

Frege meets Belnap: Basic Law V in a Relevant Logic

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Abstract

Abstractionism in the philosophy of mathematics aims at deriving large fragments of mathematics by combining abstraction principles (i.e. the abstract objects $\xi e_1, \xi e_2$, are identical if, and only if, an equivalence relation Eq_ξ holds between the entities e_1, e_2) with logic. Still, as highlighted in work on the semantics for relevant logics, there are different ways theories might be combined. In exactly what ways must logic and abstraction be combined in order to get interesting mathematics? In this paper, we investigate the matter by deriving the axioms of second-order Peano Arithmetic from Frege's Basic Law V (the extension of F is identical with the extension of G if, and only if, F and G are extensionally equivalent) in the presence of a relevant higher-order logic. The results are interesting. Not only must we take on logic as true, and not only must we apply our logic to abstraction principles, but also we have to apply our theory of abstraction back to the logic in order to arrive at arithmetic. Thus, what Abstractionism gives us is not simply what we get from abstraction via logic, but also what we get from logic via abstraction.

Keywords: Relevant Logics, Abstraction Principles, Basic Law V, Second-Order Peano Arithmetic

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1 Introduction

Abstractionism in the philosophy of mathematics is a varied logico-philosophical program, aiming at deriving large fragments of mathematics from (more often than not, classical higher-order) logic and abstraction principles. Informally, an abstraction principle states that the abstract objects $\xi e_1, \xi e_2$ are identical if, and only if, an equivalence relation Eq_ξ holds between e_1, e_2 , where e_1, e_2 are entities of some kind. Syntactically, ξ is thus a functor that is defined on expressions of

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whatever type the e 's have, and which 'returns' as it were an expression of the type of singular terms. Semantically, \S picks out a function that maps the set of equivalence classes induced by Eq_{\S} to the first-order domain.

Some famous (or infamous) examples of abstraction principles trace back to Frege [7, 8]:

(DIR) the direction of line a is identical with the direction of line b if, and only if, a and b are parallel, for all a, b ;

(HP) the number of the F s is identical with the number of the G s if, and only if, F and G are equinumerous, for all F, G ;¹

(BLV) the extension of F is identical with the extension of G if, and only if, F and G are extensionally equivalent, for all F, G .

Notice that DIR (the Direction Principle) is a first-order abstraction principle, since both lines and directions are first-order individuals. To the contrary, HP (Hume's Principle) and BLV (Basic Law V) are *second-order* abstraction principles, since they are formulated by quantifying over a second-order domain F, G belong to. Regardless of the interest first-order abstraction principles might have, in this article we will focus on second-order abstraction principles. We can then formulate a schematic second-order abstraction principle, of which HP and BLV are instances:

(ABS) for all F, G , the abstracts $\S F, \S G$ are identical if, and only if, an equivalence relation Eq_{\S} holds between F, G .

There are several mathematical or philosophical aims an abstraction principle can be required to achieve. Especially the latter may vary from one abstractionist program to the other. Still, it is fair to say that all such programs:

- aim at deriving large fragments of mathematics by combining abstraction with logic;²
- argue that such a combination guarantees the existence of the objects abstraction governs, and provides identity and distinctness conditions for those objects.

This latter is crucial also as for the former, especially as regards those abstraction principles expected to generate denumerably many objects, e.g. natural numbers.

In order to examine ABS's commitments from above, it is helpful to factorize it in its two conjuncts:

(ABS_{L2R}) If $\S F$ and $\S G$ are identical, then Eq_{\S} holds between F and G ,

¹Famously, HP is the non-logical principle that is central to so-called *Neologicism* of Hale and Wright—see [9, 20].

²See [4] for an effective survey.

meaning that a partition of the second-order domain is necessary for the identity of abstracts;

(**ABS**_{R2L}) If Eq_{\S} holds between F and G , then $\S F$ are identical $\S G$,

i.e., a partition of the second-order domain is sufficient to deliver the existence of abstracts.

So far, we have formulated ABS, and some of its most famous instances such as HP and BLV, informally. This choice is dictated by our reluctance to take a stand on what kind of conditional “if...then” is yet. In what follows we will argue that, in case the conditional is relevant, the relation between the underlying logic and abstraction can be effectively brought to light in a very general fashion.

In order to appreciate how logic and abstraction are closely intertwined, and why such a relation is worth investigating under a respect as general as the one we are proposing, say that we read the conditional classically (\supset).

The right-to-left direction of ABS, $Eq_{\S}(F, G) \supset \S F = \S G$, states that, once we are committed to the existence of a certain partition of the second-order domain in a classical theory, we are thereby committed to the existence of the corresponding abstracts in that *very same theory*. On the other hand, the left-to-right direction, $\S F = \S G \supset Eq_{\S}(F, G)$, states that, only if we are committed to the existence of a certain partition of the second-order domain in a classical theory, we are thereby committed to the identity of abstracts in that *very same theory*.

There is nothing wrong with the material reading of ABS *per se*, but it can be questioned. As far as ABS_{R2L} is concerned, it might be objected that the existence of the individuals secured by a partition of the second-order domain is elicited by the underlying classical logic, which postulates the non-vacuity of the first-order domain.³

As for ABS_{L2R}, what does it mean that a certain partition be necessary for the identity of abstracts, since their identity is fixed in the very same theory in which the equivalence relation is required to hold? After all, abstracts have to be already identical or distinct, because they abide by classical logic.⁴

Still, in work that has come out on the semantics for relevant logics, there are different ways two theories might be combined. Disentangling the role logic and abstraction play both mathematically and philosophically is not only interesting *per se*, but leads to rather surprising results. More precisely, in this paper, we will tackle the following questions:

Q1. Can noncontractive *relevant* logics be used to derive a nontrivial amount of arithmetic from abstraction principles alone?

³Some authors complain about this feature of classical abstraction, which is crucial in e.g. Wright and Hale’s Neologicism—see for instance Tennant [17, 18].

⁴Further authors complain about this feature of (extensionally) classical abstraction, which is also central in e.g. Wright and Hale’s Neologicism—see e.g. Linnebo [10, 11].

We will show that versions of the axioms of second-order Peano arithmetic PA^2 can be derived by relevant abstraction.⁵ But there are interesting caveats. In particular, certain of the Peano axioms cannot be derived in their expected forms, but instead only in a form that makes explicit the various ways in which the abstractionist must rely on the logic. Thus, in the course of answering Q1, we will raise and begin to answer:

Q2. In exactly what ways must logic and abstraction principles be combined, in order to get those mathematical results?

Q2 will sound nonsensical to classical logicians. But relevance logicians will be quite familiar with the phenomenon in question: in relevance logics, one cannot conclude $A \rightarrow (B \rightarrow C)$ from $B \rightarrow (A \rightarrow C)$. Of particular importance in what follows is that even the so-called Ackermann constant \mathfrak{t} —a sentential constant that loosely interprets the conjunction of all the truths of the logic—is subject to this constraint: $\mathfrak{t} \rightarrow (B \rightarrow C)$ does not follow from $B \rightarrow (\mathfrak{t} \rightarrow C)$, nor do either of these follow from $(\mathfrak{t} \wedge B) \rightarrow C$. Thus there is a rich diversity of different ways logic (in the form of \mathfrak{t}) might be combined with other information to arrive at a conclusion.

But also classical abstractionists (if not abstractionists in general) should be sensitive to this issue. Abstraction principles are often deemed non-logical (and also non-analytic, see e.g. [1]) because of their ontological consequences. And that’s a fair enough accusation. But investigating how exactly the logic and abstraction have to be combined to that aim will shed light on the ontological responsibilities of the logic and abstraction respectively. As it will turn out, some of the patterns we see below suggest there is a connection between ‘combinatorial’ facts about how logic and abstraction combine to give results and facts about the existence and identity of abstract objects, and therefore on the relation between logic and abstraction. In particular, we will see how the existence of abstract objects is delivered by the application of the underlying logic to abstraction; whereas the identity of abstracts objects is obtained by the application of abstraction to logic. More in general: not only must we take on logic as true, and not only must we apply our logic to abstraction principles, but also we have to apply our theory of abstraction back to the logic in order to arrive at arithmetic. Thus, what Abstractionism gives us is not simply what we get from abstraction via logic, but also what we get from logic via abstraction.⁶

All of this will be said again—and said more precisely and more clearly—below. And on the note of what’s below, here is a rough outline of what’s to come. In §2, we will present the axiomatic system for a relevant logic and abstraction, and provide explanations of its main features. In §3, we will provide a derivation of (appropriate formulations of) the axioms of PA^2 in the system previously presented. In §4, we will comment on what such derivations reveal, i.e. what the order of application of the logic to abstraction, and of abstraction

⁵It is worth noticing that we will also get usual number-theoretic results from PA^2 axioms. This is nontrivial, modulo the well-known limitations of relevant arithmetic. See §5 below.

⁶See especially §§2.4 and 4 below.

to the logic, is in order to derive PA^2 , and what the ontological responsibilities of the logic and abstraction are respectively. Furthermore, we will provide a comparison with somewhat related views in [14] and [19]. In §5, we will show how the recursiveness of addition is proved in our system. Finally, in §6, we provide closing remarks.

2 Explanation of the Apparatus and Formal Details

In this section, we will provide an axiomatization of the base second-order relevant logic we will use in the remainder. The particular choices we make will, at first, look a bit ad hoc, so we also take a moment to justify the design choices in play.

2.1 The Logic

We work in a second-order language with connectives \neg , \wedge , and \rightarrow and quantifier \forall . With respect to these we adopt the usual definitions for \vee , \leftrightarrow , and \exists . Where A is a formula and F^n is an n -ary predicate variable, we write $A[F^n/B(y_1, \dots, y_n)]$ for the formula that results from replacing each instance of a formula of the form $Ft_1 \dots t_n$ in A with an instance of the formula $B(y_1/t_1, \dots, y_n/t_n)$, where we assume that in the latter the substitution is done simultaneously via some device or other that allows one to avoid collision.

Language in place, the logic $DL2Q^{t,fc}$ that we work in is axiomatized as follows:

- A1. t
- A2. $A \rightarrow A$
- A3. $(A \wedge B) \rightarrow A$; $(A \wedge B) \rightarrow B$
- A4. $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$
- A5. $[A \wedge (B \vee C)] \rightarrow [(A \wedge B) \vee (A \wedge C)]$
- A6. $\neg\neg A \rightarrow A$
- A7. $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$
- A8. $((A \rightarrow B) \wedge (B \rightarrow C)) \rightarrow (A \rightarrow C)$
- A9. $(A \rightarrow \neg A) \rightarrow \neg A$
- A10. $\forall x A \rightarrow A(x/y)$ provided y is free for x in A .
- A11. $\forall x(A \rightarrow B) \rightarrow (A \rightarrow \forall x B)$ provided x is not free in A .
- A12. $\forall x(A \vee B) \rightarrow (A \vee \forall x B)$ provided x is not free in A .

A13. $\exists F^n \forall y_1 \dots \forall y_n (A \leftrightarrow A(F^n/B(y_1, \dots, y_n)))$ where F doesn't occur free in B and B is free for F in A .

$$\text{R1. } \frac{A \quad A \rightarrow B}{B}$$

$$\text{R2. } \frac{A \quad B}{A \wedge B}$$

$$\text{R3. } \frac{A \rightarrow B \quad C \rightarrow D}{(B \rightarrow C) \rightarrow (A \rightarrow D)}$$

$$\text{R4. } \frac{A}{\forall x A}$$

$$\text{R5. } \frac{A}{\mathbf{t} \rightarrow A}$$

As usual, we write ' $\vdash A$ ' to mean that A is provable from the given axioms using the given rules. We note that in axioms A9-A11 and rule R4, the quantification can be either first- or second-order. The 'official' (but unweildy) name ' $\text{DL2Q}^{\mathbf{t},fc}$ ' reflects the fact that the logic is a second-order quantified, includes the so-called Ackermann constant ' \mathbf{t} ' and incorporates *full* comprehension (axiom scheme A13).

We'll now turn to answering some of the obvious questions one might ask about this logic, we'll note one further 'logical' abbreviation that will be helpful for our purposes: for $n > 0$ define $A \xrightarrow{n} B$ as follows:

- $A \xrightarrow{0} B := A \rightarrow B$
- $A \xrightarrow{n+1} B := A \rightarrow (\mathbf{t} \xrightarrow{n} B)$

So, e.g. $A \xrightarrow{1} B$ abbreviates $A \rightarrow (\mathbf{t} \rightarrow B)$ and $A \xrightarrow{2} B$ abbreviates $A \rightarrow (\mathbf{t} \rightarrow (\mathbf{t} \rightarrow B))$, etc.

2.2 Answers to Some of the Obvious Questions

$\text{DL2Q}^{\mathbf{t},fc}$ certainly has a weird enough name. Readers of a suspicious bent are likely to find it suspicious on these grounds alone. The purpose of this sidebar is to allay such suspicions. The unsuspecting can thus safely skip it.

We allay the suspicions from left to right, as it were. First up, then, is DL. On its own, DL is a fairly well-known logic—it's axiomatized (see e.g. [3]) by A2-A8 together with R1-R3. DL *is* on the weaker end of the family of relevant logics. But it's also a dues-paying member in good standing of the relevance community. So those who found their suspicions aroused by the first two letters of our logic's name are invited to peruse e.g. [15] or other parts of the relevant literature to find ways to allay their worries.

Next up left-to-right is '2Q'. To extend from DL to DL2Q, we add A9-A11 and R4. That this is 'the right' way to extend is justified by the fact that it's

the straightforward second-orderization of ‘the right’ way to extend from DL to first-order quantified DL, DLQ (see e.g. [6] or [12]).

The next bit—the so-called Ackermann constant \mathbf{t} —is, proof-theoretically at least, straightforward. We’ve added \mathbf{t} by adding axiom A1 and rule R5. These are the completely standard moves—see e.g. [15] again. What it means and why it’s needed, on the other hand, requires a bit of explanation.

Intuitively, \mathbf{t} represents the conjunction of all the truths of the logic, whatever it might be. As mentioned above, one main focus of the paper is determining exactly what ways logic must be applied to abstraction, and abstraction to logic in order to arrive at various bits of arithmetic. The use of \mathbf{t} , whether in definitions or derivations of theorems, will make visible the order of both kinds of application. This should feel a bit loose at the moment, but when we find \mathbf{t} playing this role below, we’ll highlight it so that the reader can see exactly what we mean.

Finally is *fc*—full comprehension. This requires the most explanation. We’ve accomplished ‘full’ comprehension by adding axiom scheme A13. But the form of this axiom scheme might (should?) surprise the reader. Worth noting is that the instances of A13 where A is just $Fy_1 \dots y_n$, are exactly the instances of the ‘usual’ full comprehension axiom scheme:

$$\exists F \forall y_1 \dots \forall y_n (Fy_1 \dots y_n \leftrightarrow B)$$

So clearly by adopting all instances of the scheme we’ve given as axioms, we’ve also adopted all instances of the usual full (and fully impredicative) comprehension scheme. And as we leave to the reader to verify, in classical logic, once one has on hand all instances of the usual comprehension scheme, one can prove every instance of A13. Thus, in exactly the sense just spelled out, the two axiom schemes are classically equivalent. But in the setting at hand, this equivalence fails: the scheme we’ve added here is strictly stronger than the usual axiom scheme. One might wonder the reason for adopting the stronger option. Our reply is quite simple: it’s necessary in order for universal elimination to work in the way one would hope; see Theorem 1 for the precise details.

We trust this to have allayed the suspicions of all those readers who held allayable suspicions, and thus move on.

2.3 A Few Important Facts

Our first task is to prove a few things about our system. We begin with a few straightforward results:

Lemma 1.

- $\vdash A \rightarrow B$ iff $\vdash \neg B \rightarrow \neg A$. (Contraposition)
- If $\vdash \forall x(A \rightarrow B)$ and x is not free in A , then $\vdash A \rightarrow \forall xB$. (Intensional Confinement)
- If $\vdash A \rightarrow B$ and $\vdash B \rightarrow C$, then $\vdash A \rightarrow C$ (Transitivity)

- $\vdash \forall x(A \rightarrow B) \rightarrow (\exists xA \rightarrow B)$ for all B in which x does not occur free. (Existential Elimination)
- $\vdash \forall F(Fx \leftrightarrow Fx)$. (Identity)
- $\vdash A(c) \rightarrow \exists xA(x)$, where here we write ' $A(c)$ ' to mean A with one or more occurrences of ' c ' highlighted in it and by ' $A(x)$ ' we mean the corresponding formula in which the highlighted instances of ' c ' have been replaced by ' x '. (Existential Introduction)
- If $\vdash A \rightarrow (B_1 \rightarrow \dots \rightarrow (B_n \rightarrow C) \dots)$ and $\vdash C \rightarrow D$, then $\vdash A \rightarrow (B_1 \rightarrow \dots \rightarrow (B_n \rightarrow D))$. (Generalized Transitivity)
- If $\vdash C_i \rightarrow B_i$ for $1 \leq i \leq n$, then for all A and D , $\vdash [A \rightarrow (B_1 \rightarrow (\dots \rightarrow (B_n \rightarrow D) \dots))] \rightarrow [A \rightarrow (C_1 \rightarrow (\dots \rightarrow (C_n \rightarrow D) \dots))]$ (Embedded Prefixing)

Proof. The first five are proved in exactly the way one would expect, though a few come with an extra helping of tedium because of the second-order quantifiers. The last two follow by induction on n , though the second induction is a bit trickier than the first. \square

In the remainder we will write ' $a = b$ ' as shorthand for ' $\forall F(Fa \leftrightarrow Fb)$ '. We note that by Identity, $\vdash x = x$.

Corollary 1 (Generalized Cut). If $\vdash C_i \rightarrow B_i$ for $1 \leq i \leq n$ and $\vdash A \rightarrow (B_1 \rightarrow (\dots \rightarrow (B_n \rightarrow D) \dots))$, then $\vdash A \rightarrow (C_1 \rightarrow (\dots \rightarrow (C_n \rightarrow D) \dots))$.

Proof. Immediate from Embedded Prefixing \square

Corollary 2. If $\vdash B_i$ for $1 \leq i \leq n$ and $\vdash A \rightarrow (B_1 \rightarrow (\dots \rightarrow (B_n \rightarrow D) \dots))$, then $\vdash A \xrightarrow{n} D$.

Proof. Since $\vdash B_i$, R5 gives $\vdash t \rightarrow B_i$, so the result follows by Generalized Cut. \square

And, because the paper would be incomplete without going through the details in at least one case, we'll prove the following important result in all its glory:

Theorem 1 (Instantiation). If $B(\bar{y}) = B(y_1, \dots, y_n)$ is free for F^n in A and does not contain F^n free, then $\vdash \forall F^n A \rightarrow A(F^n/B(\bar{y}))$.

Proof. To begin, note that by an instance of A3,

$$\vdash (A \leftrightarrow A[F^n/B(\bar{y})]) \rightarrow (A \rightarrow A[F^n/B(\bar{y})]) \quad (1)$$

Also where $\bar{z} = z_1, \dots, z_k$ are the free variables in A ,

$$\vdash \forall \bar{z}(A \leftrightarrow A[F^n/B(\bar{y})]) \rightarrow (A \leftrightarrow A[F^n/B(\bar{y})]) \quad (2)$$

Together (1) and (2) give that

$$\vdash \forall \bar{z}(A \leftrightarrow A[F^n/B(\bar{y})]) \rightarrow (A \rightarrow A[F^n/B(\bar{y})]) \quad (3)$$

Thus by Contraposition,

$$\vdash \neg(A \rightarrow A[F^n/B(\bar{y})]) \rightarrow \neg \forall \bar{z}(A \leftrightarrow A[F^n/B(\bar{y})]) \quad (4)$$

Next observe that by an instance of A9 we have

$$\vdash \forall F^n \neg(A \rightarrow A[F^n/B(\bar{y})]) \rightarrow \neg(A \rightarrow A[F^n/B(\bar{y})]) \quad (5)$$

Together (4) and (5) then give

$$\vdash \forall F^n \neg(A \rightarrow A[F^n/B(\bar{y})]) \rightarrow \neg \forall \bar{z}(A \leftrightarrow A[F^n/B(\bar{y})]) \quad (6)$$

Thus, by an application of R4 we have that

$$\vdash \forall F^n [\forall F^n \neg(A \rightarrow A[F^n/B(\bar{y})]) \rightarrow \neg \forall \bar{z}(A \leftrightarrow A[F^n/B(\bar{y})])] \quad (7)$$

So since $\forall F$ does not occur free in $\forall F^n \neg(A \rightarrow A[F^n/B(\bar{y})])$, Intensional Confinement gives

$$\vdash \forall F^n \neg(A \rightarrow A[F^n/B(\bar{y})]) \rightarrow \forall F^n \neg \forall \bar{z}(A \leftrightarrow A[F^n/B(\bar{y})]) \quad (8)$$

Thus, again by Contraposition,

$$\vdash \exists F^n \forall \bar{z}(A \leftrightarrow A[F^n/B(\bar{y})]) \rightarrow \exists F^n (A \rightarrow A[F^n/B(\bar{y})]) \quad (9)$$

But also by A13, $\vdash \exists F^n \forall \bar{z}(A \leftrightarrow A[F^n/B(\bar{y})])$. Thus

$$\vdash \exists F^n (A \rightarrow A[F^n/B(\bar{y})]) \quad (10)$$

Now observe that since by A9 $\forall F A \rightarrow A$, an instance of A2 and an application of R3 give that

$$\vdash (A \rightarrow A[F^n/B(\bar{y})]) \rightarrow (\forall F^n A \rightarrow A[F^n/B(\bar{y})]) \quad (11)$$

But then by R4,

$$\vdash \forall F^n [(A \rightarrow A[F^n/B(\bar{y})]) \rightarrow (\forall F^n A \rightarrow A[F^n/B(\bar{y})])] \quad (12)$$

But now observe that since X^n does not occur free in $\forall F^n A \rightarrow A[F^n/B(\bar{y})]$, it follows from (12) and Existential Elimination that

$$\vdash \exists F^n (A \rightarrow A[F^n/B(\bar{y})]) \rightarrow (\forall F^n A \rightarrow A[F^n/B(\bar{y})]) \quad (13)$$

Thus, finally, by (10) and (13) we have our desired result:

$$\vdash \forall F^n A \rightarrow A[F^n/B(\bar{y})] \quad (14)$$

□

Corollary 3. If $B(\bar{y}) = B(y_1, \dots, y_n)$ is free for F^n then $\vdash \forall F^n A \rightarrow A(F^n/B(y_1, \dots, y_n))$.

Proof. If B does not contain F^n free, the result follows from the previous theorem. If it does, then consider $B' = B(F^n/G^n)$ where G^n does not occur in A . By the previous theorem $\vdash \forall F^n A \rightarrow A(F^n/B'(y_1, \dots, y_n))$. Thus by R4, $\vdash \forall G^n (\forall F^n A \rightarrow A(F^n/B'(y_1, \dots, y_n)))$. So by an instance of A9 and R1, we can instantiate at F^n and get $\vdash \forall F^n A \rightarrow A(F^n/B)$. □

2.4 Adding Abstraction

The final move is to augment $\text{DL2Q}^{\mathfrak{t},fc}$ by an abstraction principle. The abstraction principle we add should satisfy the following two desiderata:

- It should be strong enough to allow for a derivation of Peano axioms, but
- Not so strong that it interferes with the logic’s ability to track ‘combinatorial’ details about how logic and abstraction need to be mixed to deliver these results.

Consider ABS again: $\forall F\forall G(\S F = \S G \leftrightarrow Eq_{\S}(F, G))$. We have two options here:

1. keep ABS as the sole non-logical axiom added to $\text{DL2Q}^{\mathfrak{t},fc}$, and derive significant second-order abstraction principles such as HP or BLV from it, and then derive PA^2 one way or another;
2. assume a particular abstraction principle, e.g. HP or BLV, directly as the sole non-logical axiom added to $\text{DL2Q}^{\mathfrak{t},fc}$, and derive PA^2 from it.

For the sake of simplicity, we will pursue the second option. In this respect, a further choice has to be made. Peano arithmetic can be recovered either by HP or BLV. The former would involve massive use of second-order definitional resources in order to recover so-called *Frege Arithmetic*, namely second-order logic plus HP which, via appropriate definitions of Fregean notions such as *predecessor*, *(weak) ancestral*, and *natural number*, interprets PA^2 —this result is now known as *Frege’s theorem*.⁷

The problem is that, at least for agents like us, the complexity this adds ends up interfering with the second desideratum recorded above, as the formulas we end up at are so complex as to obscure rather than reveal the combinatorially interesting facts we’re after. Thus, in the end, we settle on augmenting $\text{DL2Q}^{\mathfrak{t},fc}$ with one additional (and intuitively non-logical) axiom:

$$\text{(BLV)} \quad \forall F\forall G(\{F\} = \{G\} \leftrightarrow F \equiv G),$$

where \equiv is defined as $\forall x(Fx \leftrightarrow Gx)$.⁸

An important initial observation is that BLV is formulated without any occurrences of \mathfrak{t} . More helpfully, contrast BLV in the form we’ve added it with axioms like the following:

- $\forall F\forall G(\{F\} = \{G\} \wedge \mathfrak{t} \leftrightarrow F \equiv G)$;
- $\forall F\forall G(\{F\} = \{G\} \leftrightarrow (F \equiv G \wedge \mathfrak{t}))$;

⁷A further path also worth investigating would be the derivation of HP from BLV as the sole non-logical axiom added to $\text{DL2Q}^{\mathfrak{t},fc}$, but that would involve at least the same amount of second-order resources that a derivation of PA^2 from HP as the sole non-logical axiom requires.

⁸Clearly, the functional abstraction operator $\{\}$, (semantically) taking entities in the second-order domain as arguments and individuals from the first-order domain as values, is to be added to the underlying language.

- $\forall F \forall G (\{F\} = \{G\} \leftrightarrow F \equiv G)$; and so on.

In such axioms, the ‘blame’ (as it were) meted out for the presence of the abstracta in question (extensions in the case at hand) is being divided in various ways between logic and abstraction.

But it is often argued that abstraction principles are axioms of infinity that cannot be deemed logical, because of their ontological consequences. In fact, by axioms such as HP or BLV, it is possible to prove the existence of denumerably many first-order individuals, i.e. the *denotata* of abstract terms appearing on their left-hand sides. Still,

[i]n logic we ban the empty domain as a concession to technical convenience but draw the line there: We firmly believe that the existence of even two objects, let alone infinitely many, cannot be guaranteed by logic alone. ([1, 199])

In this respect, the lack of **t** in formulating BLV makes the non-logicality of the objects it guarantees the existence of apparent, in particular as regards its ontological consequences as we will see below. So, in a sense, BLV is *pure* abstraction—or, some might say, pure mathematics.

This latter remark also provides some historical justification for choosing BLV over HP. Famously, Frege argued, not without reservations, that BLV is a logical principle—see [8, *Foreword*, p. VII]. Later, he bitterly deplored BLV and its alleged logical status, because of Russell’s paradox—[8, *Afterword*]; and eventually, he abandoned it altogether. In our framework, not only BLV requires no logic (i.e. no **t**) to be formulated, but furthermore, in what follows, we will show where exactly BLV fails to be a logical principle: in differentiating among abstracts delivered by a given equivalence relation, and therefore, when appropriate, in delivering denumerably many abstracts.

In doing so, we will also show what the responsibility of the underlying logic is: it is the underlying logic that delivers (or has to deliver) the existence of at least an individual, starting from which abstraction works like an assembly line manufacturing denumerably many objects via identity and distinctness statements. Without the input of the logic, abstraction would run on empty.

By detaching the role BLV and the logic play in the derivation of Peano axioms (and the recursiveness of addition) from below, it will be visible that BLV is no logical principle, regardless of Russell’s paradox and, most of all, *pace* Frege.

3 The Derivations of Peano Axioms

We’ll now turn to deriving the Peano Axioms. But first, of course, we need numbers. Since they can be defined quite cleanly, we’ll use (essentially) the von Neumann ordinals to do the job. Thus we adopt the following two definitions:

- $\bar{0} : \{x. \neg(x = x)\}$;

- $\{w\} := \{x.w = x\}$

We then define the (von Neumann) numbers as follows:

$$\mathbb{N}x := \forall F((\forall w(Fw \rightarrow F\{w\}) \wedge \mathfrak{t}) \rightarrow (F\bar{0} \rightarrow Fx))$$

Note that classically, F is inductive along the von Neumann successor relation and holds of the initial von Neumann ordinal just if the following is true:

$$(F\bar{0} \wedge \forall w(Fw \rightarrow F\{w\}))$$

Thus, again speaking classically, one can say ‘if F is inductive and holds of 0 , then F holds of x ’ as follows:

$$(F\bar{0} \wedge \forall w(Fw \rightarrow F\{w\})) \rightarrow Fx$$

And of course, using classically-acceptable reasoning, the above is equivalent to

$$\forall w(Fw \rightarrow F\{w\}) \rightarrow (F\bar{0} \rightarrow Fx)$$

But the transformations this relies on are not permissible in $\text{DL2Q}^{\mathfrak{t},f^c}$. This raises two natural questions:

- Why use the latter formulation instead of the initial (conjunctive) formulation for capturing the numbers?
- Why the ‘ \mathfrak{t} ’?

The answer to the first question is this: since we lack conjunctive modus ponens (viz. the following: $(A \wedge (A \rightarrow B)) \rightarrow B$) as a theorem in our setting, the initial, conjunctive interpretation of ‘inductive and holds of 0 ’ is largely impotent.

For the second question, the answer is even easier: absent the \mathfrak{t} , $\bar{0}$ doesn’t count as a number. This is fairly easy to see: what $\forall w(Fw \rightarrow F\{w\})$ guarantees is that, given anything F holds of, F also holds of the (von Neumann; we’re going to stop saying that in the remainder) successor of F . But since 0 isn’t a successor (see Lemma 2), this can’t help us learn anything at all about what’s going on with 0 . Logic, however, *can* help us out on that front: by axiom A2, the following is a theorem of the logic: $F\bar{0} \rightarrow F\bar{0}$. Being a theorem, it’s also implied by \mathfrak{t} . Thus, in order to include 0 among the numbers (which, as the proof of Peano1 below shows, crucially requires \mathfrak{t} to be carried out and will be commented on in §4), we have to add an explicit call to the logic (in the form of a ‘ $\wedge\mathfrak{t}$ ’) to our definition of numbers. Arguably, it is because the logic is necessary to prove that indeed 0 is a number, namely that the set of natural numbers is nonempty because it contains at least an individual i.e. 0 , that \mathfrak{t} is needed in the definition of \mathbb{N} . So, the logic is necessary to guarantee that there is at least a natural number, in particular, the one the ω -sequence starts from.

With definitions now expressed and discussed, we turn to deriving the Peano axioms, beginning with an explicit reconstruction of the argument given in the previous paragraph:

Theorem 2 (Peano1). $\vdash \mathbb{N}\bar{0}$

Proof. $\vdash F\bar{0} \rightarrow F\bar{0}$ by A2 so by R5, $\vdash \mathbf{t} \rightarrow (F\bar{0} \rightarrow F\bar{0})$. Also by A3, $\vdash (\forall y(Fy \rightarrow F\{y\}) \wedge \mathbf{t}) \rightarrow \mathbf{t}$. So by Lemma 1 part (b), $\vdash (\forall y(Fy \rightarrow F\{y\}) \wedge \mathbf{t}) \rightarrow (F\bar{0} \rightarrow F\bar{0})$. Thus $\vdash \mathbb{N}\bar{0}$. \square

Lemma 2. $\vdash \neg\{x\} = \bar{0}$.

Proof. First note that by BLV, we have that

$$\vdash \{x\} = \bar{0} \rightarrow \forall z(z = x \rightarrow \neg z = z) \quad (15)$$

Next, note that by instances of A9 and A10 we have

$$\vdash \forall z(z = x \rightarrow \neg z = z) \rightarrow (x = x \rightarrow \neg x = x) \quad (16)$$

$$\vdash (x = x \rightarrow \neg x = x) \rightarrow \neg x = x \quad (17)$$

Together with transitivity (b), (15), (16), and (17) then give $\vdash \{x\} = \bar{0} \rightarrow \neg x = x$. So by A7 and R1, $\vdash x = x \rightarrow \neg\{x\} = \bar{0}$. But by Identity, $\vdash x = x$ as well. So $\vdash \neg\{x\} = \bar{0}$ \square

Corollary 4 (Peano2). $\vdash (\mathbb{N}x \wedge \mathbf{t}) \rightarrow \neg\{x\} = \bar{0}$.

Theorem 3. $\vdash (\mathbb{N}x \wedge \mathbf{t}) \rightarrow \mathbb{N}\{x\}$.

Proof. Note that by BLV and Instantiation, we have

$$\vdash \mathbb{N}x \rightarrow \underbrace{[(\forall y(Fy \rightarrow F\{y\}) \wedge \mathbf{t}) \rightarrow (F\bar{0} \rightarrow Fx)]}_A \quad (18)$$

Clearly we also have that

$$\vdash (\forall y(Fy \rightarrow F\{y\}) \wedge \mathbf{t}) \rightarrow (Fx \rightarrow F\{x\}) \quad (19)$$

Thus by R5, we also have

$$\vdash \mathbf{t} \rightarrow \underbrace{((\forall y(Fy \rightarrow F\{y\}) \wedge \mathbf{t}) \rightarrow (Fx \rightarrow F\{x\}))}_B \quad (20)$$

Thus we also have

$$\vdash (\mathbb{N}x \wedge \mathbf{t}) \rightarrow (A \wedge B) \quad (21)$$

But now observe that by A4 we get the following:

$$\vdash (A \wedge B) \rightarrow [(\forall y(Fy \rightarrow F\{y\}) \wedge \mathbf{t}) \rightarrow ((F\bar{0} \rightarrow Fx) \wedge [Fx \rightarrow F\{x\}])] \quad (22)$$

And by Generalized Transitivity we also have

$$\vdash ((F\bar{0} \rightarrow Fx) \wedge [Fx \rightarrow F\{x\}]) \rightarrow (F\bar{0} \rightarrow F\{x\}) \quad (23)$$

So, by two further instances of Generalized Transitivity, (first involving 22 and 23, then involving the resulting formula and 21) we have

$$\vdash (\mathbb{N}x \wedge \mathbf{t}) \rightarrow [(\forall y(Fy \rightarrow F\{y\}) \wedge \mathbf{t}) \rightarrow (F\bar{0} \rightarrow F\{x\})] \quad (24)$$

Thus by A4,

$$\vdash \forall F[(\mathbb{N}x \wedge \mathbf{t}) \rightarrow [(\forall y(Fy \rightarrow F\{y\}) \wedge \mathbf{t}) \rightarrow (F\bar{0} \rightarrow F\{x\})]] \quad (25)$$

And since F is not free in $\mathbb{N}x \wedge \mathbf{t}$, it follows that $\vdash (\mathbb{N}x \wedge \mathbf{t}) \rightarrow \mathbb{N}\{x\}$. \square

It's worth noting that the above result does *not* seem to go through without the \mathbf{t} . We suspect that a clever application of MaGIC should suffice to show that this is the case. But we couldn't quite be clever enough, so we'll instead offer the following heuristic reason to believe this: (18) and (19) are alone insufficient for showing that $\mathbb{N}x \rightarrow [(F\bar{0} \rightarrow Fx) \wedge (Fx \rightarrow F\{x\})]$ is a theorem. And absent that, there's no way to get to the desired result.

At any rate, with the above lemma in hand, we can quite easily prove the following:

Corollary 5 (Peano3). $\vdash \forall x((\mathbb{N}x \wedge \mathbf{t}) \rightarrow \exists y(\mathbb{N}y \wedge y = \{x\}))$

Lemma 3. $\vdash x = y \rightarrow \{x\} = \{y\}$

Proof. It follows from the definition of '=' that $x = y \rightarrow \forall z(z = x \leftrightarrow z = y)$. But then by an instance of BLV, $x = y \rightarrow \{x\} = \{y\}$. \square

Note that in all the results above, the needed ' \mathbf{t} 's were all added *conjunctively*. In contrast, the next Peano axiom we derive requires an *intermediary* \mathbf{t} —that is, rather than simply conjoining addition \mathbf{t} 's, we instead have to rely on the defined conditional $\xrightarrow{1}$.

Lemma 4. $\vdash \{x\} = \{y\} \xrightarrow{1} x = y$

Proof. By BLV, Instantiation, and Transitivity, $\vdash \{x\} = \{y\} \rightarrow (x = x \rightarrow x = y)$. Thus since $\vdash \mathbf{t} \rightarrow x = x$ by Identity and R5, Corollary 2 2 gives that $\vdash \{x\} = \{y\} \rightarrow (\mathbf{t} \rightarrow x = y)$ as required. \square

Corollary 6 (Peano4). $\vdash \forall x \forall y((\mathbb{N}x \wedge \mathbb{N}y \wedge \{x\} = \{y\}) \xrightarrow{1} x = y)$

This leaves us with just Peano 5. But the astute reader probably observed already that there's nothing to do here: our definition of ' $\mathbb{N}x$ ' is essentially exactly ' x satisfies Peano 5'.

4 Philosophical Commentary

In this section, we will investigate how logic and abstraction interact in the derivation of Peano axioms from above, in order to shed light on the application both of logic to abstraction and of abstraction to logic in recovering them.

First, let us tackle the rather unequivocal cases: those in which only the \mathbf{t} is involved in the proofs; and those in which only BLV is needed. Peano1 and Peano2 are the cases in point.

As anticipated in the previous section, in order to prove Peano1, the application of the background logic to abstraction is necessary. By BLV, the definition of $\bar{0}$ is provided, but BLV *per se* does not necessarily secure that there is an individual that $\bar{0}$ can be assigned to. Now consider two facts. First, $F\bar{0} \rightarrow F\bar{0}$ would be an instance of A2, even if $\bar{0}$ were an empty term. Secondly, it is via R5 that we eventually get to $N\bar{0}$. So, in $DL2Q^{\mathbf{t},fc}$ what the proof of Peano1 shows is that the non-vacuity of the set N of natural numbers, and in particular that N contains the individual 0, relies on the \mathbf{t} . This presupposes that it is the background logic that guarantees (or has to guarantee) the existence of an individual which $\bar{0}$ is assigned to.⁹

Peano2, which captures the identity fact that 0 is no singleton, is proved as a corollary of Lemma 2, which states the functionality of the singleton. This latter result is proved on the basis on BLV—and by axioms and lemmas *not* concerning \mathbf{t} . Consequently, BLV, but not the logic, is necessary to prove an identity fact concerning the abstract objects it governs.

The relation between the logic and abstraction is more nuanced as for Peano3, which states the existence of the successor, and Peano4, stating the injectivity of the singleton.

More specifically, Peano3 follows as a corollary from Theorem 3, stating that if x is a natural number (and the logic is around) so is its singleton. The proof of Theorem 3 requires both BLV and \mathbf{t} , but in the proof BLV seems to provide just the raw material to which the logic has to be applied in order to get the desired result. In particular, by BLV if anything at all is F so will be its singleton, so that if 0 is F so will be any x , where x is a natural number—from which it also follows by Instantiation that, if anything at all is F so will be its singleton, so that if x is F so will be $\{x\}$. But *that* would be trivially true even if there were no numbers at all. As already mentioned, we suspect that without the logic we would not be able to move on from that to the proof that, if x is a natural number (and the logic is around), so will also be $\{x\}$. Consequently, without \mathbf{t} it would not be possible to prove Peano3, which means that without the logic it would not be possible to prove the existence of the successor of any natural number.

Finally, let us come to Peano4. This latter is a corollary of Lemma 4, which states the injectivity of the singleton, *provided the logic is around*. At a closer look of the proof of Lemma 4, it is clear why this happens. BLV, by Transitivity, can only guarantee that $\{x\} = \{y\} \rightarrow (x = x \rightarrow x = y)$ is a theorem of $DL2Q^{\mathbf{t},fc}$. But this theorem does *not* imply that $\vdash x = x \rightarrow (\{x\} = \{y\} \rightarrow x = y)$, since in $DL2Q^{\mathbf{t},fc}$ $A \rightarrow (B \rightarrow C)$ does not follow from $B \rightarrow (A \rightarrow C)$. If this *were* the case, the injectivity of the singleton would follow by BLV and the definition of ‘=’ (and Transitivity) alone. But this not being the case, we have

⁹In classical Peano arithmetic, $\bar{0}$ is referential by assumption; in a setting with a free logic as the background logic, that $\bar{0}$ is indeed referential has to be proved, see e.g. [17, 18]. See also fn. 10 below.

to apply abstraction, i.e. $\{x\} = \{y\}$, to the logic, i.e. $\{x\} = \{y\} \rightarrow (\mathbf{t} \rightarrow \dots)$, in order to deliver the desired result, i.e. $x = y$. This means that BLV is insufficient to prove the injectivity of the abstraction operator, at the very least as for singletons, which might sound rather surprising since BLV is the sole non-logical axiom governing the abstraction operator $\{\}$. Still, this makes sense as long as one considers that applying abstraction back to the logic is a completely different matter than applying the logic to abstraction: the logic *per se* does not deliver abstraction (sufficiently fine-grained identity conditions to distinguish between different individuals of a given kind), whereas abstraction can only be added on top of a logic (which has to secure the existence of at least one individual) in order to be productive.

Given what’s been argued so far, the logic, whether in isolation or as applied to abstraction, is needed in order to accomplish ontological results, (i.e. Peano1 and Peano3), whereas abstraction, either in isolation or as applied to the logic, is necessary when identity and distinctness facts have to be proved (i.e. Peano2 and Peano4). More precisely, once the existence of at least an individual is brought about by the logic, BLV starts proving (or at least is necessary to prove) the existence of denumerably many of them, via identity and, crucially, distinctness facts—as expected and as Lemmas 3 and 4 guarantee. But without the logic, BLV would run on empty.

Finally, our approach strikes some significant differences to [14] and [19]. First, there are some technical dissimilarities between our framework and the ones in [14] and [19] we deem important. For instance, both [14] and [19] are first-order systems with identity, so they suffer from the usual issues concerning identity in relevant settings. Being second-order, our framework can afford to define identity explicitly. This has consequences also as for the formulation of BLV, in particular in [19] where BLV has to be schematic. Since classical first-order logic plus schematic BLV is consistent, in order to bring about Russell’s paradox [19] adds membership as a primitive—while we can define it explicitly *à la* Frege as $x \in y \leftrightarrow_{def} \exists F(y = \{F\} \wedge Fx)$. This highlights that Fregean extensions in our framework, unlike [19]’s, are delivered by a conception that is more faithful to Frege’s (i.e. as objects strictly related to concepts) and, so, is remarkably different from the iterative (and even Cantorian) conception. Also, the system in [14] is modal, whereas our framework is plainly extensional.

Secondly, both in [14] and [19], the (respective) logics are meant to provide either *the* or at the very least *a* foundation for mathematics—or even all science as in [14]. This is immaterial in our perspective. Our use of the Ackermann constant \mathbf{t} delivers a significantly different philosophical result. In any model, \mathbf{t} is true at exactly those points that verify the logic of the model—that is, the set of sentences that all the theories in the model are closed under. Thus, those sentences that are a consequence of \mathbf{t} in a given model are exactly the sentences that are contained in the logic of that model. *Theorems* involving \mathbf{t} , then (such as the theorems we’ve proved above and will prove below) tell us something about what must be true in all logics—that is, what must be the case in all of our theory building endeavors whatsoever, whether they be scientific, mathematical or of any other nature.

Thus, what we are offering is a clear picture of the relation between the logic of a model (*whatever that might be*) and abstraction in deriving mathematical results. Our proposal embeds the bulk of logic and abstraction that is common to *any* logic and abstraction. What is common to any logic in our framework is that the logic has to provide at least an individual, so that abstraction does not ‘run on empty’ as it were.¹⁰

5 Addition

Before launching into the next section, it’s worth noting why it’s worth doing at all. The thing to observe is that it’s one thing to show how to derive (versions of) the axioms of some system S in a novel logical environment and another thing to show how to arrive at (versions of) all the theorems of S in that environment. This is especially the case in derivations like those we’re considering, since theories derived using classical logic are closed under an incredible array of rules that theories derived using relevant logics are not.

In the case at hand, what we need to now look at is the extent to which the above derivation of analogues of the Peano axioms also allow us to define and derive various bits of arithmetic. But there’s rather a lot of arithmetic, so we’ll only in fact be able to look one small but important piece of the picture.

To begin, we (roughly) follow the standard practice (see e.g. [5] and [21]) in defining the three-place addition relation $Sxyz$ as follows:

$$Sxyz := \forall F([\forall u\forall w(Fuw \rightarrow F\{u\}\{w\}) \wedge \mathbf{t}] \rightarrow [F\bar{0}x \rightarrow Fyz])$$

As with our definition of \mathbb{N} , our definition of S is classically equivalent to the usual definition, but modified to accommodate the vagaries of the system we’re using here. And, as we trust the reader to pause and verify for themselves, $Sxyz$ intuitively holds of those triples where the third member is the sum of the first two.

Definition in hand, we can now state that our aim is to show that

$$\vdash \forall x\forall y(\mathbb{N}x \xrightarrow{2} (\mathbb{N}y \xrightarrow{2} \exists zSxyz))$$

As an overview, here’s roughly how we go about the task.

1. We begin by noting that the following is immediate from Theorem 1:

$$\vdash \mathbb{N}y \rightarrow ((\forall w((S\bar{0}ww \wedge \mathbf{t}) \rightarrow (S\bar{0}\{w\}\{w\} \wedge \mathbf{t})) \wedge \mathbf{t}) \rightarrow ((S\bar{0}\bar{0}\bar{0} \wedge \mathbf{t}) \rightarrow (S\bar{0}xx \wedge \mathbf{t})))$$

2. Thus, by Corollary 2, to show that $\vdash \mathbb{N}x \xrightarrow{2} S\bar{0}xx$, it suffices to show that $\vdash \forall w((S\bar{0}ww \wedge \mathbf{t}) \rightarrow (S\bar{0}\{w\}\{w\} \wedge \mathbf{t})) \wedge \mathbf{t}$ and $\vdash S\bar{0}\bar{0}\bar{0} \wedge \mathbf{t}$.

¹⁰This would rule out free logics, as pointed out in [16]. In our framework, it becomes clear why free logics don’t work in this respect, unless the existence of at least an individual is proved in some way: the relation between the logic and abstraction is to be such that, for abstraction to work in the way we want it to, the underlying logic has to deliver the existence of at least an individual.

3. Since A1 gives $\vdash \mathbf{t}$, it suffices to show that $\vdash \forall w((S\bar{0}ww \wedge \mathbf{t}) \rightarrow (S\bar{0}\{w\}\{w\} \wedge \mathbf{t}))$ and $\vdash S\bar{0}\bar{0}$.
4. The second of these is essentially immediate from the fact that $F\bar{0}\bar{0} \rightarrow F\bar{0}\bar{0}$ is an axiom.
5. The first is demonstrated in Lemma 5 by an argument very like the argument we gave in Theorem 3.
6. We repeat the above procedure ‘one level up’ as it were. That is, let Fx abbreviate $\mathbb{N}y \xrightarrow{2} \exists zSxyz$. Then by Theorem 1, we have

$$\vdash \mathbb{N}x \rightarrow ((\forall w(Fw \rightarrow F\{w\}) \wedge \mathbf{t}) \rightarrow (F\bar{0} \rightarrow Fx))$$

7. Thus, by Corollary 2, to show that $\vdash \mathbb{N}x \xrightarrow{2} Fx$, it suffices to show that $\vdash \forall w((F \rightarrow F\{w\}) \wedge \mathbf{t})$ and $\vdash F\bar{0}$.
8. Since steps 1-5 will have already demonstrated (as recorded in Lemma 6) that $\vdash F\bar{0}$, it will suffice to show that $\vdash Fw \rightarrow F\{w\}$. And, having proved this in Lemma 8, two applications of R4 finish the job.

Plan in hand, we set to the job. The reader may find it useful to refer back to this summary to remind themselves why exactly we’re doing the things we’re doing.

Lemma 5. $\vdash \forall w((S\bar{0}ww \wedge \mathbf{t}) \rightarrow (S\bar{0}\{w\}\{w\} \wedge \mathbf{t}))$

Proof. It will suffice to show $\vdash (S\bar{0}yy \wedge \mathbf{t}) \rightarrow S\bar{0}\{y\}\{y\}$. To see this, note first that Instantiation gives that

$$\vdash S\bar{0}yy \rightarrow [(\forall u\forall w(Fuw \rightarrow F\{u\}\{w\}) \wedge \mathbf{t}) \rightarrow (F\bar{0}\bar{0} \rightarrow Fyy)] \quad (26)$$

and also gives

$$\vdash (\forall u\forall w(Fuw \rightarrow F\{u\}\{w\}) \wedge \mathbf{t}) \rightarrow (Fyy \rightarrow F\{y\}\{y\}) \quad (27)$$

Thus, exactly as in Theorem 3 we get

$$\vdash (S\bar{0}yy \wedge \mathbf{t}) \rightarrow [(\forall u\forall w(Fuw \rightarrow F\{u\}\{w\}) \wedge \mathbf{t}) \rightarrow (F\bar{0}\bar{0} \rightarrow F\{y\}\{y\})] \quad (28)$$

From which our desired result follows exactly as it did in Theorem 3. \square

Thus, after a helpful change of variables, we have

Lemma 6. $\vdash \mathbb{N}y \xrightarrow{2} S\bar{0}yy$

Corollary 7. $\vdash \mathbb{N}y \xrightarrow{2} \exists zS\bar{0}yz$

For this, we rely on the following Lemma:

Lemma 7. $\vdash Swyz \rightarrow S\{w\}y\{z\}$

Proof. Note that $Swyz[F/Gx\{y\}(x, y)]$ is

$$(\forall u\forall w(Gu\{w\} \rightarrow G\{u\}\{\{w\}\}) \wedge \mathbf{t}) \rightarrow (G\bar{0}\{w\} \rightarrow Gy\{z\})$$

Thus, by Instantiation, we have that

$$\vdash Swyz \rightarrow [(\forall u\forall w(Gu\{w\} \rightarrow G\{u\}\{\{w\}\}) \wedge \mathbf{t}) \rightarrow (G\bar{0}\{w\} \rightarrow Gy\{z\})] \quad (29)$$

Also note that

$$\vdash (\forall u\forall w(Guw \rightarrow G\{u\}\{w\} \wedge \mathbf{t}) \rightarrow (\forall u\forall w(Gu\{w\} \rightarrow G\{u\}\{\{w\}\})) \quad (30)$$

Essentially by instantiation and then a few applications of Intensional Confinement. Thus by Generalized Cut and another application of confinement, we get our result. \square

Corollary 8. $\vdash \forall w(Fw \rightarrow F\{w\})$

Proof. Since $\vdash Swyz \rightarrow S\{w\}y\{z\}$, by Lemma 1, $\vdash \neg S\{w\}y\{z\} \rightarrow \neg Swyz$. Also by Theorem 1, $\vdash \forall z\neg S\{w\}yz \rightarrow \neg S\{w\}y\{z\}$. Thus by A2, R3, and R1, $\vdash \forall z\neg S\{w\}yz \rightarrow \neg Swyz$. So by R4, $\vdash \forall z[\forall z\neg S\{w\}yz \rightarrow \neg Swyz]$. And since z is not free in $\forall z\neg S\{w\}yz$, R1 and an instance of A10 then gives $\vdash \forall z\neg S\{w\}yz \rightarrow \forall z\neg Swyz$. So again by Lemma 1, $\exists zSwyz \rightarrow \exists zS\{w\}yz$.

From here we get that $[\mathbb{N}y \xrightarrow{2} \exists zSwyz] \rightarrow [\mathbb{N}y \xrightarrow{2} \exists zS\{w\}yz]$ by R3 applied to a few instances of A2. R4 then gives $\forall w(Fw \rightarrow F\{w\})$, as required. \square

Theorem 4. $\vdash \forall x\forall y(\mathbb{N}x \xrightarrow{2} (\mathbb{N}y \xrightarrow{2} \exists zSxyz))$

Proof. By stringing together the pieces in exactly the way described in the plan. \square

So there we have it: again, an analogue of a piece of arithmetic is available. But this time the ‘combinatorial’ complexity required to achieve it from logic and BLV is yet more. Such complexity notwithstanding and analogously to the proof of Peano3 above, though both BLV and the logic are involved in the proof of the existence of the sum, it is the (multiple!) application of the logic to BLV that guarantees such an existential result—in particular, initially by Corollary 7. Then, by providing denumerably many natural numbers, and identity and distinctness conditions for (distinct) pairs of those natural numbers, BLV delivers denumerably many natural numbers that are the sums of those (distinct) pairs. So, again, the logic guarantees that at least an individual is around for BLV to work with as an assembly line delivering denumerably many sums.

We’ll end the section by noting that repeated investigations have strongly suggested to us that the corresponding result for multiplication will require more complexity yet. This is, in one sense not too surprising—after all, the complexity we see here is, in some sense, a straightforward reflection of the complexity present in Peano 4, iterated across the induction required to get from Peano

4 to here. Thus, since the usual recursive definition of multiplication involves both the successor and the addition operation, we expect that the complexity involved in expressing the appropriate analogue here will be at least the sum of these.

That, however, is at best a loose intuition. Fleshing it out in greater detail, and recovering larger fragments of arithmetic than this, will have to await future work.

6 Concluding Remarks

We started our investigation from two (related) questions:

Q1. Can noncontractive *relevant* logics be used to derive a nontrivial amount of arithmetic from abstraction principles alone?

The answer to Q1 seems to be in the positive. In particular, by a rather weak relevant second-order logic augmented by BLV, formulations of second-order Peano axioms (and the recursiveness of addition, which is a nice amount of arithmetic proper) are derivable. We are quite hopeful that the same can be accomplished for multiplication, though the combinatorial complexity might rise even more.

Q2. In exactly what ways must logic and abstraction be combined, in order to get those mathematical results?

As shown above, the combination of logic and abstraction can be investigated, so that the order of application of logic to abstraction and abstraction to logic required to deliver the above mathematical results can finally surface. Interestingly, both as for the derivation of Peano axioms and the recursiveness of addition, it seems that ontological results are due to the application of the logic to abstraction; whereas identity results are brought about by the application of abstraction to logic. In a sense, the logic has to ‘feed’ at least an individual to abstraction, for this latter to start answering questions of identity and distinctness in a meaningful way such that, by abstraction, denumerably many individuals are made available.

Furthermore, as a consequence of answering Q2, we also anticipated we could answer a further question: Is BLV a logical principle? Well, unsurprisingly and *pace* Frege, BLV is not, but in a rather peculiar sense. It’s not logical since it delivers the existence of denumerably many individuals by the identity and distinctness conditions it incorporates. But, as mentioned above, it does so only if the background logic secures the existence of at least an individual, which, once is ‘fed’ to BLV, allows BLV to produce denumerably many more. So, BLV is not a logical principle, if the background logic ‘feeds’ it with at least an individual. But without the logic, BLV runs on empty: *per se* it guarantees the existence of no individual at all, so, if in logic the ban on the empty domain really just is a “concession to technical convenience”, ironically, without the logic, BLV is trivially logical.

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