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LP, K3, and FDE as Substructural Logics

LIONEL SHAPIRO[1]

Abstract: Building on recent work, I present sequent systems for the non-classical logics LP, K3, and FDE with two main virtues. First, derivations closely resemble those in standard Gentzen-style systems. Second, the systems can be obtained by reformulating a classical system using nonstandard sequent structure and simply removing certain structural rules (relatives of exchange and contraction). I clarify two senses in which these logics count as "substructural."

Keywords: Non-classical logics, Substructural logics, Many-valued logics, Many-sided sequent systems, Structural rules

1 Introduction

The non-classical propositional logics LP, K3, and FDE, standardly presented using many-valued semantics, have long enjoyed cut-free sequent proof systems.[2] The paracomplete and paraconsistent logic FDE is in fact introduced in Anderson and Belnap (1963) using a two-sided sequent system; the extensions of this system to the paraconsistent LP and the paracomplete K3 are straightforward (Avron, 1991; Beall, 2011). Three-sided systems for LP and K3 are given by Ripley (2012) and Hjortland (2013).[3] Most recently, several authors have independently proposed alternative sequent systems for LP and K3 (Fjellstad, 2016; Shapiro, 2016) or all three logics (Wintein, 2016). These systems, which are closely related, are motivated by a variety of formal and philosophical considerations.

[1]For invaluable help with these topics, I am indebted to Dave Ripley. I also benefited greatly from discussion with Jc Beall and the participants of Logica 2016, where a version of the earlier paper on which I here elaborate was given.

[2]FDE is the logic of "first-degree entailments" of Anderson and Belnap (1963, 1975). LP is the "logic of paradox" of Priest (1979); its propositional fragment is due to Asenjo (1966). The "Strong Kleene" logic K3 derives from Kleene (1952). For an overview, see Priest (2008).

[3]These are based on the methods of Baaz, Fermüller, and Zach (1993), which yield a four-sided system for FDE (as Hjortland notes).

My purpose here is to present a more explicit version of the *substructural* approach proposed in Shapiro (2016), and to show how it can be improved by incorporating the key insight of Fjellstad and Wintein. The result is a class of sequent systems for these sub-classical logics whose use is especially intuitive: derivations look almost like familiar classical derivations of the same sequents. To obtain these systems, I start with a Gentzen-style system for classical logic and introduce new sequent structure, together with a set of structural rules. By including different subsets of those rules, I arrive at sequent systems for FDE, LP, K3 or (when all are included) classical logic CL. The systems coincide in their initial sequents, their operational rules, and in how derivable sequents correspond to true consequence claims. They differ only in their structural rules.

The paper is organized as follows. §2 introduces the consequence relations FDE, LP, and K3 and the respective two-sided sequent systems. §3 motivates and elaborates the three-sided substructural approach from Shapiro (2016). In the process, two senses are distinguished in which a logic can count as substructural. Finally, §4 modifies this approach to yield four-sided substructural systems that amount to variants of the systems of Wintein and Fjellstad. The resulting systems possess the advantages of the three-sided substructural systems without the disadvantage these share with two-sided systems—namely, the need for rules involving more than one connective.

2 Two-sided semantics and sequent systems

2.1 Many-valued semantics

To start, I present the four propositional logics semantically (cf. Priest, 2008, §8.4). Truth-values will be the members of $\{t, n, b, f\}$, partially ordered as reflected in this lattice:

$$\begin{tikzcd} & t & \\ n \arrow[ur] & & b \arrow[ul] \\ & f \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Definition 1 *For each of our logics X, an X-valuation is a function from the set of sentences to $\{t, n, b, f\}$. An FDE-valuation has as its range the full set $\{t, n, b, f\}$, while a LP-valuation has range $\{t, b, f\}$, a K3-valuation has range $\{t, n, f\}$, and a CL-valuation a function has range $\{t, f\}$. Every X-valuation is subject to the following conditions:*

- *if* $v(\alpha) = t$ *then* $v(\neg\alpha) = f$,
  if $v(\neg\alpha) = f$ *then* $v(\alpha) = t$,
  if $v(\alpha) = n$, *or* $v(\alpha) = b$, *then* $v(\neg\alpha) = v(\alpha)$,
- $v(\alpha \wedge \beta)$ *is the greatest lower bound of* $v(\alpha)$ *and* $v(\beta)$,
- $v(\alpha \vee \beta)$ *is the least upper bound of* $v(\alpha)$ *and* $v(\beta)$.

For each logic X, we can now define its multiple-conclusion consequence relation by taking the values t and b as designated:

Definition 2 *For any sets* Γ *and* Δ *of sentences,* $\Gamma \models_X \Delta$ *just in case there is no* X*-valuation such that each sentence in* Γ *receives either* t *or* b*, and each sentence in* Δ *receives either* f *or* n*.*

It's easy to check that

- $\Gamma \models_{LP} \Delta, \alpha \vee \neg\alpha$, whereas this consequence (excluded middle) fails for FDE and K3, which are thus *paracomplete*,
- $\Gamma, \alpha \wedge \neg\alpha \models_{K3} \Delta$, whereas this consequence (explosion) fails for FDE and LP, which are thus *paraconsistent*.

2.2 Sequent systems

The following proof system $\mathbf{S2}_{FDE}$ (for 'two-sided') is a variant of one of Anderson and Belnap's systems (1963; 1975, pp. 179–80). Sequents are of form $\Gamma \rhd \Delta$, where (as throughout this paper) upper-case Greek letters stand for sets of sentences.[4] However, as I'll later be generalizing the sequent structure, I write the rules using $\mathfrak{A} \rhd \mathfrak{B}$ instead. In the present case, $\mathfrak{A}(\alpha)$ abbreviates some Γ, α, which in turn stands for the set $\Gamma \cup \{\alpha\}$.

1. *Initial sequents*

   For atomic α: (Id) $\alpha \rhd \alpha$ (NegId) $\neg\alpha \rhd \neg\alpha$

2. *Operational rules*

   Conjunction and disjunction rules

$$(\wedge\mathrm{L})\ \frac{\mathfrak{A}(\alpha, \beta) \rhd \mathfrak{B}}{\mathfrak{A}(\alpha \wedge \beta) \rhd \mathfrak{B}} \qquad (\wedge\mathrm{R})\ \frac{\mathfrak{A} \rhd \mathfrak{B}(\alpha) \qquad \mathfrak{A} \rhd \mathfrak{B}(\beta)}{\mathfrak{A} \rhd \mathfrak{B}(\alpha \wedge \beta)}$$

[4] I follow Ripley (2012) in using $\rhd$ as sequent separator.

$$(\vee L)\ \frac{\mathfrak{A}(\alpha) \rhd \mathfrak{B} \qquad \mathfrak{A}(\beta) \rhd \mathfrak{B}}{\mathfrak{A}(\alpha \vee \beta) \rhd \mathfrak{B}} \qquad (\vee R)\ \frac{\mathfrak{A} \rhd \mathfrak{B}(\alpha, \beta)}{\mathfrak{A} \rhd \mathfrak{B}(\alpha \vee \beta)}$$

Negated connective rules

$$(\neg\wedge L)\ \frac{\mathfrak{A}(\neg\alpha) \rhd \mathfrak{B} \qquad \mathfrak{A}(\neg\beta) \rhd \mathfrak{B}}{\mathfrak{A}(\neg(\alpha \wedge \beta)) \rhd \mathfrak{B}} \quad (\neg\wedge R)\ \frac{\mathfrak{A} \rhd \mathfrak{B}(\neg\alpha, \neg\beta)}{\mathfrak{A} \rhd \mathfrak{B}(\neg(\alpha \wedge \beta))}$$

$$(\neg\vee L)\ \frac{\mathfrak{A}(\neg\alpha, \neg\beta) \rhd \mathfrak{B}}{\mathfrak{A}(\neg(\alpha \vee \beta)) \rhd \mathfrak{B}} \quad (\neg\vee R)\ \frac{\mathfrak{A} \rhd \mathfrak{B}(\neg\alpha) \qquad \mathfrak{A} \rhd \mathfrak{B}(\neg\beta)}{\mathfrak{A} \rhd \mathfrak{B}(\neg(\alpha \vee \beta))}$$

$$(\neg\neg L)\ \frac{\mathfrak{A}(\alpha) \rhd \mathfrak{B}}{\mathfrak{A}(\neg\neg\alpha) \rhd \Delta} \qquad (\neg\neg R)\ \frac{\mathfrak{A} \rhd \mathfrak{B}(\alpha)}{\mathfrak{A} \rhd \mathfrak{B}(\neg\neg\alpha)}$$

3. *Structural rule*

$$(\text{Weak})\ \frac{\mathfrak{A}(\Gamma) \rhd \mathfrak{B}(\Delta)}{\mathfrak{A}(\Gamma, \Gamma') \rhd \mathfrak{B}(\Delta, \Delta')}$$

To obtain $\mathbf{S2}_{LP}$ and $\mathbf{S2}_{K3}$ simply add, respectively, the initial sequents LEM or Explosion, where α is atomic:

$$(\text{LEM})\ \emptyset \rhd \alpha, \neg\alpha \qquad (\text{Explosion})\ \alpha, \neg\alpha \rhd \emptyset$$

Including LEM as well as Explosion yields the system $\mathbf{S2}_{CL}$.

Proposition 1 (Beall, 2011) *For each logic X, the sequent $\Gamma \rhd \Delta$ is derivable in $\mathbf{S2}_X$ iff $\Gamma \vDash_X \Delta$.*

3 Three-sided substructural systems

3.1 Motivation

Shapiro (2016) argues that the systems $\mathbf{S2}_X$ are unsatisfactory when evaluated from the perspective according to which it is desirable to preserve as much as possible of the familiar ways of deriving consequence claims using a classical multiple-conclusion sequent system.[5] Such a system includes rules that give negation its usual "flip-flop behavior" (Beall, 2016).

[5] Fjellstad too argues that it is "easier to adopt the sequent calculus as a tool for reasoning and thus perhaps also the logic as such" when "the rules are relatively familiar" (2016).

$$(\neg\text{L})\ \frac{\Gamma \rhd \Delta, \alpha}{\Gamma, \neg\alpha \rhd \Delta} \qquad (\neg\text{R})\ \frac{\Gamma, \alpha \rhd \Delta}{\Gamma \rhd \Delta, \neg\alpha}$$

When we consider the material conditional $\supset$, where $\alpha \supset \beta$ is $\neg\alpha \vee \beta$, this behavior corresponds to central aspects of a conditional's behavior. Conditional proof, the rule of right-$\supset$ introduction, is derivable using $\neg$R. Likewise, the $\supset$-left introduction rule that yields the *modus ponens* sequent $\alpha, \alpha \supset \beta \rhd \beta$ is derivable using $\neg$L.

$$\dfrac{\dfrac{\Gamma, \alpha \rhd \Delta, \beta}{\Gamma \rhd \Delta, \beta, \neg\alpha}\ \neg\text{R}}{\Gamma \rhd \Delta, \alpha \supset \beta}\ \vee\text{R} \qquad \dfrac{\dfrac{\Gamma \rhd \alpha, \Delta}{\Gamma, \neg\alpha \rhd \Delta}\ \neg\text{L} \qquad \Gamma, \beta \rhd \Delta}{\Gamma, \alpha \supset \beta \rhd \Delta}\ \vee\text{L}$$

How much of this familiar inferential behavior must we give up when employing one of our non-classical logics? Beall argues that we renounce it entirely if we adopt FDE as our logic. Since that's what he proposes doing, he concludes that "there is no logical negation." What he means, he clarifies, is that while there is a "logical connective called *negation*, . . . logic imposes no interesting constraint on it . . . aside from what logic demands of its interaction with other logical connectives." By the same token, he should hold, FDE contains no "logical conditional."

One aim of Shapiro (2016) was to give sequent systems for LP and K3 that preserve negation's flip-flop behavior together with familiar rules for conjunction and disjunction.[6] The lesson, when applied to FDE, is that Beall's conclusion is problematic.[7] As I'll now show, the way to preserve familiar classical derivations of consequence claims of our non-classical logics is to modify the standard structure of a consequence relation—in a way that goes beyond Beall's own embrace of multiple-conclusion sequents.

3.2 Systems distinguished by negation rules

The systems in Shapiro (2016) weren't designed for their formal virtues, but to make that paper's philosophical point. Here I give a slightly modified formulation, which recovers a version of the subformula property: for each

[6] The presentation there focuses on LP, but the system for K3 is given in footnote 20.

[7] This is so even if we understand him as demanding of a "logical negation" that it coexist with other connectives obeying familiar operational rules. In other words, his conclusion is problematic even when interpreted so as not to already be refuted by a many-sided system based on Baaz et al. (1993), whose derivations look very unlike classical ones. Such systems do display "stand-alone negation behavior" involving flip-flops.

formula in a sequent's derivation, either it, or a formula of which it is the negation, is a subformula of some sentence appearing in that sequent.[8]

Sequents will take the form $\Gamma; \Sigma \rhd \Delta$. The two-sided $\Gamma \rhd \Delta$ is to be understood as a convenient notation for $\Gamma; \emptyset \rhd \Delta$.[9] The system $\mathbf{S3}_{FDE}$ consists of the following initial sequents and rules. Where one of the rules from §2.2 is included, $\mathfrak{A}(\alpha)$ on the left may be either $\Gamma, \alpha; \Sigma$ or $\Gamma; \Sigma, \alpha$ and $\mathfrak{B}(\alpha)$ on the right is Δ, α.

1. *Initial sequents*

   For atomic α: (Id) $\alpha \rhd \alpha$ (NegId*) $\neg\alpha; \alpha \rhd \emptyset$

2. *Operational rules*

   Negation rules

$$(\neg\text{L})\ \frac{\Gamma; \Sigma \rhd \Delta, \alpha}{\Gamma; \Sigma, \neg\alpha \rhd \Delta} \qquad (\neg\text{R})\ \frac{\Gamma; \Sigma, \alpha \rhd \Delta}{\Gamma; \Sigma \rhd \Delta, \neg\alpha}$$

   All above conjunction, disjunction and negated connective rules

3. *Structural rule*

   Weak

We then obtain $\mathbf{S3}_{K3}$ by including an additional rule ¬L*, and $\mathbf{S3}_{LP}$ by including instead its dual rule ¬R*. The system $\mathbf{S3}_{CL}$ includes both rules.

$$(\neg\text{L*})\ \frac{\Gamma; \Sigma \rhd \Delta, \alpha}{\Gamma, \neg\alpha; \Sigma \rhd \Delta} \qquad (\neg\text{R*})\ \frac{\Gamma, \alpha; \Sigma \rhd \Delta}{\Gamma; \Sigma \rhd \Delta, \neg\alpha}$$

Shapiro (2016, note 16) describes how to extend the three-valued semantics for LP to the three-sided sequents of these systems. I present this semantics in §4.2, generalized to four-sided sequents following Fjellstad and Wintein. Using it and Proposition 1, I show that $\mathbf{S3}_X$ is sound and complete with respect to X-consequence as defined above.

[8] In place of a left introduction rule for negation, Shapiro (2016) uses the elimination rule that is the inverse of a right introduction rule. Initial sequents are of form Id, where α isn't restricted to atomic sentences.

[9] Shapiro (2016) uses a slightly different format. Sequents there include three-sided ones $\Gamma; \Sigma \vdash \Delta$ as well as two-sided "ordinary sequents" $\Gamma \vdash \Delta$. The notations $\emptyset; \Gamma \vdash \Delta$ and $\Gamma; \emptyset \vdash \Delta$ are *both* stipulated to stand for $\Gamma \vdash \Delta$, which behaves like $\Gamma; \emptyset \rhd \Delta$ in $\mathbf{S3}_X$ below. This necessitates restricting ¬L and ¬R below to instances with nonempty Γ. Furthermore, in the system for LP, it makes possible a weakening rule that is more general than Weak in that it covers the move from $\Gamma \vdash \Delta$ to $\Gamma'; \Gamma \vdash \Delta$. The sequent $\emptyset; \Gamma \rhd \Delta$ of $\mathbf{S3}_X$ corresponds to no sequent in the systems of Shapiro (2016).

Proposition 2 $\Gamma \rhd \Delta$ *is derivable in* $\textbf{\textit{S3}}_X$ *iff* $\Gamma \vDash_X \Delta$.

Proof. This follows from Corollary 1 in §4.2 below.

The systems $\mathbf{S3}_{FDE}$, $\mathbf{S3}_{LP}$ and $\mathbf{S3}_{K3}$ count as "substructural" in one common sense of that term: each renders inadmissible one or more versions of the standard structural rules of weakening, exchange, contraction or cut.

Fact 1 *The following rule is inadmissible in* $\textbf{\textit{S3}}_{FDE}$, $\textbf{\textit{S3}}_{LP}$ *and* $\textbf{\textit{S3}}_{K3}$.

$$(\text{Exch})\ \dfrac{\Gamma, \alpha; \Sigma, \beta \rhd \Delta}{\Gamma, \beta; \Sigma, \alpha \rhd \Delta}$$

Proof. From Proposition 2, since adding Exch derives $\beta \rhd \alpha, \neg\alpha$, which is invalid in K3, and $\neg\alpha, \alpha \rhd \neg\beta$, which is invalid in LP.

$$\dfrac{\dfrac{\dfrac{\alpha \rhd \alpha}{\alpha; \beta \rhd \alpha}\,\text{Weak}}{\beta; \alpha \rhd \alpha}\,\text{Exch}}{\beta \rhd \alpha, \neg\alpha}\,\neg\text{R} \qquad \dfrac{\dfrac{\dfrac{\neg\alpha; \alpha \rhd \emptyset}{\neg\alpha, \beta; \alpha \rhd \emptyset}\,\text{Weak}}{\neg\alpha, \alpha; \beta \rhd \emptyset}\,\text{Exch}}{\neg\alpha, \alpha \rhd \neg\beta}\,\neg\text{R}$$

Fact 2 *The following rules are inadmissible in* $\textbf{\textit{S3}}_{FDE}$ *and* $\textbf{\textit{S3}}_{LP}$.

$$(\text{Cont1})\ \dfrac{\Gamma, \alpha; \Sigma, \alpha \rhd \Delta}{\Gamma, \alpha; \Sigma \rhd \Delta} \qquad (\text{Cut})\ \dfrac{\Gamma; \Sigma \rhd \Delta, \alpha \qquad \Gamma; \Sigma, \alpha \rhd \Delta}{\Gamma; \Sigma \rhd \Delta}$$

Proof. Again, adding Cont1 or Cut derives Explosion.

$$\dfrac{\dfrac{\dfrac{\alpha \rhd \alpha}{\alpha; \neg\alpha \rhd \emptyset}\,\neg\text{L}}{\alpha, \neg\alpha; \neg\alpha \rhd \emptyset}\,\text{Weak}}{\alpha, \neg\alpha \rhd \emptyset}\,\text{Cont1} \qquad \dfrac{\dfrac{\alpha \rhd \alpha}{\alpha, \neg\alpha \rhd \alpha}\,\text{Weak} \qquad \dfrac{\neg\alpha; \alpha \rhd \emptyset}{\alpha, \neg\alpha; \alpha \rhd \emptyset}\,\text{Weak}}{\alpha, \neg\alpha \rhd \emptyset}\,\text{Cut}$$

Fact 3 *The following rule is inadmissible in* $\textbf{\textit{S3}}_{FDE}$ *and* $\textbf{\textit{S3}}_{K3}$.

$$(\text{Cont2})\ \dfrac{\Gamma, \alpha; \Sigma, \alpha \rhd \Delta}{\Gamma; \Sigma, \alpha \rhd \Delta}$$

Proof. Adding Cont2 derives LEM.[10]

$$\dfrac{\dfrac{\dfrac{\alpha \rhd \alpha}{\alpha; \alpha \rhd \alpha}\,\text{Weak}}{\emptyset; \alpha \rhd \alpha}\,\text{Cont2}}{\emptyset \rhd \alpha, \neg\alpha}\,\neg\text{R}$$

[10] In fact, Cont2 is inadmissible in $\mathbf{S3}_{LP}$ too, since $\alpha; \alpha \rhd \alpha$ is derivable but $\emptyset; \alpha \rhd \alpha$ is not. However, Cont2 will be admissible in the revised system $\mathbf{S3}'_{LP}$ I introduce next.

3.3 Systems distinguished by shift rules

The term 'substructural' is sometimes used in a different sense. In this sense, the failure of a standard structural rule doesn't suffice to make the logic specified by a sequent system count as substructural. Rather, the logic must be obtainable from classical logic *solely* by restricting structural rules. Došen, who coined the term, introduces it thus:

> Our proposal is to call logics that can be obtained ... by restricting structural rules, *substructural logics*.... Canonically, we should assume the same rules for logical constants in the logic whose structural rules we restrict, i.e. classical logic, and in the resulting substructural logic.... We don't insist that the Gentzen formulation of classical logic whose structural rules we restrict in order to obtain a substructural logic should be the standard one. It could as well be a nonstandard formulation with sequents whose left-hand and right-hand sides are not sequences of formulae of L but some other sort of structure involving formulae of L.... (Došen, 1993, p. 6)

Notice that $\mathbf{S3}_{FDE}$, $\mathbf{S3}_{LP}$ and $\mathbf{S3}_{K3}$ differ from classical logic, and from each other, in their negation rules. Hence they don't qualify as substructural in Došen's sense. But they can be modified so as to meet his condition.

$\mathbf{S3}'_{FDE}$ is just $\mathbf{S3}_{FDE}$ without the rules ¬∧R, ¬∨R and ¬¬R, which are easily seen to be admissible in light of Lemma 1 of the Appendix. Each of the systems $\mathbf{S3}'_{LP}$ and $\mathbf{S3}'_{K3}$ is then obtained by simply adding a *structural shift* rule, which is a relative of the usual structural rules of contraction and exchange. For $\mathbf{S3}'_{LP}$, we add the rule RightShift, while for $\mathbf{S3}'_{K3}$ we add the dual rule LeftShift. Adding both shift rules yields $\mathbf{S3}'_{CL}$.

$$\text{(RightShift)}\ \frac{\Gamma, \alpha; \Sigma \rhd \Delta}{\Gamma; \Sigma, \alpha \rhd \Delta} \qquad \text{(LeftShift)}\ \frac{\Gamma; \Sigma, \alpha \rhd \Delta}{\Gamma, \alpha; \Sigma \rhd \Delta}$$

Using RightShift and ¬R we can immediately derive ¬R*, while using ¬L and Left Shift we can derive ¬L*.

Proposition 3 $\Gamma \rhd \Delta$ *is derivable in* $\mathbf{S3}'_X$ *iff* $\Gamma \vDash_X \Delta$.

Proof. This follows from Corollary 1 in §4.2 below.

This means that FDE, LP, and K3 can be obtained from classical logic (as formulated using a sequent framework in which the left-hand sides are pairs of sets of sentences) by removing one or more of this framework's structural rules. Classical logic thus relates to LP, K3, and FDE in a manner analogous to the way it relates to *affine logic*, *distribution-free relevant logic*, and *linear logic*. In a sequent system that uses multisets of sentences, these three logics can be obtained from classical logic by dropping (respectively) contraction, weakening, and both contraction and weakening.[11]

We have seen how $\mathbf{S3}_{FDE}$ and $\mathbf{S3}'_{FDE}$ partly recover the flip-flop behavior of negation in the form of rules $\neg$L and $\neg$R. They also partly recover the classical behavior of $\supset$, as the following are both derivable.[12]

$$(\supset\text{L*})\ \frac{\Gamma; \Sigma \triangleright \alpha, \Delta \qquad \Gamma; \Sigma, \beta \triangleright \Delta}{\Gamma; \Sigma, \alpha \supset \beta \triangleright \Delta} \qquad (\supset\text{R*})\ \frac{\Gamma; \Sigma, \alpha \triangleright \beta, \Delta}{\Gamma; \Sigma \triangleright \alpha \supset \beta, \Delta}$$

So is the *modus ponens* sequent $\alpha \supset \beta; \alpha \triangleright \beta$.

$$\frac{\neg\alpha; \alpha \triangleright \beta \qquad \beta; \alpha \triangleright \beta}{\alpha \supset \beta; \alpha \triangleright \beta}\ \vee\text{L1}$$

Consequently, many classical derivations go through virtually unchanged. For example, here is a derivation using the conditional rules admissible in all the above three-sided systems:

$$\dfrac{\dfrac{\dfrac{\dfrac{\cancel{\alpha \supset \beta; \overset{\gamma}{\alpha}} \triangleright \cancel{\overset{\gamma}{\beta}}}{\alpha \supset \beta; \gamma, \cancel{\alpha} \triangleright \gamma, \beta}\ \text{Weak} \qquad \dfrac{\alpha \supset \beta; \alpha \triangleright \beta}{\alpha \supset \beta; \gamma, \alpha \triangleright \beta}\ \text{Weak}}{\alpha \supset \beta; \gamma, \gamma \supset \alpha \triangleright \beta}\ \supset\text{L}*}{\alpha \supset \beta; \gamma \supset \alpha \triangleright \gamma \supset \beta}\ \supset\text{R}*}{\alpha \supset \beta \triangleright (\gamma \supset \alpha) \supset (\gamma \supset \beta)}\ \supset\text{R}*$$

4 Four-sided substructural systems

Nonetheless, with a view to preserving as much classical behavior as possible, the present systems still aren't optimal. As did $\mathbf{S2}_X$, they require negated connective rules.[13] Also, while $\alpha \supset \beta; \alpha \triangleright \beta$ is derivable in $\mathbf{S3}'_X$, it isn't directly derivable using any $\supset$-L-like rule admissible in $\mathbf{S3}'_{LP}$. By this I mean any rule that is an instance of the following general form.

$$(\text{COND})\ \frac{\mathfrak{A} \triangleright \mathfrak{B}(\alpha) \qquad \mathfrak{A}(\beta) \triangleright \mathfrak{B}}{\mathfrak{A}(\alpha \supset \beta) \triangleright \mathfrak{B}}$$

[11] See Paoli (2002, §2.2). Multisets are like sets except that they keep track of how many times a given member appears.

[12] In the systems for LP and K3, two additional conditional rules each are derivable; I omit them here for reasons of space.

[13] Both Fjellstad (2016) and Wintein (2016) take it as a desideratum that each operational rule should refer to only one connective.

4.1 Systems with dual negation and shift rules

Both defects can be remedied by implementing the key idea underlying the systems of Fjellstad (2016) and Wintein (2016). This is to allow *dual* flip-flop behavior by altering the sequent structure. Fjellstad presents his system as a "dual two-sided sequent calculus," whereas Wintein presents his as a four-sided signed calculus. Here I follow Wintein, though I use notation that displays a "left" vs. "right" distinction implicit in his formulation, and replace his extra initial sequents with shift rules. (See the Appendix.)

To get systems $\mathbf{S4}_X$, let sequents take form $\Gamma;\Sigma \rhd \Delta;\Theta$. As before, $\Gamma \rhd \mathfrak{B}$ abbreviates $\Gamma;\emptyset \rhd \mathfrak{B}$, but now $\mathfrak{A} \rhd \Delta$ abbreviates $\mathfrak{A} \rhd \Delta;\emptyset$. We modify $\mathbf{S3}_X$ to take advantage of left-right symmetry. $\mathbf{S4}_{FDE}$ consists of the following initial sequents and rules. Where a rule from §2.2 is included, $\mathfrak{A}(\alpha)$ on the left may be either $\Gamma,\alpha;\Sigma$ or $\Gamma;\Sigma,\alpha$ and $\mathfrak{B}(\alpha)$ on the right is $\Delta,\alpha;\Theta$ or $\Delta;\Theta,\alpha$.

1. *Initial sequents*

   For atomic α: (Id) $\alpha \rhd \alpha$ (NegId**) $\emptyset;\alpha \rhd \emptyset;\alpha$

2. *Operational rules*

   Negation rules

$$(\neg\text{L1})\ \frac{\Gamma;\Sigma \rhd \Delta;\Theta,\alpha}{\Gamma,\neg\alpha;\Sigma \rhd \Delta;\Theta} \qquad (\neg\text{L2})\ \frac{\Gamma;\Sigma \rhd \Delta,\alpha;\Theta}{\Gamma;\Sigma,\neg\alpha \rhd \Delta;\Theta}$$

$$(\neg\text{R1})\ \frac{\Gamma;\Sigma,\alpha \rhd \Delta;\Theta}{\Gamma;\Sigma \rhd \Delta,\neg\alpha;\Theta} \qquad (\neg\text{R2})\ \frac{\Gamma,\alpha;\Sigma \rhd \Delta;\Theta}{\Gamma;\Sigma \rhd \Delta;\Theta,\neg\alpha}$$

   Conjunction and disjunction rules

   $\wedge$L, $\wedge$R, $\vee$L, $\vee$R

3. *Structural rule*

   Weak

Wintein's systems for LP, K3, and classical logic are distinguished by additional initial sequents (as are Fjellstad's systems, once translated into the present format as indicated in the Appendix). Here, I instead use structural shift rules. To get $\mathbf{S4}_{LP}$, we add two *inward shift* rules.

$$\text{(RightShiftL)}\ \frac{\Gamma, \alpha; \Sigma \rhd \Delta; \Theta}{\Gamma; \Sigma, \alpha \rhd \Delta; \Theta} \quad \text{(LeftShiftR)}\ \frac{\Gamma; \Sigma \rhd \Delta; \Theta, \alpha}{\Gamma; \Sigma \rhd \Delta, \alpha; \Theta}$$

To get **S4**$_{K3}$, we add instead the dual pair of *outward shift* rules.

$$\text{(LeftShiftL)}\ \frac{\Gamma; \Sigma, \alpha \rhd \Delta; \Theta}{\Gamma, \alpha; \Sigma \rhd \Delta; \Theta} \quad \text{(RightShiftR)}\ \frac{\Gamma; \Sigma \rhd \Delta, \alpha; \Theta}{\Gamma; \Sigma \rhd \Delta; \Theta, \alpha}$$

Finally, to get **S4**$_{CL}$, we include all four shift rules.

By allowing two pairs of flip-flop rules, **S4**$_{FDE}$ derives the negated connective rules of **S3**$_{FDE}$. Likewise, it derives these conditional rules:

$$(\supset\text{L1**})\ \frac{\Gamma; \Sigma \rhd \alpha, \Delta; \Theta \qquad \Gamma; \Sigma, \beta \rhd \Delta; \Theta}{\Gamma; \Sigma, \alpha \supset \beta \rhd \Delta; \Theta}$$

$$(\supset\text{L2**})\ \frac{\Gamma; \Sigma \rhd \Delta; \Theta, \alpha \qquad \Gamma, \beta; \Sigma \rhd \Delta; \Theta}{\Gamma, \alpha \supset \beta; \Sigma \rhd \Delta; \Theta}$$

And $\supset$L2**, which is an instance of the general form COND above, may be used to derive $\alpha \supset \beta; \alpha \rhd \beta$.

$$\frac{\emptyset; \alpha \rhd \beta; \alpha \qquad \beta; \alpha \rhd \beta}{\alpha \supset \beta; \alpha \rhd \beta}\ \supset\text{L2**}$$

The structural rules inadmissible in **S4**$_X$ include, besides those we saw are inadmissible in **S3**$_X$, dual rules involving a sequent's right-hand side.

$$\text{(ExchR)}\ \frac{\Gamma; \Sigma, \rhd \Delta, \alpha; \Theta, \beta}{\Gamma; \Sigma \rhd \Delta, \beta; \Theta, \alpha} \quad \text{(Cut*)}\ \frac{\Gamma; \Sigma \rhd \Delta; \Theta, \alpha \qquad \Gamma, \alpha; \Sigma \rhd \Delta}{\Gamma; \Sigma \rhd \Delta}$$

$$\text{(ContR1)}\ \frac{\Gamma; \Sigma \rhd \Delta, \alpha; \Theta; \alpha}{\Gamma; \Sigma \rhd \Delta, \alpha; \Theta} \qquad \text{(ContR2)}\ \frac{\Gamma; \Sigma \rhd \Delta, \alpha; \Theta; \alpha}{\Gamma; \Sigma, \rhd \Delta; \Theta, \alpha}$$

Examples parallel to those in Facts 1-3 show that ExchR is inadmissible in **S4**$_{FDE}$, **S4**$_{LP}$ and **S4**$_{K3}$, while Cut* and ContR1 are inadmissible in **S4**$_{K3}$, and ContR2 is inadmissible in **S4**$_{LP}$.[14]

[14]For more on the status of cut rules, cf. Wintein (2016, pp. 526-8).

4.2 Many-valued semantics

Wintein gives a four-valued semantics for four-sided sequents. If we require that Θ be empty, and note that LP-valuations have range $\{t, f, b\}$, the result in the case of LP is essentially the semantics for three-sided sequents given in Shapiro (2016, note 16).[15]

Definition 3 *For any sets Γ, Σ, Δ, and Θ of sentences, $\Gamma; \Sigma \vDash_{4X} \Delta; \Theta$ iff there is no X-valuation v such that*
$v(\gamma) = t$ or $v(\gamma) = b$ for all $\gamma \in \Gamma$, and
$v(\sigma) = t$ or $v(\sigma) = n$ for all $\sigma \in \Sigma$ and
$v(\delta) = f$ or $v(\delta) = n$ for all $\delta \in \Delta$ and
$v(\theta) = f$ or $v(\theta) = b$ for all $\theta \in \Theta$

The following soundness and completeness results obtain.

Proposition 4 $\Gamma; \Sigma \rhd \Delta; \Theta$ *is derivable in* ***S4***$_X$ *iff* $\Gamma; \Sigma \vDash_{4X} \Delta; \Theta$.

Proof. To establish the soundness direction, it suffices to check that the initial sequents of **S4**$_X$ correspond to 4X-consequences and that the rules preserve 4X-consequence. The completeness direction follows from the completeness of Wintein's systems (2016, Theorem 1), since his initial sequents and rules are all derivable in **S4**$_X$.

Proposition 5 *(a)* $\Gamma; \emptyset \rhd \Delta$ *is derivable in* ***S3***$_X$ *iff* $\Gamma; \emptyset \vDash_{4X} \Delta; \emptyset$ *and (b)* $\Gamma; \Sigma \rhd \Delta$ *is derivable in* ***S3***$'_X$ *iff* $\Gamma; \Sigma \vDash_{4X} \Delta; \emptyset$.

Important note: it's *not* the case that $\Gamma; \Sigma \rhd \Delta$ is derivable in **S3**$_X$ iff $\Gamma; \Sigma \vDash_{4X} \Delta; \emptyset$. For example, $\emptyset; \alpha \vDash_{4LP} \alpha; \emptyset$, yet $\emptyset; \alpha \rhd \alpha$ isn't derivable in **S3**$_{LP}$. This shows that RightShift is inadmissible. And since $\emptyset \rhd \alpha, \neg\alpha$ is derivable in **S3**$_{LP}$, it also means that the inverse of $\neg$R isn't admissible, whereas by Lemma 1 of the Appendix it is admissible in **S3**$'_{LP}$.

Proof. Again, soundness is easily checked, with $\Gamma; \Sigma \rhd \Delta$ interpreted as the four-sided $\Gamma; \Sigma \rhd \Delta; \emptyset$. Rather than show the completeness directions directly, or via Proposition 4, I give an argument that invokes the completeness of the two-sided **S2**$_X$ (Proposition 1).

Completeness direction for (a): Suppose $\Gamma; \emptyset \vDash_{4X} \Delta; \emptyset$. Then by Definition 2 we have $\Gamma \vDash_X \Delta$. Proposition 1 now entails that $\Gamma \rhd \Delta$ is derivable in **S2**$_X$. But the initial sequents of **S2**$_X$ are derivable in **S3**$_X$, and the rules

[15] The only difference is that, as explained in note 9 above, the sequent structure employed in that paper has no sequent corresponding to $\emptyset; \Gamma \rhd \Delta$.

of $\mathbf{S2}_X$ are included in $\mathbf{S3}_X$, when two-sided sequents $\Gamma \triangleright \Delta$ are interpreted as $\Gamma; \emptyset \triangleright \Delta$. Hence $\Gamma; \emptyset \triangleright \Delta$ is derivable in $\mathbf{S3}_X$.

Completeness direction for (b): Suppose $\Gamma; \Sigma \vDash_{4X} \Delta; \emptyset$. Then it follows from Definitions 1 and 3 that $\Gamma; \emptyset \vDash_{4X} \Delta, \neg\Sigma; \emptyset$,[16] whence by Definition 2 also $\Gamma \vDash_X \Delta, \neg\Sigma$. Completeness of $\mathbf{S2}_X$ now entails that $\Gamma \triangleright \Delta, \neg\Sigma$ is derivable in $\mathbf{S2}_X$. But the initial sequents of $\mathbf{S2}_X$ are again derivable in $\mathbf{S3}'_X$, and the rules of $\mathbf{S2}_X$ are again included, derivable or admissible in $\mathbf{S3}'_X$. To show that $\neg\wedge$R and $\neg\vee$R are admissible, it suffices to show that the inverse of $\neg$R is admissible. (See Lemma 1 of the Appendix.) Hence $\Gamma; \emptyset \triangleright \Delta, \neg\Sigma$ is derivable in $\mathbf{S3}'_X$. By Lemma 1, so is $\Gamma; \Sigma \triangleright \Delta$.

Corollary 1 $\Gamma \triangleright \Delta$ *is is derivable in* $\boldsymbol{S4}_X$ *iff* $\Gamma; \emptyset \vDash_{4X} \Delta; \emptyset$. *And the same holds for* $\boldsymbol{S3}_X$ *and* $\boldsymbol{S3}'_X$.

5 Conclusion

This paper has investigated sequent systems for LP, K3, and FDE that combine two features. First, they preserve versions of all classical rules. Second, the truth of a consequence claim $\Gamma \vDash \Delta$ corresponds to the derivability of a sequent with Γ alone on the left-hand side and Δ alone on the right-hand side. This allows derivations in the sub-classical systems to correspond closely to standard classical derivations. Moreover, we've seen that in some formulations, the systems for LP, K3, FDE, and classical logic differ only in the inclusion of structural rules that combine features of exchange and contraction. Including each such rule amounts to eliminating some distinction drawn by the nonstandard sequent structure. Including the full complement of structural rules accordingly results in a system for classical logic.

In discussion of logically revisionary approaches to paradox, those based on LP, K3, and FDE are often *contrasted* with "substructural approaches" (e.g. Beall & Ripley, 2011). Must we conclude that they should be reclassified as substructural approaches, given the senses we have seen in which these three logics qualify as substructural? If so, the substructural/structural distinction, when applied to approaches to paradox, would appear less interesting than it has been thought to be.[17]

[16] Here $\neg\Sigma$ is the set whose members are the negations of the members of Σ.
[17] See Shapiro (2016) for an extended discussion of this topic.

Appendix

A. Translating between systems

There are straightforward mappings between sequents derivable in the **S4** systems and those derivable in the systems **SK** of Wintein (2016) and $\mathbf{SC}_{RT}$ of Fjellstad (2016).

In **SK**, the **S4**-sequent $\Gamma; \Sigma \rhd \Delta; \Theta$ corresponds to the set of sets of signed sentences $\{\{t, b\} : \Gamma, \{t, n\} : \Sigma, \{f, n\} : \Delta, \{f, b\} : \Theta\}$. Wintein shows that for LP, K3, FDE, and classical logic, $\Gamma \vDash \Delta$ iff $\Gamma \rhd \Delta$ is derivable in the respective fragment of **SK**, where these are distinguished by their initial sequents. In the case of FDE, the initial sequents correspond to those of our **S4**. For LP, he adds the counterpart of $\emptyset; \alpha \rhd \alpha; \emptyset$, while for K3 he adds instead the counterpart of $\alpha; \emptyset \rhd \emptyset; \alpha$. For classical logic, he includes all of the above.

In $\mathbf{SC}_{RT}$, the **S4**-sequent $\Gamma; \Sigma \rhd \Delta; \Theta$ corresponds to different "dual two-sided sequents," depending on whether we are considering its derivability in $\mathbf{S4}_{LP}$ or in $\mathbf{S4}_{K3}$. Considered with regard to $\mathbf{S4}_{LP}$ it corresponds to $\Gamma \Rightarrow_R \Theta \parallel \Sigma \Rightarrow_T \Delta$. Considered with regard to $\mathbf{S4}_{K3}$ it corresponds to $\Sigma \Rightarrow_R \Delta \parallel \Gamma \Rightarrow_T \Theta$. Thus consequences in the two logics correspond to different derivable sequents: $\Gamma \vDash_{LP} \Delta$ iff $\Gamma \Rightarrow_R \emptyset \parallel \emptyset \Rightarrow_T \Delta$ is derivable in $\mathbf{SC}_{RT}$, whereas $\Gamma \vDash_{K3} \Delta$ iff $\emptyset \Rightarrow_R \Delta \parallel \Gamma \Rightarrow_T \emptyset$ is derivable. Finally, $\Gamma \vDash_{CL} \Delta$ iff $\emptyset \Rightarrow_R \emptyset \parallel \Gamma \Rightarrow_T \Delta$ is derivable.[18]

This allows Fjellstad to use the same three initial sequents for LP, K3, and classical logic (I ignore his building in of weakening and his first-order setting).

$$\emptyset \Rightarrow_R \emptyset \parallel \alpha \Rightarrow_T \alpha \qquad \alpha \Rightarrow_R \emptyset \parallel \emptyset \Rightarrow_T \alpha \qquad \emptyset \Rightarrow_R \alpha \parallel \alpha \Rightarrow_T \emptyset$$

The translation reveals that FDE-consequence (which Fjellstad doesn't discuss) can be recovered by removing one of these initial sequents: the first when consequence is read off derivable sequents in the manner that yields LP, or the second when it's read off in the manner that yields K3.

B. The rule $\neg$R in $\mathbf{S3}'_X$ is invertible

Lemma 1 *If* $\Gamma; \Sigma \rhd \Delta, \neg\alpha$ *is derivable in* $\mathbf{S3}'_X$*, then so is* $\Gamma; \Sigma, \alpha \rhd \Delta$.

[18]To be precise, Fjellstad's discussion concerns not a propositional language but a first-order language with a transparent truth predicate. Derivability of $\emptyset \Rightarrow_R \emptyset \parallel \Gamma \Rightarrow_T \Delta$ then corresponds to the nontransitive truth-theoretic consequence $\Gamma \vDash^{st}_{+} \Delta$ of Ripley (2012), which coincides with classical consequence over the truth-free fragment.

Proof. Let $\vdash_{r,X} S$ mean sequent S is derivable in $\mathbf{S3}'_X$ with derivation height at most r. We show by induction that for all r, if $\vdash_{r,X} \Gamma; \Sigma \rhd \Delta, \neg\alpha$ then $\vdash_{r,X} \Gamma; \Sigma, \alpha \rhd \Delta$. The case $r = 0$ is easy, as no initial sequent has a negation on the right. Assume as inductive hypothesis that the conditional holds for all $n < r$, and suppose $\vdash_{r,X} \Gamma; \Sigma \rhd \Delta, \neg\alpha$. We show $\vdash_{r,X} \Gamma; \Sigma, \alpha \rhd \Delta$ by considering the cases.

Suppose the last step in deriving $\Gamma; \Sigma \rhd \Delta, \neg\alpha$ doesn't have $\neg\alpha$ on the right as principal formula. We can use the inductive hypothesis on the rule's premise(s), and then apply the same rule again. The only special case to consider is where the last step uses LeftShift of $\mathbf{S3}'_{K3}$ to derive $\Gamma', \alpha; \Sigma' \rhd \Delta, \neg\alpha$ from $\Gamma'; \Sigma', \alpha \rhd \Delta, \neg\alpha$. Here, the inductive hypothesis gives us $\vdash_{r-1,K3} \Gamma'; \Sigma', \alpha \rhd \Delta$, and a use of Weak shows $\vdash_{r,K3} \Gamma', \alpha; \Sigma', \alpha \rhd \Delta$.

Suppose the last step is by Weak, with $\neg\alpha$ on the right a principal formula. If the step is the trivial one that derives $\Gamma; \Sigma \rhd \Delta, \neg\alpha$ from itself, the inductive hypothesis suffices. In the non-trivial case, use an instance of Weak that differs only in introducing α to the right of the semicolon.

Finally, suppose the step is by $\neg$R with $\neg\alpha$ as principal formula. If it comes from $\Gamma; \Sigma, \alpha \rhd \Delta$, we're done. The other possibility is that it's from $\Gamma; \Sigma, \alpha \rhd \Delta, \neg\alpha$, in which case the inductive hypothesis suffices.

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Lionel Shapiro
University of Connecticut
USA
E-mail: `lionel.shapiro@uconn.edu`