

**Aggregating Sets of Judgments: Two Impossibility Results Compared<sup>1</sup>**  
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**Abstract.** The “doctrinal paradox” or “discursive dilemma” shows that propositionwise majority voting over the judgments held by multiple individuals on some interconnected propositions can lead to inconsistent collective judgments on these propositions. List and Pettit (2002) have proved that this paradox illustrates a more general impossibility theorem showing that there exists no aggregation procedure that generally produces consistent collective judgments and satisfies certain minimal conditions. Although the paradox and the theorem concern the aggregation of judgments rather than preferences, they invite comparison with two established results on the aggregation of preferences: the Condorcet paradox and Arrow’s impossibility theorem. We may ask whether the new impossibility theorem is a special case of Arrow’s theorem, or whether there are interesting disanalogies between the two results. In this paper, we compare the two theorems, and show that they are not straightforward corollaries of each other. We further suggest that, while the framework of preference aggregation can be mapped into the framework of judgment aggregation, there exists no obvious reverse mapping. Finally, we address one particular minimal condition that is used in both theorems – an independence condition – and suggest that this condition points towards a unifying property underlying both impossibility results.

## 1. Introduction

May’s celebrated theorem (1952) shows that, if a group of individuals wants to make a choice between two alternatives (say  $x$  and  $y$ ), then majority voting is the unique decision procedure satisfying a set of attractive minimal conditions. The conditions are (i) universal domain: the decision procedure should produce an outcome ( $x$ ,  $y$  or tie) for any logically possible combination of individual votes over  $x$  and  $y$ ; (ii) anonymity: the collective choice should be invariant under permutations of the individual votes, i.e. all individual votes should have equal weight; (iii) neutrality: if the individual votes for  $x$  and  $y$  are swapped, then the outcome should be swapped in the same way, i.e. the labels of the alternatives should not matter; (iv) positive responsiveness: supposing all other votes remain the same, if one individual changes his or her vote in favour of a winning alternative, then this alternative should remain the outcome; if there was previously a tie, a change of one individual vote should break the tie in the direction of that change.

May’s theorem is often interpreted as a vindication of majoritarian democracy when a collective decision between two alternatives is to be made. Many collective decision problems are, however, more complex. They may not be confined to a binary choice between two alternatives, or between the acceptance or rejection of a single proposition.

Suppose there are three or more alternatives (say  $x$ ,  $y$  and  $z$ ). In that case, it may seem natural to determine an overall collective preference ranking of these alternatives by applying majority voting to each pair of alternatives. But, unfortunately, pairwise majority voting may lead to cyclical collective preferences. Suppose individual 1 prefers  $x$  to  $y$  to  $z$ , individual 2 prefers  $y$  to  $z$  to  $x$ , and individual 3 prefers  $z$  to  $x$  to  $y$ .

Then there are majorities of two out of three for  $x$  against  $y$ , for  $y$  against  $z$ , and for  $z$  against  $x$ , a cycle. This is Condorcet's paradox.

But a greater number of alternatives is not the only way in which a collective decision problem may deviate from the single binary choice framework of May's theorem. A collective decision problem may also involve simultaneous decisions on the acceptance or rejection of multiple interconnected propositions. For instance, a policy package or a legal decision may consist of multiple propositions which mutually constrain each other. To ensure consistency, the acceptance or rejection of some of these propositions may constrain the acceptance or rejection of others. Once again, a natural suggestion would be to apply majority voting to each proposition separately. As we will see in detail below, however, this method also generates a paradox, sometimes called the 'doctrinal paradox' or 'discursive dilemma': propositionwise majority voting over multiple interconnected propositions may lead to inconsistent collective sets of judgments on these propositions.

We have thus identified two dimensions along which a collective decision problem may deviate from the single binary choice framework of May's theorem: (a) the number of alternatives, and (b) the number of interconnected propositions on which simultaneous decisions are to be made. Deviations along each of these dimensions lead to a breakdown of the attractive properties of majority voting highlighted by May's theorem. Deviations along dimension (a) can generate Condorcet's paradox: pairwise majority voting over multiple alternatives may lead to cyclical collective preferences. And deviations along dimension (b) can generate the 'doctrinal paradox' or 'discursive dilemma': propositionwise majority voting over multiple interconnected propositions may lead to inconsistent collective sets of judgments on these propositions.

In each case, we can ask whether the paradox is just an artefact of majority voting in special contrived circumstances, or whether it actually illustrates a more general problem. Arrow's impossibility theorem (1951/1963) famously affirms the latter for dimension (a): Condorcet's paradox brings to the surface a more general impossibility problem of collective decision making between three or more alternatives. But Arrow's theorem does not apply straightforwardly to the case of dimension (b). List and Pettit (2002) have shown that the 'doctrinal paradox' or 'discursive dilemma' also illustrates a more general impossibility problem, this time regarding simultaneous collective decisions on multiple interconnected propositions. The two impossibility theorems are related, but not identical. Arrow's result makes it less surprising to find that an impossibility problem pertains to the latter decision problem too, and yet the two theorems are not trivial corollaries of each other.

The aim of this paper is to compare these two impossibility results and to explore their connections and dissimilarities. In sections 2 and 3 we briefly introduce, respectively, Arrow's theorem and the new theorem on the aggregation of sets of judgments. In section 4 we address the question of whether the two generalizations of May's single binary choice framework – the framework of preferences over three or more options and the framework of sets of judgments over multiple interconnected propositions – can somehow be mapped into each other. In section 5, by reinterpreting preferences as ranking judgments, we derive a simple impossibility theorem on the aggregation of preferences from the theorem on the aggregation of sets of judgments,

and compare the result with Arrow's theorem. A formal proof of the result is given in an appendix. In section 6 we discuss escape-routes from the two impossibility results, and indicate their parallels. In section 7, finally, we explore the role of two crucial conditions underlying the two impossibility theorems – independence of irrelevant alternatives and systematicity – and identify a unifying mechanism generating both impossibility problems.

## 2. Arrow's Impossibility Theorem

May's theorem concerns a single collective decision between two alternatives. Arrow's theorem concerns a single collective decision between more than two alternatives. Such decision problems arise, for instance, in elections with three or more candidates, or when the task is to rank several restaurants, several candidates or several policy alternatives in an order of collective preference.

Suppose there is a set of individuals,  $N$ , each named by a numeral: 1, 2, 3, ...,  $n$ ; we assume that  $N$  has at least two members. Suppose further there is a set of alternatives,  $X$ , each named by a letter  $x, y, z, \dots$ ; we assume that  $X$  has at least three members. This is the crucial deviation from the binary choice framework of May's theorem.

Each individual,  $i$ , has a personal preference ordering over the alternatives in  $X$ , labelled  $R_i$ , where  $xR_iy$  is interpreted to mean "from individual  $i$ 's perspective,  $x$  is at least as good as  $y$ ". The ordering  $R_i$  is assumed to satisfy three rationality properties: reflexivity, transitivity and connectedness.  $R_i$  is reflexive: for any alternative  $x$  in  $X$ , we have  $xR_ix$  ("an alternative is always at least as good as itself").  $R_i$  is transitive: for any alternatives  $x, y$  and  $z$  in  $X$ , if  $xR_iy$  and  $yR_iz$ , then we must also have  $xR_iz$  ("if  $x$  is at least as good as  $y$ , and  $y$  is at least as good as  $z$ , then  $x$  is at least as good as  $z$ ").  $R_i$  is connected: for any alternatives  $x$  and  $y$  in  $X$ , we must have either  $xR_iy$  or  $yR_ix$  ("two alternatives are always comparable: either  $x$  is at least as good as  $y$ , or  $y$  is at least as good as  $x$ "). The ordering  $R_i$  also induces a strong ordering,  $P_i$ . This is defined as follows: for all  $x$  and  $y$  in  $X$ ,  $xP_iy$  if and only if  $xR_iy$  and not  $yR_ix$ , where  $xP_iy$  is interpreted to mean "from individual  $i$ 's perspective,  $x$  is strictly better than  $y$ ".

We can now define the *profile of personal preference orderings* that are held across the group,  $N$ , as the  $n$ -tuple of those personal preference orderings, containing one for each individual:  $\{R_1, R_2, \dots, R_n\}$  or in short  $\{R_i\}_{i \in N}$ .

Now the question is whether there is any aggregation procedure which takes as its input a profile of reflexive, transitive and connected personal preference orderings across the individuals in  $N$ , and which produces as its output a collective preference ordering,  $R$ , which is also reflexive, transitive and connected. Such a procedure is called a *social welfare function* and labelled  $F$ .  $xRy$  is interpreted to mean "from the perspective of the group  $N$ ,  $x$  is at least as good as  $y$ ". In analogy with the personal preference orderings,  $R$  also induces a strong ordering  $P$ .

The problem Arrow's theorem addresses is whether there is a social welfare function,  $F$ , that satisfies some minimal conditions. Let  $\{R_i\}_{i \in N}$  be a profile of personal preference orderings, and let  $R$  be the corresponding collective preference ordering produced by the social welfare function  $F$ , i.e.  $R = F(\{R_i\}_{i \in N})$ . The conditions are:

**UNIVERSAL DOMAIN (U).** The domain of  $F$ , i.e. the set of admissible inputs to  $F$ , is the set of all logically possible profiles of reflexive, transitive and connected personal preference orderings.

Condition (U) is the requirement that any logically possible combination of personal preference orderings across individuals be admissible as input to the aggregation, provided that each individual holds a personal preference ordering which satisfies the conditions of reflexivity, transitivity and connectedness.

**WEAK PARETO PRINCIPLE (P).** If, for all individuals  $i$  in  $N$ ,  $xP_iy$ , then  $xPy$ .

Condition (P) is the requirement that, if all individuals unanimously prefer one alternative to another, then this alternative should also be collectively preferred to the other.

**INDEPENDENCE OF IRRELEVANT ALTERNATIVES (I).** Suppose  $\{R_i\}_{i \in N}$  and  $\{R^*_i\}_{i \in N}$  are two profiles of personal preference orderings in the domain of  $F$  and  $x$  and  $y$  are two alternatives in  $X$  such that, for all individuals  $i$  in  $N$ ,  $xR_iy$  if and only if  $xR^*_iy$ . Then  $xRy$  if and only if  $xR^*y$ .

Condition (I) is the requirement that the collective ranking of any pair of alternatives should depend exclusively on the individual rankings over that pair.

**NON-DICTATORSHIP (D).**  $F$  is not dictatorial: there does not exist an individual  $i$  in  $N$  such that, for all profiles of personal preference orderings in the domain of  $F$ , if  $xP_iy$ , then  $xPy$ .

Condition (D) is the requirement that there should not be one individual, a *dictator*, whose preference (except possibly in cases of indifference) always determines the collective preference.

Pairwise majority voting over the alternatives in  $X$  constitutes a social welfare function which satisfies (U), (P), (I) and (D), but which sometimes fails to generate transitive collective preference orderings, as Condorcet's paradox shows. Arrow's theorem shows that this problem is not just an artefact of pairwise majority voting.

**Theorem 1.** (Arrow, 1951/1963) There exists no social welfare function (generating reflexive, transitive and connected collective preference orderings) which satisfies (U), (P), (I) and (D).

A large literature addresses the question of what Arrow's theorem does and does not show (see, amongst many others, Riker, 1982; Saari, 1998; Bossert and Weymark, 1996; Dryzek and List, 2003). As we here aim to compare Arrow's theorem with another impossibility theorem, we will now turn to the exposition of that new result.

### 3. An Impossibility Theorem Regarding the Aggregation of Sets of Judgments

Arrow's theorem addresses decision problems that differ from the ones addressed by May's theorem in the number of alternatives. We will now address decision problems that preserve the binary choice structure of May's framework, but that differ from

May's framework in that they involve simultaneous decisions on multiple interconnected propositions. Such problems will be called problems of *aggregating sets of judgments*.

Judgments are modelled on acts of acceptance or rejection of certain propositions, and differ from credences in not allowing of *degrees* of confidence. Under the present concept of judgment, a judgment is always an on or off affair; either someone accepts a certain proposition, or they do not. The problem of aggregating sets of judgments arises when a group has to come up with an overall set of judgments on a set of interconnected propositions – on a set of propositions where the acceptance or rejection of certain propositions constrains the acceptance or rejection of others – and when it has to do so on the basis of the judgments individuals make on these propositions.

Just as Condorcet's paradox serves to introduce the aggregation problem addressed by Arrow's theorem, the present aggregation problem can be introduced by a paradox, sometimes called the 'doctrinal paradox' or 'discursive dilemma'.

Suppose a three-judge court has to make a decision on whether a defendant is liable under a charge of breach of contract (Kornhauser and Sager, 1993, p. 11). According to legal doctrine, the court is required to find that the defendant is liable ( $R$ ) if and only if it finds, first that a valid contract was in place ( $P$ ), and second that the defendant's behaviour was such as to breach a contract of that kind ( $Q$ ), formally ( $R \leftrightarrow (P \wedge Q)$ ). Now suppose that the three judges, 1, 2 and 3, vote as follows on those issues and on the derivable matter of whether the defendant is indeed liable. The 'true' or 'false' on any row represents the disposition of the relevant judge to accept or reject the corresponding proposition.

	valid contract $P$	breach $Q$	legal doctrine $(R \leftrightarrow (P \wedge Q))$	defendant liable $R$
1	yes	yes	yes	yes
2	yes	no	yes	no
3	no	yes	yes	no

**Table 1**

Note that each judge holds a perfectly consistent set of judgments in light of the proposition that  $R$  should be accepted if and only if both  $P$  and  $Q$  are accepted.

Now if we apply majority voting to each proposition then  $P$  ("there was a valid contract in place") and  $Q$  ("there was a breach") will each be accepted by a majority of two out of three, but  $R$  ("the defendant is liable") will be rejected by a majority of two out of three, in spite of the unanimous acceptance of the proposition that  $R$  should be accepted if and only if both  $P$  and  $Q$  are accepted, a contradiction. This shows that propositionwise majority voting can generate an inconsistent collective set of judgments, even when each individual holds a perfectly consistent set of judgments.

Like Condorcet's paradox, the paradox on propositionwise majority voting raises the question of whether the paradox is just a contrived theoretical artefact, or whether it illustrates a more general problem of aggregation. We shall now turn to the

presentation of a more general impossibility result, first presented in List and Pettit (2002).

Suppose, again, there is a set of individuals,  $N$ , each named by a numeral: 1, 2, 3, ...,  $n$ , where  $N$  has at least two members. Let  $\mathcal{E}$  be the set of propositions on which judgments are to be made, including atomic propositions like  $P$  and  $Q$  and compound propositions like  $(P \wedge Q)$ ,  $(P \vee Q)$ ,  $(R \leftrightarrow (P \wedge Q))$ , and so on; we assume that  $\mathcal{E}$  contains at least two distinct atomic propositions,  $P$ ,  $Q$ , their conjunction,  $(P \wedge Q)$ , and the negation of their conjunction,  $\neg(P \wedge Q)$ .

We identify the set of judgments made by an individual,  $i$ , over the propositions in  $\mathcal{E}$  – we call them  $i$ 's *personal set of judgments* – with the set of those propositions in  $\mathcal{E}$  which individual  $i$  accepts. The individual's personal set of judgments is thus a subset  $\Phi_i$  of  $\mathcal{E}$ . Note that non-acceptance of a proposition (the non-inclusion of that proposition in  $\Phi_i$ ) does not involve making a judgment in respect of it; in the case of non-acceptance of a proposition, individual  $i$  will make a judgment only if he or she accepts the negation of the proposition (i.e. if the negation of the proposition is included in  $\Phi_i$ ).

The personal set of judgments  $\Phi_i$  is assumed to satisfy three rationality properties: *completeness*, *consistency* and *deductive closure*.  $\Phi_i$  is *complete*: for all propositions  $\phi$  in  $\mathcal{E}$ , either  $\phi$  or its negation  $\neg\phi$  (or unnegated form  $\psi$  in case  $\phi = \neg\psi$ ) is contained in  $\Phi_i$  ("an individual always accepts a proposition or its negation").  $\Phi_i$  is *consistent*: there is no proposition  $\phi$  in  $\mathcal{E}$  such that both  $\phi$  and its negation  $\neg\phi$  are contained in  $\Phi_i$  ("an individual never accepts a proposition and its negation simultaneously").  $\Phi_i$  is *deductively closed*: whenever  $\Phi_i$  logically entails some other proposition  $\psi$  in  $\mathcal{E}$ ,  $\psi$  is also contained in  $\Phi_i$  ("an individual always accepts the logical consequences of what he or she accepts").

In analogy to a profile of personal preference orderings, we define the *profile of personal sets of judgments* that are held across the group,  $N$ , as the  $n$ -tuple of those personal sets of judgments, containing one for each individual:  $\{\Phi_1, \Phi_2, \dots, \Phi_n\}$  or in short  $\{\Phi_i\}_{i \in N}$ .

This time the question is whether there is any aggregation procedure which takes as its input a profile of complete, consistent and deductively closed personal sets of judgements across the individuals in  $N$ , and which produces as its output a collective set of judgments,  $\Phi$ , which is also complete, consistent and deductively closed. Such a procedure is called a *judgment aggregation function* and labelled  $F$ .  $\Phi$  is also a subset of  $\mathcal{E}$ , and is interpreted as the set of those propositions in  $\mathcal{E}$  which are collectively accepted.

Specifically, the question is whether there is a judgment aggregation function,  $F$ , that satisfies some minimal conditions. Let  $\{\Phi_i\}_{i \in N}$  be a profile of personal sets of judgments, and let  $\Phi$  be the corresponding collective set of judgments produced by the judgment aggregation function  $F$ , i.e.  $\Phi = F(\{\Phi_i\}_{i \in N})$ . The conditions are:

**UNIVERSAL DOMAIN (U).** The domain of  $F$ , i.e. the set of admissible inputs to  $F$ , is the set of all logically possible profiles of complete, consistent and deductively closed personal sets of judgments.

Condition (U) is the requirement that any logically possible combination of personal sets of judgments across individuals be admissible as input to the aggregation, provided that each individual holds a personal set of judgments which satisfies the conditions of completeness, consistency and deductive closure.

**ANONYMITY (A).** For any  $\{\Phi_i\}_{i \in N}$  in the domain of  $F$  and any permutation  $\sigma: N \rightarrow N$ ,  $F(\{\Phi_i\}_{i \in N}) = F(\{\Phi_{\sigma(i)}\}_{i \in N})$ .

Condition (A) is the requirement that the collective set of judgments be invariant under any permutation of the individuals in  $N$ , or, informally, that all individuals be given equal weight in the aggregation.

**SYSTEMATICITY (S).** There exists a function  $f: \{0, 1\}^n \rightarrow \{0, 1\}$  such that, for any  $\{\Phi_i\}_{i \in N}$  in the domain of  $F$ ,  $F(\{\Phi_i\}_{i \in N}) = \{\phi \in \mathcal{E} : f(\delta_1(\phi), \delta_2(\phi), \dots, \delta_n(\phi)) = 1\}$ , where, for each  $i$  in  $N$  and each  $\phi$  in  $\mathcal{E}$ ,  $\delta_i(\phi) = 1$  if  $\phi \in \Phi_i$  and  $\delta_i(\phi) = 0$  if  $\phi \notin \Phi_i$ .

Condition (S) is the requirement that (i) the collective judgment on each proposition should depend exclusively on the pattern of individual judgments on that proposition and (ii) the same pattern of dependence should hold for all propositions.

Propositionwise majority voting over the propositions in  $\mathcal{E}$  is a judgment aggregation function which satisfies (U), (A) and (S), but which sometimes fails to generate complete, consistent and deductively closed collective sets of judgments, as the 'doctrinal paradox' or 'discursive dilemma' shows. List and Pettit have shown that this problem is not just an artefact of propositionwise majority voting.

**Theorem 2.** (List and Pettit, 2002) There exists no judgment aggregation function  $F$  (generating complete, consistent and deductively closed collective sets of judgments) which satisfies (U), (A) and (S).

Every procedure, not just propositionwise majority voting, will fail to satisfy all the conditions of theorem 2 simultaneously.

#### **4. Can the Framework of Preferences and the Framework of Sets of Judgments be Mapped into Each Other?**

To address the question of how Arrow's theorem and the theorem on the aggregation of sets of judgments are related to each other, it is useful to ask whether the two frameworks -- the framework of preference orderings and the framework of sets of judgments -- can be mapped into each other. We shall first address the question of whether the framework of sets of judgments can be interpreted as a special case of the framework of preferences, and we shall then address the question of whether the framework of preferences can be interpreted as a special case of the framework of sets of judgments. We will argue that the latter is the more promising approach towards connecting the two frameworks.

### Transcribing into the framework of preferences

To ask whether the framework of sets of judgments can be interpreted as a special case of the framework of preferences, we need to specify the set of alternatives,  $X$ , over which the relevant preferences are to be held. The most obvious possibility is the following.<sup>2</sup> Each logically possible *set of judgments* over the propositions in  $\mathcal{E}$  is interpreted as a single alternative. Then the set of alternatives,  $X$ , is the set of all logically possible sets of judgments over the relevant propositions. The alternatives (or 'candidates') are thus not individual propositions or their negations, but rather entire *sets* of judgments, each in one 'package'. Under this approach, each individual's most preferred alternative is precisely the set of judgments accepted by that individual. But, apart from this top preference, we have no further information about an individual's preferences. For instance, given two sets of judgments both of which are different from the individual's most preferred set of judgments, we have no information about how the individual would rank these two sets. Of course, we might make certain *assumptions* about when an individual would prefer one set of judgments to another, using, for instance, the symmetrical distance of a set of judgments from the individual's most preferred set of judgments as a measure of (dis)preference for that set.<sup>3</sup> But such assumptions would be stipulations beyond the information contained in the framework of sets of judgments. Hence, if the task is to genuinely *map* the framework of sets of judgments into the framework of preferences, each individual's set of judgments induces only an *incomplete* preference ordering over all logically possible sets of judgments. The ordering would determine a most highly ranked alternative, but it would be silent on the relative rankings of all other alternatives.

This observation does not imply that a problem of determining a collective set of judgments can never be interpreted as an appropriate problem of preference aggregation. In particular, suppose we define the set  $X$  to be the set of all logically possible sets of judgments over the propositions in  $\mathcal{E}$ . And suppose, further, that we assign, to each individual, not just a single most preferred set of judgments, i.e. a single most preferred alternative in  $X$ , but rather a complete preference ordering over the set  $X$ . Then the problem of determining a collective set of judgments becomes the problem of aggregating the individual preference orderings over  $X$  into a collectively most preferred alternative. But this strategy clearly requires a richer informational basis than the one used by the theorem on the aggregation of sets of judgments and is therefore unsuitable for mapping the framework of this theorem into the framework of preferences.<sup>4</sup>

It is worth noting, in connection with this strategy, that when transcribed into the framework of preferences instances of the discursive dilemma do not always constitute instances of the Condorcet paradox; and equally instances of the Condorcet paradox do not always constitute instances of the discursive dilemma (see Kornhauser 1992b, 453-57). Consider the situation represented in Table I where a discursive dilemma arises and imagine that the orderings of the parties, 1, 2 and 3, over sets of judgments on the questions of validity and breach, respectively — on other questions we may assume that the answers go accordingly — are as follows:



- 1: Yes-No > No-Yes > Yes-Yes > No-No
- 2: Yes-Yes > No-Yes > Yes-No > No-No
- 3: No-Yes > Yes-Yes > Yes-No > No-No.

If the orderings of the parties follow this pattern, then the No-Yes pair — and therefore the conclusion in favour of No — is a Condorcet winner. It is preferred by a majority to each of the other three pairs: it is preferred by 1 and 3 to Yes-Yes; it is preferred by 2 and 3 to Yes-No; and it is preferred by all to No-No. Thus in such a scenario, there would be a discursive dilemma but no Condorcet paradox.

It is equally easy to illustrate the second possibility, in which we have a Condorcet paradox but no discursive dilemma. We will have a Condorcet paradox if the orderings of the parties in a certain scenario have this structure:

- 1: No-No > Yes-No > No-Yes > Yes-Yes
- 2: No-Yes > No-No > Yes-No > Yes-Yes
- 3: Yes-No > No-Yes > No-No > Yes-Yes.

Given these orderings, a majority (1 and 2) will prefer No-No to Yes-No; a majority (1 and 3) will prefer Yes-No to No-Yes; and yet, paradoxically, a majority (2 and 3) will prefer No-Yes to No-No. But despite the presence of a Condorcet paradox in such a scenario, there will be no discursive dilemma. The parties will vote on the premises according to their most preferred pairs: that is, No-No; No-Yes; and Yes-No. And that will not give rise to the dilemma.

#### Transcribing into the framework of judgments

We will now address the converse question of whether the framework of preferences can be interpreted as a special case of the framework of judgments. A preference ordering is a binary relation, which in turn can be represented as a set of propositions of *first-order predicate calculus*. For instance, the reflexive, complete and transitive ordering  $R_i$  on the set of alternatives  $\{x, y, z\}$  which ranks the alternatives in the order  $x > y > z$ , can be represented as the set of propositions  $P_i := \{xRy, yRz, xRz, xRx, yRy, zRz\}$  or, to make it complete, consistent and deductively closed and to include the rationality conditions of orderings as well,  $P_i^* := \{xRy, yRz, xRz, xRx, yRy, zRz, \neg yRx, \neg zRy, \neg zRx, \forall w(wRw)$  ("reflexivity"),  $\forall u\forall v\forall w((uRv \vee vRw) \rightarrow uRw)$  ("transitivity"),  $\forall v\forall w(vRw \vee wRv)$  ("connectedness")) (or, if necessary, the deductive closure of the latter set). In this fashion, every ordering  $R_i$  can be represented as a set of propositions from the predicate calculus  $P_i^*$ .  $P_i^*$  can be interpreted as a set of ranking judgments.

The Condorcet paradox can now be rendered as follows:

	$xRy$	$yRz$	$xRz$	$yRx$	$zRy$	$zRx$	$\forall w(wRw)$ "reflexivity"	$\forall u\forall v\forall w$ $((uRv \vee vRw) \rightarrow uRw)$ "transitivity"	$\forall v\forall w$ $(vRw \vee wRv)$ "connectedness"
1 $(x>y>z)$	yes	yes	yes	no	no	no	yes	yes	yes
2 $(y>z>x)$	no	yes	no	yes	no	yes	yes	yes	yes
3 $(z>x>y)$	yes	no	no	no	yes	yes	yes	yes	yes
majority	yes	yes	no	no	no	yes	yes	yes	yes

Table 2

Each individual holds a perfectly consistent set of ranking judgments in light of the propositions of reflexivity, transitivity and connectedness.

If we apply majority voting to each proposition, then  $xRy$ ,  $yRz$  and  $zRx$  will each be accepted by a majority of two out of three, but  $xRz$  will be rejected by a majority of two out of three, in spite of the unanimous acceptance of the proposition that the conjunction of  $xRy$  and  $yRz$  implies  $xRz$  (transitivity), a contradiction. Propositionwise majority voting over individual ranking judgments generates an inconsistent collective set of ranking judgments, although each individual holds a perfectly consistent set of ranking judgments.

Condorcet's paradox can thus be interpreted as a specific version of the 'doctrinal paradox' or 'discursive dilemma', applied to ranking judgments. Under this interpretation, the aggregation problem is indeed one of aggregating judgments over multiple interconnected propositions. The propositions are ranking propositions expressed in first-order predicate calculus, and their logical interconnections are specified by the conditions of reflexivity, transitivity and connectedness, also expressed as propositions of first-order predicate calculus. Likewise, Arrow's theorem can be interpreted as a result about the aggregation of sets of ranking judgments, interconnected by the conditions of reflexivity, transitivity and connectedness, all expressed in first-order predicate calculus.

Does this make Arrow's theorem a logical consequence of the theorem on the aggregation of sets of judgments? Not quite. As we have seen, the expression of ranking judgments requires the use of first-order predicate calculus. The theorem on the aggregation of sets of judgments, on the other hand, concerns propositions of the propositional calculus. The latter cannot fully represent the rich internal structure of ranking propositions and of the connection rules of reflexivity, transitivity and connectedness, all of which contain quantifiers.<sup>5</sup>

But in the next section we will see that the mathematical mechanism underlying the proof of the propositional-calculus version of the theorem on the aggregation of sets of judgments can be generalized so as to apply to sets of propositions of first-order predicate calculus, and can thus be used to prove a simple impossibility theorem on the aggregation of ranking judgments. The new theorem is closely related to, but not identical with, Arrow's theorem, and serves as an illustration of the exact logical relation between the two theorems we here aim to compare.

## 5. A Simple Impossibility Theorem on the Aggregation of Ranking Judgments

The basic framework is similar to, but not identical with, the framework of Arrow's theorem. As before, let  $N$  be the set of individuals, where  $N$  has at least two members. Let  $X$  be the set of alternatives, including at least three alternatives, say  $x, y, z$ . Here we assume that each individual,  $i$ , has a *strong* personal preference ordering,  $P_i$ , over the set of alternatives, where  $P_i$  satisfies *irreflexivity*, *transitivity* and *completeness*.  $P_i$  is *irreflexive*: there exists no alternative  $x$  in  $X$  such that  $xP_i x$  ("an alternative is never better than itself").  $P_i$  is *transitive*: for any alternatives  $x, y$  and  $z$  in  $X$ , if  $xP_i y$  and  $yP_i z$ , then we must also have  $xP_i z$  ("if  $x$  is better than  $y$ , and  $y$  is better than  $z$ , then  $x$  is better than  $z$ ").  $P_i$  is *complete*: for any two *distinct* alternatives  $x$  and  $y$  in  $X$ , we must have either  $xP_i y$  or  $yP_i x$  ("two distinct alternatives are always comparable: either  $x$  is better than  $y$ , or  $y$  is better than  $x$ "). The requirement that each individual should have a *strong* personal preference ordering is more demanding than the corresponding assumption of Arrow's theorem.

The following interpretation illustrates how the strong personal preference orderings can be interpreted in terms of ranking judgments. The set of propositions on which judgments are to be made,  $\mathcal{E}$ , is the set of all logically possible ranking propositions over pairs of distinct alternatives in  $X$ , i.e.  $\mathcal{E} = \{xPy : x, y \in X, x \neq y\}$ . For any pair of alternatives in  $X$ ,  $x$  and  $y$ , individual  $i$  accepts the judgment  $xPy$  if that individual's preference ordering  $P_i$  ranks  $x$  above  $y$ , i.e. if  $xP_i y$ . The negation of a proposition  $xPy$  in  $\mathcal{E}$  is simply  $yPx$ .

The conditions of irreflexivity, transitivity and completeness can be reinterpreted in terms of the conditions of completeness, consistency and deductive closure, as defined in the context of sets of judgments. The completeness of each  $P_i$  entails the completeness of individual  $i$ 's set of judgments over  $\mathcal{E}$ : an individual always accepts either a proposition or its negation, i.e. at least one of  $xPy$  or  $yPx$  for each such proposition-negation pair. The irreflexivity and transitivity of each  $P_i$  entails the consistency of individual  $i$ 's set of judgments over  $\mathcal{E}$ : an individual never accepts both a proposition and its negation, i.e. never both  $xPy$  and  $yPx$ . The transitivity of each  $P_i$ , finally, entails the deductive closure of individual  $i$ 's set of judgments over  $\mathcal{E}$ : interpreting  $xPz$  as a logical consequence of  $xPy$  and  $yPz$ , an individual always accepts the logical consequences of other propositions he or she accepts.

A few more definitions are due. The *profile of strong personal preference orderings* that are held across the group,  $N$ , is the  $n$ -tuple of those strong personal preference orderings, containing one for each individual:  $\{P_1, P_2, \dots, P_n\}$  or in short  $\{P_i\}_{i \in N}$ . Further, define a ranking judgment aggregation function to be a function,  $F$ , which takes as its input a profile of irreflexive, transitive and complete strong personal preference orderings across the individuals in  $N$ , and which produces as its output a strong collective preference ordering,  $P$ , which is also irreflexive, transitive and complete.

The conditions of universal domain, anonymity and systematicity, as stated for the theorem on the aggregation of sets of judgments, can now be restated for the context of ranking judgments. Let  $\{P_i\}_{i \in N}$  be a profile of strong personal preference

orderings, and let  $P$  be the corresponding collective preference ordering produced by the ranking judgment aggregation function  $F$ , i.e.  $P = F(\{P_i\}_{i \in N})$ .

Condition (U) entails the requirement that any logically possible combination of strong personal preference orderings across individuals be admissible as input to the aggregation, provided that each individual holds a strong personal preference ordering which satisfies the conditions of irreflexivity, transitivity and completeness. Note that this condition is equivalent to the condition of universal domain in the framework of Arrow's theorem, except that the input and output to the aggregation is restricted to *strong* orderings.

**UNIVERSAL DOMAIN (U).** The domain of  $F$  is the set of all logically possible profiles of irreflexive, transitive and complete strong personal preference orderings.

Condition (A) entails the requirement that the strong collective preference ordering be invariant under any permutation of the individuals in  $N$ , or, informally, that all individuals be given equal weight in the aggregation. Note that this condition is more demanding than the condition of non-dictatorship in the framework of Arrow's theorem. It entails, but is not entailed by, non-dictatorship.

**ANONYMITY (A).** For any  $\{P_i\}_{i \in N}$  in the domain of  $F$  and any permutation  $\sigma : N \rightarrow N$ ,  $F(\{P_i\}_{i \in N}) = F(\{P_{\sigma(i)}\}_{i \in N})$ .

Condition (S) entails the requirement that (i) each collective pairwise ranking judgment should depend exclusively on the pattern of individual ranking judgments over the same pair of alternatives and (ii) the same pattern of dependence should hold for all pairwise ranking judgments. Part (i) of this requirement is equivalent to the condition of independence of irrelevant alternatives in the framework of Arrow's theorem. However, while independence of irrelevant alternatives still permits differences in the dependencies that hold for ranking judgments over different pairs of alternatives, part (ii) of the present requirement disallows such differences. Thus condition (S) entails what is called independence of non-welfare characteristics in the framework of Arrowian social choice. Independence of non-welfare characteristics entails, but is not entailed by, independence of irrelevant alternatives.

**INDEPENDENCE OF NON-WELFARE CHARACTERISTICS (INW).** Suppose  $\{P_i\}_{i \in N}$  and  $\{P_i^*\}_{i \in N}$  are two profiles of personal preference orderings in the domain of  $F$  and  $x_1, y_1$  ( $x_1 \neq y_1$ ) and  $x_2, y_2$  ( $x_2 \neq y_2$ ) are two pairs of alternatives in  $X$  such that, for all individuals  $i$  in  $N$ ,  $x_1 P_i y_1$  if and only if  $x_2 P_i^* y_2$ . Then  $x_1 P y_1$  if and only if  $x_2 P^* y_2$ .

The theorem on the aggregation of sets of judgments can now be restated for ranking judgments. The new theorem 3 is a variant of theorem 2 for the special case of the aggregation of ranking judgments. A formal proof of the new theorem is given in an appendix.

**Theorem 3.** There exists no ranking judgment aggregation function  $F$  (generating irreflexive, transitive and complete strong collective preference orderings) which satisfies (U), (A) and (INW).

Comparing theorem 1 (Arrow's theorem) and theorem 3, we make the following observations. First, Arrow's theorem imposes a less stringent requirement upon the nature of the input and the output of the aggregation: while Arrow's theorem allows weak orderings in both the input and the output to the aggregation, the new theorem requires strong orderings. Second, Arrow's theorem uses the less demanding condition of non-dictatorship instead of anonymity, although the conditions are of course similar in spirit and the condition of anonymity is arguably very compelling. Third, Arrow's theorem uses the less demanding condition of independence of irrelevant alternatives instead of independence of non-welfare characteristics. This difference may be less innocent. Independence of irrelevant alternatives is often already considered to be implausibly demanding, so that a further strengthening of the independence condition, such as the one in theorem 3, may seem to weaken the force of the resulting impossibility theorem. In section 7 below, we shall turn to the question of how much potentially relevant information about the individual input an aggregation procedure ignores when it respects each of the two independence conditions. Fourth, however, Arrow's theorem requires the weak Pareto principle, while the new theorem requires no such condition at all. In spite of the three respects in which the new theorem requires more demanding conditions than Arrow's theorem – namely, strong instead of weak orderings, (A) instead of (D), and (INW) instead of (I) – the fourth point is striking. Theorem 3 demonstrates that an impossibility result on the aggregation of preferences can be derived without invoking any version of the Pareto principle or any other unanimity condition. In other words, not even Pareto-inefficient aggregation procedures will satisfy the conditions of theorem 3. It should further be noted that, since theorem 3 holds for any number of individuals, in particular an odd number, the new result is not just a trivial result about the occurrence of ties, which strong orderings cannot handle.

## **6. Escape-Routes from The Two Impossibility Results**

As is well known in the case of Arrow's theorem and as List and Pettit (2002) have shown in the case of the theorem on the aggregation of sets of judgments, each condition of each of the two theorems can be associated with a corresponding escape-route from the impossibility problem. The main interest of the theorems — in particular, the main interest of our result — lies in the fact that they direct us to the such escape-routes. We shall here focus only on a few escape-routes that serve to illustrate the connections between the two theorems.

### Relaxation of universal domain

Suppose the input to the aggregation is relatively 'cohesive' across individuals. Specifically, suppose that an aggregation procedure is required to accept as admissible input not all logically possible profiles of personal preference orderings, or of personal sets of judgments, but only those profiles satisfying a certain structure condition: single-peakedness or value-restriction in the framework of preferences, unidimensional alignment in the framework of sets of judgments. Given such a modified domain condition, pairwise majority voting will always generate reflexive, transitive and connected collective preference orderings, while also satisfying the weak Pareto principle, independence of irrelevant alternatives and non-dictatorship. And propositionwise majority voting will always generate complete, consistent and deductive closed collective sets of judgments, while also satisfying anonymity and

systematicity. The result for the framework of preferences is well-known (Black, 1948; Sen, 1966). The new result for the framework of sets of judgments is presented in List (2003).

If empirical circumstances are such that the requisite level of cohesion obtains, or that it can be brought about, the present escape-route is obviously attractive, in that it sacrifices none of the collective rationality of the output of the aggregation, and none of its responsiveness to the input.

#### Relaxation of non-dictatorship or anonymity

Suppose we allow the appointment of one individual as a dictator: the preference ordering, or the set of judgments, held by this individual will always be adopted as the collective preference ordering. We assume further that the dictator's personal preference ordering satisfies reflexivity, transitivity and connectedness; and that the dictator's personal set of judgments satisfies completeness, consistency and deductive closure. Then the dictatorial aggregation procedure, while violating some basic democratic intuitions, certainly satisfies universal domain, the weak Pareto principle and independence of irrelevant alternatives in the framework of preferences, and universal domain and systematicity in the framework of sets of judgments.

As the strategy of relaxing universal domain can be associated with the idea of bringing about sufficient cohesion across individuals to make rational collective outcomes possible through democratic means, the strategy of relaxing non-dictatorship or anonymity can be associated with the idea of giving authority to a chief executive in a group to ensure that, however diverse the preferences or judgments of the individual group members may be, the group as a whole does not fall into collective irrationality.

#### Relaxation of independence of irrelevant alternatives or systematicity

Suppose we do not insist that individual pairwise rankings of alternatives alone should determine the corresponding collective pairwise rankings; and we do not insist that the collective acceptance or rejection of a proposition should depend exclusively on the individual acceptance/rejection pattern on that proposition.

Then the well known method of the Borda count is a procedure for aggregating profiles of personal preference orderings into reflexive, transitive and connected collective preference orderings, where the procedure satisfies universal domain, the weak Pareto principle and non-dictatorship.

The Borda count is an attractive procedure in those cases in which agreement on the set of relevant alternatives can be reached, and in which violations of independence of irrelevant alternatives are not associated with threats of agenda-manipulation or intractable third-alternative dependencies.

In the framework of sets of judgments, a different type of escape-route becomes available: the prioritization strategy. Choose a subset of logically independent propositions from the set of propositions  $\mathcal{E}$  on which collective judgments are to be made. The subset might, for instance, contain all the 'premises', such as  $P$  and  $Q$  in the

case of the 'doctrinal paradox', as shown in table 1, as well as the connection rule ( $R \leftrightarrow (P \wedge Q)$ ). The subset is interpreted as the set of those propositions which are given priority or a particular weight in the collective decision process. The collective judgments on the propositions in this subset are then determined by the familiar method of propositionwise majority voting. The result, in the case of table 1, would be the collective acceptance of  $P$ ,  $Q$  and ( $R \leftrightarrow (P \wedge Q)$ ). But now the collective judgments on propositions *outside this subset*, in the present example  $R$ , are determined not by propositionwise majority voting, but rather by logical inference from the collective judgments on the prioritized propositions. In particular, since  $P$ ,  $Q$  and ( $R \leftrightarrow (P \wedge Q)$ ) are collectively accepted and they logically entail  $R$ ,  $R$  will also be collectively accepted, in spite of the fact that  $R$  would not be accepted under propositionwise majority voting. This approach of prioritizing the 'premises' can be called the premise-centred approach. Prioritizing a different set of propositions will lead to a different outcome. For example, a conclusion-centred approach is also possible: here  $R$  as well as ( $R \leftrightarrow (P \wedge Q)$ ) would be given priority. Suitably defined, the prioritization strategy will provide an aggregation procedure that generates complete, consistent and deductively closed collective sets of judgments, and that satisfies universal domain as well as anonymity.

The prioritization strategy provides a promising escape-route from the impossibility result on the aggregation of sets of judgments whenever there is a prospect of agreement on which propositions to prioritize. It is important to note, however, that the collectively accepted set of judgments will then depend crucially on the choice of the prioritized propositions (for a discussion of path-dependencies resulting from this problem, see List, 2002).

#### Relaxation of the collective rationality conditions

Suppose we relax the requirement that collective preference orderings be reflexive, transitive and connected; and we relax the requirement that collective sets of judgments be complete, consistent and deductively closed. Then we can certainly find aggregation procedures satisfying the other conditions of Arrow's theorem or of the theorem on the aggregation of sets of judgments (even pairwise majority voting and propositionwise majority voting fall into this category). However, going far in relaxing the collective rationality conditions may have the undesirable consequence of the occurrence of collective outcomes that are outright inconsistent, such as the cyclical collective preferences of Condorcet's paradox or the inconsistent collective set of judgments of the 'doctrinal paradox' or 'discursive dilemma'.

The question thus arises whether the collective rationality conditions can be relaxed only slightly, so as to preserve a certain minimal form of collective rationality. There are two strategies which have precisely this property. In the framework of preferences, the requirement that collective preference orderings be transitive can be replaced with the less demanding requirement that they be *quasi-transitive* -- only the strong ordering  $P$ , but not the weak ordering  $R$  must be transitive --, and the criterion of Pareto-dominance can then be used to define an aggregation procedure that satisfies all the other conditions of Arrow's theorem. Specifically, an alternative  $x$  Pareto-dominates an alternative  $y$  if all individuals in  $N$  unanimously prefer  $x$  to  $y$ . Now  $x$  will be collectively at least as good as  $y$  if  $y$  does *not* Pareto-dominate  $x$ ; and  $x$  will be collectively better than  $y$  if  $x$  Pareto-dominates  $y$ . The analogous escape-route

in the framework of sets of judgments involves the relaxation of the requirement that collective sets of judgments be complete. Then the criterion of unanimity can be used to define an aggregation procedure that satisfies all the other conditions of the theorem on the aggregation of sets of judgments. Specifically, a proposition will be collectively accepted if it is unanimously accepted by all individuals in  $N$ .

Such unanimity strategies are attractive in those cases in which unanimous support for the resulting collective outcomes is considered important, and where the collectivity can afford to withhold judgment whenever no unanimous support can be crafted.

### **7. Independence of Irrelevant Alternatives and Systematicity: The Conditions Driving the Paradoxes?**

Independence of irrelevant alternatives requires that the collective ranking of any pair of alternatives should depend exclusively on the individual rankings over that pair. Systematicity requires that (i) the collective judgment on each proposition should depend exclusively on the pattern of individual judgments on that proposition and (ii) the same pattern of dependence should hold for all propositions.

As we have seen, in the context of ranking judgments, systematicity entails the requirement that (i) each collective pairwise ranking judgment should depend exclusively on the pattern of individual ranking judgments over the same pair of alternatives and (ii) the same pattern of dependence should hold for all pairwise ranking judgments. Part (i) of that requirement is equivalent to independence of irrelevant alternatives. Thus systematicity is more demanding than independence of irrelevant alternatives.

But both conditions have requirement (i) in common, namely the requirement that a collective judgment on a proposition (a pairwise ranking proposition in the case of preferences) should depend exclusively on the pattern of individual judgments on that proposition. Drawing on an observation made by Saari (1998) in the case of preferences, we will now see that, if the condition of anonymity is also invoked, requirement (i) has a striking implication, which might even be seen as a driving force behind the two paradoxes and impossibility problems.

According to requirement (i), the collective judgment on a proposition (including a pairwise ranking proposition) must depend exclusively on the individual acceptance/rejection pattern on that proposition. Further, according to anonymity, that collective judgment must be invariant under permutations of the individuals. The conjunction of requirement (i) and anonymity therefore entails that

(\*) The collective judgment on a proposition must depend exclusively on the number of individuals accepting that proposition, and the number of individuals rejecting it.

To explore the implications of condition (\*), we will consider two examples. First, consider the profile of personal preference orderings that gives rise to Condorcet's paradox:



	$xRy$	$yRz$	$xRz$	$yRx$	$zRy$	$zRx$
1	yes	yes	yes	no	no	no
2	no	yes	no	yes	no	yes
3	yes	no	no	no	yes	yes
number of yes : number of no	2:1	2:1	1:2	1:2	1:2	2:1

**Table 3**

Of course, each individual's personal preference ordering is reflexive, transitive and connected. The paradox is that pairwise majority voting over these reflexive, transitive and connected individual preferences leads to cyclical collective preferences.

But, given condition (\*), all that matters from the perspective of the aggregation procedure is that there are precisely 2 individuals ranking  $x$  above  $y$ , precisely 2 individuals ranking  $y$  above  $z$ , and precisely 2 individuals ranking  $z$  above  $x$ . These total numbers can be brought about not only by the famous profile of Condorcet's paradox, but also by the following alternative profile:

	$xRy$	$yRz$	$xRz$	$yRx$	$zRy$	$zRx$
1	yes	yes	no	no	no	yes
2	yes	yes	no	no	no	yes
3	no	no	yes	yes	yes	no
number of yes : number of no	2 : 1	2 : 1	1 : 2	1 : 2	1 : 2	2 : 1

**Table 4**

Note that the personal preference orderings of all three individuals are intransitive as well as outright cyclical. In short, no individual satisfies the rationality conditions of Arrow's theorem. Again, pairwise majority voting leads to cyclical collective preferences, but this time this result seems less paradoxical, in so far as the individual preference orderings are cyclical themselves.

Strikingly, though, the profile in table 3 and the one in table 4 have the same informational content from the perspective of any aggregation procedure satisfying condition (\*). This condition forces the aggregation procedure to ignore all the information contained in each table except the row labelled "number of yes : number of no". The two tables are identical with respect to that row. In particular, the

aggregation procedure is of necessity insensitive to the crucial information of whether or not individual preferences satisfy reflexivity, transitivity and connectedness.

Second, consider the profile of personal sets of judgments that gives rise to the doctrinal paradox or discursive dilemma:

	valid contract $P$	breach $Q$	legal doctrine $(R \leftrightarrow (P \wedge Q))$	defendant liable $R$
1	yes	yes	yes	yes
2	yes	no	yes	no
3	no	yes	yes	no
number of yes : number of no	2 : 1	2 : 1	3 : 0	1 : 2

**Table 5**

Here, each individual's personal set of judgments is complete, consistent and deductively closed. The paradox is that propositionwise majority voting over these complete, consistent and deductively closed personal sets of judgments leads to a collective set of judgments that violates these consistency conditions.

But, again, given condition (\*), all that matters from the perspective of the aggregation procedure is that there are precisely 2 individuals accepting  $P$ , precisely 2 individuals accepting  $Q$ , precisely 3 individuals accepting  $(R \leftrightarrow (P \wedge Q))$ , and precisely 2 individuals rejecting  $R$ . These total numbers can be brought about not only by the profile of the doctrinal paradox or discursive dilemma, but also by an alternative profile:

	valid contract $P$	breach $Q$	legal doctrine $(R \leftrightarrow (P \wedge Q))$	defendant liable $R$
1	yes	yes	yes	no
2	yes	yes	yes	no
3	no	no	yes	yes
number of yes : number of no	2 : 1	2 : 1	3 : 0	1 : 2

**Table 6**

Note that the personal sets of judgments of all three individuals violate the consistency conditions of completeness, consistency and deductive closure. Again, propositionwise majority voting leads to an inconsistent collective set of judgments, once again an intuitively less surprising result, given that the individual sets of judgments are themselves inconsistent.

But, here too, the profile in table 5 and the one in table 6 have the same informational content from the perspective of any aggregation procedure satisfying condition (\*). As before, this condition forces the aggregation procedure to ignore all the information contained in each table except the row labelled "number of yes : number of no". Again, the two tables are identical with respect to that row. Here the aggregation procedure is insensitive to the crucial information of whether or not the individual sets of judgments satisfy completeness, consistency and deductive closure.

Both examples illustrate that two profiles can be *informationally identical* from the perspective of an aggregation procedure satisfying condition (\*), even though the individual preferences or sets of judgments in one profile satisfy the relevant rationality conditions, while the ones in the other violate them. Intuitively, an irrational collective outcome seems a lot less paradoxical when the input to the aggregation was itself irrational. The surprising situation is that, of two profiles that are *informationally identical* from the perspective of an aggregation procedure satisfying condition (\*), only one strikes us as leading to a paradox, while the other does not, even though both profiles generate an identical collective outcome.

There are two different interpretations of this finding. On one interpretation, there is nothing wrong with imposing independence of irrelevant alternatives or systematicity, jointly with anonymity as minimal conditions on aggregation. On this interpretation, the heart of the paradoxes lies simply in the fact that, from the perspective of an aggregation procedure satisfying these minimal conditions (and by implication satisfying condition (\*)), a profile of individual inputs satisfying certain rationality conditions can be informationally identical to a profile violating these rationality conditions.

On the other interpretation, it is considered a bad idea to use an aggregation procedure satisfying condition (\*) with the aim of generating rational collective outcomes. On this interpretation, it is unsurprising, and even expected, that an aggregation procedure which ignores any information about the rationality of its input cannot ensure the rationality of its output.

On both interpretations, condition (\*) appears as a driving force behind the paradoxes of aggregation, both Condorcet's paradox and the 'doctrinal paradox' or 'discursive dilemma'. On the first interpretation, condition (\*) explains why the paradoxes occur, and why they are *paradoxical*. On the second interpretation, condition (\*) explains why the paradoxes are *unsurprising*, given independence of irrelevant alternatives or systematicity, and why invoking these conditions is a bad idea. In the framework of preferences, Saari (1998) rejects the plausibility of independence of irrelevant alternatives, or of any minimal condition that implies condition (\*). Saari's argument is that any aggregation procedure whose informational perspective places rational and irrational individual input indistinguishably into the same equivalence class will of necessity fail to ensure the rationality of its output; and further that as soon as the informational perspective of an aggregation procedure allows distinguishing between rational and irrational inputs then it may avoid the paradoxes.

Whichever of the two interpretations we adopt, condition (\*) can be seen as a unifying mechanism underlying the two at first sight distinct paradoxes.

## 8. Conclusion

The aim of this paper has been to compare two impossibility theorems, Arrow's theorem on the aggregation of preferences, and a new theorem on the aggregation of sets of judgments over multiple interconnected propositions. We began this paper with a possibility result, May's theorem on majority voting over two alternatives, and with the observation that May's positive result breaks down as soon as a collective decision problem deviates from the binary choice framework along one of two

dimensions: (a) the number of alternatives, or (b) the number of interconnected propositions on which simultaneous decisions are to be made. Deviations along dimension (a) give rise to Condorcet's paradox and Arrow's theorem, and deviations along dimension (b) give rise to the 'doctrinal paradox' or 'discursive dilemma' and the new theorem on the aggregation of sets of judgments.

Although the two impossibility theorems are not corollaries of each other, we have seen that they are not completely orthogonal either. While the framework of sets of judgments cannot be easily mapped into the framework of preferences, the converse mapping is possible, and plausible, and we have reinterpreted Condorcet's paradox as a result about the aggregation of ranking judgments. We have further derived a new simple impossibility theorem on the aggregation of ranking judgments from the more general theorem on the aggregation of sets of judgments, and we have compared this result with Arrow's theorem.

We have identified escape-routes from Arrow's theorem and the theorem on the aggregation of sets of judgments, and have shown not only that the two results have similar escape-routes, but also that they are, at least in part, driven by a unifying mechanism, namely the informational limitations which the conditions of independence of irrelevant alternatives and systematicity place upon the input to the aggregation. The identification not only of possible escape-routes but also of informational limitations resulting from the minimal conditions of the theorems should encourage us to exercise care in interpreting the two impossibility theorems. In particular, the theorems should be interpreted not as proofs of the impossibility of a certain kind of democratic aggregation, but rather as guidance in approaching the question of when collective decisions can be successfully reached, and what the scopes and limits of mechanical aggregation are.

### **Appendix: A Simple Impossibility Theorem on the Aggregation of Ranking Judgments**

Christian List

Let  $N$  be a set of individuals, and  $X$  a set of alternatives, where  $|N| \geq 2$  and  $|X| \geq 3$ . To each  $i \in N$ , there corresponds a strong personal preference ordering,  $P_i$ , over  $X$ , where  $P_i$  is irreflexive, transitive and complete. A *profile of strong personal preference orderings* is an  $n$ -tuple  $\{P_i\}_{i \in N} = \{P_1, P_2, \dots, P_n\}$ . A *ranking judgment aggregation function* is a function,  $F$ , whose input is a profile of strong personal preference orderings and whose output is a strong collective preference ordering,  $P$ .

Let  $F$  be a ranking judgment aggregation function,  $\{P_i\}_{i \in N}$  a profile of strong personal preference orderings, and  $P = F(\{P_i\}_{i \in N})$ .

**UNIVERSAL DOMAIN (U).** The domain of  $F$  is the set of all logically possible profiles of irreflexive, transitive and complete strong personal preference orderings.

**ANONYMITY (A).** For any  $\{P_i\}_{i \in N}$  in the domain of  $F$  and any permutation  $\sigma: N \rightarrow N$ ,  $F(\{P_i\}_{i \in N}) = F(\{P_{\sigma(i)}\}_{i \in N})$ .

**INDEPENDENCE OF NON-WELFARE CHARACTERISTICS (INW).** Suppose  $\{P_i\}_{i \in N}$  and  $\{P_i^*\}_{i \in N}$  are two profiles of personal preference orderings in the domain of  $F$  and  $x_1, y_1$  ( $x_1 \neq y_1$ ) and  $x_2, y_2$  ( $x_2 \neq y_2$ ) are two pairs of alternatives in  $X$  such that, for all individuals  $i$  in  $N$ ,  $x_1 P_i y_1$  if and only if  $x_2 P_i^* y_2$ . Then  $x_1 P y_1$  if and only if  $x_2 P^* y_2$ .

**Theorem 3.** There exists no ranking judgment aggregation function  $F$  (generating irreflexive, transitive and complete strong collective preference orderings) which satisfies (U), (A) and (INW).

**Proof.** Suppose  $F$  is a ranking judgment aggregation function satisfying (U), (A) and (INW) and generates irreflexive, transitive and complete strong collective preference orderings. By assumption, we may assume that  $x, y, z \in X$ .

Step (1). For each  $x, y \in X$ , define  $N_{xPy} := \{i \in N : x P_i y\}$ . Since  $F$  satisfies (INW), we have, for any two pairs of alternatives  $x_1, y_1$  ( $x_1 \neq y_1$ ) and  $x_2, y_2$  ( $x_2 \neq y_2$ ) in  $X$ , if  $N_{x_1 P y_1} = N_{x_2 P y_2}$ , then  $x_1 P y_1$  if and only if  $x_2 P^* y_2$ . Further, since  $F$  satisfies (A) in addition to (INW), for any two pairs of alternatives  $x_1, y_1$  ( $x_1 \neq y_1$ ) and  $x_2, y_2$  ( $x_2 \neq y_2$ ) in  $X$ , if  $|N_{x_1 P y_1}| = |N_{x_2 P y_2}|$ , then  $x_1 P y_1$  if and only if  $x_2 P^* y_2$ .

Step (2). By assumption,  $n \geq 2$ . Consider a profile of strong personal preference orderings with the following properties:

	$x P_i y$	$y P_i z$	$x P_i z$	$z P_i x$
$i = 1$ ( $x > y > z$ )	yes	yes	yes	no
$i = 2$ ( $z > x > y$ )	yes	no	no	yes
$i = 3$ ( $y > z > x$ )	no	yes	no	yes
$i > 3$ and $i$ is even ( $x > y > z$ )	yes	yes	yes	no
$i > 3$ and $i$ is odd ( $z > y > x$ )	no	no	no	yes

**Table 7**

Let  $P = F(\{P_i\}_{i \in N})$ .

First note that, if, for two distinct alternatives  $x$  and  $y$  in  $X$ ,  $x P y$  and  $y P x$ , then this entails a contradiction; for, if we have  $x P y$  and  $y P x$ , then by transitivity we also have  $x P x$ , which contradicts the assumption that  $P$  is irreflexive.

Case (i).  $n$  is even. We have  $|N_{xPz}| = |N_{zPx}|$ , whence  $x P z$  if and only if  $z P x$ . By the completeness of  $P$ , at least one of  $x P z$  or  $z P x$  must hold. But then we must have both  $x P z$  and  $z P x$ , a contradiction.

Case (ii).  $n$  is odd. We have  $|N_{xPy}| = |N_{yPz}| = |N_{zPx}|$ , whence either all, or none, of  $x P y$ ,  $y P z$  and  $z P x$  must hold.

If  $x P y$ ,  $y P z$  and  $z P x$ , then, since  $x P y$  and  $y P z$  and  $P$  is transitive,  $x P z$ . But then both  $z P x$  and  $x P z$  hold, a contradiction.

If not  $xPy$ , not  $yPz$  and not  $zPx$ , then, by the completeness of  $P$ , we must have  $yPx$ ,  $zPy$  and  $xPz$ . But since  $zPy$  and  $yPx$  and  $P$  is transitive,  $zPx$ . But then, once again, both  $zPx$  and  $xPz$  hold, a contradiction. **Q.E.D.**

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<sup>1</sup> Although a joint project in all ways that matter, the technical results of this paper are the work of Christian List. A closely related companion paper, List and Pettit (2002), was presented at a conference in honour of Isaac Levi at Columbia University, New York, October 2000. The present paper addresses issues that developed from discussion at the conference and afterwards, and we hope that it can stand as a tribute to the lead given by Isaac Levi in developing a conception of the intentional mind under which it becomes possible to inquire into the mindedness of groups; this, in essence, is what we are doing here. We were greatly helped by the comments and questions of conference participants and other friends and colleagues, in particular, Akeel Bilgrami, Steven Brams, Geoffrey Brennan, John Collins, John Dryzek, Mark Fey, Robert Goodin, James Johnson, Marek Kaminski, Isaac Levi, Carol Rovane, Teddy Seidenfeld and Michael Wagner.

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<sup>2</sup> Another, less plausible possibility is this. Each proposition-negation pair constitutes a separate set of alternatives, and each individual has a separate preference ordering for each such set of alternatives. For each proposition-negation pair, the corresponding ordering would rank a proposition above its negation if the individual accepts the proposition, and the negation above the proposition if the individual rejects the proposition. This approach is not particularly promising as a way of mapping the aggregation problem addressed by the new impossibility theorem into the aggregation problem addressed by Arrow's theorem. Under the present approach, a single problem of simultaneous aggregation over multiple interconnected propositions would simply be decomposed into several separate preference aggregation problems, one corresponding to each proposition-negation pair. The approach would be silent on how the logical interconnections between the preferences on different such proposition-negation pairs could be captured. A further problem regarding the identification of a separate preference ordering for each proposition-negation pair is that an individual's preferences on different proposition-negation pairs may not be separable from each other. Supposing that  $P$  is collectively accepted, the question of whether an individual prefers  $(P \wedge Q)$  to  $\neg(P \wedge Q)$  may still depend on whether or not  $Q$  is also collectively accepted. An individual may place a particular value on consistency that may reverse his or her preference ordering on certain proposition-negation pairs, depending on the judgments collectively reached on other proposition-negation pairs. In short, the present approach fails to reduce the problem of simultaneous aggregation over multiple interconnected propositions to a single aggregation problem over multiple alternatives.

<sup>3</sup> The symmetrical distance of one set of judgments from another is simply the total number of proposition-negation pairs on which the two sets fail to coincide. For example, the symmetrical distance between  $\{P, Q, R, (P \wedge Q)\}$  and  $\{P, \neg Q, R, \neg(P \wedge Q)\}$  is 2.

<sup>4</sup> Ostrogorski's paradox can be seen as a result about the aggregation of preferences over sets of multiple propositions. Specifically, Ostrogorski's paradox shows that proposition-by-proposition aggregation and aggregation of preferences over entire sets of propositions can lead to different outcomes, with the former method leading to collective sets of propositions that are dispreferred according to the latter method. See Anscombe (1976) and Kelly (1989).

<sup>5</sup> While pairwise ranking propositions like  $xRy, yRz$  etc. might seem representable as atomic propositions, the propositions stating reflexivity, transitivity and connectedness all contain quantifiers and cannot easily be represented as quantifier-free propositions of the propositional calculus. However, given a finite set of alternatives  $X$ , all logically possible pairwise ranking propositions can be exhaustively enumerated, and all substitution instances of the reflexivity, transitivity and connectedness conditions can also be exhaustively enumerated. This would provide a way of representing all the ranking propositions from the framework of Arrow's theorem as propositions of the propositional calculus. Unlike a predicate calculus representation of ranking propositions, this representation does not explicitly display the internal structure of these ranking propositions, and does not explicitly mark compound propositions as being instances of reflexivity, transitivity or connectedness.  $\square$