|_w = \{u \in W : \text{ for all } a \in U_w, u \in |A(p)[g(a/p)]|_w\}$, where $g(a/p)$ is the assignment that is obtained from g by assigning the proposition a to the propositional variable p and letting $g(a/p)(q) = g(q)$ for all other propositional variables q.

(vii) $|\Box A[g]|_w = W$, if $|A[g]|_w = W$; and $|\Box A[g]|_w = \emptyset$, otherwise;

(viii) $|\dagger A[g]|_w = \{u \in W, u \in |A[g]|_u\}$.

Observe that the only role for the context parameter w is to fix the domain U_w over which the propositional variables range. Once, the domain of quantification is fixed by context, it is taken to be constant for all points of evaluation.

For each formula A, model M, pair of worlds u, v in M and assignment g into M, we define:

$$M \models_{u,v} A[g] iff v \in |A[g]|_u.$$

We say that A is *true at* v *with reference to* M, g *and* u iff $M \models_{u,v} A[g]$. We have:

$$M \models_{u,v} \bot [g];$$

$$M \models_{u,v} (A \to B)[g] \text{ iff either } M \models_{u,v} A \text{ or } M \models_{u,v} B[g];$$

$$M \models_{u,v} \Box A[g] \text{ iff for all } w \in W, M \models_{u,v} A[g];$$

$$M \models_{u,v} (A = B)[g] \text{ iff } |A[g]|_u = |B[g]|_u;$$

$$M \models_{u,v} \forall pA[g] \text{ iff for all } a \in U_u, M \models_{u,v} A[g(a/p)];$$

$$M \models_{u,v} \dagger A \text{ iff } M \models_{v,v} A.$$

We define:

$$M \models_u A[g] \text{iff} M \models_{u,u} A[g] (\text{i.e., iff} u \in |A[g]|_u).$$

We say that the formula A is *valid* in the model M (in symbols, $M \models A$), if for every $u \in W$ and every assignment g into M, $M \models_u A[g]$. A is *satisfied* by the model M if for some $u \in W$ and some assignment g into M, $M \models_u A[g]$.

Notes

1. The fullest and most recent presentation of the argument is Kaplan (1994). Kaplan who has been communicating the argument since the middle 70's, presented it as part of his John Locke lectures in Oxford during the spring of 1980 and at the *7th International Congress of Logic Methodology and Philosophy of Science* in Salzburg, Austria, July 11-16, 1983 (An abstract of the argument appeared in *Abstracts of Sections 5 and 12: 7th International Congress of Logic Methodology and Philosophy of Science*). It has been reported and discussed in (Cresswell, 1980), (Davies, 1981), (Lewis, 1986) and (Jubien, 1988).

2. It may be instructive to consider Russell's informal explanation in (Whitehead and Russell, 1910–1913, vol. I, p. 62) of the resolution of the Liar paradox within the ramified theory of types:

> When a man says "I am lying", we may interpret his statement as: "There is a proposition which I am affirming and which is false". That is to say, he is asserting the truth of some value of the function "I assert p, and p is false". But we saw that the word "false" is ambiguous, and that, in order to make it unambiguous, we must specify the order of falsehood, or what comes to the same thing, the order of the proposition to which falsehood is ascribed. We saw also that, if p is a proposition of the nth order, a proposition in which p occurs as an apparent variable is not of the nth order but of a higher order. Hence the kind of truth or falsehood which can belong to the statement "there is a proposition p which I am affirming and which has falsehood of the nth order" is a truth or falsehood of a order higher than the nth. Hence, the statement of Epimenides does not fall within its own scope, and therefore no contradiction emerges.

See also (Russell, 1908) and (Church, 1976) for discussions of the so-called semantical paradoxes within the context of the ramified theory of types.

3. Cf. (Quine, 1953) and (Church 1951, 1974). The distinction between the direct-discourse approach and the indirect-discourse approach to intensional logic was clearly stated in (Kaplan, 1964). In that work, Kaplan is concerned with developing the direct-discourse approach of Alonzo Church and providing it with a model-theoretic semantics of the possible worlds variety. The model-theoretic study of Church's logic of sense and denotation within possible worlds semantics is pursued further in (Kaplan, 1975) and (Parsons, 1982). For an early discussion of different methods of semantic analysis in intensional logic, including Church's, see (Carnap, 1956).

4. The reader might wonder about the exact formalization of the principle of plenitude. Perhaps, we should rather write it as:

$$\forall p\,\Box\,\exists q(q = p \wedge Qq \wedge \forall r(Qr \rightarrow r = q)), \qquad \text{(P')}$$

or as

$$\forall p\,\Box\,\forall q(Qq \leftrightarrow q = p). \qquad \text{(P")}$$

Given, a possible worlds semantics where the quantifiers range over a constant domain containing all propositions, the three formulations are equivalent. But what about the case when we allow for the domain of quantification to vary from one world to another? Even then, the three formulations are equivalent provided that the operator Q satisfies the requirement:

$$\forall p\,\Box\,(Qp \rightarrow \exists q(q = p \wedge Qq)).$$

In its absence, (P') will be our official formalization of the principle of plenitude.

5. Myhill (1979) is of the opposite opinion: he seems to view the intensional Liar as a genuine antinomy that requires a division of propositions into types along the lines of Russell's Ramified Theory of Types.

6. This variant formulation of the paradox is, of course, due to Kaplan (1994).

7. The †-operator is discussed in (Lewis, 1973, section 2.8). See also (Segerberg, 1973).

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