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# Bicollective Ground

## Towards a (Hyper)Graphic Account

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## 1 Introduction

Most authors on grounding hold that grounding is *left-collective*: there are some truths $\gamma_0, \gamma_1, \ldots$ such that, without any one of the $\gamma_i$ grounding $\phi$ on its own, *taken together* the truths $\gamma_0, \gamma_1, \ldots$ nevertheless ground $\phi$. (A standard example is the grounding of a conjunctive truth in its conjuncts.) Could grounding also be *right*-collective? In the simplest case: could a truth $\phi$ ground some truths $\gamma_0, \gamma_1, \ldots$ *taken together* without the truth $\phi$ grounding any of the truths $\gamma_i$ on its own? More generally, let us say that grounding is *bicollective* if it is both left- and right-collective.[1]

If bicollective ground is intelligible an interesting kind of metaphysical coherentism becomes a live option. Just like an epistemological coherentist may say that it is a mistake to ask, about a particular belief, what makes *it* justified, a metaphysical coherentist may say that it is a mistake to ask, of a particular truth, what grounds *it*. In the epistemological case we should rather ask of some beliefs, taken together, what makes *them* justified; in the metaphysical case, we should rather ask, of some truths, what grounds *them*. A considerable advantage of this version of metaphysical coherentism is that one does not have to countenance circles of ground in order to be a metaphysical coherentist.[2]

The intelligibility of bicollective ground was first argued for by Dasgupta (2014b) in the course of defending various structuralist theses. Recently, Litland (2016) showed how some sense can be made of the notion by developing a logic of bicollective ground using Fine's truthmaker semantics. While the truthmaker semantics is by far

[1] A note about terminology. Dasgupta (2014b) speaks of grounding being "irreducibly plural" and Litland (2016) speaks of grounding being "many–many". I believe the present terminology is better; the issue is not whether there are grounding operators that take many (or a plurality) of arguments on the right; the issue is whether grounding is collective (non-distributive) on the right. The point is purely terminological—nobody has been confused on this point.

[2] Compare the discussion of "reciprocal" essence in (Fine 1994, pp. 65–6).

the most developed approach to the logic of ground,[3] it is nevertheless problematic as a semantics for ground.[4] These problems, as we will see, are particularly pronounced in the case of bicollective ground.

A different—in many ways more natural—approach to the logic of ground is (hyper)graph-theoretic.[5] The main contribution of this paper lies in showing how to extend the (hyper)graph-theoretic account of ground to the bicollective case. (Along the way we correct some minor infelicities in previous presentations of the graph-theoretical account of ground.) We also—on the philosophical side—sketch how bicollective ground is naturally applied to mathematical structuralism.

### *1.1 Overview*

We begin in §2 by introducing the central notion of immediate strict full ground. In §3 we develop some ways of making sense of the characteristic non-distributivity of bicollective ground and argue that mathematical structuralists should avail themselves of bicollective ground. In §4 we rehearse the truthmaker semantics for bicollective ground and point out some problems that arise in the bicollective case. In §5 we recall the graph-theoretic account for the left-collective case and argue against Fine's principle of Amalgamation. The main contribution of the paper comes in §6 where we develop the graph-theoretic account of bicollective ground. We discuss how to define acyclic graphs, mediate ground, the notions of partial ground, and what it is for two collections of truths to be ground-theoretically equivalent. We conclude with some questions for future research (§7).

## 2 Notions of Ground

The central notions of ground for the graph-theoretic approach are strict full mediate ($<$) and immediate ground ($\ll$).[6] For notational convenience, we here treat ground as a binary relation between multisets of truths. Claims of ground are then generated as follows: whenever $\Gamma$, $\Delta$ are *multisets* of truths $\Gamma < \Delta$ and $\Gamma \ll \Delta$ are claims of ground.[7]

[3] Apart from its applications to ground (Fine 2012b,c Litland 2016) the truthmaker theory has an impressive range of further applications: to counterfactuals (Fine 2012a), intuitionistic logic (Fine 2014), and partial content (Fine 2016).

[4] For the case of left-collective ground this has been forcefully argued by deRosset (2013, 2015).

[5] This is the approach taken (for the case of partial ground) by Schaffer (2009) (a more general approach is sketched in (Schaffer 2016), by deRosset (2015), and by Litland (2015, 2017). The talk of "trees" in (Correia 2014) and (Rosen 2010) is, as we shall see, closely related to the graph-theoretical approach.

[6] We introduce further notions of ground later.

[7] Three notes on grammar. First, note that $\Gamma$ and $\Delta$ are *multisets*, not sets. Unlike a set a multiset is sensitive to repetion: while the set $\{a, a\}$ is identical to the set $\{a\}$, the multiset $\{a, a\}$ differs from the set $\{a\}$ in that it has two occurrences of the member $a$. It turns out that it is important to work with multisets. (See §5.1 below.) Second, we allow both $\Gamma$ and $\Delta$ to be empty. As in (Fine 2012b, 47–8) we distinguish sharply between a truth's being ungrounded and a truth's being zero-grounded, that is, grounded, but by the empty collection of truths. (While zero-grounding will play no positive role in this paper it is important to develop the theory leaving room for zero-grounding.) Third, grounding, of course, is not a relation between

These notions of ground are to be understood as follows. If $\Gamma < \Delta$ then the truths $\Gamma$ provide a *full* explanation of the truths $\Delta$—nothing needs be added to the truths $\Gamma$ in order fully to explain why the truths $\Delta$ are the case. The explanation is *strict* in the sense that $\Delta$ cannot in turn be part of an explanation of $\Gamma$. It is strict full ground in its left-collective form that has been the focus of most of the recent work on ground.

$\Gamma$ is an *immediate* strict full ground for $\Delta$ if $\Gamma$'s grounding $\Delta$ "does not have to be seen to be mediated". The guarded phrase is required since there are cases where a truth $\phi$ is both a mediate *and* an immediate ground for a truth $\psi$. A standard example is $\phi$ and the disjunction $\phi \vee (\phi \vee \phi)$. The ease with which we can treat immediate ground is one of the main advantages of the graph-theoretic approach.

## 3 Distribution Failure

That some sense can be made of grounding many truths is not in question.[8] What might be problematic about bicollective ground is that it is *non-distributive*: how can some truths $\Gamma$ ground some truths $\delta_0, \delta_1, \ldots$ even though for each $i$ and every $\Gamma' \subseteq \Gamma$, the truths $\Gamma'$ do not ground $\delta_i$? But there are pictures of grounding that make it comprehensible how distributivity could fail.

### *3.1 The wall*

One way of thinking about grounding is that the grounded "rests on" the grounds; the grounds "support" the grounded. It is widely accepted that the grounds have to be "relevant" to the grounded. (Consider, e.g. the consensus that mere necessitation is not sufficient for grounding.) If the grounds "support" the grounded the support has to be "relevant".

Seen in light of this metaphor non-distributive grounding arises quite naturally. Figure 7.1 depicts a wall made of bricks, the upper row being supported by the lower row. There is sense in which no one brick in the upper row rests on any collection of bricks from the lower row. Consider, for example, the bricks **a**, **b**, and **c**. **b** and **c** provide support for **a**; but since **b**, **c** take up more space than is needed stably to support **a** they do not *relevantly* support **a**. Taken as a whole, however, the upper row is relevantly supported by the lower row (taken as a whole).

(multi)sets; it is either a multigrade relation between truths or (better) it should be treated as a variable arity sentential operator. We can accommodate this by reading the notation as follows. "If $\Gamma$ and $\Delta$ are multisets of sentences then the result of writing the sentences in $\Gamma$ (in any order) followed by a grounding operator followed by the sentences in $\Delta$ (in any order) is a sentence." (Similarly, if one prefers to treat ground as a multigrade relation between truths.)

[8] The notions of *simultaneous* and *distributive* many–many ground are both definable in terms of left-collective ground (Fine 2012b, p. 54). $\Gamma$ *simultaneously* grounds $\psi_0, \psi_1, \ldots$ iff $\Gamma$ grounds each $\psi_i$. $\Gamma$ *distributively* grounds $\Delta$ if $\Gamma$ and $\Delta$ have decompositions into $\Gamma_0, \Gamma_1, \ldots$ and $\delta_0, \delta_1, \ldots$ respectively such that $\Gamma_i$ grounds $\delta_i$ for each $i$.

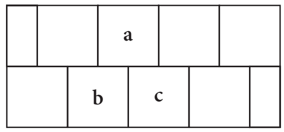



**Figure 7.1.** Grounds as support

This is obviously quite metaphorical, but the metaphor is useful for forming the right intuitions about bicollective ground. As we will see, it provides a useful way of thinking about the hypergraphs.

### *3.2 Ground as (holistic) explanation*

Many philosophers have tied grounding closely to explanation.[9] It will be useful to think of the explanations associated with ground as constituted by a special class of explanatory arguments, where the explanatory arguments in turn are understood as composed of explanatory inferences. (The explanatory inferences correspond to immediate ground, while the explanatory arguments correspond to mediate ground.) In the bicollective setting we have to generalize our conception of argument and inference: we have to allow inferences with many conclusions:

- $q_0, q_1, \ldots$. Therefore. $p_0, p_1, \ldots$

Multiple conclusion inference is, of course, not new; what is distintive about the use to which we will put it here is that the conclusions are read conjunctively.

If there is "holistic explanation" then this connection between grounding and explanation should alert us to the possibility of bicollective grounding. By "holistic explanation" I do not mean circular explanations; what I have in mind is rather the following. We have some truths $\phi_0, \phi_1, \ldots$ (that describe some complex system $S$, say) and it is impossible to explain these truths one by one; rather, what we have to do is to explain the truths $\phi_0, \phi_1, \ldots$ together, and all at once.

A natural place to look for the right sort of holistic explanation is in various structuralist metaphysical views. Indeed, when Dasgupta introduced bicollective ground it was in order to make sense of structuralist views. His chosen examples were qualitativism about individuals and relationalism about quantities; to my mind, however, it is mathematical structuralism that provides the cleanest motivation for bicollective ground.

[9] Some—Dasgupta (2014a,c), Litland (2015, 2017)—hold that to say that $\Gamma$ grounds $\phi$ simply is to say that $\Gamma$ explains $\phi$ in a distinctive way; others—Schaffer (2012, 2016), Audi (2012)—hold that grounding is a relation "underlying" or "underwriting" a distinctive type of explanation. While the difference is important nothing said here will turn on this.

### *3.3 Mathematical structuralism and bicollective grounding*

There are many positions that can be described as structuralist, but a natural thesis for a grounding theorist is that all mathematical truths are grounded in wholly structural truths. To state this more precisely we require some terminology. Consider a mathematical system $S$ and a truth $\phi(a_0, a_1, \ldots, a_n)$ that concerns some objects in $S$.[10] We need a notion of two truths being identical.[11] Say that a truth $\phi$ is *wholly structural* if for all automorphisms $\pi$ of $S$ the truth (expressed by) $\phi(\pi(a_0), \ldots, \pi(a_n))$ is identical to the truth (expressed by) $\phi(a_0, a_1, \ldots, a_n)$.

The precise thesis is then:

(Structuralism) For every system $S$ and every truth $\phi$ about some objects in $S$, either $\phi$ is wholly structural or $\phi$ is grounded in wholly structural truths.

A famous problem for structuralist views is presented by the complex field (Burgess 1999; Keränen 2001). The two square roots of $-1$, that is, $i$ and $-i$, are *indiscriminable* in the sense that there are automorphisms interchanging them. This makes it difficult to see how the truth that $i$ exists can be grounded in wholly structural truths: there is no wholly structural truth about the complex field that bears on the existence of $i$ (as opposed to the existence of $-i$). For the same reason, it is very plausible that there is no wholly structural truth about the complex field that grounds the existence of $-i$.

The structuralist has two ways out. The first way out invokes bicollective ground: while neither the truth that $i$ exists nor the truth that $-i$ exists are individually grounded in wholly structural truths, the truths that $i$ exists, $-i$ exists, *taken together*, are grounded in purely structural truths. There is no problem about how wholly structural truths may bear on the pair of truths $i$ exists, $-i$ exists: since $i$ and $-i$ are the unique square roots of $-1$, the *pair* of objects $(i, -i)$ is uniquely characterized in structural terms. The second way out holds that the truths that $i$ exists and $-i$ exists have exactly the same wholly structural grounds: they are, we may say, commonly grounded in the wholly structural.[12]

There are, I believe, strong reasons to opt for the first view and accept bicollective ground; I will indicate some of those reasons below. A full defense of this claim, however, lies beyond the scope of this paper; for present purposes it suffices that the first view has some plausibility and that its formulation requires bicollective ground.[13]

[10] We make may take a system to be a tuple $S = \langle D, c_0, c_1, \ldots, f_0, f_1, \ldots, R_0, R_1, \ldots \rangle$. Here $D$ is the domain of objects, $c_0, c_1, \ldots$ some designated objects, $f_0, f_1, \ldots$ some functions, and $R_0, R_1, \ldots$, some relations. For more detail about this way of setting things up and some related discussion of the problems raised by non-trivial automorphisms, see (Linnebo and Pettigrew 2014).

[11] We do not strictly speaking need to reify truths: instead of a relation of equality between truths we could use a sentential operator like "what it is for . . . to be the case just is for . . . to be the case".

[12] We should also add that they do not have any non-structural grounds that are not themselves grounded in the wholly structural.

[13] One might think that there is a third possibility. One might think that there is a *single* truth: $(i, -i)$ exists; and that it is this truth that is grounded in the wholly structural. (One may understand this single truth either in terms of a variably polyadic existence predicate or as ascribing existence to the plurality

# 4 The Truthmaker Account of Bicollective Ground

## 4.1 *The theory presented*

The truthmaker semantics is based on the idea that truths can obtain in different (determinate) *ways*: the ways in which a truth can obtain are its metaphysical verifiers. The metaphysical verifiers are mereologically structured. For every set of verifiers $\{f_0, f_1, \ldots\}$ there is a verifier $\Pi(\{f_0, f_1, \ldots\})$ that is the fusion of $f_0, f_1, \ldots$. We can then say that two sets of verifiers $\{f_0, f_1, \ldots\}$, and $\{g_0, g_1, \ldots\}$ are *equivalent* if the fusion of $f_0, f_1, \ldots$ is identical to the fusion of $g_0, g_1, \ldots$.

This allows us to characterize a notion of bicollective (weak) ground. We say that $\phi_0, \phi_1, \ldots$ ground $\psi_0, \psi_1, \ldots$ iff for any ways $f_0, f_1, \ldots$ for $\phi_0, \phi_1, \ldots$ to be the case there are ways $g_0, g_1, \ldots$ for $\psi_0, \psi_1, \ldots$ to be the case such that $\{f_0, f_1, \ldots\}$ is equivalent to $\{g_0, g_1, \ldots\}$.

More precisely, a *state space* is a pair $\langle F, \Pi \rangle$. Here $F$ is a set of objects and $\Pi : \mathcal{P}(F) \to F$ is a total function from the powerset of $F$ to $F$. The members of $F$ represent metaphysical verifiers and $\Pi$ represents the fusion operator. The order and the number of times one fuses some verifiers makes no difference to the resulting fusion; more formally, $\Pi$ is associative in the sense that when each $X_i$ is a collection of verifiers and each $X_j$ is a collection of verifiers, then

$$\Pi(\bigcup_{j \in J} X_j \cup \{\Pi(X_i) : i \in I\}) = \Pi(\bigcup_{k \in I \cup J} X_k)$$

If $X = \{a_0, a_1, \ldots\}$ we write $\Pi(X)$ as $a_0 \cdot a_1 \cdot \ldots$.

Let $V \subseteq F$. We say that $V$ is *closed* if for all non-empty $V_0 \subseteq V$, $\Pi(V_0) \in V$. A *proposition* over $\langle F, \Pi \rangle$ is simply a closed subset of $F$. Let $\{P_i\}_{i \in I}$ be a collection of propositions (that is, closed subsets of $F$). We define $\Pi(P_i)$—the fusion of the propositions $P_i$—as the set of pointwise fusions:

$$\Pi_{i \in I}(P_i) = \{\Pi(\{a_0, a_1, \ldots\}) : a_i \in P_i\}$$

We may write $\Pi(P_0, P_1, \ldots)$ for $\Pi_{i \in I}(P_i)$.

We can now define the notions of weak full ($\leq$), strict full ($<$), weak partial ($\preceq$), and strict partial ($\prec$) ground familiar from Fine (2012c). (In the following definition we assume that $P_0, P_1, \ldots$ and $Q_0, Q_1, \ldots$ are propositions over $F$.)

**Definition 4.1.**

(i) $P_0, P_1, \ldots \leq Q_0, Q_1, \ldots$ iff $\Pi(P_0, P_1, \ldots) \subseteq \Pi(Q_0, Q_1, \ldots)$.

$(i, -i)$.) While I have no objection to there being a single truth that $(i, -i)$ exists, I do not think this gives us a third alternative. For the question arises: what is the relationship between the *one* truth that $(i, -i)$ exist and the *two* truths that $i$ exists and that $-i$ exists? Again we either have to hold that the truths that $i$ exists, that $-i$ exists are commonly grounded in the wholly structural or we have to hold that *they* are grounded in the wholly structural, even though they are not individually grounded in the wholly structural.

(ii) $P_0, P_1, \ldots \preceq Q_0, Q_1, \ldots$ iff there are $R_0, R_1, \ldots$ such that $P_0, P_1, \ldots, R_0, R_1, \ldots \leq Q_0, Q_1, \ldots$

(iii) $P_0, P_1, \ldots < Q_1, Q_2, \ldots$ iff $P_0, P_1, \ldots \leq Q_0, Q_1, \ldots$ and it is not the case that $Q_0, Q_1, \ldots \preceq P_0, P_1, \ldots$.

(iv) $P_0, P_1, \ldots \prec Q_0, Q_1, \ldots$ iff $P_0, P_1, P_2, \ldots \preceq Q_0, Q_1, \ldots$ and it is not the case that $Q_0, Q_1, \ldots \preceq P_0, P_1, \ldots$.

Just as in the left-collective case the basic notion of ground for the truthmaker semantics is weak full ground. Strict full ground is understood in terms of weak full ground—as "irreversible" weak full ground.[14] $\Gamma$ strictly fully grounds $\Delta$ iff $\Gamma$ weakly fully grounds $\Delta$ and $\Delta$ does not in turn contribute to grounding $\Delta$; that is, $\Delta$ does not weakly partially ground $\Gamma$.

## *4.2 Structuralism and the truthmaker semantics*

The truthmaker semantics identifies a truth $\phi$ with the set of states verifying $\phi$. It is worth mentioning that if one accepts the truthmaker semantics for bicollective ground one is forced to adopt the bicollective development of mathematical structuralism. For suppose one held that the truths that $i$ exists, $-i$ exists were commonly grounded. (That is, that the grounds for the truth that $i$ exists are exactly the same as the grounds for the truth that $-i$ exists.) One would then be forced to accept the absurd conclusion that the truth that $i$ exists is identical to the truth that $-i$ exists.

By going bicollective one avoids this problem: on their own the truths that $i$ exist and that $-i$ exist are not grounded in the wholly structural: it is only together that the two truths are grounded in the wholly structural.

While this is a pleasing result the problematic features of the truthmaker semantics mean that one should not put too much weight on it. Let us now turn to discussing some of these problematic features.

## *4.3 Problems with the truthmaker semantics*

The most important problem with the truthmaker semantics is its incapability of dealing with immediate ground. We can see this as follows (working in the left-collective case). For let $R$ be a truth having as its verifiers $p$, $q$, and $p \cdot q$. Let $P$ be verified by $p$. Is $R$ immediately or only mediately grounded in $P$? There is no way of telling. If $R$ is the proposition $P \vee (P \vee Q)$ then $R$ is immediately *and* mediately grounded in $P$; if $R$ is the proposition $(P \vee P) \vee Q$, then $R$ is only mediately grounded in $P$. As we will see, the graph-theoretical approach deals with cases like this with ease.

In case one is dubious about the distinction between immediate and mediate ground, however, it is worth noting that the truthmaker semantics also is problematic as an account of mediate ground.

[14] For more details about the truthmaker semantics see (Litland 2016).

An initial worry arises from the extensional character of the semantics. The *content* of a collection of propositions $\Gamma$ is the set of fusions of the verifiers of the $\gamma \in \Gamma$. By inspection of the clauses for $\leq$ we see that $\Gamma \leq \Delta$ is true if the content of $\Gamma$ is included in the content of $\Delta$. But from the fact that the content of $\Gamma$ is included in the content of $\Delta$ why think that the obtaining of $\Gamma$ in any way *explains* the obtaining of $\Delta$? In particular, why think that one can get from $\Gamma$ to $\Delta$ by means of a sequence of explanatory inferences?

A decent response to this worry is to hold that while the truthmaker semantics may not give us the intended semantics of the logic of ground it still provides a serviceable model theory for the (pure) logic of ground. This response might work in the left-collective case.[15] In the bicollective case, however, the truthmaker semantics validates some problematic principles.

Let me begin with the principle of Self-Ground. As one can easily check, this principle is valid:

$$\frac{\Gamma < \Delta}{\Gamma < \Delta, \Gamma}\ \text{Self-Ground}$$

In other words, a collection of truths $\Gamma$ can strictly ground collections of truths that contain $\Gamma$.

That *some* notion of strict ground allows for instances of Self-Ground is not absurd. After all, there is no conflict with the definition of asymmetry. To say that strict full ground is asymmetric is to say that if $\Gamma < \Delta$, then $\Delta$ does not ground $\Gamma$. It follows that for no $\Gamma$ do we have that $\Gamma$ strictly fully grounds $\Gamma$. But it does not follow from this that $\Gamma$ does not strictly fully ground some $\Sigma$, where $\Gamma \subset \Sigma$.

What *is* problematic is that in the truthmaker semantics one cannot hope to define a notion of strict ground that does not validate Self-Ground.[16] If one ties grounding to explanatory arguments one would not want Self-Ground to be validated: if there is an explanatory argument from $\Gamma$ to $\Delta$ why think that there is an explanatory argument from $\Gamma$ to $\Gamma, \Delta$?

Much the same problem arises with the principle of Squeezing:[17]

$$\frac{\Gamma < \Delta \qquad \Gamma < \Delta, \Sigma, \Theta}{\Gamma < \Delta, \Sigma}\ \text{Squeezing}$$

In words: if a collection of truths $\Delta, \Sigma$ is "squeezed" between two collections $\Delta$ and $\Delta, \Sigma, \Theta$ that both are strictly fully grounded in $\Gamma$, then $\Delta, \Sigma$ is grounded in $\Gamma$.

[15] Though, as we will see, there are worries about Amalgamation.

[16] Strengthening the requirement of asymmetry to require that if $\Gamma$ strictly fully grounds $\Theta$ then there are no $\Delta_0, \Delta_1$ such that $\Theta, \Delta_0$ weakly fully grounds $\Gamma, \Delta_1$ will not work. Since we may always set $\Delta_0 = \Delta_1$ to be the totality of all propositions, this strengthening would ensure that there are no cases of strict full ground.

[17] For a proof of Squeezing, see (Litland 2016).

But this is problematic. This is, again, perhaps most easily seen if we tie grounding to explanation. Why think that there is an explanatory argument from $\Gamma$ to $\Delta, \Sigma$ when we have explanatory arguments from $\Gamma$ to $\Delta$ and also to $\Delta, \Sigma, \Theta$? What guarantees that there is such an argument?

All these problems can be overcome by developing a hypergraph-theoretic account of bicollective ground.

## 5 The (Hyper)Graphic Approach: The Left-Collective Case

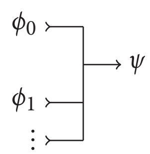
Since the bicollective case leads to complications we first rehearse the theory for the left-collective case. In passing we correct some minor infelicities in previous presentations of the graph-theoretic approach for left-collective ground (deRosset 2015; Litland 2015). If $V$ is a set let $\mathcal{P}(V)$ be the set of *multisets* of $V$.[18]

**Definition 5.1.** A *left-collective directed hypergraph* is a tuple $\mathcal{G} = \langle V_{\mathcal{G}}, \mathcal{A}_{\mathcal{G}}, t_{\mathcal{G}}, h_{\mathcal{G}} \rangle$. Here $V_{\mathcal{G}}$ is a collection of *vertices*. $\mathcal{A}_{\mathcal{G}}$ is a collection of hyperarcs. $t_{\mathcal{G}}, h_{\mathcal{G}}$ are functions $\mathcal{A}_{\mathcal{G}} \to \mathcal{P}(V_{\mathcal{G}})$. If $A \in \mathcal{A}_{\mathcal{G}}$, $t_{\mathcal{G}}(A)$ is the *tail* of $A$ and $h_{\mathcal{G}}(A)$ is the *head* of $A$. We demand that the cardinality of $h_{\mathcal{G}}(A)$ is 1 for each $A \in \mathcal{A}_{\mathcal{G}}$.

We allow $t(A)$ to be of any cardinality; in particular, we allow $t(A) = \emptyset$, in this way we make room for Fine's notion of zero-grounding. When no confusion is likely to arise we freely drop the subscripts from $\langle V_{\mathcal{G}}, \mathcal{A}_{\mathcal{G}}, h_{\mathcal{G}}, t_{\mathcal{G}} \rangle$.

Intuitively, the vertices represent truths and an arc $A$ represents that the truths $t(A)$ immediately ground the truth $h(A)$.[19] We occasionally refer to heads and tails as *limbs*. In having $h(A)$ and $t(A)$ be *multi*sets the present account differs from the previous ones. Since we take $V$ to consist of truths we will often use $\phi, \psi, \ldots$ to range over the elements of $V$ and use $\Gamma, \Delta, \Sigma, \ldots$ to range over multisets of elements of $V$.

The following graphical depiction of a hyperarc with tail $\phi_0, \phi_1, \ldots$ and head $\psi$ might be useful:




Since we are interested in capturing notions of *strict* ground we impose an *acyclicity* condition.

[18] For cardinality reasons we impose a limit on the amount of repetition that is allowed. (What limit we impose does not matter as long as we allow arbitrary finite repetition of elements.) For definiteness, letting $\lambda$ be the least strongly inaccessible cardinal greater than the cardinality of $V$, we allow an element in a multiset to have $\kappa$-many occurrences for each $\kappa < \lambda$.

[19] Strictly speaking $h(A)$ is a multiset containing a single truth, but for ease of expression we will refer to $h(A)$ as a truth. No confusion should ensue.

**Definition 5.2.** Let $\mathcal{G} = \langle V, \mathcal{A}, t, h\rangle$ be a graph. A *path* in $\mathcal{G}$ is either a sequence $\langle \phi \rangle$ where $\phi \in V$ or a sequence $\langle \phi_0, A_0, \phi_1, A_1, \ldots, A_{n-1}, \phi_n\rangle$ such that for each $i$, $\phi_i \in t(A_i)$ and $\{\phi_{i+1}\} = h(A_i)$. The length of a path $\langle \phi_0, A_0, \phi_1, A_1, \ldots, A_{n-1}, \phi_n\rangle$ is $n$.

We say that $\phi$ lies on a path from $\psi$ if there is path $\langle \phi_0, A_0, \phi_1, A_1, \ldots, A_{n-1}, \phi_n\rangle$ where $n \geq 1$ and $\phi_0 = \psi$ and $\phi_n = \phi$.

**Definition 5.3.** Let $\mathcal{G} = \langle V, \mathcal{A}, h, t\rangle$ be a graph. We say that $\mathcal{G}$ is *acyclic* if for all $\phi \in V$ there is no path from $\phi$ to $\phi$.

We extend this to an account of mediate ground. Rather than working directly with the grounding graphs we extract certain labeled trees and read ground off of those trees. This is convenient for two reasons. First, the graphs themselves are quite messy: many truths have several (immediate) grounds, and there will be a multitude of ways for a given truth to be mediately grounded. Each tree, on the other hand, isolates a unique way for a given truth to be grounded. Second, the trees may be viewed as representing explanatory arguments *from* the grounds *to* the grounded.

For our purposes a tree is a certain type of hypergraph.

**Definition 5.4.** A hypergraph $\mathcal{T} = \langle T, \mathcal{A}, h, t, \rangle$ is a *tree* if $\mathcal{T}$ satisfies the following conditions:

(i) $t(A)$ is a set (not a multiset) for each $A \in \mathcal{A}$.
(ii) For every $u \in T$ there is at most one $A \in \mathcal{A}$ such that $u \in t(A)$.
(iii) For every $u \in T$ there is at most one $A \in \mathcal{A}$ such that $u \in h(A)$.
(iv) There is a unique $c \in T$ such that each $u \in T$ lies on a path ending in $c$.
(v) There is $P \subseteq T$ such each $p \in P$ is not in the head of any $A \in \mathcal{A}$ and such that $T$ is the closure of $P$ under $\mathcal{A}$.

We refer to $P$ as the *premiss nodes* of the tree and $c$ as its *conclusion node*. For short we will say that $P$ are the premisses of $T$ and $c$ its conclusion. To have an explicit notation we use $\langle T, P, c, \mathcal{A}, h, t\rangle$ to indicate that $\langle T, \mathcal{A}, t, h\rangle$ is a tree with premisses $P$ and conclusion $c$.

Suppose we label the nodes of a tree $\langle T, P, c, \mathcal{A}, h, t\rangle$ with propositions. We may then take the label on $c$ to represent a grounded truth; the labels on $P$ represent the truths that ground (the truth that labels) $c$. The tree as a whole, then, depicts an "explanatory derivation" of the grounded truth $c$ from its grounds $P$.

Of particular interest are the labeled trees that arise from grounding graphs.

**Definition 5.5.** Let $\mathcal{G} = \langle V, \mathcal{A}, h, t\rangle$ be a grounding graph. A labeled (directed) tree over $\mathcal{G}$ is a tuple $\mathcal{T} = \langle T, P, c, \mathcal{A}_T, L, t_T, h_T\rangle$. Here $\langle T, P, c, \mathcal{A}_T, t_T, h_T\rangle$ is a tree as above; $L\colon T \to V$ is a function assigning labels from $V$ to the nodes of $T$. The sets of arcs $\mathcal{A}$ and $\mathcal{A}_T$ are related as follows.

- For all $A \in \mathcal{A}_T$ there is $A' \in \mathcal{A}$ such that for all $\phi \in V$,
  - the cardinality of $\{\psi \in t(A') \colon \psi = \phi\}$ is the cardinality of $\{u \in t_T(A) \colon L(u) = \phi\}$.
  - If $u \in h_T(A)$ then $L(u) \in h(A')$.

(Remember that while $t_T(A)$ is a set of nodes; $t(A')$ is multiset.)

We refer to the labeled trees over $\mathcal{G}$ as $\mathcal{G}$-trees. We now define the notion of strict full mediate ground.

**Definition 5.6.** Let $\mathcal{G} = \langle V, \mathcal{A}, h, t\rangle$ be a grounding-graph. Let $\Gamma \cup \{\phi\} \subseteq V$. We say that $\Gamma < \phi$ if there is $\mathcal{G}$-tree $\mathcal{T} = \langle T, P, c, \mathcal{A}_T, L\rangle$ such that $L(P) = \Gamma$ and $L(c) = \phi$.

Since we think of trees as arguments from the grounds to the grounded it is convenient to adopt some standard proof-theoretic notion. We write

$$\begin{array}{c}\Gamma\\ \mathcal{E}\\ \phi\end{array}$$

to indicate that $\mathcal{E}$ is a tree with premisses exactly $\Gamma$ and conclusion $\phi$. ("$\mathcal{E}$" for "explanatory argument".)

To understand the definitions it is helpful to consider Figure 7.2. The trees show perspicuously how both $(\phi, \theta)$ and $(\psi, \theta)$ and $(\phi, \psi, \theta)$ mediately ground $(\phi \vee \psi) \wedge \theta$. (We here assume that if both $\phi, \psi$ is true, then $\phi, \psi$ together ground $\phi \vee \psi$.)

## *5.1 Amalgamation failure*

The principle of (Strict) Amalgamation says that if $\Gamma_i < \phi$ for each $i \in I$ then $\bigcup_{i \in I} \Gamma_i < \phi$. This principle is invalid—according to Definition 5.6. In fact, we can establish something stronger.[20]

Say that a grounding graph $\mathcal{G} = \langle V, \mathcal{A}, t, h\rangle$ is *amalgamating* if whenever we have $A_i \in \mathcal{A}$ such that $h(A_i) = h(A_j)$ for all $i, j \in I$, then there is $A \in \mathcal{A}$ such that $h(A) = h(A_i)$ for each $i \in I$ and such that $t(A) = \bigcup_{i \in I} t(A_i)$. If a grounding graph $\mathcal{G}$ is amalgamating then the principle of Amalgamation holds for *immediate* ground. However, Amalgamation for mediate ground does not follow from Amalgamation for immediate ground.

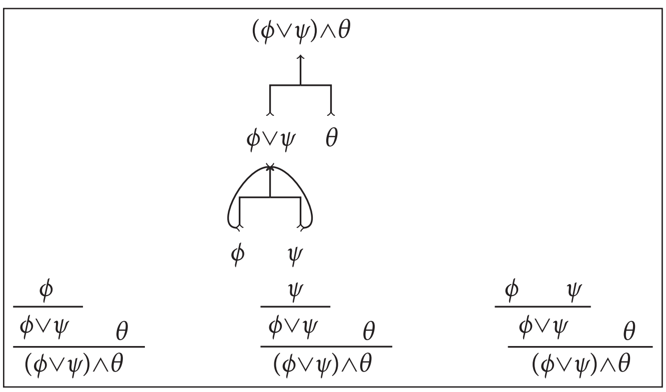



**Figure 7.2.** A $\mathcal{G}$-graph and (some of) its $\mathcal{G}$-trees

[20] Correia (2014, n. 17) gives a similar counterexample.

To see this let $\phi$ be a truth immediately grounding a truth $\psi$. Contrast the following two cases. First, the truth $\psi$ immediately grounds the truth $\psi \wedge \psi$; second, the truth $\psi$ grounds the truth $\psi \vee \theta$, where $\theta$ is some unrelated truth.

In the first case $\phi$ is a mediate ground for $\psi \wedge \psi$; moreover, $\phi, \psi$ taken together also form a mediate ground for $\psi \wedge \psi$. In the second case, $\phi$ is a mediate ground for $\psi \vee \theta$, but $\phi, \psi$ taken together do not constitute a mediate ground for $\psi \vee \theta$.

The reason is that in grounding $\psi \wedge \psi$, $\psi$ is used twice; we can elect to use $\phi$ to ground only one of the occurrences of $\psi$, leaving the other occurrence of $\psi$ available to make a contribution to grounding $\psi \wedge \psi$. In the grounding of $\psi \vee \theta$, on the other hand, $\psi$ is used only once. If we use $\phi$ to ground $\psi$ there is no occurrence of $\psi$ "left over" to make a separate contribution to grounding $\psi \vee \theta$. The difference between the two situations is depicted in Figure 7.3.

Amalgamation for immediate ground expresses a choice about how to deal with overdetermination. In cases of overdetermination not only are the individual candidate immediate grounds in fact grounds, they are also immediate grounds taken together. But if two grounds are "on the same path" to the grounded we should not count them as *together* forming a ground, which is what Amalgamation in full generality forces us to do.[21]

It is not just Amalgamation that fails for this reason. The following "structural" principles of ground also fail:

$$\frac{\Gamma, \phi < \psi}{\Gamma, \phi, \phi < \psi}\ \text{Mingle} \qquad\qquad \frac{\Gamma, \phi, \phi < \psi}{\Gamma, \phi < \psi}\ \text{Contraction}$$

But all of Amalgamation, Contraction, and Mingle are validated by the truthmaker semantics for the left-collective pure logic of ground. The truthmaker semantics cannot even serve the instrumental role of characterizing the validities of the pure logic of ground.


**Figure 7.3.** Amalgamation failures

[21] Consider the causal parallel. If Billy and Suzy each throw a rock at the window we might count the event of them both throwing as a cause of the window's shattering. In contrast, consider just the rock Billy threw. If the rock's momentum at $t_0$ is a cause of its momentum at $t_1$ and its momentum at $t_1$ is a cause of the window's shattering we might reasonably count both the rock's momentum at $t_0$ and its momentum at $t_1$ as causes for the window's shattering. But it would be wrong to count the momentum of the rock as $t_0$ together with its momentum at $t_1$ as a *joint* cause of the window's shattering.

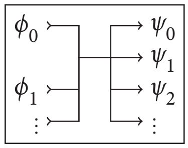



**Figure 7.4.** A hyperarc

## 6 The Hypergraphic Account: The Bicollective Case

To study bicollective ground proper we consider grounding graphs $\langle V, \mathcal{A}, h, t\rangle$ where both $t(A)$ and $h(A)$, for $A \in \mathcal{A}$, may have any cardinality. A graphical representation of a hyperarc $A$ with $t(A) = \{\phi_0, \phi_1, \ldots\}$ and $h(A) = \{\psi_0, \psi_1, \ldots\}$ is given in Figure 7.4.

The definition of immediate ground is the obvious one: $\Gamma \ll \Delta$ holds in a graph $\mathcal{G} = \langle V, \mathcal{A}, h, t\rangle$ if there is $A \in \mathcal{A}$ such that $t(A) = \Gamma$ and $h(A) = \Delta$.

Since we allow both $h(A)$ and $t(A)$ to be empty we have to ensure that the notion of the empty ground behaves properly. Whatever sense can be made of the empty ground one thing is at least clear: the empty ground is *minimal*. We therefore impose the following condition:

(∅-Minimality) If $\mathcal{G} = \langle V, \mathcal{A}, h, t\rangle$ is a graph then for all $A$, if $h(A) = \emptyset$, there is $B$ such that $t(B) = \emptyset$ and $h(B) = t(A)$.[22]

### 6.1 *Acyclicity*

Since we are interested in *strict* immediate ground we impose an acyclicity condition on the graphs. Unlike in the left-collective case there is some choice about what the right notion of acyclicity is.

First, we must define the notion of a path.

**Definition 6.1.** Let $\mathcal{G} = \langle V, \mathcal{A}, t, h\rangle$ be a graph. A *path* in $\mathcal{G}$ is either a sequence $\langle V_0\rangle$, where $V_0 \subseteq V$ or an alternating sequence of (sets of) vertices and arcs $\langle V_0, A_0, V_1, A_1 \ldots, A_{n-1}, V_n\rangle$ such that $t(A_0) = V_0$ and $h(A_0) = V_1$, $t(A_1) \cap V_1 \neq \emptyset$ and $h(A_1) = V_2 \ldots$, $t(A_{n-1}) \cap V_{n-1} \neq \emptyset$ and $h(A_{n-1}) = V_n$. The length of a path $\langle V_0, A_0, V_1, A_1 \ldots, A_{n-1}, V_n\rangle$ is $n$. If there is a path $\langle V_0, A_0, V_1, A_1 \ldots, A_{n-1}, V_n\rangle$ of length $n \geq 1$ we say that there is a path *from* $V_0$ to $V_n$.

To see what a path looks like, it helps to return to (a variation on) the picture of the wall (in Figure 7.1):

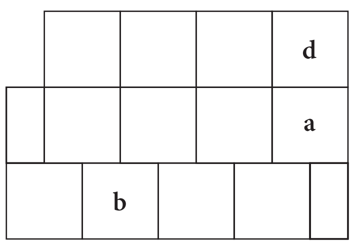



**Figure 7.5.**

[22] This says that if $t(A)$ grounds the null fact then $t(A)$ is *immediately* zero-grounded.

We may consider Figure 7.5 as a graph by letting each small box be a vertex. We let there be an arc the tail of which is the entirety of the bottom row; and the head of which is the entirety of the middle row. For each box in the middle row, we let there be an arc from that box to the box directly above it on the upper row. For instance, there is an arc with tail {**b**} and head {**a**}. In this case there is a path from the box labeled **b** to the box labeled **d**. **d** does rest on the lower row as a whole, and so in part on **b**.

**Definition 6.2.** Let $\mathcal{G} = \langle V, \mathcal{A}, t, h\rangle$ be a graph.

(i) $\mathcal{G}$ is *weakly acyclic* if for all $V_1 \subseteq V_0 \subseteq V$ there is no path from $V_0$ to $V_1$;
(ii) $\mathcal{G}$ is *strongly acyclic* if $\mathcal{G}$ is weakly acyclic and for all $V_0$ and $V_1$ such that $V_0 \cap V_1 \neq \emptyset$ there is no path from $V_0$ to $V_1$.

In (weakly or strongly) acyclic graphs nothing grounds the empty ground.

**Observation 6.3.** Let $\mathcal{G} = \langle V, \mathcal{A}, h, t\rangle$ be a (weakly, strongly) acyclic graph. Then for no $A \in \mathcal{A}$ do we have $h(A) = \emptyset$.

*Proof.* Suppose $h(A) = \emptyset$. Then by ($\emptyset$-Minimality) there is $B$ such that $h(B) = t(A)$ and $t(B) = \emptyset$. We then have a path from $\emptyset$ to $\emptyset$, contradicting weak (strong) acyclicity. ■

Weak acyclicity is too weak to capture what we want in a strict notion of ground. To see this consider the graph $\mathcal{H}$ depicted in Figure 7.6. $\mathcal{H} = \langle V, \mathcal{A}, h, t\rangle$ where $V = \{a, b, c, d\}$ and $\mathcal{A} = \{A, B, C\}$ where $t(A) = a$ and $h(A) = c$; $t(B) = b$ and $h(B) = d$ and $t(C) = \{c, d\}$ and $h(C) = \{a, b\}$. $\mathcal{H}$ is weakly acyclic according to Definition 6.2, but intuitively $\mathcal{H}$ is cyclic: it is natural to think that $\mathcal{H}$ represents that $a, b$ (mediately) grounds $a, b$.

I am inclined to think that strong acyclicity is the notion we are after, but some might think that it demands too much. For consider the plurality of all truths: if we require strong acyclicity we rule out that the totality of all truths is grounded. (If the totality of all truths is grounded it has to be grounded in a subplurality, but this would contradict strong acyclicity.) This might be an unwelcome result for someone who wants to defend a version of the principle of sufficient reason for bicollective ground.

Once we have defined mediate ground we can define a notion of acyclicity intermediate between strong and weak. Before I go on to characterize mediate ground, let me return to structuralism—and $i$ and $-i$.

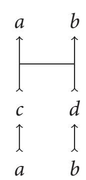



**Figure 7.6.** The insufficiency of weak acyclicity

## *6.2 Structuralism again*

In §4.2 above we noted that the truthmaker semantics rules out that the truths that $i$ exists and $-i$ exists are commonly grounded. (Their being commonly grounded would make the truth that $i$ exists identical to the truth that $-i$ exists.) This type of argument fails in the graph-theoretical framework. For it is possible for two truths to have exactly the same immediate grounds, but be grounded in them in different *ways*. Formally, this comes down to there being distinct co-tailed arcs. (As a plausible example of this one might want to distinguish the truths $\phi \vee \phi$ and $\phi \wedge \phi$.[23]) One might think that one way to be a structuralist is to hold that while the truths that $i$ exists and $-i$ exists are commonly grounded, they are nevertheless grounded in different *ways*.[24]

However, one might argue that this is impossible. For $i$ and $-i$ are *absolutely* indiscriminable, there is no feature that tells $i$ apart from $-i$. Suppose now that the truths that $i$ exists and $-i$ exists are commonly grounded, but that the truths are grounded in different ways. Then it would be possible to distinguish $i$ and $-i$ after all; for $i$ is characterized by the fact that the truth that $i$ exists is grounded in way $w_0$ (rather than way $w_1$); whereas $-i$ is characterized by the fact that the truth that $-i$ exists is grounded in way $w_1$ (rather than way $w_2$). But then we could discriminate between $i$, $-i$ after all. Since they are not discriminable, we conclude that the truths that $i$ exists, $-i$ exists are not grounded in different ways. If this argument succeeds the structuralist has no choice but to accept bicollective ground.[25]

## *6.3 Mediate ground*

We begin by generalizing the notion of a tree.

**Definition 6.4.** An *edifice* is a hypergraph $\mathcal{E} = \langle E, \mathcal{A}, h, t \rangle$ such that

(i) $t(A)$ and $h(A)$ are sets (not multisets) for every $A \in \mathcal{A}$.
(ii) For every $u \in E$ there is at most one $A \in \mathcal{A}$ such that $u \in t(A)$.
(iii) For every $u \in E$ there is at most one $A \in \mathcal{A}$ such that $u \in h(A)$.
(iv) There is a set $C$ such that if $c \in C$ then $c$ is not in the tail of any $A \in \mathcal{A}$ and such that each path through $E$ terminates in some $C_0 \subseteq C$.
(v) There is $P \subseteq E$ such each $p \in P$ is not in the head of any $A \in \mathcal{A}$ and such that $E$ is the closure of $P$ under $\mathcal{A}$.

Note that any graph with the empty collection of arcs counts as an edifice.

We may think of an edifice as an argument from the premisses $P$ to the conclusions $C$. Whereas a tree represents an argument with a single conclusion an edifice

[23] Admittedly, the distinction here can be drawn in terms of the multiplicity of grounds; but there are more complicated (infinitary) examples where this move will not work.

[24] We here invoke the notion of a *way of grounding*. Putting this notion on a rigorous footing is, I believe, one of the most pressing issues in the theory of ground and one to which I hope to return elsewhere.

[25] Much more needs to be said, of course, to make this rigorous, but the above should be sufficient for the present, largely motivational, purposes.

represents an argument with several conclusions. (Recall that the conclusions of an argument are to be read conjunctively.) We write $\mathcal{E} = \langle E, P, C, \mathcal{A}, h, t\rangle$ to make it explicit that $\mathcal{E}$ has premisses $P$ and conclusions $C$. A *labeled edifice* is an edifice where every node has been assigned a label. Just as in the left-collective case we are mainly interested in the labeled edifices that are generated by grounding graphs.

**Definition 6.5.** Let $\mathcal{G} = \langle V, \mathcal{A}, h, t\rangle$ be a graph. An *edifice over* $\mathcal{G}$ is a tuple $\mathcal{E} = \langle E, P, C, \mathcal{A}_E, L, h_E, t_E\rangle$ such that $\langle E, P, C, \mathcal{A}_E, h_E, t_E\rangle$ is an edifice and $L : E \to V$ is a labeling function such that

- For all $A \in \mathcal{A}_E$ there is $A' \in \mathcal{A}$ such that for all $\phi \in V$,
  - the cardinality of $\left\{\psi \in t(A') \colon \psi = \phi\right\}$ is the cardinality of $\{u \in t_E(A) \colon L(u) = \phi\}$.
  - the cardinality of $\left\{\psi \in h(A') \colon \psi = \phi\right\}$ is the cardinality of $\{u \in h_E(A) \colon L(u) = \phi\}$.

If $\mathcal{E}$ is an edifice over $\mathcal{G}$ we also say that $\mathcal{E}$ is a $\mathcal{G}$-edifice.

It might help to think of the relationship between a graph $\mathcal{G}$ and the $\mathcal{G}$-edifices in the following way. Think of the vertices of a graph $\mathcal{G} = \langle V, \mathcal{A}, h, t\rangle$ as a collection of various specialized bricks. Think of the arcs in $\mathcal{A}$ as specifying how the bricks may be combined to produce more complicated structures (e.g. walls). One can think of the edifices as representing the particular ways in which the blocks have been put together in accordance with the rules.

How can we use edifices to define mediate ground? The natural definition is to say that $\Gamma < \Delta$ holds in $\mathcal{G}$ iff there is a $\mathcal{G}$-edifice $\mathcal{E} = \langle E, P, C, \mathcal{A}_E, L, h, t\rangle$ such that $L(P) = \Gamma$ and $L(C) = \Delta$. However, even in strongly acyclic graphs this does not define a notion of *strict* ground.

To see this consider the grounding graph $\mathcal{Z} = \langle V, \mathcal{A}, h, t\rangle$ where $V = \{\phi_i \colon i \in \mathbb{Z}\}$, and where for each $i \in \mathbb{Z}$ there is $A_i$ with $t(A_i) = \{\phi_i\}$ and $h(A_i) = \{\phi_{i+1}\}$. $\mathcal{Z}$ has the following infinitely ascending and descending chain of immediate ground:

$$\ldots \phi_{-2} \ll \phi_{-1} \ll \phi_0 \ll \phi_1 \ll \phi_2 \ldots$$

Consider the two sets:

$$\Phi_{\text{even}} = \{\phi_{-4}, \phi_{-2}, \phi_0, \phi_2, \ldots\}$$

and

$$\Phi_{\text{odd}} = \{\phi_{-3}, \phi_{-1}, \phi_1, \phi_3, \ldots\}$$

These sets give us infinitely ascending and descending chains of *mediate* ground in the obvious way.

It is easy to see that there is an edifice with premisses $\Phi_{\text{even}}$ and conclusions $\Phi_{\text{odd}}$ and also one with premisses $\Phi_{\text{odd}}$ and conclusions $\Phi_{\text{even}}$. The following edifice witnesses the first case.

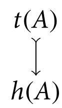
$$\begin{array}{ccccc} \ldots & \phi_{-2} & \phi_0 & \phi_2 & \ldots \\ \ldots & \phi_{-1} & \phi_1 & \phi_3 & \ldots \end{array}$$

But then we get both $\Phi_{\text{even}} < \Phi_{\text{odd}}$ and $\Phi_{\text{odd}} < \Phi_{\text{even}}$, contradicting the asymmetry of strict ground.

To define mediate ground we define the classes of weak, strict, and immediately strict edifices. The idea is that the strict edifices are generated from the immediately strict edifices by composing them with weak edifices in a constrained way. This allows us to capture the idea that strict ground is the closure of immediate strict ground.

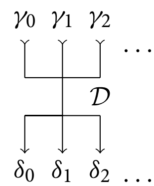
To state the next definitions perspicuously we need some notation. Let us use $\mathcal{D}, \mathcal{E}, \ldots$ to range over edifices. Consider the edifice $\langle \{t(A)\} \cup \{h(A)\}, \{A\}, h, t\rangle$. We may depict this edifice both proof-theoretically and graph-theoretically. (Typographic considerations dicate which depiction we choose.)

$$\frac{t(A)}{h(A)}\,A$$




More generally if $\mathcal{D}$ is an edifice with premisses $\{\gamma_0, \gamma_1, \ldots\}$ and conclusions $\{\delta_0, \delta_1, \ldots\}$ we may write this in either of the following ways.

$$\begin{array}{c} \gamma_0, \gamma_1, \ldots \\ \mathcal{D} \\ \delta_0, \delta_1, \ldots \end{array}$$



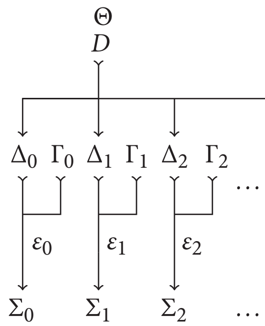

Having both notations allows us to conveniently express how to compose edifices. Suppose

$$\begin{array}{c} \Theta \\ \mathcal{D} \\ \Delta_0, \Delta_1, \ldots, \quad \Upsilon \end{array}$$

is an edifice with conclusions $\Delta_0, \Delta_1, \ldots, \Upsilon$. Suppose further that for each $i$,

$$\begin{array}{c} \Delta_i, \Gamma_i \\ \mathcal{E}_i \\ \Sigma_i \end{array}$$

is an edifice with premisses $\Delta_i, \Gamma_i$ and conclusions $\Sigma_i$. These edifices may be composed to yield an edifice with premisses $\Theta, \Gamma_0, \Gamma_1, \ldots$ and conclusions $\Sigma_0, \Sigma_1, \ldots, \Upsilon$. We will depict such an edifice as follows.




**Definition 6.6.** Let $\mathcal{G} = \langle V_{\mathcal{G}}, \mathcal{A}_{\mathcal{G}}, t_{\mathcal{G}}, h_{\mathcal{G}} \rangle$ be a grounding graph. The immediately strict, strict, and weak edifices over $\mathcal{G}$ are the least classes of $\mathcal{G}$-edifices satisfying the conditions in Figure 7.7.

To see the idea behind Definition 6.6 look first at the immediately strict edifices. In an immediately strict edifice the conclusions are all in the head of a single arc $A$. We move from the many premisses $t(A)$ to the many conclusions $h(A)$ by a single step of ground. (Many things are grounded, but in a single step of ground.)

Suppose that $\mathcal{E}(s)$ is a strict edifice with conclusions $C$. Suppose that $C = C_0 \cup C_1$ and we have $t(A_0) = C_0$ and $t(A_1) = C_1$. Then the edifice

$$A_0 \, \frac{C_0}{h(A_0)} \qquad \frac{C_1}{h(A_1)} \, A_1$$

is not strict. However, the edifice

$$\begin{array}{ccc} & \mathcal{E}(s) & \\ A_0 \, \dfrac{C_0}{h(A_0)} & & \dfrac{C_1}{h(A_1)} \, A_1 \end{array}$$

is strict.

The following proposition pinpoints an important difference between the strict and the merely weak edifices.

**Proposition 6.7.** Let $\mathcal{E} = \langle E, P, C, \mathcal{A}, L \rangle$ be a strict edifice. Then for every $p \in P$ and $c \in C$ there is some $P' \subseteq P$ and some $C' \subseteq C$ such that there is a path from $P'$ to $C'$ and such that $p \in P'$ and $c \in C'$.

*Proof.* The proof proceeds by induction on the complexity of the strict edifice $\mathcal{E} = \langle E, P, C, \mathcal{A}, L \rangle$. If $\mathcal{E}$ is immediately strict the result is immediate. Suppose that $\mathcal{E}$ is the result of an application of (Strict Right Composition). Then $\mathcal{E}$ is of the form:

(Weakness) Every edifice over $\mathcal{G}$ is a weak edifice

(Inclusion) Every immediately strict edifice is a strict edifice

(Immediacy) $\dfrac{t(A)}{h(A)}$ $A$ is an immediately strict edifice for each $A \in A$.

(Strict Left Composition) If $\begin{matrix}\mathcal{E}\\ \Gamma\end{matrix}$ is a weak edifice and $\begin{matrix}\Gamma\\ \mathcal{D}\\ \Sigma\end{matrix}$ is a strict edifice then any edifice of the form $\begin{matrix}\mathcal{E}\\ \Gamma\\ \mathcal{D}\\ \Sigma\end{matrix}$ is a strict edifice.

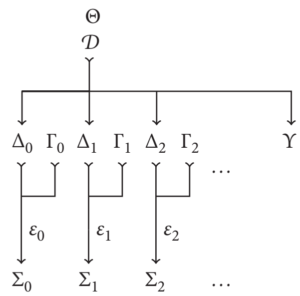
(Strict Right Composition) If

$$\begin{matrix}\Theta\\ \mathcal{D}\\ \Delta_0, \Delta_1, \ldots, \ \Upsilon\end{matrix}$$

is a strict edifice and

$$\begin{matrix}\Delta_i, \Gamma_i\\ \mathcal{E}_i\\ \Sigma_i\end{matrix}$$

is a strict edifice for each $i$ then any edifice of the form




is a strict edifice

(Weak Composition) If

$$\begin{matrix}\Theta\\ \mathcal{D}\\ \Delta_o, \Delta_1, \ldots\end{matrix}$$

is a strict edifice and

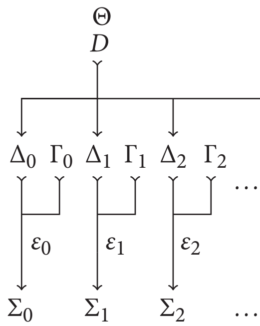
$$\begin{matrix}\Delta_i\\ \mathcal{E}_i\\ \Sigma_i\end{matrix}$$

is a weak edifice for each $i$ then any edifice of the form

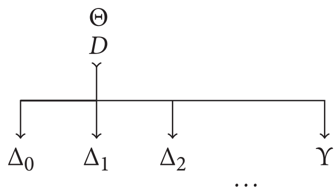
$$\begin{matrix} & \Theta & \\ & \mathcal{D} & \\ \Delta_0 & \Delta_1 & \ldots\\ \mathcal{E}_0 & \mathcal{E}_1 & \ldots\\ \Sigma_0 & \Sigma_1 & \ldots\end{matrix}$$

is a strict edifice

**Figure 7.7.** Strict and weak edifices




So let $p \in P$ and $c \in C$ be given. If $p \in \Gamma_i$ for some $i$, the result follows by the induction hypothesis applied to $\mathcal{E}_i$. If $p \in \Theta$ we reason as follows. Suppose first that $c \in \Upsilon$. The edifice $\mathcal{D}$, that is,




is strict. It then follows by the induction hypothesis that there is some $P' \subseteq P$ and $C'$ such that there is a path from $P'$ to $C'$ and such that $p \in P'$ and $c \in C'$.

Suppose then that $c \in \Sigma_i$, for some $i$. Let $d \in \Delta_i$ be given. Since $\mathcal{E}_i$ is a strict edifice there is $D' \subseteq \Delta_i \cup \Gamma_i$ and $C' \subseteq \Sigma_i$ such that there is path from $D'$ to $C'$ where $d \in D'$ and $c \in C'$. Since $\mathcal{D}$ is strict there is also $D'' \subseteq \Delta_0 \cup \Delta_1 \cup \cdots \cup \cdots$ such that there is a path from $P'$ to $D''$ where $p \in P' \subseteq P$ and $d \in D''$. But since $D' \cap D'' \neq \emptyset$ this shows that there is a path from $P'$ to $C'$ which is what we have to show.

To prove the cases of (Weak Composition) and (Strict Left Composition) we observe that if $\mathcal{D} = \langle D, P, C, \mathcal{A}_{\mathcal{D}} \rangle$ is a weak edifice then (i) for all $c \in C$, there is some $p \in P' \subseteq P$ such that for some $C' \subseteq C$ there is a path from $P'$ to $C'$; and (ii) for all $p \in P$, there is some $c \in C$ such that for some $P' \subseteq P$ there is a path from $P'$ to $C'$. Having made this observation the proof proceeds similarly to the case of (Strict Right Composition). ■

We can finally define an intermediate notion of acyclicity.

**Definition 6.8.** A graph $\mathcal{G} = \langle V, \mathcal{A}, h, t \rangle$ is *acyclic* if for all $\Gamma_0 \subseteq V$ there is no strict $\mathcal{G}$-edifice $\mathcal{E} = \langle E, P, C, \mathcal{A}_E, L, h, t \rangle$ such that $L(P) = \Gamma_0$ and $L(C) = \Gamma_1 \subseteq \Gamma_0$.

**Proposition 6.9.** If $\mathcal{G}$ is strongly acyclic, $\mathcal{G}$ is acyclic.

*Proof.* Suppose $\mathcal{E} = \langle E, P, C, L, \mathcal{A}, h, t \rangle$ is a strict $\mathcal{G}$-edifice such that $L(P) = \Gamma_0$ and $L(C) = \Gamma_1$, where $\Gamma_1 \subseteq \Gamma_0$. Let $\gamma \in \Gamma_0$. Let $p \in P$ and $c \in C$ be such that $L(p) = L(c) = \gamma$. By Proposition 6.7 there is $P' \subseteq P$ and $C' \subseteq \Gamma$ such that there is a path from $P'$ to $C'$. This path in $\mathcal{E}$ induces, in $\mathcal{G}$, a path from $L(P')$ to $L(C')$. Since $\gamma \in L(P') \cap L(C')$ this contradicts the strong acyclicity of $\mathcal{G}$. ■

There are, however, acyclic graphs that are not strongly acyclic. Consider for instance the graph $\mathcal{G}$ defined as follows. $V = \{a, b\}$, $\mathcal{A} = \{A\}$, where $t(A) = \{a\}$ and $h(A) = \{a, b\}$. $\mathcal{G}$ is not strongly acyclic but it is acyclic. It would be of some interest to determine under what conditions acyclic graphs are strongly acyclic.

We can finally define the various notions of full ground.

**Definition 6.10.** Let $\mathcal{G} = \langle V, \mathcal{A}, h, t\rangle$ be a graph.

(i) $\Gamma \ll \Delta$ in $\mathcal{G}$ if there is $A \in \mathcal{A}$ such that $t(A) = \Gamma$ and $h(A) = \Delta$.
(ii) $\Gamma < \Delta$ holds in $\mathcal{G}$ if there is a strict $\mathcal{G}$-edifice with premisses $\Gamma$ and conclusions $\Delta$.
(iii) $\Gamma \leq \Delta$ holds in $\mathcal{G}$ if there is a $\mathcal{G}$-edifice with premisses $\Gamma$ and conclusions $\Delta$.

We note the following principles about the interaction of weak and strict full ground.

**Proposition 6.11.**

*(i)* $\Gamma \leq \Gamma$ *for all* $\Gamma$. (*Identity*)
*(ii) (a) If* $\Gamma \ll \Delta$ *then* $\Gamma < \Delta$. (*Subsumption* ($\ll/<$))
*(b) If* $\Gamma < \Delta$ *then* $\Gamma \leq \Delta$. (*Subsumption*($</\leq$))
*(iii) If* $\Gamma < \Delta_0, \Delta_1, \ldots$ *and* $\Delta_i \leq \Sigma_i$, *for each i, then* $\Gamma < \Sigma_0, \Sigma_1, \ldots$ (*Weak Right Cut*)
*(iv) If* $\Gamma < \Delta_0, \Delta_1, \ldots$ *and* $\Delta_i, \Theta_i < \Sigma_i$, *for each i, then* $\Gamma, \Theta_0, \Theta_1, \ldots < \Sigma_0, \Sigma_1, \ldots$ (*Strict Right Cut*)
*(v) If* $\Sigma_0, \Sigma_1, \ldots < \Delta$ *and* $\Gamma_i \leq \Sigma_i$ *for each i, then* $\Gamma_0, \Gamma_1, \ldots < \Delta$. (*Left-Cut*)
*(vi) For no* $\Gamma$ *and* $\Delta$ *do we have* $\Gamma, \Delta < \Gamma_0$ *where* $\Gamma_0 \subseteq \Gamma$. (*Irreflexivity*)
*(vii) If* $\mathcal{G}$ *is strongly acyclic: for no* $\Gamma, \Delta$ *do we have* $\Gamma < \Delta, \gamma$, *where* $\gamma \in \Gamma$.
*(viii) if* $\Gamma < \Delta$ *holds in* $\mathcal{G}$, *then for no* $\Sigma$ *do we have* $\Delta, \Sigma \leq \Gamma_0$, *where* $\Gamma_0 \subseteq \Gamma$. (Irreversibility)

*Proof.* We prove some of the cases. The proofs are by and large just unpacking the definitions.

Identity is immediate. Consider the edifice with vertices (labeled by) $\Gamma$ and the empty set of arcs.

To prove Weak Right Cut suppose that $\Gamma < \Delta_0, \Delta_1, \ldots$. There is then a strict edifice $\mathcal{E}$ with premisses $\Gamma$ and conclusions $\Delta_0, \Delta_1, \ldots$. For each $i = 0, 1, 2, \ldots$ let $\mathcal{D}_i$ be a weak edifice with premisses $\Delta_i$ and conclusions $\Sigma_i$. Then by (Weak Composition) the following is a strict edifice with premisses $\Gamma$ and conclusions $\Sigma_0, \Sigma_1, \ldots$.

$$\begin{array}{ccc} & \Gamma & \\ & \mathcal{D} & \\ \Delta_0 & \Delta_1 & \ldots \\ \mathcal{E}_0 & \mathcal{E}_1 & \ldots \\ \Sigma_0 & \Sigma_1 & \ldots \end{array}$$

Strict Right Cut is immediate from (Strict Right Composition).

Left Cut follows immediately from (Strict Left Composition)

To prove (Irreversibility) we reason as follows. Suppose, $\Gamma < \Delta$. Let $\mathcal{E}$ be a strict edifice witnessing this. Suppose, for contradiction, that there is a weak edifice $\mathcal{F}$ with premisses $\Delta, \Sigma$ and conclusions $\Gamma_0$, where $\Gamma_0 \subseteq \Gamma$. Let $\Gamma_1 = \Gamma \setminus \Gamma_0$. Then the following is a strict edifice with premisses $\Delta, \Sigma, \Gamma_1$ and conclusions $\Delta$, contradicting the acyclicity of $\mathcal{G}$.

$$\begin{array}{ccccc} \Delta & & \Sigma & & \\ & \mathcal{F} & & & \\ & \Gamma_0 & & & \Gamma_1 \\ & & & \mathcal{E} & \\ & & & \Delta & \end{array}$$

■

## 6.4 *Partial ground*

We can also define various notions of partial ground.

**Definition 6.12.** Let $\mathcal{G}$ be a graph.

(i) $\Gamma \preceq \Delta$ holds in $\mathcal{G}$ if there is a $\mathcal{G}$-edifice with premisses $\Gamma, \Sigma$ and conclusions $\Delta$, for some $\Sigma$.
(ii) $\Gamma \prec^* \Delta$ holds in $\mathcal{G}$ if there is a strict $\mathcal{G}$-edifice with premisses $\Gamma, \Sigma$ and conclusions $\Delta$, for some $\Sigma$.
(iii) $\Gamma \prec \Delta$ holds in $\Gamma$ if $\Gamma \preceq \Delta$ holds in $\mathcal{G}$ and $\Delta \preceq \Gamma$ does not hold in $\mathcal{G}$.

$\prec^*$ is the notion of partial strict ground. $\Gamma$ is a partial strict ground for $\Delta$ when $\Gamma$ is part of a strict full ground for $\Delta$. $\prec$ is the notion of strict partial ground. $\Gamma$ is a partial strict ground for $\Delta$ when $\Gamma$ weakly partially grounds $\Delta$ but $\Delta$ does not weakly partially ground $\Gamma$. (Strict partial ground is irreversible weak ground.)

In the truthmaker semantics strict partial and partial strict ground notoriously come apart. Interestingly, they come apart in the graphical approach as well. To see this, consider again the graph $\mathcal{Z}$ from above. Recall that we have the following ascending and descending sequence of immediate ground:

$$\ldots \phi_{i-2} \ll \phi_{i-1} \ll \phi_i \ll \phi_{i+1} \ll \ldots$$

Let $\Phi = \{\phi_i \colon i \in \mathbb{Z}\}$. Then we have $\phi_i \prec \Phi$, for each $i$, but we do not have $\phi_i \prec^* \Phi$, for any $i$.

$\mathcal{Z}$ also provides a counterexample to the principle Reverse Subsumption. This principle says that if $\gamma_0, \gamma_1, \ldots \leq \Phi$ and $\gamma_i \prec \Phi$, for each $i$, then $\gamma_0, \gamma_1, \ldots < \Phi$.

The following principles about partial ground are easily established.

**Proposition 6.13.**

- *If* $\Gamma, \Delta \leq \Theta$ *then* $\Gamma \preceq \Theta$ (*Subsumption*$(\leq/\preceq)$)
- *If* $\Gamma, \Delta < \Theta$ *then* $\Gamma \prec \Theta$. (*Subsumption*$(</\prec)$)
- *If* $\Gamma \preceq \Delta$ *and* $\Delta \preceq \Theta$ *then* $\Gamma \preceq \Theta$. (*Transitivity*$(\preceq/\preceq)$)
- *If* $\Gamma \prec \Delta$ *and* $\Delta \preceq \Theta$ *then* $\Gamma \prec \Theta$ (*Transitivity*$(\prec/\preceq)$)
- *If* $\Gamma \preceq \Delta$ *and* $\Delta \prec \Theta$ *then* $\Gamma \prec \Theta$ (*Transitivity*$(\preceq/\prec)$)

## 6.5 *Equivalent collections*

It is tempting to use the notion of weak full ground to express identity in the object language. This, however, will not work in the bicollective case. Let us use $\Gamma \approx \Delta$ to mean that $\Gamma \leq \Delta$ and $\Delta \leq \Gamma$. There are collections $\Gamma, \Delta$ such that $\Gamma \approx \Delta$ but $\Gamma$ and $\Delta$ differ in what they immediately ground.

Consider, for example, the collection $\Phi_{\text{even}}$ and $\Phi_{\text{odd}}$ from above. In §6.3 above we noted, in effect, that we have $\Phi_{\text{Even}} \approx \Phi_{\text{Odd}}$. Consider the conjunctions $\bigwedge \Phi_{\text{Even}}$ and $\bigwedge \Phi_{\text{Odd}}$. On the plausible assumption that the immediate grounds for a conjunction are all and only the conjuncts, $\bigwedge \Phi_{\text{Even}}$ and $\bigwedge \Phi_{\text{Odd}}$ differ in what they immediately ground. Since any notion of identity of collections of propositions has to satisfy (the analogue of) Leibniz's law this shows that mutual weak full ground is not the right notion of identity between collections of propositions.

Fortunately, the correct notion of identity between collections of propositions is not hard to come by. Say that $\Gamma \approx_I \Delta$ if for all $\Sigma, \Theta$, if $\Gamma, \Sigma \ll \Theta$ then $\Sigma, \Delta \ll \Theta$ (and vice versa). Equivalent collections of propositions agree on what they (help to) immediately ground.

## 7 Conclusion

In this paper I have shown how we can develop a graph-theoretic account of bicollective ground and indicated how it might be useful in formulating mathematical structuralism. As should be apparent, bicollective ground is much more complicated than left-collective ground. The main contribution of this paper has been finding the right definitions and establishing the (fairly rudimentary) results showing that the definitions work. While lots of work remains to be done, both on the technical and philosophical side, I hope to have done enough to convince the reader both that bicollective ground might be useful and also that it can be developed rigorously.

In closing, let me briefly mention two outstanding technical issues. First, can one devise a calculus for a pure logic of bicollective ground and establish soundness and completeness with respect to the hypergraph models constructed here? Second, assuming one has done this, what is the relationship between the graphical semantics for bicollective ground and the truthmaker semantics for bicollective ground?[26]

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[26] Relatedly, what is the relationship between the graphical semantics and the truthmaker semantics for left-collective ground?.

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