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Notes on Stratified Semantics

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Abstract
In 1988, Kit Fine published a semantic theory for quantified relevant logics. He referred to this theory as *stratified semantics*. While it has received some attention in the literature (see, e.g. Mares, *Studia Logica 51*(1), 1–20, 1992; Mares & Goldblatt, *Journal of Symbolic Logic 71*(1), 163–187, 2006), stratified semantics has overall received much less attention than it deserves. There are two plausible reasons for this. First, the only two dedicated treatments of stratified semantics available are (Fine, *Journal of Philosophical Logic 17*(1), 27–59, 1988; Mares, *Studia Logica 51*(1), 1–20, 1992), both of which are quite dense and technically challenging. Second, there are a number of prima facie reasons to be worried about stratified semantics. The purpose of this paper is to revitalize research on stratified semantics. I will do so by giving a 'user friendly' presentation of the semantics, and by giving reasons to think that the prima facie reasons to be worried about it are too simplistic.

Keywords Relevance logics · Quantification · Arbitrary objects · Quantified relevance logics · Stratified semantics · Varying domain semantics · Constant domain semantics

1 Introduction

With his usual flair, Bob Meyer summarized the pre-1973 state of research in relevant logics as follows:

> Yea, every year or so Anderson & Belnap turned out a new logic, and they did call it **E**, or **R**, or $\mathbf{E}_{\bar{\mathbf{I}}}$, or **P**−**W**, and they beheld each such logic, and they were called relevant. And these logics were looked upon with favor by many, for they captureth the intuitions, but by many more they were scorned, in that they hadeth no semantics. [27, p. 199]

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In a series of three papers beginning with the one containing this passage, Routley and Meyer 'gaveth' semantics for relevant logics.[1] The resulting theory came to be known as Routley-Meyer semantics. While it was fairly successful when it came to *sentential* relevant logics, it turned out that for strong-enough *quantified* relevant logics, the naïve way to extend Routley-Meyer semantics resulted in a semantics for which these logics were incomplete.[2] Over the following decades, several semantic theories that avoided this problem were produced.[3] The first of these, Kit Fine's *stratified* semantics, will be the subject of this paper.

That stratified semantics is a *technically* fascinating semantics has never been in doubt. But it has received relatively little attention in the literature. What little attention it has received has tended to come either in the form of perfunctory acknowledgements that there *are* semantic theories for quantified relevant logics or in the form of generalized complaints about it as a semantics. The complaints, in turn, have centered on three issues.

The first issue is that, in Restall's words, stratified semantics is "formally astounding but philosophically opaque".[4] I take it that the issue being highlighted here and in other versions of this complaint is that it's unclear what semantic phenomena stratified semantics is supposed to be capturing.

The second complaint is that stratified semantics seems to be *too complex*. As an example, in [10, p. 86], we hear that "it is not altogether clear whether the rich and complex structure of Fine's semantics is necessary to give a semantics for quantified relevant logics." Ed Mares and Rob Goldblatt are more direct in their assessment:

> Fine's semantics is very complicated. Since it was produced it in the mid 1980s relevant logicians have wanted to simplify it. [20, p. 163]

The final complaint is more technical and centers on the fact that the domains of quantification in stratified models vary from world to world. This could be easily justified if stratified semantics was meant to be a model theory for a modal language. But stratified models are models for quantified but *non*-modal language. Varying domains might thus seem quite odd.

I respond to these worries in both Sections 2 and 5. In Section 2 we give a general overview of stratified semantics, along the way explaining why stratified models are structured the way they are. Then, in Sections 3 and 4, we build the logics and the varying-domain semantic theory that we require for the discussion in the final section to make sense. I've put special emphasis in these sections on presenting stratified semantics in an accessible and user-friendly way. In Section 5, we finally turn to explicitly addressing the three worries highlighted above. The four attached appendices provide soundness and completeness proofs as well as a broad tour of the methods used in this general region of philosophical logic. My hope is that presenting all of this material in one place for the first time will help make further work on stratified semantics possible.

[1] See [27, 28], and [29].

[2] See [14] for the original, and one of the only, discussions of this.

[3] See [5, 6, 13]. Later there was also [20], which has been treated in an accessible way in [17].

[4] See [24, p. 5]

2 Stratified Semantics, Generally

We first need to settle some linguistic matters. We will be discussing two languages in this paper. The first, LZ, is a polyadic sentential (unquantified) language. The second, LQ, is a standard first-order language.

The vocabulary of LZ consists of countably many individual constants; for each i, countably many i-ary predicates; and the connectives $\wedge$, $\neg$, and $\rightarrow$. We take '$\vee$' to be a defined connective (in both LZ and LQ), with $A \vee B =_{df} \neg(\neg A \wedge \neg B)$.

The vocabulary of LQ contains the vocabulary of LZ. In addition, it contains the quantifier $\forall$ and countably many individual variables.[5] If τ is a variable or a constant, we say τ is a *term*. Formation rules (for both languages) are entirely standard as are the definitions of free and bound occurrences of variables. For the sake of explicitness, it's worth stating that in LZ no variables occur at all (free or bound), and that we take LQ to be the set of all wffs, whether or not they contain free occurrences of variables.

With linguistic matters out of the way, in the remainder of the Section I give a general-purpose recipe for building stratified models. For now, I'll avoid technicality and instead focus on giving a big-picture overview of what makes these theories tick.

2.1 The Recipe

Stratified models are constructed by gluing together models for LZ. It's useful to picture this gluing together as following a four-step procedure:

- Step 1: Stack up a family of LZ-models.
- Step 2: Within the domains of the models in the stack, single out a nondecreasing stack of sets of objects. Call these objects 'AOs'.
- Step 3: Put conditions on the stack of models that ensure the AOs behave like *arbitrary objects*.
- Step 4: Use the AOs to define truth for quantified sentences, so that the stack of LZ-models becomes a (single) LQ-model

A caveat: the 'stack' of LZ-models is not linearly ordered, but partially ordered. So perhaps it's more of a pile than a stack. Be that as it may, the recipe is useful in thinking through how stratified models generally work. The basic idea is straightforward: once we've built a semantic theory for LZ that captures whatever logic we're after, we know all there is to know about the logic of unquantified sentences. In stratified semantics, the semantic clause for universally quantified formulas reduces questions about truth of quantified sentences to questions about truth of unquantified sentences by the trick Kit Fine describes as follows:

> ...a universal sentence $\forall x \psi(x)$ is true just in case $\psi(x)$ is true of an arbitrary or generic individual. But let me not be misunderstood. My saying that $\psi(x)$ is true of an arbitrary individual is not a fancy way of saying that $\psi(x)$ is true of

[5] We could do without individual constants in LQ, as Fine did in [13]. I chose to include them primarily for the sake of variety, as having both approaches available in the literature seems valuable.

> every individual. I mean to be taken literally; for the universal sentence $\forall x \psi(x)$ to be true, there must actually be an arbitrary individual of which the condition $\psi(x)$ is true. [13, p. 29]

Applying this trick, we expect that $\forall x \psi(x)$ will be true in a stratified model just if there is an AO ω so that $\psi(\omega)$ is true. With minor variations, this is correct.

It should be clear at this point that all the action in our recipe is in Step 3, and in particular in deciding exactly what it means for an AO to 'behave like' an arbitrary object. In general, the intuitions one should be guided by are these:

Suppose ω is an object that is arbitrary in a given situation S. Then we expect that

I1 Within the situation S, the arbitrary object ω shouldn't 'look like' anything, but

I2 For any non-arbitrary object d in the domain of S, there should be an allowable extension of S in which ω 'looks like' d.[6]

I3 Extending S with the assumption that ω 'looks like' any non-arbitrary object d_1 has no impact on what S is compatible with.

It's natural to wonder at this point why *these* intuitions are the intuitions we should be guided by. Briefly, the answer is that these intuitions are the ones that will guide us to the semantic theory we want. This answer is, of course, unsatisfactory. One would want – and someone ought to give – a thorough justification of these intuitions as capturing the 'essence' of an appropriate sort of arbitrariness. But this isn't the place to do so. I take it as clear, however, that they at least capture *some part* of the notion of arbitrariness.

None of that should be taken as gospel, or as giving *the* definitive view on stratified semantics. It's meant only as vague, motivational material, and I hope the reader takes it in that light. In any event, hopefully this discussion makes clear how we ought to proceed from here: first we need to build LZ-models; after that we need only figure out how to stack them up appropriately. We tackle the first of these tasks in the next section.

3 RWZ

This section will give an overview of the sentential relevant logic **RWZ**.[7] Before getting to the details, it's worth spelling out why I chose to focus on this particular logic. My reasons are essentially pragmatic. Focusing on this particular logic allowed for a presentation of stratified semantics that had all of the complexity these theories allowed, but which was free from much of the clutter that came along with the presentations found in [13] or [18].

[6] In his work on arbitrary objects generally (see [12]), Fine considers arbitrary objects that have different 'value ranges'. We will not be discussing this possibility here, but acknowledge that taking account of this would make for a more interesting theory.

[7] We identify logics with their sets of theorems. A logic is *contractionless* when it does not contain every instance of $(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$. What exactly it takes for a logic to be relevant is a matter of some contention that we won't get into, though a useful starting point is Bimbó's dictum (see [3, p. 729]) that 'An implication is relevant if the antecedent and the consequent are appropriately related.'

One reason for the amount of clutter in [13] is that it undertook an ambitious project. Overall [13] gives three distinct semantic theories for each of the standard quantified relevant logics between the very weak **BQ** to the very strong **RQ**. To mitigate the type of clutter this gives rise to, we've focused in this paper on just one logic. And while the discussion in Section 2 makes clear that stratified semantics for, e.g. classical logic makes perfectly good sense, in order to put the full power of stratified semantics on display, we really needed to focus our attention on a relevant logic of some sort or other. On the other hand, in order to make the presentation as simple as possible, it helped to focus our attention on a quantified relevant that possessed a particularly clean and elegant semantics for its sentential fragment. The semantics for **RWZ** in [23] fit the bill, so I focussed the paper on **RWQ**.

There's still one matter left: a variety of axiom-systems for (what amounts to) **RWZ** can be found in the literature. I've chosen to use the one given in [4]. Again, this choice was made on entirely pragmatic grounds: the main party to whom the philosophical part of the paper, found below, is addressed, is Ross Brady. So I used Ross Brady's axiomatization. On this axiomatization, **RWZ** has nine axioms and two rules:

A1 $\alpha \to \alpha$
A2 $(\alpha \wedge \beta) \to \alpha$
A3 $(\alpha \wedge \beta) \to \beta$
A4 $((\alpha \to \beta) \wedge (\alpha \to \gamma)) \to (\alpha \to (\beta \wedge \gamma))$
A5 $(\alpha \wedge (\beta \vee \gamma)) \to ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$
A6 $\neg\neg\alpha \to \alpha$
A7 $(\alpha \to \neg\beta) \to (\beta \to \neg\alpha)$
A8 $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$
A9 $\alpha \to ((\alpha \to \beta) \to \beta)$

R1 $\frac{\alpha, \alpha \to \beta}{\beta}$
R2 $\frac{\alpha, \beta}{\alpha \wedge \beta}$

Formally, we will take **RWZ** to be the smallest subset of LZ that contains every LZ-instance of **A1**-**A9**, and which is closed under **R1** and **R2**.

3.1 Model Theory for RWZ

The model theory given here is inspired by what is called 'the natural four-valued semantics' in [23].[8] As mentioned above, this model theory allows enough complexity for the full strength of stratified semantics to become visible without being so complex as to be inaccessible.

[8][23] has something of a bad reputation since, as pointed out in [25], there are errors in its predecessor paper, [22]. I think this bad reputation is not entirely deserved. The errors are, once one knows how to look for them, easy to spot and, in any event, don't infect most of the good ideas to be found in either paper. We avoid the problematic bits here by not aiming to build a simplified semantics.

3.1.1 Premodels

An **RWZ**-premodel is a 7-tuple $\langle D, S, N, R, \delta, \mathcal{E}^+, \mathcal{E}^- \rangle$ where

- D is a set called the *domain* of the model.
- S is a set whose elements are called *setups*.[9]
- $N \subseteq S$ are the *normal setups*.
- R is a ternary relation that holds among setups. At several points, it will help to follow the suggestion in [2] to read '$Rabc$' as 'a is compatible with b relative to c'.
- δ is a *denotation function* that maps each constant c to an object $\delta(c)$ in D.
- $\mathcal{E}^+$ and $\mathcal{E}^-$ are (respectively) the *extension* and *antiextension functions*. Each maps each pair consisting of an i-ary predicate P and a setup a to a subset of D^i.

A few standard notational definitions will greatly simplify the presentation: '$x \leq y$' abbreviates 'for some $n \in N$, $Rnxy$;' '$Rabcd$' abbreviates 'for some x, $Rabx$ and $Rxcd$;' and '$Ra(bc)d$' abbreviates 'for some x, $Rbcx$ and $Raxd$.'

3.1.2 Models

An **RWZ**-model is an **RWZ**-premodel that satisfies the following conditions:

Ordering: $\leq$ is a partial ordering – that is, is reflexive, transitive, and antisymmetric.
Monotonicity: If $a' \leq a$, $b' \leq b$, and $c \leq c'$, then if $Rabc$ then $Ra'b'c'$.
Closure: If $n \in N$ and $n \leq m$, then $m \in N$.
B: If $Rabcd$, then $Ra(bc)d$.
B': If $Rabcd$, then $Rb(ac)d$
C: If $Rabcd$, then $Racbd$
Horizontal Atomic Heredity: If P is a predicate and $a \leq b$, then $\mathcal{E}^+(P, a) \subseteq \mathcal{E}^+(P, b)$ and $\mathcal{E}^-(P, a) \subseteq \mathcal{E}^-(P, b)$.

It's worth highlighting one important consequence of these conditions:

Permutation If $Rabc$, then $Rbac$.

Proof By Ordering, $a \leq a$. So for some $n \in N$, $Rnaa$. So since $Rnaa$ and $Rabc$, $Rnabc$. Thus by C, $Rnbac$. So for some x, $Rnbx$ and $Rxac$. Since $Rnbx$, $b \leq x$. Thus by Monotonicity, from $Rxac$ we can conclude $Rbac$ as well. □

3.1.3 Truth

Given an **RWZ**-model $M = \langle D, S, N, R, \delta, \mathcal{E}^+, \mathcal{E}^- \rangle$ and $a \in S$ we assign each $\alpha \in$ LZ a truth value $M^a(\alpha)$ in $\{\{1\}, \{0\}, \emptyset, \{1, 0\}\}$ – following [9], we think of these as (respectively) *true*, *false*, *neither*, and *both* – in the following way:

- $1 \in M^a(Pc_1 \ldots c_n)$ iff $\langle \delta(c_1), \ldots, \delta(c_n) \rangle \in \mathcal{E}^+(P, a)$

[9] The term 'setup' is due to Routley and Routley, see [30]. A useful discussion of setups generally can be found in Sections 16.2.1 of [1]. We prefer it to the alternative term 'world' because the latter comes with a good deal of baggage. But it's also preferable to, e.g. 'index' in that it at least brings some intuitive content with it.

- $0 \in M^a(Pc_1 \ldots c_n)$ iff $\langle \delta(c_1), \ldots, \delta(c_n) \rangle \in \mathcal{E}^-(P, a)$
- $1 \in M^a(\phi \wedge \psi)$ iff $1 \in M^a(\phi)$ and $1 \in M^a(\psi)$.
- $0 \in M^a(\phi \wedge \psi)$ iff $0 \in M^a(\phi)$ or $0 \in M^a(\psi)$.
- $1 \in M^a(\neg\phi)$ iff $0 \in M^a(\phi)$
- $0 \in M^a(\neg\phi)$ iff $1 \in M^a(\phi)$
- $1 \in M^a(\phi \rightarrow \psi)$ iff for all b and c, if $Rabc$ then if $1 \in M^b(\phi)$, then $1 \in M^c(\psi)$, and if $0 \in M^b(\psi)$, then $0 \in M^c(\phi)$.
- $0 \in M^a(\phi \rightarrow \psi)$ iff for some b and c with $Rbca$, $1 \in M^b(\phi)$ and $0 \in M^c(\psi)$.

We say α is true in M when $1 \in M^n(\alpha)$ for all $n \in N$. α is **RWZ**-valid when α is true in every **RWZ**-model. In Appendices A and B (respectively) we prove that **RWZ** is sound and complete for this semantics – that is, that the set of **RWZ**-valid sentences is exactly **RWZ**.

4 RWQ

RWQ extends **RWZ** to LQ. Again (and for the same reasons as before) we lift our axiomatic characterization from [4]. In addition to the axioms and rules for **RWZ** – now taken to range over LQ – **RWQ** has three additional axioms and one additional rule:

QA1 $\forall \nu \phi \rightarrow \phi(\tau/\nu)$ where τ is a term that is free for ν in ϕ.
QA2 $\forall \nu(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \forall \nu \psi)$ where ν is not free in ϕ.
QA3 $\forall \nu(\phi \vee \psi) \rightarrow (\phi \vee \forall \nu \psi)$ where ν is not free in ϕ.
QR1 $\frac{\phi}{\forall \nu \phi}$

Formally, **RWQ** is the smallest subset of LQ that contains every LQ-instance of **A1-A9** and **QA1-QA3** and which is closed under **R1**, **R2**, and **QR1**.

4.1 Model Theory for RWQ

Loosely speaking, the model theory presented here is what results when all the bells and whistles from [22] and [23] are incorporated into the third semantic theory presented in [13]. Since that's quite an imprecise description and because some details differ from what this would lead you to believe, I'll take the time to go through all the details.

4.1.1 Premodels

An **RWQ**-premodel is a 5-tuple $\langle D, \Omega, \delta, \mathcal{M}, \Downarrow \rangle$ where

- D is a set called the base domain of the model.
- $\Omega = \{\omega_i\}_{i=1}^{\infty}$ is a set of objects called AOs. We require $\Omega \cap D = \emptyset$.
- δ is a function from constants to D.
- $\mathcal{M}$ is a function mapping each finite set X of numbers[10] to an **RWZ**-model M_X of the form $\langle D_X, S_X, N_X, R_X, \delta, \mathcal{E}^+_X, \mathcal{E}^-_X \rangle$ where $D_X = D \cup \{\omega_i\}_{i \in X}$.

[10] Here and throughout the rest of the paper, we take 'number' to mean 'natural number'. From here on, we also take 'set of numbers' to mean 'finite set of numbers', though essentially nothing depends on this.

– $\Downarrow$ is a family of *restriction functions* $\downarrow^X_Y : S_X \to S_Y$ for each pair of sets of numbers $X \supseteq Y$. We will write $\downarrow^X_Y$ with postfix notation, and we require that $a\downarrow^X_Y\downarrow^Y_Z = a\downarrow^X_Z$ (whenever these are all defined) and that $\downarrow^X_X$ be the identity function on S_X.

If $a \in S_X$ and m and n are in D_X, then we say that a is symmetric in m and n when a does not extensionally distinguish m from n; that is, when for each i-ary predicate P, $\langle d_1, \ldots, m, \ldots, d_i\rangle \in \mathcal{E}^+_X(P, a)$ iff $\langle d_1, \ldots, n, \ldots, d_i\rangle \in \mathcal{E}^+_X(P, a)$ and $\langle d_1, \ldots, m, \ldots, d_i\rangle \in \mathcal{E}^-_X(P, a)$ iff $\langle d_1, \ldots, n, \ldots, d_i\rangle \in \mathcal{E}^-_X(P, a)$. Note that if a is symmetric in m and n, then at a the object m 'looks like' the object n in every way.

At this point, the 'stacking up' of LZ-models from Step 1 of our recipe and the nondecreasing sets of AOs from Step 2 are clearly visible. What remains is to adopt policies to ensure the AOs behave like arbitrary objects (Step 3), and then to use the AOs to define truth for quantified sentences (Step 4). We accomplish the first of these in the next part.

4.1.2 Models

An **RWQ**-model is an **RWQ**-premodel that satisfies the following:

Vertical Atomic Heredity: If $a\downarrow^X_Y = b$ and P is an i-ary predicate, then $\mathcal{E}^+_X(P, a) \cap D^i_Y = \mathcal{E}^+_Y(P, b)$, and $\mathcal{E}^-_X(P, a) \cap D^i_Y = \mathcal{E}^-_Y(P, b)$.

Normality: $a\downarrow^X_Y \in N_Y$ if and only if $a \in N_X$.

Lifting: If $a \in S_X$, $b \in S_Y$, and $a\downarrow^X_{X\cap Y} = b\downarrow^Y_{X\cap Y}$ then for some $c \in S_{X\cup Y}$, $c\downarrow^{X\cup Y}_X \leq a$ and $b \leq c\downarrow^{X\cup Y}_Y$.

Homomorphism: If a, b, and c are in S_X, $X \supseteq Y$, and $R_X abc$, then $R_Y a\downarrow^X_Y b\downarrow^X_Y c\downarrow^X_Y$.

Extension: If a, b, and c are in S_Y, $X \supseteq Y$, and $R_Y abc$, then[11]

- if $d \in S_X$ and $d\downarrow^X_Y = a$ then there are e and f such that $e\downarrow^X_Y = b$ and $f\downarrow^X_Y = c$ and $R_X def$;
- if $e \in S_X$ and $e\downarrow^X_Y = b$ then there are d and f such that $d\downarrow^X_Y = a$ and $f\downarrow^X_Y = c$ and $R_X def$; and
- if $f \in S_X$ and $f\downarrow^X_Y = c$ then there are d and e such that $d\downarrow^X_Y = a$ and $e\downarrow^X_Y = b$ and $R_X def$.

From here on we will almost always write 'R' instead of 'R_X'.

Symmetry: If $a \in S_Y$, $X \supseteq Y$, $m \in D_Y$ and $n \in D_X - D_Y$, then there is a $b \in S_X$ that is symmetric in m and n such that $b\downarrow^X_Y = a$.

For the purpose of accomplishing Step 3 in our recipe, the crucial conditions here are the Symmetry condition and the Heredity condition. The Heredity condition roughly states that if $a\downarrow^X_Y = b$, then a is an *atomic extension* of b – the atomic truths at a include all the atomic truths at b. As one probably expects (and as is proved in

[11] We technically only need one of the first two clauses here since each M_X satisfies the Permutation condition. But having all three clauses will simplify things at several points so we'll keep them all around.

Lemma C.5), this extends – the truths (atomic and otherwise) at a include the truths at b. So a really is an extension of b.

Recall that in Section 2, we identified three intuitions (**I1**, **I2**, and **I3**) that would guide our understanding of what it means to 'behave like' an arbitrary object. In **RWQ**-models, **I1** – which required that arbitrary objects not 'look like' anything – is accomplished by simply leaving ω_i out of the domain of M_X when $i \notin X$. **I2** – which required that we can always extend so that our arbitrary object 'looks like' any other object we have on hand – is accomplished by the symmetry condition. It's a worthwhile exercise (which we leave to the reader) to think through exactly how the symmetry condition does this. Finally, **I3** required that extending situations in this way not impact what they are compatible with. **RWQ**-models accomplish this by the homomorphism condition.

4.1.3 Variable Assignments

Finally, we turn to Step 4, which requires defining truth. To do so, we'll first need variable assignments. These are mildly more complex in stratified semantics than they generally are elsewhere. If ν is a variable and X is a set of numbers then a variable assignment maps the *pair* $\langle \nu, X\rangle$ to an element of D_X. We say the variable assignment $\mathtt{va}$ is X-*coherent* when for all ν and all $Y \supseteq X$, $\mathtt{va}(\nu, X) = \mathtt{va}(\nu, Y)$. Clearly if $\mathtt{va}$ is X-coherent, then $\mathtt{va}$ is also Y-coherent for any $Y \supseteq X$. If $\mathtt{va}$ is a variable assignment and ν is a variable, then for each $d \in D \cup \Omega$ we define a variable assignment $\mathtt{va}^{\nu}_{d}$ as follows:

$$\mathtt{va}^{\nu}_{d}(\chi, X) = \begin{cases} \mathtt{va}(\chi, X) & \text{if } \chi \neq \nu \ \text{ or } d \notin D_X \\ d & \text{if } \chi = \nu \ \text{ and } d \in D_X \end{cases}$$

Notice if $\mathtt{va}$ is X-coherent then $\mathtt{va}^{\nu}_{d}$ is Y-coherent if and only if $D_X \cup \{d\} \subseteq D_Y$. Finally, if τ is a term and X is a set of numbers, then by $\varepsilon^{\mathtt{va}}_{X}(\tau)$ we mean whichever of $\delta(\tau)$ and $\mathtt{va}(\tau, X)$ is appropriate.

4.1.4 Truth

Given an **RWQ**-model $M = \langle D, \Omega, \delta, \mathcal{M}, \Downarrow\rangle$, a set of numbers X, and $a \in S_X$, we assign each pair $\langle \mathtt{va}, \alpha\rangle$ – where $\mathtt{va}$ is an X-coherent variable assignment and $\alpha \in$ LQ – a truth value $M^a_X(\mathtt{va}, \alpha)$ in $\{\{1\}, \{0\}, \emptyset, \{1, 0\}\}$ in the following way:

- $1 \in M^a_X(\mathtt{va}, P\tau_1 \ldots \tau_n)$ iff $\langle \varepsilon^{\mathtt{va}}_X(\tau_1), \ldots, \varepsilon^{\mathtt{va}}_X(\tau_n)\rangle \in \mathcal{E}^{+}_X(P, a)$
- $0 \in M^a_X(\mathtt{va}, P\tau_1 \ldots \tau_n)$ iff $\langle \varepsilon^{\mathtt{va}}_X(\tau_1), \ldots, \varepsilon^{\mathtt{va}}_X(\tau_n)\rangle \in \mathcal{E}^{-}_X(P, a)$
- $1 \in M^a_X(\mathtt{va}, \phi \wedge \psi)$ iff $1 \in M^a_X(\mathtt{va}, \phi)$ and $1 \in M^a_X(\mathtt{va}, \psi)$.
- $0 \in M^a_X(\mathtt{va}, \phi \wedge \psi)$ iff $0 \in M^a_X(\mathtt{va}, \phi)$ or $0 \in M^a_X(\mathtt{va}, \psi)$.
- $1 \in M^a_X(\mathtt{va}, \neg\phi)$ iff $0 \in M^a_X(\mathtt{va}, \phi)$
- $0 \in M^a_X(\mathtt{va}, \neg\phi)$ iff $1 \in M^a_X(\mathtt{va}, \phi)$
- $1 \in M^a_X(\mathtt{va}, \phi \to \psi)$ iff for all b and c, if $Rabc$ then if $1 \in M^b_X(\mathtt{va}, \phi)$ then $1 \in M^c_X(\mathtt{va}, \psi)$, and if $0 \in M^b_X(\mathtt{va}, \psi)$ then $0 \in M^c_X(\mathtt{va}, \phi)$.
- $0 \in M^a_X(\mathtt{va}, \phi \to \psi)$ iff for some b and c with $Rbca$, $1 \in M^b_X(\mathtt{va}, \phi)$ and $0 \in M^c_X(\mathtt{va}, \psi)$.

- $1 \in M^a_X(\mathtt{va}, \forall \nu \phi)$ iff for some $Y \supsetneq X$ and $i \in Y - X$, for all $b \in S_Y$, if $b{\downarrow}^Y_X = a$, then $1 \in M^b_Y(\mathtt{va}^{\nu}_{\omega_i}, \phi)$.
- $0 \in M^a_X(\mathtt{va}, \forall \nu \phi)$ iff for every $Y \supsetneq X$ and $i \in Y - X$ there is a $b \in S_Y$ such that $b{\downarrow}^Y_X = a$ and $0 \in M^b_Y(\mathtt{va}^{\nu}_{\omega_i}, \phi)$.

Say that M makes α true when for all for all X, if $\mathtt{va}$ is X-coherent, then for all $n \in N_X$, $1 \in M^n_X(\mathtt{va}, \alpha)$. Say α is **RWQ**-valid when α is true in every **RWQ**-model. In Appendices C and D (respectively) we prove that **RWQ** is sound and complete for this semantics; that is, that the set of **RWQ**-valid wffs is exactly **RWQ**.

4.2 Discussion of the Semantics

The semantics just presented is essentially what results when we follow the recipe in Section 2, using the LZ-models from Section 3 as our building-blocks. But the end result is a semantics that is different in significant ways from any of the semantic theories found in [13]. In particular, where Fine's semantics was an 'Australian-style' two-valued semantic theory with negation interpreted via the Routley star, the semantics I've given is an 'American-style' four-valued semantic theory with negation interpreted classically.[12] Aside from the fact that the four-valued semantics will be more familiar-looking to many non-relevant logicians (a benefit that ought not be overlooked), the chief benefit to American-style semantic theories in this context is that they make the transition to *constant domain* stratified models (examined in Section 5.1) possible. In fact, while I won't argue for it here (though see footnote 18), it seems clear to me that a natural Australian-style constant domain stratified semantics is impossible.

There are two further, more technical differences between the semantics given here and the semantics found in [13]. The first comes in the lifting condition. Fine uses a slightly stricter condition: rather than requiring that $c{\downarrow}^{X\cup Y}_X \leq a$ and $b \leq c{\downarrow}^{X\cup Y}_Y$, he requires that $c{\downarrow}^{X\cup Y}_X = a$ and $b \leq c{\downarrow}^{X\cup Y}_Y$. The interested reader will easily verify that the semantics given is sound for Fine's condition. But completeness *seems to* not go through with Fine's condition.[13]

The second difference concerns the use of finite subsets of the natural numbers as indices for the strata. Fine instead uses any family $\mathcal{F}$ of sets satisfying the following three conditions:[14]

Extendibility: For all $X \in \mathcal{F}$ there is $Y \in \mathcal{F}$ with $X \subsetneq Y$.
Upper Bound: The union of any two members of $\mathcal{F}$ is in $\mathcal{F}$
Reversibility: If X, Y, and Z are in $\mathcal{F}$ and $X \subseteq Y \subseteq Z$, then $X \cup (Z - Y) \in \mathcal{F}$.

[12] See [26] for a discussion of the differences between 'American-' and 'Australian-'style semantic theories.

[13] I've modified the notation in this paragraph to match my own. What I'm calling 'Fine's condition', Fine calls Condition IV.(i)(b) [13, p.46]. The only place in the soundness proof where lifting plays a role is in the proof of Lemma C.5, and either condition will do the job equally well there, with only minor changes required. The problems creep up in the proof of Lemma D.12; again it's not hard to see them once you know where to look.

[14] Again, notation has been modified. These are conditions II.(i)-II.(iii) [13, p.33].

But it is clear from Fine's completeness proof that taking $\mathcal{F}$ to be the set of finite sets of numbers is sufficient. Since doing so also leaves the semantics a good bit cleaner, I decided this simplification was worthwhile.

5 Prolegomena to any Future Objection

I will confess to being quite taken with stratified semantics. So I would love to spend this section vigorously defending it. The problem is that it's not clear who to defend it *from* – what objections there are to stratified semantics have, for the most part, been presented halfheartedly. So what I hope to do in this final section of the paper proper is present a roadmap that will help guide anyone who wants to make a strenuous case against stratified semantics for one reason or another.

I will orient this roadmap toward the three broad complaints against stratified semantics I identified in the introduction. The first of these was that stratified semantics is 'philosophically opaque'. As far as I can tell there are two plausible origins for the perceived opacity of stratified semantics: the density with which the technical details of the semantics has been so far presented, and the scant effort that has been put in to philosophically motivating the semantics. I take the user-friendly presentation of the semantics given above to address the first part of this.

For the second part, we might distinguish three potential ways the worry could go:

- The worry could amount to no more than a request for the philosophical content of stratified semantics to be made clear;
- The worry could amount to a claim that there just *isn't* philosophical content to be found in stratified semantics; or
- The worry could amount to a claim that, while there is philosophical content, stratified semantics just gets things wrong.

I take the discussion in Section 2 and at the end of Section 4.1.2 to have addressed the first two of these. Thus the objector who wants to take this road would seem to be left with the third option. On this front, it seems that since the philosophical explanation we've given for stratified semantics is in terms of arbitrary objects, there are again three options:

- One could object to the interpretation of quantified sentences in terms of arbitrary reference;
- One could object to arbitrary objects as appropriate accounts of what's going on in arbitrary reference; or
- One could object to the way stratified semantics deals with arbitrary objects.

The first two have been deal with by Kit Fine, who has given a variety of defenses of arbitrary objects; see e.g. [16] and [12] for early versions and [15] for something more recent. Further, only the third version of the complaint is targeted at stratified semantics proper. One can imagine two ways of fleshing the complaint out – first, by claiming that intuitions **I1**-**I3** fail to capture arbitrary objects; second, by claiming that stratified semantics fails to live up to the demands these intuitions put on it. I'm unmoved by either version, but I can imagine it possible that a compelling case for either can be made. In any event, I take the terrain surrounding this particular complaint to now be well-mapped.

The second complaint I identified in the introduction was a complaint about the *complexity* of stratified semantics. Here is a (recent but) typical expression of this sort of worry:

> [My] general concern with complexity is as follows. Put oneself in the mind of a reasoner conducting a simple inference step and ask the question: what is the rationale or justification for the inference? ... [A] reasoner is not going to embrace much complexity in making and justifying a single inference step. The logic governing the step would be clean and clear, based on well-understood concepts. [8, p. 757]

For this to amount to a serious reason to be worried about stratified semantics, it will need to see some substantial fleshing-out. In particular, one would like to see three things: an explicit enunciation of the variety of complexity at issue, an argument that this type of complexity is philosophically significant, and a demonstration that stratified semantics is indeed complex in the specified way.

Focusing only on the second and third issues, I think that this looks like a particularly *un*promising route for an objector to take. One reason for this is that it's just not clear stratified semantics really is that complex. In particular, it's worth noting the following:

- **RWQ**-models are made by gluing together **RWZ**-models along the poset of finite subsets of the natural numbers.
- **RWZ**-models are simple enough that they've gained widespread acceptance.
- Making models by gluing together simpler models along a poset is exactly how e.g. Kripke models for intuitionistic logic are constructed.
- The semantic clause for universally quantified sentences in stratified models is no more complex than the semantic clause for universally quantified sentences in Kripke semantics for intuitionistic logic.

So, in brief, **RWQ**-models are made of simple pieces (**RWZ**-models) put together in a simple way (roughly, in the Kripke-models-for-intuitionistic-logic way). This doesn't, as a referee helpfully points out, *guarantee* that the resulting models are simple. After all, if we take 'simple enough' to mean 'decidable', then both the addition-free fragment of arithmetic and the multiplication-free fragments of arithmetic are simple enough even though the structure we get by gluing these together (full arithmetic) is not. Nonetheless, since **RWQ**-models *can* be described as being made of simple pieces put together in a simple way, it seems that the pressure is on the objector to provide some reason to think that, despite this, they really are too complex. That is, it seems that if the objection is to carry any weight, we need some reason to believe that the analogy with arithmetic is the correct analogy in this case.

Another reason for thinking the complexity route is unpromising is that it's hard to imagine a measure of complexity that might carry the philosophical weight needed to make the objection stand. Examining Brady's version of the complexity objection seems to reveal several fallacies that are likely the reason for many people feeling otherwise.

First, while I grant that it's true that a reasoner is unlikely 'to embrace much complexity in making and justifying a single inference step', all this tells us is that inferences that are likely to *strike us as basic* tend to be cognitively simple. But being cognitively simple is a different matter from being *semantically* simple, and striking us as basic is different from being basic, which is in turn different from being valid. So I simply don't see how to make any use of this observation.

More to the point, there's just no reason I can see for us expect that logic comes in cognitively bite-sized chunks in the first place. If a given inference preserves truth (or preserves meaning or content or whatever else you might think suffices for making an inference logical) then that inference is a logical inference. Perhaps all such inferences can be broken down into cognitively bite-sized chunks. Perhaps not. There's certainly nothing in any plausible definition of logical consequence that seems to *demand* this, and I, for one, am certainly open to the possibility that it fails.

Generally speaking, what I've identified here are three hazards on the complexity-objection path: the relevant notion of complexity must be spelled out, it's not obvious that stratified models actually *are* complex, and it's not obvious that it would be a problem if they were. Again, I can imagine it possible that a compelling case can be made for an objection based on complexity. But I don't think such a case *has* been made, and the terrain in the area doesn't look promising.

So much for the complexity worries. Now let's turn to worries about varying domains. Again, the concern is well-stated by Brady:

> It is understandable that for quantified modal logics that possible worlds might have differing domains from world to world, but this is not clear for practical non-modalized examples such as Peano arithmetic. Indeed, logical applications generally have fixed domains of objects, such as numbers or sets, and one should not have to vary such a domain when replacing classical logic by a supposedly superior logic. [8, p. 757]

As with the complexity objection, one is left wanting here: the specific problem that varying domains seem to bring with them should be made clear. But it's not clear that enunciating such a problem will an easy.

The reason I say this is that there is a well-known logic whose semantics tends to include varying domains even when discussing arithmetic, but which we generally don't find problematic: intuitionistic logic with Kripke semantics.[15] Thus, it seems that any issue one might find with stratified semantics being a varying domain semantics would equally well be an issue one could raise against Kripke semantics. Yet Kripke semantics for intuitionistic logic is generally taken to be on pretty solid footing, with a natural interpretation in terms of stages of proof or construction.

Of course what makes the varying domains of intuitionistic semantics acceptable at all is the interpretation of this semantics in terms of stages of construction. But as I've emphasized in this paper, stratified models are also naturally interpreted in a way that makes varying domains acceptable. We can explain this by simply elaborating

[15]Of course, in philosophy there's rarely a complete consensus. In Section 2.3 of [31], nearly the same complaint made by Brady is raised against Heyting Arithmetic. For example, Shapiro claims that "We should not need to invoke non-standard numbers in order to capture intuitionistic arithmetic."

a theme from earlier in the paper: the higher strata in a stratified model should be seen as extensions of the lower strata.[16] I take it that the notion of an 'extension' of a model is a fairly clear one: it's a model that adds additional information to the model one has. In the case of stratified semantics, we are extending our model by adding to it information about certain arbitrary objects existing. For it to be the case that the objects we are adding to the model are *arbitrary*, it must be the case that (a) they don't have the features of any particular object, but (b) that they *could* have the features of any particular object, yet (c) any of these ways the objects could be is compatible with it having been any of the other ways. The stratified semantics exactly captures this. If ω is an AO that doesn't occur in the domain of w, then ω has, at w, no features at all. But, by the symmetry condition, ω *could* have the features of any particular object that *is* in the domain of w. Yet, by the homomorphism feature (and by taking $Rabc$ to mean 'a and b are compatible relative to c'; again see [2] for a defense of this reading), any of these ways for ω to be is compatible with any of the others. Finally, if we take '$\forall x\phi$' to mean 'ϕ is true of an arbitrary object', then the semantic clause for the universal in a stratified model is exactly what one expects.

So there are good reasons for the domains in stratified models to vary. And, indeed, if we take these reasons seriously, a *constant domain* semantics would seem very odd, since it would be lacking in arbitrary objects, and thus (on this view) would incorrectly interpret quantification.

Of course, this might seem to leave varying domain semantics open to the charge of being ontologically profligate. That is, it looks like in order for us to take the above defense of varying domain semantics seriously, one has to also take the genuine existence of arbitrary objects seriously. But this is, again, not obviously correct. Higher strata are *extensions of* lower strata. What the semantic clause for the universal requires for the truth of a universally quantified sentence is that there be an extension in which certain things happen. But a different and seemingly less ontologically committing way to say the same thing is that a universal is true at w when it is a fact about w that it *can be extended* in certain ways. But this just a fact about w, and it's not at all obvious that it engenders ontological commitment to the extensions being mentioned, let alone to the objects living in the extensions. If this is correct, it would follow that not only does the varying domain semantics capture something perfectly intelligible, it does so in a non-ontologically-profligate way.

But let's put all that aside for now, and let's suppose that some genuine objection to varying-domain semantic theories *has been* enunciated. For this to be an objection to stratified semantics *generally*, rather than to the particular version of stratified semantics I've presented here, one would need it to be the case that stratified semantic theories always have varying domains. But that's just untrue, as I'll now show.

5.1 Constant Domain Models

Before giving the technical details, it's worth pausing to paint an intuitive picture. There are basically two changes to be made. First, rather than setting $D_X = D_\emptyset \cup$

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[16] I owe thanks to John Carroll for pointing out that this will serve as a fairly viable defense of these models.

$\{\omega_i\}_{i \in X}$, we set $D_X = D_\emptyset \cup \Omega$. Thus, intuitively, the domain of each M_X contains all of the objects in the base model and *all of the* arbitrary objects one might have found at any of the strata of a varying-domain model.

So in a constant domain model each M_X contains (when we compare it to a corresponding varying-domain M_X) a bunch of 'extra' arbitrary objects. Absent further restrictions, these objects will, of course, tend to muck things up by making all kinds of sentences true or false that oughtn't be true or false. Thus the second change we make is to add conditions on models that keep the extra arbitrary objects we've got on hand from getting into this sort of trouble. Rather than fleshing this out here, I'll instead just give the actual model theory, then annotate which parts are doing what afterwards.

Let $\Omega = \{\omega_i\}_{i=1}^{\infty}$ be a countable set. A constant-domain Ω-premodel is a 4-tuple $\langle \mathcal{D}, \delta, \mathcal{M}, \Downarrow \rangle$ where

- $\mathcal{D} \supset \Omega$ is a set called the domain of the model.
- δ is a function from constants to $\mathcal{D} - \Omega$.
- $\mathcal{M}$ is a function mapping each set X of numbers to an **RWZ**-model M_X of the form $\langle \mathcal{D}, S_X, N_X, R_X, \delta, \mathcal{E}_X^+, \mathcal{E}_X^- \rangle$.
- $\Downarrow$ is a family of *restriction functions* $\downarrow_Y^X : S_X \to S_Y$. These behave exactly as in the varying-domain case.

A constant-domain Ω-model is a constant-domain Ω-premodel that satisfies the following seven conditions:

Featurelessness: If P is an n-ary predicate, $i \notin X$, $a \in S_X$, and $\langle d_1, \ldots, d_n \rangle \in \mathcal{E}_X^+(P, a) \cup \mathcal{E}_X^-(P, a)$, then $\omega_i \neq d_j$ for $1 \leq j \leq n$. (In words: if $i \notin X$, then ω_i does not occur in the extension or antiextension of any predicate at any setup in S_X.)

Atomic Heredity: If $a\downarrow_Y^X = b$ and P is a predicate, then

- The elements of $\mathcal{E}_Y^+(P, a)$ are exactly the elements of $\mathcal{E}_X^+(P, b)$ in which no member of $\{\omega_i\}_{i \in X-Y}$ occurs, and
- The elements of $\mathcal{E}_Y^-(P, a)$ are exactly the elements of $\mathcal{E}_X^-(P, b)$ in which no member of $\{\omega_i\}_{i \in X-Y}$ occurs.

Normality, Lifting, Homomorphism, and Extension: All as before.

Symmetry: If $a \in S_X$, $Y \supseteq X$, $m \in \mathcal{D} - \{\omega_i\}_{i \notin X}$ and $n \in \{\omega_i\}_{i \in Y-X}$, then there is a $b \in S_Y$ that is symmetric in m and n such that $b\downarrow_X^Y = a$.

Variable assignments can now be taken to be functions from variables to (all of!) $\mathcal{D}$. Variants, too, can then be given their usual definition. With respect to these models and variable assignments, truth values are assigned in exactly the same way as they were in the varying domain semantics.[17]

Ω-models are clearly stratified models. But every setup in an Ω-model has the same domain. This domain contains all the objects, arbitrary or not. Of course, dif-

[17] This particular way of presenting the constant-domain semantics (as quadruples rather than quintuples) was suggested by Shawn Standefer.

ferent parts of this domain play different roles in different setups. But there's nothing at all unusual about *that* type of variance, and it doesn't, in any event, mean that the *domain* is varying.

Most of the conditions are either exactly as they were in varying domain models or are straightforward adaptations of those conditions. The obvious exception to this is the Featurelessness condition. But the role this condition is playing is also easily explained: it's what keeps the 'extra' arbitrary objects from making true or false anything they shouldn't. Somewhat more concretely, the Featurelessness condition ensures that, if $i \notin X$, then ω_i doesn't 'look like' anything in a setup in S_X. And it does this by 'stashing' any sentence that *tries to* say something about one of the extra arbitrary objects in the null truth value.[18] This sort of featurelessness, you will recall, is one half of what we intuitively expect of arbitrary objects. The other half is again guaranteed by the symmetry condition.

Given the forgoing work, we can define Ω-validity in the obvious way: $\phi \in$ LQ is Ω-valid when ϕ is true in every Ω-model. But it's obvious from the construction of Ω-models that, if Ω_1 and Ω_2 are countable sets, then ϕ is Ω_1-valid iff ϕ is Ω_2-valid. Thus, we can define *constant-domain* validity in the obvious way: ϕ is constant-domain **RWQ**-valid iff for some (and hence every) countable set Ω, ϕ is Ω-valid.

RWQ, it turns out, is sound and complete for the constant-domain semantics. Thus, while it might *seem* quite odd for a semantic theory to require varying domains in order to interpret something like Peano Arithmetic, this is *at best* a complaint that applies to varying-domain stratified semantic theories, not to stratified semantic theories generally.

5.2 Discussion

The constant domain semantics *does* display several odd features worth dwelling on for a moment. The first of these is the most obvious: while the domains *do* stay constant across worlds in the constant domain semantics, the roles played by various objects in these domains changes. One *might* feel there's something fishy about this. Indeed, an anonymous referee of this paper expressed feeling regarding Ω-models that 'while the letter of the law for constant domains has been upheld, the spirit appears to have been violated.' I'm sympathetic with this view, in part because I'm sympathetic to the varying-domain models in the first place, and think that the runaround the constant-domain reconstruction has forced us to do is a bit unnecessary and inelegant. Nonetheless, if the letter of the law has been upheld, it would seem that the ball is in the objector's court. That is, I leave it up to the objector to state what, if it *isn't* varying domains, is the problem with stratified models that they were trying to point to. I'm leaving the door open for philosophical dialogue here in part because I'm optimistic about the chance for it to be philosophically fruitful. Perhaps there really is something wrong with stratified models. Perhaps what's wrong

[18]This sort of stashing is difficult to accomplish in a two-valued semantics. The Routley star might seem to offer a way to do it, but absent some very unnatural assumptions, the technical details don't seem to work out.

with them is whatever it is that is upsetting the folks who think they are upset about varying domains. If so, hopefully the discussion in this paper will help us get clear on just what this problem is.

Another oddity of the constant-domain semantics is that in any given setup, there are countably many featureless objects lingering about. And the featurelessness of these objects is really quite deep – they contribute neither to the truth nor to the falsity of any sentence in the entire language. So they are blank in every possible way. But as we pointed out above, this is an oddity one can learn to live with. The featureless objects are, after all, supposed to be *arbitrary* objects. We intuitively expect featurelessness of them.

A final oddity is this: while models of (e.g.) arithmetic will not contain setups with different domains, it will nonetheless be the case that in a model of arithmetic, the domain of each setup will contain, in addition to all the good old numbers, a countable infinity of *arbitrary numbers* as well. So there are extra 'things' around not just in *some* models of arithmetic – the upward Löwenheim-Skolem Theorem ensures that, e.g., classical first-order Peano Arithmetic has a similar problem to that, after all – but in *every* model. What's more: it's not just theories with infinite domains that run into this problem. Even when the non-arbitrary part of the domain is finite, the Ω in an Ω-model must still be infinite in order to allow for the correct interpretation of formulas with arbitrarily-many quantifiers in them. So our constant-domain models *always* require there to be infinitely many arbitrary objects around – even when we construct theories meant to account for explicitly finite-domain phenomena!

This, it seems to me, is a bullet that will have to be bitten by anyone tempted by the naïve formulation of constant-domain stratified semantics I've just outlined. So perhaps there's an objection to be made that ends here. Perhaps, that is, one can give reasons for rejecting varying-domain semantics generally, then give reasons for rejecting the extravagant ontology of constant-domain stratified semantics and, by combining these arguments, have at last a good reason for rejecting stratified semantics. But I'm not overly optimistic about either part of this being easy, and can't myself see how the objections should go. Nonetheless, for anyone tempted by this line, I hope this has at least mapped the terrain in the area.

5.3 Concluding Remarks on Constant-Domain Models

So at least for **RWQ**, we can in fact build a stratified semantic theory that is constant-domain and still fairly natural. While I think there is (and take myself to have given) good reason to ignore the usual worries about varying domain semantics, examining this theory shows that even if we take them seriously, they can be addressed within the stratified semantics framework.

I will conclude by saying a few words about the extension of this approach to other logics (and, in particular, to logics that allow contraction). As pointed out in [23], four-valued semantics seems uniquely well-suited to contractionless logics. We might worry, then, about how much the approach outlined here generalizes. I have two responses.

First, a variety of work on four-valued semantics for a range of other relevant logics exists – see for example [26] or [19]. Perhaps the constant-domain version of the stratified semantics outlined here can be extended by adapting these semantic theories. This is, I hasten to emphasize, *pure* conjecture – I've put no work into seeing whether this is in fact possible.

My second response, though, is that I'm just not too concerned about how the extension business turns out. Even if it were to turn out that no plausible constant-domain semantic theory can be built for a logic other than **RWQ**, then so be it – **RWQ** is a pretty nice logic anyways. At a bare minimum, it's worth seeing where it can take us.

6 Conclusion

My goal is for this paper to lead to more discussion of stratified semantics. If the lone result of this paper is that it aids in the production of some knockdown argument demonstrating a fatal flaw in these theories, I will count that as a success. I also hope that anyone interested in writing *that* paper will find the roadmap to an objection given above helpful, and will find the four attached technical appendices helpful as well.

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Appendix A: Soundness For RWZ

Let $M = \langle D, S, N, R, \delta, \mathcal{E}^+, \mathcal{E}^- \rangle$ be an **RWZ**-model.

Lemma 1 (Horizontal Heredity) *If* $a \leq b$ *then if* $1 \in M^a(\phi)$ *then* $1 \in M^b(\phi)$ *and if* $0 \in M^a(\phi)$ *then* $0 \in M^b(\phi)$.

Proof The proof is by induction on the complexity of ϕ. If ϕ is atomic, then the result follows from atomic horizontal heredity. Most of the induction cases are trivial. We present only the conditional cases here, and leave the rest to the reader.

$1 \in M^a(\phi \to \psi)$ iff for all c and d, if $Racd$ and $1 \in M^c(\phi)$, then $1 \in M^d(\psi)$, and if $Racd$ and $0 \in M^c(\psi)$, then $0 \in M^d(\phi)$. But if $Rbcd$ then since $a \leq b$, $Racd$ as well. The desired result follows from this observation almost immediately.

On the other hand, $0 \in M^a(\phi \to \psi)$ iff for some c and d with $Rcda$, $1 \in M^c(\phi)$ and $0 \in M^d(\psi)$. Since $a \leq b$, there is an $n \in N$ such that $Rnab$. Thus, by Permutation, $Ranb$ as well. But from $Rcda$ and $Ranb$ it follows by B' that there is a z so that $Rcnz$ and $Rdzb$. From $Rdzb$ and $Rcnz$ it follows by permutation that $Rzdb$ and $Rncz$. From the latter, we see that $c \leq z$. Thus, by the inductive hypothesis,

$1 \in M^z(\phi)$. But since we already know that $0 \in M^d(\psi)$ and that $Rzdb$, this tells us that $0 \in M^b(\phi \to \psi)$. □

Theorem 1 *If* $\alpha \in$ **RWZ**, *then* α *is* **RWZ**-*valid.*

Proof The proof is straightforward: we show each axiom is valid, then that the rules preserve validity. For the axioms, we present here only the argument for **A8** and leave the rest to the reader. For rules, we only examine **R1**.

A8: $M \vDash (\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$ iff $1 \in M^n[(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))]$ for all $n \in N$. This happens, in turn, iff if $Rnab$, then (a) if $1 \in M^a(\alpha \to \beta)$ then $1 \in M^b[(\beta \to \gamma) \to (\alpha \to \gamma)]$, and (b) if $0 \in M^a[(\beta \to \gamma) \to (\alpha \to \gamma)]$ then $0 \in M^b(\alpha \to \beta)$.

For (a) suppose $1 \in M^a(\alpha \to \beta)$. Then by heredity $1 \in M^b(\alpha \to \beta)$. Note that $1 \in M^b[(\beta \to \gamma) \to (\alpha \to \gamma)]$ iff for all c and d if $Rbcd$ then (i) if $1 \in M^c(\beta \to \gamma)$ then $1 \in M^d(\alpha \to \gamma)$, and (ii) if $0 \in M^c(\alpha \to \gamma)$ then $0 \in M^d(\beta \to \gamma)$.

Suppose $Rbcd$. For (i) let $1 \in M^c(\beta \to \gamma)$. To show that $1 \in M^d(\alpha \to \gamma)$ we need to show that for all e and f if $Rdef$ then (i†) if $1 \in M^e(\alpha)$ then $1 \in M^f(\gamma)$ and (i††) if $0 \in M^e(\gamma)$ then $0 \in M^f(\alpha)$. For (i†), suppose $Rdef$ and $1 \in M^e(\alpha)$. Since $Rbcd$ and $Rdef$, there is some z so that $Rczf$ and $Rbez$. Since $Rbez$, $1 \in M^b(\alpha \to \beta)$ and $1 \in M^e(\alpha)$, we get that $1 \in M^z(\beta)$. Then since $Rczf$, $1 \in M^c(\beta \to \gamma)$ and $1 \in M^z(\beta)$, we get that $1 \in M^f(\gamma)$ as required. For (i††), suppose $Rdef$ and $0 \in M^e(\gamma)$. Since $Rbcd$, $Rcbd$ as well. With this and $Rdef$ we get that there is some z so that $Rcez$ and $Rbzf$. From $Rcez$, $1 \in M^c(\beta \to \gamma)$ and $0 \in M^e(\gamma)$, we get that $0 \in M^z(\beta)$. Finally, from $Rbzf$, $1 \in M^b(\alpha \to \beta)$ and $0 \in M^z(\beta)$, we get that $0 \in M^f(\alpha)$ as required.

For (ii), suppose $Rbcd$ (so by permutation $Rcbd$ as well) and $0 \in M^c(\alpha \to \gamma)$. Then there are e and f so that $Refc$, $1 \in M^e(\alpha)$ and $0 \in M^f(\gamma)$. By B', since $Refc$ and $Rcbd$ there is a z so that $Rebz$ and $Rfzd$. From $Rebz$, permutation gives $Rbez$. This together with $1 \in M^b(\alpha \to \beta)$ and $1 \in M^e(\alpha)$ gives us that $1 \in M^z(\beta)$. From $Rfzd$ permutation gives $Rzfd$. Since $1 \in M^z(\beta)$ and $0 \in M^f(\gamma)$, we get that $0 \in M^d(\beta \to \gamma)$ as required.

For (b), suppose $0 \in M^a[(\beta \to \gamma) \to (\alpha \to \gamma)]$. It follows by heredity that $0 \in M^b[(\beta \to \gamma) \to (\alpha \to \gamma)]$. So for some c and d, $Rcdb$ and $1 \in M^c(\beta \to \gamma)$ and $0 \in M^d(\alpha \to \gamma)$. Thus, for some e and f, $Refd$, $1 \in M^e(\alpha)$ and $0 \in M^f(\gamma)$. From $Rcdb$ we get $Rdcb$ while from $Refd$ we get $Rfed$. Together $Rfed$ and $Rdcb$ give that there is a z so that $Rfcz$ and $Rezb$. But from $Rfcz$ we get $Rcfz$, and since $1 \in M^c(\beta \to \gamma)$ and $0 \in M^f(\gamma)$, we get that $0 \in M^z(\beta)$. Then from $Rezb$, $1 \in M^e(\alpha)$ and $0 \in M^z(\beta)$ we get $0 \in M^b(\alpha \to \beta)$ as required.

R1: Suppose $M \vDash \alpha$ and $M \vDash \alpha \to \beta$. Let $n \in N$. Since $n \leq n$, for some $m \in N$, $Rmnn$. Since $m \in N$ and $M \vDash \alpha \to \beta$, $1 \in M^m(\alpha \to \beta)$. Since $n \in N$, and $M \vDash \alpha$, $1 \in M^n(\alpha)$. Thus, since $Rmnn$, $1 \in M^n(\beta)$. Since n was an arbitrary element of N, $M \vDash \beta$ as required.

□

Appendix B: Completeness For RWZ

Our completeness proof is via the Post-Lindenbaum route. Explicitly, this means we will use the proof theory itself to construct a canonical model. Before we get going, we'll need a variety of fairly standard definitions. Let $\Sigma \cup \Pi \cup \Delta \cup \{\phi\} \subseteq \text{LZ}$.

- Write $\Sigma \vdash \phi$ when there is a sequence $\phi_1, \ldots, \phi_n = \phi$ such that each ϕ_i is either in **RWZ**$\cup\,\Sigma$ or follows from previous members of the sequence by **R1** or by **R2**.
- Write $\Pi_{\rightarrow}$ for the set of sentences in Π that are of the form $\alpha \rightarrow \beta$.
- Write $\Sigma \vdash_{\Pi} \phi$ when $\Sigma \cup \Pi_{\rightarrow} \vdash \phi$.
- Say Σ is a Π-theory when (i) $\alpha \in \Sigma$ and $\vdash_{\Pi} \alpha \rightarrow \beta$ only if $\beta \in \Sigma$ and (ii) $\alpha \in \Sigma$ and $\beta \in \Sigma$ only if $\alpha \wedge \beta \in \Sigma$.
- Say Σ is prime when $\alpha \vee \beta \in \Sigma$ only if $\alpha \in \Sigma$ or $\beta \in \Sigma$.
- Say that Σ is closed under disjunction when $\alpha \in \Sigma$ and $\beta \in \Sigma$ only if $\alpha \vee \beta \in \Sigma$.
- Write $\Sigma \vdash_{\Pi} \Delta$ when there are $\delta_1, \ldots, \delta_n$ all in Δ such that $\Sigma \vdash_{\Pi} \delta_1 \vee \cdots \vee \delta_n$.
- Write $\vdash_{\Pi} \Sigma \rightarrow \Delta$ when there are $\sigma_1, \ldots, \sigma_n$ all in Σ and $\delta_1, \ldots, \delta_m$ all in Δ such that $\vdash_{\Pi} (\sigma_1 \wedge \cdots \wedge \sigma_n) \rightarrow (\delta_1 \vee \cdots \vee \delta_m)$.
- Say that $\langle \Sigma, \Delta \rangle$ is a Π-partition when $\Sigma \cup \Delta = \text{LZ}$ and $\nvdash_{\Pi} \Sigma \rightarrow \Delta$.
- Say that Π is *normal* when $\textbf{RWZ} \subseteq \Pi$.

B.1 Facts About Provability

The first thing we need are some facts about the provability relation. Let $\Pi \cup \{\alpha, \beta, \gamma, \rho\} \subseteq \text{LZ}$. Then, as can be verified by tediously constructing derivations, all of the following hold:

Fact 1: $\vdash_{\Pi} \alpha \rightarrow \neg\neg\alpha$
Fact 2: If $\vdash_{\Pi} \alpha \rightarrow \beta$ and $\vdash_{\Pi} \beta \rightarrow \gamma$, then $\vdash_{\Pi} \alpha \rightarrow \gamma$
Fact 3: $\vdash_{\Pi} (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\beta \rightarrow (\alpha \rightarrow \gamma))$
Fact 4: $\vdash_{\Pi} (\alpha \rightarrow \beta) \rightarrow (\neg\beta \rightarrow \neg\alpha)$
Fact 5: $\vdash_{\Pi} \neg\alpha \rightarrow \neg(\alpha \wedge \beta)$
Fact 6: $\vdash_{\Pi} \neg\beta \rightarrow \neg(\alpha \wedge \beta)$
Fact 7: $\vdash_{\Pi} \neg(\alpha \wedge \beta) \rightarrow (\neg\alpha \vee \neg\beta)$
Fact 8: $\vdash_{\Pi} (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\alpha \rightarrow (\neg\gamma \rightarrow \neg\beta))$
Fact 9: $\vdash_{\Pi} (\neg\alpha \rightarrow \beta) \rightarrow (\neg\beta \rightarrow \alpha)$
Fact 10: $\vdash_{\Pi} (\gamma \rightarrow \rho) \rightarrow ((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\alpha \rightarrow (\beta \rightarrow \rho)))$
Fact 11: $\vdash_{\Pi} ((\alpha \rightarrow \beta) \wedge (\delta \rightarrow \rho)) \rightarrow ((\alpha \wedge \delta) \rightarrow (\beta \wedge \rho))$
Fact 12: $\vdash_{\Pi} ((\alpha \rightarrow \delta) \wedge (\beta \rightarrow \delta)) \rightarrow ((\alpha \vee \beta) \rightarrow \delta)$

B.2 Partition and Primality Lemmas

The proofs of these results are fairly standard (see for example Section 11.3 of [24]). But the techniques they introduce are ubiquitous, so they are worth quickly going through.

Lemma 2 *If* $\langle \Sigma, \Delta \rangle$ *is a* Π*-partition, then* Σ *is a prime* Π*-theory.*

Proof Suppose α and β are in Σ. If $\alpha \wedge \beta \notin \Sigma$, then since $\Sigma \cup \Delta = \text{LZ}$, $\alpha \wedge \beta \in \Delta$. But then since $(\alpha \wedge \beta) \to (\alpha \wedge \beta) \in \mathbf{RWZ}$, $\vdash_\Pi (\alpha \wedge \beta) \to (\alpha \wedge \beta)$, so $\vdash_\Pi \Sigma \to \Delta$. This contradicts $\langle \Sigma, \Delta \rangle$ being a Π-partition, so $\alpha \wedge \beta \in \Sigma$. Now suppose $\alpha \in \Sigma$ and $\vdash_\Pi \alpha \to \beta$. Then if $\beta \in \Delta$, then $\vdash_\Pi \Sigma \to \Delta$. So since $\nvdash_\Pi \Sigma \to \Delta$, we must have $\beta \notin \Delta$. But then $\beta \in \Sigma$. So Σ is a theory.

Now suppose $\alpha \vee \beta \in \Sigma$ but that $\alpha \notin \Sigma$ and $\beta \notin \Sigma$. Then α and β are in Δ. Since $(\alpha \vee \beta) \to (\alpha \vee \beta) \in \mathbf{RWZ}$, $\vdash_\Pi (\alpha \vee \beta) \to (\alpha \vee \beta)$. But then $\vdash_\Pi \Sigma \to \Delta$. So since $\nvdash_\Pi \Sigma \to \Delta$, either $\alpha \in \Sigma$ or $\beta \in \Sigma$. So Σ is prime. $\square$

Lemma 3 *If $\nvdash_\Pi \Sigma \to \Delta$ then there is a Π-partition $\langle \Sigma', \Delta' \rangle$ with $\Sigma' \supseteq \Sigma$ and $\Delta' \supseteq \Delta$.*

Proof Let $\phi_1, \phi_2, \ldots$ be an enumeration of LZ, $\Sigma_0 = \Sigma$, $\Delta_0 = \Delta$, and for $i > 0$ define Σ_i and Δ_i as follows:

If $\nvdash_\Pi \Sigma_{i-1} \cup \{\phi_{i-1}\} \to \Delta_{i-1}$, then $\Sigma_i = \Sigma_{i-1} \cup \{\phi_{i-1}\}$ and $\Delta_i = \Delta_{i-1}$.
If $\vdash_\Pi \Sigma_{i-1} \cup \{\phi_{i-1}\} \to \Delta_{i-1}$, then $\Sigma_i = \Sigma_{i-1}$ and $\Delta_i = \Delta_{i-1} \cup \{\phi_{i-1}\}$.

Let $\Sigma' = \cup_{i=0}^{\infty} \Sigma_i$ and $\Delta' = \cup_{i=0}^{\infty} \Delta_i$. It's clear that $\Sigma' \cup \Delta' = \text{LZ}$. It claim that in fact $\langle \Sigma', \Delta' \rangle$ is a Π-partition. To see this, note that if it isn't, then $\vdash_\Pi \Sigma' \to \Delta'$. So for some $\phi_{i_1}, \ldots, \phi_{i_n}$ all in Σ and $\phi_{j_1}, \ldots, \phi_{j_m}$ all in Δ, $\vdash_\Pi (\phi_{i_1} \wedge \cdots \wedge \phi_{i_n}) \to (\phi_{j_1} \vee \cdots \vee \phi_{j_m})$. Letting $M = \max\{i_1, \ldots, i_n, j_1, \ldots, j_m\}$, we then have that $\vdash_\Pi \Sigma_M \to \Delta_M$. Thus there is a least k for which $\vdash_\Pi \Sigma_k \to \Delta_k$. Call this number k_0.

By assumption, $k_0 \neq 0$. So Σ_{k_0-1} and Δ_{k_0-1} are defined. By construction either $\Sigma_{k_0} = \Sigma_{k_0-1}$ or $\Delta_{k_0} = \Delta_{k_0-1}$. The latter case only occurs if we also have that $\Sigma_{k_0} = \Sigma_{k_0-1} \cup \{\phi_{k_0-1}\}$ and $\nvdash_\Pi \Sigma_{k_0-1} \cup \{\phi_{k_0-1}\} \to \Delta_{k_0-1}$. But then clearly $\nvdash_\Pi \Sigma_{k_0} \to \Delta_{k_0}$, which is a contradiction.

On the other hand, $\Sigma_{k_0} = \Sigma_{k_0-1}$ only if both

(a) $\vdash_\Pi \Sigma_{k_0-1} \cup \{\phi_{k_0-1}\} \to \Delta_{k_0-1}$, and
(b) $\Delta_{k_0} = \Delta_{k_0-1} \cup \{\phi_{k_0-1}\}$.

It then follows from $\vdash_\Pi \Sigma_{k_0} \to \Delta_{k_0}$, (b), and the fact that $\Sigma_{k_0} = \Sigma_{k_0-1}$ that

(c) $\vdash_\Pi \Sigma_{k_0-1} \to \Delta_{k_0-1} \cup \{\phi_{k_0-1}\}$.

Also, since k_0 is minimal, we have

(d) $\nvdash_\Pi \Sigma_{k_0-1} \to \Delta_{k_0-1}$.

It follows from (a) and (d) that for some $\sigma \in \Sigma_{k_0-1}$ and $\delta \in \Delta_{k_0-1}$ we have

(e) $\vdash_\Pi (\sigma \wedge \phi_{k_0-1}) \to \delta$.

And it follows from (c) and (d) that for some $\sigma' \in \Sigma_{k_0-1}$ and $\delta' \in \Delta_{k_0-1}$ we have

(f) $\vdash_\Pi \sigma' \to (\delta' \vee \phi_{k_0-1})$.

From (e) and (f) it follows (by a mildly annoying derivation) that $\vdash_\Pi \Sigma_{k_0-1} \to \Delta_{k_0-1}$, which contradicts k_0's minimality. So $\langle \Sigma', \Delta' \rangle$ is a Π-partition. $\square$

Lemma 4 *If Σ is a Π-theory, Δ is closed under disjunction, and $\Sigma \cap \Delta = \emptyset$ then there is a prime Π-theory $\Sigma' \supset \Sigma$ with $\Sigma' \cap \Delta = \emptyset$.*

Proof If $\vdash_\Pi \Sigma \rightarrow \Delta$, then $\vdash_\Pi (\sigma_1 \wedge \cdots \wedge \sigma_n) \rightarrow (\delta_1 \vee \cdots \vee \delta_m)$ for some $\sigma_1, \ldots, \sigma_n \in \Sigma$ and $\delta_1, \ldots, \delta_m \in \Delta$. Since Σ is a Π-theory and each $\sigma_i \in \Sigma$, clearly $\sigma_1 \wedge \cdots \wedge \sigma_n \in \Sigma$. It follows that $\delta_1 \vee \cdots \vee \delta_m \in \Sigma$. But Δ is closed under disjunction, so $\delta_1 \vee \cdots \vee \delta_m \in \Delta$. This contradicts $\Sigma \cap \Delta = \emptyset$, so $\not\vdash_\Pi \Sigma \rightarrow \Delta$. Lemmas 2 and 3 then give the result. □

Lemma 5 *Suppose $\Sigma \not\vdash_\Pi \alpha$. Then there is a prime Π-theory $\Sigma' \supseteq \Sigma$ with $\alpha \notin \Sigma'$.*

Proof If Σ is empty, then $\not\vdash_\Pi \Sigma \rightarrow \{\alpha\}$. On the other hand, if Σ is nonempty and $\vdash_\Pi \Sigma \rightarrow \{\alpha\}$, then $\vdash_\Pi (\sigma_1 \wedge \cdots \wedge \sigma_n) \rightarrow \alpha$ for some $\sigma_1, \ldots, \sigma_n$ in Σ. Since each $\sigma_i \in \Sigma$, clearly $\Sigma \vdash_\Pi \sigma_1 \wedge \cdots \wedge \sigma_n$. It then follows that $\Sigma \vdash_\Pi \alpha$. So since $\Sigma \not\vdash \alpha$, it is also the case that $\not\vdash_\Pi \Sigma \rightarrow \{\alpha\}$. Thus by Lemma 3, there is a $\Sigma' \supseteq \Sigma$ and $\Gamma \supseteq \{\alpha\}$ so that $\langle \Sigma', \Gamma \rangle$ is a Π-partition. But then by Lemma 2, Σ' is a prime Π-theory and since $\vdash_\Pi \alpha \rightarrow \alpha$, it follows from the definition of a Π-partition that $\alpha \notin \Sigma'$. □

B.3 Extension Lemmas

First, for $\Sigma \cup \Delta \cup \Gamma \subseteq \mathrm{LZ}$, define the relation $\mathcal{R}$ by saying $\mathcal{R}\Sigma\Delta\Gamma$ iff if $\delta \rightarrow \gamma \in \Sigma$ and $\delta \in \Delta$, then $\gamma \in \Gamma$. Each of the following lemmas proves that under certain conditions, if $\mathcal{R}\Sigma\Delta\Gamma$, then we can extend one or more of Σ, Γ, or Δ to a prime theory without breaking the relation. Each is also proved in generally the same way: first, we construct a set Θ of sentences that we want to avoid. We then complete the required theory(ies) away from Θ using Lemma 4.

Lemma 6 *If Σ, Γ, and Δ are Π-theories, $\mathcal{R}\Sigma\Gamma\Delta$ and Δ is a prime Π-theory, then there is a prime Π-theory $\Gamma' \supseteq \Gamma$ such that $\mathcal{R}\Sigma\Gamma'\Delta$*

Proof We want to extend Γ' to a prime theory without adding to it the antecedent of any conditional in Σ whose consequent isn't in Δ. So we let Θ, the set of sentences we want to avoid, be $\{\alpha : \alpha \rightarrow \beta \in \Sigma \text{ for some } \beta \notin \Delta\}$. I claim Θ is closed under disjunctions. To see this, notice that if α_1 and α_2 are in Θ, then there are β_1 and β_2 not in Δ such that $\alpha_1 \rightarrow \beta_1$ and $\alpha_2 \rightarrow \beta_2$ are in Σ and since Δ is prime and β_1 and β_2 are not in Δ, $\beta_1 \vee \beta_2 \notin \Delta$. By an instance of **A8** we have that $\vdash_\Pi (\alpha_1 \rightarrow \beta_1) \rightarrow ((\beta_1 \rightarrow (\beta_1 \vee \beta_2)) \rightarrow (\alpha_1 \rightarrow (\beta_1 \vee \beta_2)))$. So by Fact 3, $\vdash_\Pi (\beta_1 \rightarrow (\beta_1 \vee \beta_2)) \rightarrow ((\alpha_1 \rightarrow \beta_1) \rightarrow (\alpha_1 \rightarrow (\beta_1 \vee \beta_2)))$. But using Fact 2, the definition of '$\vee$' and instances of Facts 1 and 5, we get that $\vdash_\Pi \beta_1 \rightarrow (\beta_1 \vee \beta_2)$. So $\vdash_\Pi (\alpha_1 \rightarrow \beta_1) \rightarrow (\alpha_1 \rightarrow (\beta_1 \vee \beta_2))$. Since $\alpha_1 \rightarrow \beta_1 \in \Sigma$ and Σ is a Π-theory, it then follows that $\alpha_1 \rightarrow (\beta_1 \vee \beta_2) \in \Sigma$. So $\beta_1 \vee \beta_2 \in \Theta$.

Suppose $\alpha \in \Gamma \cap \Theta$. Then since $\alpha \in \Theta$, for some $\beta \notin \Delta$, $\alpha \rightarrow \beta \in \Sigma$. But from $\mathcal{R}\Sigma\Gamma\Delta$, $\alpha \rightarrow \beta \in \Sigma$ and $\alpha \in \Gamma$ it follows that $\beta \in \Delta$, a contradiction. So $\Gamma \cap \Theta$ must be empty. So by Lemma 4, there is a prime Π-theory $\Gamma' \supseteq \Gamma$ with $\Gamma' \cap \Theta = \emptyset$. Now suppose $\alpha \rightarrow \beta \in \Sigma$ and $\alpha \in \Gamma'$. If $\beta \notin \Delta$, then $\alpha \in \Theta$. But then $\alpha \in \Gamma' \cap \Theta$, which is a contradiction. So $\beta \in \Delta$. Thus $\mathcal{R}\Sigma\Gamma'\Delta$. □

Lemma 7 *If Σ, Γ, and Δ are Π-theories, $\mathcal{R}\Sigma\Gamma\Delta$, and Δ is a prime Π-theory, then there is a prime Π-theory $\Sigma' \supseteq \Sigma$ such that $\mathcal{R}\Sigma'\Gamma\Delta$*

Proof The hard part here is identifying the sentences we want to avoid. The idea to have is this: we don't want to add to Σ anything that forces us to also add to it some conditional whose antecedent is in Γ, but whose consequent isn't in Δ. Thus, this time we let Θ be $\{\alpha : \vdash_\Pi \alpha \rightarrow (\gamma \rightarrow \beta) \text{ for some } \gamma \in \Gamma \text{ and } \beta \notin \Delta\}$. By almost the same argument as in the previous lemma, we see that Θ is closed under disjunction.

Suppose that $\alpha \in \Sigma \cap \Theta$. Then $\vdash_\Pi \alpha \rightarrow (\gamma \rightarrow \beta)$ for some $\gamma \in \Gamma$ and $\beta \notin \Delta$. Since Σ is a Π-theory, it follows that $\gamma \rightarrow \beta \in \Sigma$. But then $\beta \in \Delta$ because $\mathcal{R}\Sigma\Gamma\Delta$, $\gamma \rightarrow \beta \in \Sigma$ and $\gamma \in \Gamma$. Since this is a contradiction, $\Sigma \cap \Theta = \emptyset$. So by Lemma 4, there is a prime Π-theory Σ' with $\Sigma' \supseteq \Sigma$ and $\Sigma' \cap \Theta = \emptyset$.

Finally, suppose $\alpha \rightarrow \beta \in \Sigma'$. Since $\Sigma' \cap \Theta = \emptyset$, $\alpha \rightarrow \beta \notin \Theta$. Since $\vdash_\Pi (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)$, it follows from the definition of Θ that if $\alpha \in \Gamma$ and $\beta \notin \Delta$, then $\alpha \rightarrow \beta \in \Theta$. Since this is impossible, if $\alpha \in \Gamma$, then $\beta \in \Delta$. So $\mathcal{R}\Sigma'\Gamma\Delta$. $\square$

Lemma 8 *Let Σ be a prime Π-theory and $\gamma \rightarrow \delta \notin \Sigma$. Then there are prime Π-theories Γ and Δ such that $\mathcal{R}\Sigma\Gamma\Delta$, $\gamma \in \Gamma$, and $\delta \notin \Delta$.*

Proof To begin, let $\Gamma' = \{\alpha : \vdash_\Pi \gamma \rightarrow \alpha\}$ and $\Delta' = \{\beta : \alpha \rightarrow \beta \in \Sigma \text{ for some } \alpha \in \Gamma'\}$. We leave it to the reader to verify that Γ' is a Π-theory. It's clear from these definitions that $\mathcal{R}\Sigma\Gamma'\Delta'$ and $\gamma \in \Gamma'$. To see that $\delta \notin \Delta'$, notice that if δ were in Δ', then there would be an $\alpha \in \Gamma'$ with $\alpha \rightarrow \delta \in \Sigma$. But if $\alpha \in \Gamma'$, then $\vdash_\Pi \gamma \rightarrow \alpha$. But also by **A8**, $\vdash_\Pi (\gamma \rightarrow \alpha) \rightarrow ((\alpha \rightarrow \delta) \rightarrow (\gamma \rightarrow \delta))$. So if $\vdash_\Pi \gamma \rightarrow \alpha$, then $\vdash_\Pi (\alpha \rightarrow \delta) \rightarrow (\gamma \rightarrow \delta)$. Since Σ is a Π-theory, it would then follow that $\gamma \rightarrow \delta \in \Sigma$, which is a contradiction. So $\delta \notin \Delta'$.

To see that Δ' is a Π-theory, first let $\beta_1 \in \Delta'$ and $\beta_2 \in \Delta'$. Then for some $\alpha_1 \in \Gamma'$ and $\alpha_2 \in \Gamma'$, $\alpha_1 \rightarrow \beta_1 \in \Sigma$ and $\alpha_2 \rightarrow \beta_2 \in \Sigma$. Since Γ' is a theory, $\alpha_1 \wedge \alpha_2 \in \Gamma'$. Since Σ is a theory, $(\alpha_1 \rightarrow \beta_1) \wedge (\alpha_2 \rightarrow \beta_2) \in \Sigma$. By Fact 12, $\vdash_\Pi ((\alpha_1 \rightarrow \beta_1) \wedge (\alpha_2 \rightarrow \beta_2)) \rightarrow ((\alpha_1 \wedge \alpha_2) \rightarrow (\beta_1 \wedge \beta_2))$. Thus $(\alpha_1 \wedge \alpha_2) \rightarrow (\beta_1 \wedge \beta_2) \in \Sigma$. So $\beta_1 \wedge \beta_2 \in \Delta'$. Now suppose $\beta \in \Delta'$ and $\vdash_\Pi \beta \rightarrow \rho$. Since $\beta \in \Delta'$, $\alpha \rightarrow \beta \in \Sigma$ for some $\alpha \in \Gamma'$. Applying Fact 3 to an instance of **A8**, we see that $\vdash_\Pi (\beta \rightarrow \rho) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \rho))$. Thus, since $\vdash_\Pi \beta \rightarrow \rho$, it follows that $\vdash_\Pi (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \rho)$ as well. So since Σ is a theory, $\alpha \rightarrow \rho \in \Sigma$. Thus $\rho \in \Delta'$.

Let Θ be the closure of $\{\delta\}$ under disjunction. A trivial induction using instances of Fact 12 shows that if $\theta \in \Theta$, then $\vdash_\Pi \theta \rightarrow \delta$. Thus, if $\theta \in \Delta' \cap \Theta$, then since Δ' is a theory, $\delta \in \Delta'$. So since $\delta \notin \Delta'$, $\Delta' \cap \Theta = \emptyset$. It follows by Lemma 4 that there is a prime theory $\Delta \supseteq \Delta'$ with $\Delta \cap \Theta = \emptyset$. In particular $\delta \notin \Delta$, and since $\Delta \supseteq \Delta'$, $\mathcal{R}\Sigma\Gamma'\Delta$. Thus by Lemma 6, there is a prime Π-theory $\Gamma \supset \Gamma'$ so that $\mathcal{R}\Sigma\Gamma\Delta$. Since $\gamma \in \Gamma'$, $\gamma \in \Gamma$ as well, finishing the proof. $\square$

B.4 Completeness

If Π is a prime theory, then the *canonical* premodel $\mathfrak{C}_\Pi$ is the septuple $\langle D, S, N, \mathcal{R}, \delta, \mathcal{E}^+_\Pi, \mathcal{E}^-_\Pi \rangle$ such that

- D is the set of names,

- S is the set of prime Π-theories,
- N is the set of normal prime Π-theories,
- δ is the identity function,
- If P is an i-ary predicate, then $\mathcal{E}^+_\Pi(P, a)$ is the set of all i-tuples $\langle n_1, \ldots, n_i \rangle$ such that $Pn_1 \ldots n_i \in a$, and
- If P is an i-ary predicate, then $\mathcal{E}^-_\Pi(P, a)$ is the set of all i-tuples $\langle n_1, \ldots, n_i \rangle$ such that $\neg Pn_1 \ldots n_i \in a$.

Obviously this is a premodel. To see it's a model, we must verify that it meets the ordering, monotonicity, closure, rearranging, and horizontal atomic heredity conditions. Of these, ordering, closure, and heredity follow immediately from the following lemma:

Lemma 9 *In* $\mathfrak{C}_\Pi$, $a \leq b$ *iff* $a \subseteq b$

Proof Let n be a normal prime Π-theory and suppose $\mathcal{R}nab$. Let $\alpha \in a$. Then since n is normal, $\alpha \to \alpha \in n$. So $\alpha \in b$. Thus if $a \leq b$, then $a \subseteq b$.

Now suppose $a \subseteq b$. If $\alpha \to \beta \in$ **RWZ** and $\alpha \in a$, then since a is a Π-theory, and $\vdash_\Pi \alpha \to \beta$, $\beta \in a$. Thus $\beta \in b$. So $\mathcal{R}\mathbf{RWZ}ab$, and thus $a \leq b$. So if $a \subseteq b$, then $a \leq b$. □

For the remaining conditions, we argue as follows:

Monotonicity: If $\mathcal{R}abc$, $a' \leq a$, $b' \leq b$, and $c \leq c'$, then by Lemma 9 $a' \subseteq a$, $b' \subseteq b$, and $c \subseteq c'$. So if $\phi \to \psi \in a'$ and $\phi \in b'$, then $\phi \to \psi \in a$ and $\phi \in b$, and since $\mathcal{R}abc$ it follows from these that $\psi \in c$. So since $c \subseteq c'$, $\psi \in c'$. So $\mathcal{R}a'b'c'$.

B: Suppose $\mathcal{R}abcd$ – that is, that for some Π-theory x, $\mathcal{R}abx$ and $\mathcal{R}xcd$. Let $y' = \{\alpha : \gamma \to \alpha \in b \text{ for some } \gamma \in c\}$. The reader will easily verify that y' is a theory, and it is clear by definition that $\mathcal{R}bcy'$. I claim that $\mathcal{R}ay'd$ as well.

To see this, suppose $\phi \to \psi \in a$ and $\phi \in y'$. Since $\phi \in y'$, for some $\gamma \in c$ we have that $\gamma \to \phi \in b$. Using **A8** and Fact 3, we see that $\vdash_\Pi (\phi \to \psi) \to ((\gamma \to \phi) \to (\gamma \to \psi))$. Thus since $\phi \to \psi \in a$ and a is a Π-theory, $(\gamma \to \phi) \to (\gamma \to \psi) \in a$ as well. Thus since $\mathcal{R}abx$ and $\gamma \to \phi \in b$, $\gamma \to \psi \in x$. And since $\gamma \to \psi \in x$ and $\gamma \in c$, we can conclude that $\psi \in d$ as required.

Lemma 6 then gives that there is a Π-theory $y \supseteq y'$ so that $\mathcal{R}ayd$. Clearly since $y \supseteq y'$ and $\mathcal{R}bcy'$ it is also the case that $\mathcal{R}bcy$, completing the proof.

B': Again, suppose $\mathcal{R}abcd$. This time let $y' = \{\alpha : \gamma \to \alpha \in a \text{ for some } \gamma \in c\}$. Mutatis mutandis, the same argument as before works here.

C: With the same assumptions as in the prior case, all that is required is that we show that $\mathcal{R}y'bd$. So let $\phi \to \psi \in y'$ and $\phi \in b$. Since $\phi \to \psi \in y'$, $\gamma \to (\phi \to \psi) \in a$ for some $\gamma \in c$. By Fact 3, since $\gamma \to (\phi \to \psi) \in a$ and a is a Π-theory, $\phi \to (\gamma \to \psi) \in a$ as well. Thus, since $\phi \in b$ and $\mathcal{R}abx$, $\gamma \to \psi \in x$. And thus since $\gamma \in c$, $\psi \in d$. So $\mathcal{R}y'bd$. The proof then finishes as before, replacing Lemma 6 with Lemma 7.

Lemma 10 $1 \in \mathfrak{C}^a_\Pi(\phi)$ *iff* $\phi \in a$ *and* $0 \in \mathfrak{C}^a_\Pi(\phi)$ *iff* $\neg\phi \in a$.

Proof By simultaneous induction on the complexity of ϕ in both (a) and (b). The base case and the cases involving conjunctions and negations are entirely straightforward so omitted.

For the '1' part of the conditional case, first suppose $\phi \to \psi \in a$, $\mathcal{R}abc$, and $1 \in \mathfrak{C}^b_\Pi(\phi)$. Then by the inductive hypothesis, $\phi \in b$. So since $\mathcal{R}abc$, $\psi \in c$. It follows that $1 \in \mathfrak{C}^c_\Pi(\psi)$. Now suppose instead that $0 \in \mathfrak{C}^b_\Pi(\psi)$. Then by the inductive hypothesis, $\neg\psi \in b$. By Fact 4, $\vdash_\Pi (\phi \to \psi) \to (\neg\psi \to \neg\phi)$. So since $\phi \to \psi \in a$, $\neg\psi \to \neg\phi \in a$ as well. Thus since $\mathcal{R}abc$, $\neg\phi \in c$. Thus, again by the inductive hypothesis, $0 \in \mathfrak{C}^c_\Pi(\phi)$. Together these two pieces give that $1 \in \mathfrak{C}^a_\Pi(\phi \to \psi)$ as required. For the other direction, note that if $\phi \to \psi \notin a$, then by Lemma 8 there are b and c so that $\mathcal{R}abc$, $\phi \in b$ but $\psi \notin c$. By the inductive hypothesis, then, $1 \in \mathfrak{C}^b_\Pi(\phi)$ and $1 \notin \mathfrak{C}^c_\Pi(\psi)$. So $1 \notin \mathfrak{C}^a_\Pi(\phi \to \psi)$.

For the '0' part of the conditional case, suppose $0 \in \mathfrak{C}^a_\Pi(\phi \to \psi)$. Then for some b and c with $\mathcal{R}bca$, $1 \in \mathfrak{C}^b_\Pi(\phi)$ and $0 \in \mathfrak{C}^c_\Pi(\psi)$. By the inductive hypothesis, then, $\phi \in b$ and $\neg\psi \in c$. Since $\vdash_\Pi \phi \to ((\phi \to \psi) \to \psi)$ and b is a Π-theory, $(\phi \to \psi) \to \psi \in b$. By Fact 4, $\vdash_\Pi ((\phi \to \psi) \to \psi) \to (\neg\psi \to \neg(\phi \to \psi))$, so $\neg\psi \to \neg(\phi \to \psi) \in b$ too. Thus, since $\mathcal{R}bca$ and $\neg\psi \in c$, we get that $\neg(\phi \to \psi) \in a$.

On the other hand, suppose $\neg(\phi \to \psi) \in a$. Let $b = \{\beta :\vdash_\Pi \phi \to \beta\}$ and let $c = \{\gamma :\vdash_\Pi \neg\psi \to \gamma\}$. Clearly $\phi \in b$ and $\neg\psi \in c$. For familiar reasons (see, e.g. the proof of Lemma 8), both of these are Π-theories. It is also true that $\mathcal{R}bca$. To see this, let $\beta_1 \to \beta_2 \in b$ and let $\beta_1 \in c$. Then by definition,

(a) $\vdash_\Pi \phi \to (\beta_1 \to \beta_2)$ and
(b) $\vdash_\Pi \neg\psi \to \beta_1$.

From (a), an instance of Fact 8 gives

(c) $\vdash_\Pi \phi \to (\neg\beta_2 \to \neg\beta_1)$.

From (b), an instance of Fact 9 gives

(d) $\vdash_\Pi \neg\beta_1 \to \psi$.

(c) and (d) together with an instance of, first, Fact 10 and, second, Fact 3 then give

(e) $\vdash_\Pi \neg\beta_1 \to (\phi \to \psi)$

Thus, by Fact 10 again, $\vdash_\Pi \neg(\phi \to \psi) \to \beta_1$. So since a is a Π-theory and $\neg(\phi \to \psi) \in a$, $\beta_1 \in a$ as required.

Since a is a *prime* Π-theory, Lemmas 6 and 7 then allow us to apply the inductive hypothesis and finish the proof. □

Theorem 2 *If α is* **RWZ***-valid, then $\alpha \in$* **RWZ**.

Proof Suppose $\alpha \notin$ **RWZ**. It follows from this that **RWZ** $\nvdash_\emptyset \alpha$. But then by Lemma 5, there is a prime $\emptyset$-theory $\Pi \supseteq$ **RWZ** such that $\alpha \notin \Pi$. Since $\Pi \supseteq$ **RWZ**, Π is normal. By Lemma 10, since $\alpha \notin \Pi$, it follows that $1 \notin \mathfrak{C}^\Pi_\Pi(\alpha)$. So α is not true in $\mathfrak{C}_\Pi$ and thus not **RWZ**-valid. □

Appendix C: Soundness For RWQ

Let $M = \langle D, \Omega, \delta, \mathcal{M}, \Downarrow\rangle$ be a varying-domain **RWQ**-model and $\phi \in$ LQ. Each of the first three lemmas we need can be proved with straightforward inductions which we leave to the reader.

Lemma 11 (Stability) *If $d \in D_X$ and ν does not occur freely in ϕ, then $1 \in M_X^a(\mathtt{va}, \phi)$ iff $1 \in M_X^a(\mathtt{va}_d^\nu, \phi)$ and $0 \in M_X^a(\mathtt{va}, \phi)$ iff $0 \in M_X^a(\mathtt{va}_d^\nu, \phi)$.*

Lemma 12 (Symmetry) *If $a \in S_X$ is symmetric in m and n, and $\mathtt{va}$ is X-coherent then $1 \in M_X^a(\mathtt{va}_m^\nu, \phi)$ iff $1 \in M_X^a(\mathtt{va}_n^\nu, \phi)$ and $0 \in M_X^a(\mathtt{va}_m^\nu, \phi)$ iff $0 \in M_X^a(\mathtt{va}_n^\nu, \phi)$.*

Lemma 13 (Evaluation) *If $a \in S_X$, $\mathtt{va}$ is X-coherent, and $d = \varepsilon_X^{\mathtt{va}}(\tau)$, then $1 \in M_X^a(\mathtt{va}_d^\nu, \phi)$ iff $1 \in M_X^a(\mathtt{va}, \phi(\tau/\nu))$ and $0 \in M_X^a(\mathtt{va}_d^\nu, \phi)$ iff $0 \in M_X^a(\mathtt{va}, \phi(\tau/\nu))$.*

Lemma 14 (Horizontal Heredity) *If a and b are in S_X, $\mathtt{va}$ is X-coherent, and $a \leq b$ then if $1 \in M_X^a(\mathtt{va}, \phi)$ then $1 \in M_X^b(\mathtt{va}, \phi)$, and if $0 \in M_X^a(\mathtt{va}, \phi)$ then $0 \in M_X^b(\mathtt{va}, \phi)$.*

Proof Generally as in Lemma 1. The only novelty concerns the quantifiers. For these, note that $1 \in M_X^a(\mathtt{va}, \forall\nu\phi)$ iff for some $Y \supsetneq X$ and $i \in Y - X$, for all $c \in S_Y$ if $c{\downarrow}_X^Y = a$ then $1 \in M_Y^c(\mathtt{va}_{\omega_i}^\nu, \phi)$. Since $a \leq b$, $Rnab$ for some $n \in N_X$. Suppose $d{\downarrow}_X^Y = b$. Then by the Extension Condition, there are e and f in S_Y such that $Refd$, $e{\downarrow}_X^Y = n$ and $f{\downarrow}_X^Y = a$. By the Normality Condition, $e{\downarrow}_X^Y = n$ gives that $e \in N_Y$. So $f \leq d$. Also, since $f{\downarrow}_X^Y = a$, it follows that $1 \in M_Y^f(\mathtt{va}_{\omega_i}^\nu, \phi)$. It then follows by the inductive hypothesis that $1 \in M_Y^d(\mathtt{va}_{\omega_i}^\nu, \phi)$. Since d was an arbitrary setup in S_Y such that $d{\downarrow}_X^Y = b$, it follows that $1 \in M_X^b(\mathtt{va}, \forall\phi)$. The zero case is essentially the same. □

Lemma 15 (Vertical Heredity) *If $\mathtt{va}$ is X-coherent and $b{\downarrow}_X^Y = a$, then $1 \in M_Y^b(\mathtt{va}, \phi)$ iff $1 \in M_X^a(\mathtt{va}, \phi)$ and $0 \in M_Y^b(\mathtt{va}, \phi)$ iff $0 \in M_X^a(\mathtt{va}, \phi)$.*

Proof By induction on the complexity of ϕ. The base case and the cases involving the extensional connectives are straightforward and omitted. The conditional case is simply a matter of keeping track of all the details, so is also omitted. This leaves the quantifier cases.

To begin, suppose $1 \in M_Y^b(\mathtt{va}, \forall\nu\phi)$. Then for some $Z \supsetneq Y$ and $i \in Z - Y$, for all $c \in S_Z$, if $c{\downarrow}_Y^Z = b$, then $1 \in M_Y^c(\mathtt{va}_{\omega_i}^\nu, \phi)$. Following the proof of Lemma 4 in [13], let $W = X \cup (Z - Y)$.[19] Notice that $i \in W$, $W \cap Y = X$, $W \cup Y = Z$, and that since $\mathtt{va}$ is X-coherent, $\mathtt{va}_{\omega_i}^\nu$ is W-coherent and Z-coherent.

[19]There is a diagram there that might help the reader understand what's going on here. Frankly, I was mystified by it.

Let $d \in S_W$ be such that $d{\downarrow}^W_X = a$. Then notice $d{\downarrow}^W_{W\cap Y} = d{\downarrow}^W_X = a = b{\downarrow}^Y_X = b{\downarrow}^Y_{W\cap Y}$. Thus, by the Lifting condition, there is an $e \in S_{W\cup Y} = S_Z$ such that $e{\downarrow}^Z_W \leq d$ and $b \leq e{\downarrow}^Z_Y$. Since $b \leq e{\downarrow}^Z_Y$, for some $n \in N_Y$, $Rnbe{\downarrow}^Z_Y$. By the extension and normality conditions, then, for some $m \in N_Z$ and $f \in S_Z$, $Rmfe$, $m{\downarrow}^Z_Y = n$, and $f{\downarrow}^Z_Y = b$. Since $f{\downarrow}^Z_Y = b$, $1 \in M^f_Z(\mathtt{va}^\nu_{\omega_i}, \phi)$. So by Horizontal Heredity, $1 \in M^e_Z(\mathtt{va}^\nu_{\omega_i}, \phi)$. The inductive hypothesis then gives that $1 \in M^{e\downarrow^Z_W}_W(\mathtt{va}^\nu_{\omega_i}, \phi)$. Since $e{\downarrow}^Z_W \leq d$, Horizontal Heredity then gives that $1 \in M^d_W(\mathtt{va}^\nu_{\omega_i}, \phi)$. But d was an arbitrary element of S_w such that $d{\downarrow}^W_X = a$, so all such elements have this feature. So $1 \in M^a_X(\mathtt{va}, \forall\nu\phi)$.

Now suppose $1 \in M^a_X(\mathtt{va}, \forall\nu\phi)$. Then for some $Z \supsetneq X$ and $i \in Z - X$, for all $c \in S_Z$, if $c{\downarrow}^Z_X = a$, then $1 \in M^c_Z(\mathtt{va}^\nu_{\omega_i}, \phi)$. Let $W = Y \cup Z$ and let $d \in S_W$ be such that $d{\downarrow}^W_Y = b$. Notice that $d{\downarrow}^W_Z{\downarrow}^Z_X = d{\downarrow}^W_X = d{\downarrow}^W_Y{\downarrow}^Y_X = b{\downarrow}^Y_X = a$. Thus since for all $c \in S_Z$ if $c{\downarrow}^Z_X = a$, then $1 \in M^c_Z(\mathtt{va}^\nu_{\omega_i}, \phi)$, it follows that $1 \in M^{d\downarrow^W_Z}_Z(\mathtt{va}^\nu_{\omega_i}, \phi)$. But then by the inductive hypothesis, $1 \in M^d_W(\mathtt{va}^\nu_{\omega_i}, \phi)$, so $1 \in M^b_Y(\mathtt{va}, \forall\nu\phi)$.

The zero case is analogous, replacing each instance of '$1 \in$' with '$0 \notin$'. □

Theorem 3 *If* $\alpha \in$ **RWQ**, *then* α *is* **RWQ**-*valid.*

Proof As expected, we show that every axiom is valid and that the rules preserve validity. We will only cover **QA1** and **QA2**, leaving the remaining axioms and all the rules to the reader.

QA1: $M \models \forall\nu\phi \rightarrow \phi(\tau/\nu)$ iff if $\mathtt{va}$ is X-coherent and $n \in N_X$, then $1 \in M^n_X(\mathtt{va}, \forall\nu\phi \rightarrow \phi(\tau/\nu))$, iff for all a and b, if $Rnab$ then (i) if $1 \in M^a_X(\mathtt{va}, \forall\nu\phi)$ then $1 \in M^b_X(\mathtt{va}, \phi(\tau/\nu))$, and (ii) if $0 \in M^a_X(\mathtt{va}, \phi(\tau/\nu))$ then $0 \in M^b_X(\mathtt{va}, \forall\nu\phi)$.

For (i), suppose $1 \in M^a_X(\mathtt{va}, \forall\nu\phi)$. Then for some $Y \supsetneq X$ and $y \in Y - X$, for all $c \in S_Y$, if $c{\downarrow}^Y_X = a$, then $1 \in M^c_Y(\mathtt{va}^\nu_{\omega_y}, \phi)$. Since $\mathtt{va}$ is X-coherent, $\varepsilon^{\mathtt{va}}_X(\tau) \in D_X$ and $\omega_y \in D_Y - D_X$, the symmetry condition guarantees that there is a $c \in S_Y$ that is symmetric in ω_y and $\varepsilon^{\mathtt{va}}_X(\tau)$ such that $c{\downarrow}^Y_X = a$. It follows that $1 \in M^c_Y(\mathtt{va}^\nu_{\omega_y}, \phi)$. Since c is symmetric in ω_y and $\varepsilon^{\mathtt{va}}_X(\tau)$, the Symmetry Lemma gives that $1 \in M^c_Y(\mathtt{va}^\nu_{\varepsilon^{\mathtt{va}}_X(\tau)}, \phi)$. So by the Evaluation Lemma, $1 \in M^c_Y(\mathtt{va}, \phi(\tau/\nu))$. Thus, since $c{\downarrow}^Y_X = a$, the Heredity Lemma gives that $1 \in M^a_X(\mathtt{va}, \phi(\tau/\nu))$. Since $a \leq b$, Horizontal Heredity then gives $1 \in M^b_X(\mathtt{va}, \phi(\tau/\nu))$.

For (ii), suppose $0 \in M^a_X(\mathtt{va}, \phi(\tau/\nu))$. Let $Y \supsetneq X$ and $y \in Y - X$. By the symmetry condition, there is a $c \in S_Y$ that is symmetric in ω_y and $\varepsilon^{\mathtt{va}}_X(\tau)$ such that $c{\downarrow}^Y_X = a$. By the Heredity Lemma, $0 \in M^c_Y(\mathtt{va}, \phi(\tau/\nu))$. Thus, by the Evaluation Lemma, $0 \in M^c_Y(\mathtt{va}^\nu_{\varepsilon^{\mathtt{va}}_X(\tau)}, \phi)$. So, by the Symmetry Lemma, $0 \in M^c_Y(\mathtt{va}^\nu_{\omega_y}, \phi)$. Since Y and y were arbitrary, this suffices to demonstrate that $0 \in M^a_X(\mathtt{va}, \forall\nu\phi)$. So $0 \in M^b_X(\mathtt{va}, \forall\nu\phi)$ follows by Horizontal Heredity.

QA2: $M \models \forall\nu(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \forall\nu\psi)$ iff if $\mathtt{va}$ is X-coherent and $n \in N_X$, then $1 \in M^n_X[\mathtt{va}, \forall\nu(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \forall\nu\psi)]$. This happens iff for all a and b, if

$Rnab$ then (i) if $1 \in M_X^a[\mathtt{va}, \forall\nu(\phi \to \psi)]$, then $1 \in M_X^b(\mathtt{va}, \phi \to \forall\nu\psi)$, and (ii) if $0 \in M_X^a[\mathtt{va}, \phi \to \forall\nu\psi]$, then $0 \in M_X^b[\mathtt{va}, \forall\nu(\phi \to \psi)]$

For (i), suppose $1 \in M_X^a[\mathtt{va}, \forall\nu(\phi \to \psi)]$. Then by Horizontal Heredity, $1 \in M_X^b(\mathtt{va}, \forall\nu(\phi \to \psi))$. We need to show that $1 \in M_X^b(\mathtt{va}, \phi \to \forall\nu\psi)$. To do so we must show that if $Rbcd$, then (a) if $1 \in M_X^c(\mathtt{va}, \phi)$, then $1 \in M_X^d(\mathtt{va}, \forall\nu\psi)$, and (b) if $0 \in M_X^c(\mathtt{va}, \forall\nu\psi)$, then $0 \in M_X^c(\mathtt{va}, \phi)$. In either case, we also have that $1 \in M_X^b(\mathtt{va}, \forall\nu(\phi \to \psi))$. Thus for some $Y \supsetneq X$ and $y \in Y - X$, for all $e \in S_Y$, if $e{\downarrow}_X^Y = b$, then $1 \in M_Y^e(\mathtt{va}_{\omega_y}^\nu, \phi \to \psi)$.

For (a), suppose $Rbcd$ and $1 \in M_X^c(\mathtt{va}, \phi)$. Let $f \in S_Y$ be such that $f{\downarrow}_X^Y = d$. By the extension condition, there are g and h in S_Y such that $Rghf$, $g{\downarrow}_X^Y = b$, and $h{\downarrow}_X^Y = c$. Since $g{\downarrow}_X^Y = b$, $1 \in M_Y^g(\mathtt{va}_{\omega_y}^\nu, \phi \to \psi)$. Since $h{\downarrow}_X^Y = c$ and $1 \in M_X^c(\mathtt{va}, \phi)$, Vertical Heredity gives that $1 \in M_Y^h(\mathtt{va}, \phi)$. Since ν does not occur freely in ϕ, the Stability Lemma then gives that $1 \in M_Y^h(\mathtt{va}_{\omega_y}^\nu, \phi)$. So $1 \in M_Y^f(\mathtt{va}_{\omega_y}^\nu, \psi)$. Since f was an arbitrary element of S_Y such that $f{\downarrow}_X^Y = d$, it follows from this that $1 \in M_X^d(\mathtt{va}, \forall\nu\psi)$.

For (b), suppose $Rbcd$ and $0 \in M_X^c(\mathtt{va}, \forall\nu\psi)$. It follows that for some $f \in S_Y$, $f{\downarrow}_X^Y = c$ and $0 \in M_Y^f(\mathtt{va}_{\omega_y}^\nu, \psi)$. Since $f{\downarrow}_X^Y = c$, the extension condition gives that for some g and h in S_Y, $Rgfh$, $g{\downarrow}_X^Y = b$ and $h{\downarrow}_X^Y = d$. Since $g{\downarrow}_X^Y = b$, $1 \in M_Y^g(\mathtt{va}_{\omega_y}^\nu, \phi \to \psi)$. Then, since $0 \in M_Y^f(\mathtt{va}_{\omega_y}^\nu, \psi)$, $0 \in M_Y^h(\mathtt{va}_{\omega_y}^\nu, \phi)$. Since ν does not occur free in ϕ, the Stability Lemma then gives that $0 \in M_Y^h(\mathtt{va}, \phi)$. Thus, by the Heredity Lemma, $0 \in M_X^c(\mathtt{va}, \phi)$.

For (ii) Suppose $0 \in M_X^a[\mathtt{va}, \phi \to \forall\nu\psi]$. Then for some c and d, $Rcda$, $1 \in M_X^c(\mathtt{va}, \phi)$ and $0 \in M_X^d(\mathtt{va}, \forall\nu\psi)$. Let $Y \supsetneq X$ be and $y \in Y - X$. Since $0 \in M_X^d(\mathtt{va}, \forall\nu\psi)$, there is an $e \in S_Y$ with $e{\downarrow}_X^Y = d$ and $0 \in M_Y^e(\mathtt{va}_{\omega_y}^\nu, \psi)$. Since $e{\downarrow}_X^Y = d$ and $Rcda$, there are f and g in S_Y such that $f{\downarrow}_X^Y = c$, $g{\downarrow}_X^Y = a$, and $Rfeg$.

By the Heredity Lemma, $1 \in M_Y^f(\mathtt{va}, \phi)$. Since ν does not occur freely in ϕ and $\mathtt{va}_{\omega_y}^\nu$ is Y-coherent, it follows that $1 \in M_Y^f(\mathtt{va}_{\omega_y}^\nu, \phi)$. Thus, since $Rfeg$ and $0 \in M_Y^e(\mathtt{va}_{\omega_y}^\nu, \psi)$, $0 \in M_Y^g(\mathtt{va}_{\omega_y}^\nu, \phi \to \psi)$. Since Y and y were arbitrary, it follows that $0 \in M_Y^a[\mathtt{va}, \forall\nu(\phi \to \psi)]$. Thus, by Horizontal Heredity, $0 \in M_Y^b[\mathtt{va}, \forall\nu(\phi \to \psi)]$.

□

Appendix D: Completeness For RWQ

In this appendix we construct our canonical model in a way that differs mildly from the construction found in Appendix B. There are two reasons for doing this: first, it makes the proof of Symmetry in Lemma 27 a bit easier. But second, and more importantly, it's nice to have a demonstration of the different methods available.

To begin we settle some notation. Let $\{v_1, v_2, \dots\}$ be the set of variables in LQ. For $\phi \in$ LQ, let $fv(\phi)$ be the set of variables that occur freely in ϕ. For each set of numbers V, let $\mathrm{LQ}_V = \{\phi \in \mathrm{LQ} : fv(\phi) \subseteq \{v_i\}_{i \in V}\}$. Let $\Sigma \cup \{\phi\} \subseteq$ LQ.

D.1 Definitions

- Write Σ_V for $\Sigma \cap \mathrm{LQ}_V$.
- Write $\Sigma \Vdash \phi$ when there is a sequence $\phi_1, \phi_2, \dots, \phi_n$ such that $\phi_n = \phi$ and for all $1 \leq i \leq n$, either $\phi_i \in \Sigma$; or ϕ_i follows by one of these two rules:

  DR1 For some $j < i$ and $k < i$, $\phi_i = \phi_j \wedge \phi_k$;
  DR2 For some $j < i$, $\phi_j \to \phi_i \in$ **RWQ**.

  We say that any such sequence *witnesses* $\Sigma \Vdash \phi$.[20]
- Write $c(\Sigma)$ for $\{\phi : \Sigma \Vdash \phi\}$. Write $c_V(\Sigma)$ for $(c(\Sigma))_V$.
- Say Σ is a *theory* when $\Sigma \Vdash \phi$ only if $\phi \in \Sigma$.
- Say Σ is a V-theory when $\Sigma \subseteq \mathrm{LQ}_V$ and for all $\phi \in \mathrm{LQ}_V$, $\Sigma \Vdash \phi$ only if $\phi \in \Sigma$.
- Say a V-theory Σ is *normal* when $\mathbf{RWQ}_V \subseteq \Sigma$.
- Say Σ is prime when $\alpha \vee \beta \in \Sigma$ only if $\alpha \in \Sigma$ or $\beta \in \Sigma$.

We adopt the usual abbreviations. For example, we write $\alpha \Vdash \beta$ instead of $\{\alpha\} \Vdash \beta$ and $\Delta, \alpha \Vdash \beta$ instead of $\Delta \cup \{\alpha\} \Vdash \beta$.

D.2 Basic Lemmas

Lemma 16 (Deduction Lemma) *If $\alpha \Vdash \beta$, then $\alpha \to \beta \in$* **RWQ**.

Proof By induction on the length of the shortest witness of $\alpha \Vdash \beta$. If there is a witness of length 1, then β is α. Since $\alpha \to \alpha \in$ **RWQ**, the base case is established. Now suppose $\phi_1, \dots, \beta = \phi_{n+1}$ witnesses $\alpha \Vdash \beta$. There are two cases to consider.

Case 1: There are $j < n+1$ and $k < n+1$ such that $\beta = \phi_j \wedge \phi_k$. Then $\phi_1, \dots, \phi_j$ witnesses $\alpha \Vdash \phi_j$ and $\phi_1, \dots, \phi_k$ witnesses $\alpha \Vdash \phi_k$. By the inductive hypothesis, $\alpha \to \phi_j \in$ **RWQ** and $\alpha \to \phi_k \in$ **RWQ**. From here a short derivation gives that $\alpha \to (\phi_j \wedge \phi_k) = \alpha \to \beta \in$ **RWQ** as required.

Case 2: There is a $j < n+1$ such that $\phi_j \to \beta \in$ **RWQ**. Then $\phi_1, \dots, \phi_j$ witnesses $\alpha \Vdash \phi_j$. By the inductive hypothesis, $\alpha \to \phi_j \in$ **RWQ**. From here a short derivation gives $\alpha \to \beta \in$ **RWQ** as well.

□

Lemma 17 *If $\Sigma \Vdash \phi(\nu)$, ν not free in Σ but free for x in $\phi(\nu)$, then $\Sigma \Vdash \forall x \phi(x)$.*

Proof If $\Sigma \Vdash \phi(\nu)$, then for some $\sigma_1, \dots, \sigma_n$ all in Σ, $\sigma_1 \wedge \cdots \wedge \sigma_n \Vdash \phi(\nu)$. So by the Deduction Lemma, $(\sigma_1 \wedge \cdots \wedge \sigma_n) \to \phi(\nu) \in$ **RWQ**. So by **RQ1**, $\forall \nu[(\sigma_1 \wedge \cdots \wedge$

[20] $\Vdash$ is essentially what Fine in [11] and [13] and Mares in [18] would write $\vdash_{\mathbf{RWQ}}$.

$\sigma_n) \to \phi(\nu)] \in$ **RWQ**. Since ν is not free in Σ, $\forall\nu[(\sigma_1 \wedge \cdots \wedge \sigma_n) \to \phi(\nu)] \to [(\sigma_1 \wedge \cdots \wedge \sigma_n) \to \forall\nu\phi(\nu)] \in$ **RWQ**. Thus $(\sigma_1 \wedge \cdots \wedge \sigma_n) \to \forall\nu\phi(\nu) \in$ **RWQ**, and hence $\Sigma \Vdash \forall\nu\phi(\nu)$. But $\forall\nu\phi(\nu) \to \forall x\phi(x) \in$ **RWQ**, so $\Sigma \Vdash \forall x\phi(x)$ as required. □

Lemma 18 (Lindenbaum) *Suppose* Δ *is closed under disjunction,* Γ *is a theory, and* $\Gamma \cap \Delta = \emptyset$*. Then there is a prime theory* $\Gamma' \supset \Gamma$ *such that* $\Gamma' \cap \Delta = \emptyset$*.*

Proof To begin, choose an enumeration of all the formulas of the form $\alpha \vee \beta$. Let $\alpha_i \vee \beta_i$ be the ith member of this list. Define Γ^i_j as follows:

- $\Gamma^0_0 = \Gamma$.
- $\Gamma^i_{j+1} = \begin{cases} \Gamma^i_j \text{ if } \alpha_j \vee \beta_j \notin c(\Gamma^i_j) \\ \Gamma^i_j \cup \{\alpha_j\} \text{ if } \alpha_j \vee \beta_j \in c(\Gamma^i_j) \text{ and } c(\Gamma^i_j \cup \{\alpha_j\}) \cap \Delta = \emptyset \\ \Gamma^i_j \cup \{\beta_j\} \text{ otherwise} \end{cases}$
- $\Gamma^{i+1}_0 = \cup^\infty_{j=0}\Gamma^i_j$

Let $\Gamma' = \cup^\infty_{i=0}\Gamma^i_0$. Clearly $\Gamma = \Gamma^0_0 \subseteq \Gamma'$. It remains to show that Γ' is prime, is a theory, and that $\Gamma' \cap \Delta = \emptyset$.

To see Γ' is prime, let $\alpha \vee \beta \in \Gamma$. Then for some i, $\Gamma^i_0 \Vdash \alpha_j \vee \beta_j$. But then by the definition of Γ^i_{j+1}, either $\alpha_j \in \Gamma^i_{j+1}$ or $\beta_j \in \Gamma^i_{j+1}$. Either way, $\alpha_j \in \Gamma'$ or $\beta_j \in \Gamma'$.

To see that Γ' is a theory, notice that if $\Gamma' \Vdash \alpha$, then $\Gamma' \Vdash \alpha \vee \alpha$. Thus for some i, $\Gamma^i_0 \Vdash \alpha \vee \alpha$. By construction, we then see that $\alpha \in \Gamma^i \subseteq \Gamma'$.

To see that $\Gamma' \cap \Delta = \emptyset$, suppose otherwise, and let $i_0 = \inf\{i : c(\Gamma^i_j) \cap \Delta \neq \emptyset \text{ for some } j\}$ and let $j_0 = \inf\{j : c(\Gamma^{i_0}_j) \cap \Delta \neq \emptyset\}$. Clearly either $i_0 = 0$ or $i_0 \neq 0$. In the former case, since $\Gamma^0_0 = \Gamma$ is a theory and $\Gamma \cap \Delta = \emptyset$, clearly $j_0 \neq 0$. In the latter case, since $\Gamma^{i+1}_0 = \cup^\infty_{j=0}\Gamma^i_j$, if $c(\Gamma^{i_0}_0) \cap \Delta \neq \emptyset$, then since proofs are finite, for some j, $c(\Gamma^{i_0-1}_j) \cap \Delta \neq \emptyset$, contradicting the minimality of i_0. So again we can conclude that $j_0 \neq 0$. Thus, in either case, $j_0 \neq 0$. So $\Gamma^{i_0}_{j_0-1}$ is defined. Also if $\alpha_{j_0-1} \vee \beta_{j_0-1} \notin c(\Gamma^{i_0}_{j_0-1})$, then $\Gamma^{i_0}_{j_0} = \Gamma^{i_0}_{j_0-1}$. But then $c(\Gamma^{i_0}_{j_0-1}) \cap \Delta \neq \emptyset$, contradicting j_0's minimality. So $\alpha_{j_0-1} \vee \beta_{j_0-1} \in c(\Gamma^{i_0}_{j_0-1})$.

Now suppose $c(\Gamma^{i_0}_{j_0-1} \cup \{\alpha_{j_0-1}\}) \cap \Delta = \emptyset$. Then $\Gamma^{i_0}_{j_0} = \Gamma^{i_0}_{j_0-1} \cup \{\alpha_{j_0-1}\}$. But then $c(\Gamma^{i_0}_{j_0}) \cap \Delta = \emptyset$, contradicting our assumption. Thus some sentence $\alpha \in c(\Gamma^{i_0}_{j_0-1} \cup \{\alpha_{j_0-1}\}) \cap \Delta$ It follows that $\Gamma^{i_0}_{j_0} = \Gamma^{i_0}_{j_0-1} \cup \{\beta_{j_0-1}\}$. Since $c(\Gamma^{i_0}_{j_0-1}) \cap \Delta = \emptyset$ (by j_0's minimality), but $c(\Gamma^{i_0}_{j_0}) \cap \Delta \neq \emptyset$, it follows that for some $\beta \in \Delta$, $\Gamma^{i_0}_{j_0-1}, \beta_{j_0-1} \Vdash \beta$.

By short derivations, we can establish that $\alpha \to (\alpha \vee \beta)$ and $\beta \to (\alpha \vee \beta)$ are in **RWQ**. Thus, since $\Gamma^{i_0}_{j_0-1}, \alpha_{j_0-1} \Vdash \alpha$ and $\Gamma^{i_0}_{j_0-1}, \beta_{j_0-1} \Vdash \beta$, it is also the case that $\Gamma^{i_0}_{j_0-1}, \alpha_{j_0-1} \Vdash \alpha \vee \beta$ and $\Gamma^{i_0}_{j_0-1}, \beta_{j_0-1} \Vdash \alpha \vee \beta$. By a series of tedious manipulations, one sees that it follows from these that $\Gamma^{i_0}_{j_0-1}, \alpha_{j_0-1} \vee \beta_{j_0-1} \Vdash \alpha \vee \beta$.

But $\Gamma^{i_0}_{j_0-1} \Vdash \alpha_{j_0-1} \vee \beta_{j_0-1}$ by assumption. So in fact $\Gamma^{i_0}_{j_0-1} \Vdash \alpha \vee \beta$. But $\alpha \vee \beta \in \Delta$ because Δ is closed under disjunctions. This contradicts the minimality of i_0 and j_0, completing the proof. □

The following lemma is important corollary of the Lindenbaum Lemma:

Lemma 19 *If* Γ *is a theory and* $\{\Gamma_i\}_{i\in I}$ *is the set of all prime theories that contain* Γ*, then* $\Gamma = \bigcap_{i\in I}\Gamma_i$

Proof Clearly $\Gamma \subseteq \bigcap_{i\in I}\Gamma_i$. Now suppose $\delta \notin \Gamma$. Let Δ be the closure of $\{\delta\}$ under disjunction. Since Γ is a theory and $\delta \notin \Gamma$, Γ does not intersect Δ. So by the Lindenbaum Lemma, there is a prime theory containing Γ that does not intersect Δ. So $\delta \notin \bigcap_{i\in I}\Gamma_i$. □

D.3 A Preview

The plan is for the canonical model to be the quintuple $\langle D, \Omega, \delta, \mathcal{M}, \Downarrow\rangle$ such that

- D is the set of names.
- Ω is the set of variables.
- δ is the identity function.
- $\mathcal{M} : X \mapsto M_X = \langle D_X, S_X, N_X, R_X, \delta, \mathcal{E}^+_X, \mathcal{E}^-_X\rangle$ with S_X the set of all prime X-theories; N_X the set of all normal prime X-theories; $R_X\Sigma\Gamma\Delta$ iff $\beta \in \Delta$ whenever $\alpha \to \beta \in \Sigma$ and $\alpha \in \Gamma$; and $\mathcal{E}^+_X$ and $\mathcal{E}^-_X$ exactly as in Appendix B.
- $a{\downarrow}^X_Y = a_Y = a \cap \mathrm{LQ}_Y$.

For this to do any work for us, we need to locate $\mathbf{RWQ}_X$ among the N_X. We will do this by showing **RWQ** to be prime. To show this, one standardly uses what are called *metavaluations*. This technique was pioneered in [21] and extended in [32]. A recent survey of applications of this technique can be found in [7].

D.4 Metavaluations

We take a metavaluation to be a pair of functions M and M^* mapping LQ to $\{\mathsf{T}, \mathsf{F}\}$. The intuitive picture to have is that $M(\phi) = \mathsf{T}$ just if $\phi \in \mathbf{RWQ}$ and $M^*(\phi) = \mathsf{T}$ just if $\neg\phi \notin \mathbf{RWQ}$. Explicitly, we require the functions to obey the following conditions:

MV1: If ϕ is atomic, then $M(\phi) = \mathsf{F}$ and $M^*(\phi) = \mathsf{T}$
MV2: $M(\phi \wedge \psi) = \mathsf{T}$ iff $M(\phi) = \mathsf{T}$ and $M(\psi) = \mathsf{T}$.
MV3: $M^*(\phi \wedge \psi) = \mathsf{T}$ iff $M^*(\phi) = \mathsf{T}$ and $M^*(\psi) = \mathsf{T}$.
MV4: $M(\neg\phi) = \mathsf{T}$ iff $\neg\phi \in \mathbf{RWQ}$ and $M^*(\phi) = \mathsf{F}$.
MV5: $M^*(\neg\phi) = \mathsf{T}$ iff $M(\phi) = \mathsf{F}$.
MV6: $M(\phi \to \psi) = \mathsf{T}$ iff (i) $\phi \to \psi \in \mathbf{RWQ}$, (ii) $M(\phi) = \mathsf{T}$ materially implies $M(\psi) = \mathsf{T}$, and (iii) $M^*(\phi) = \mathsf{T}$ materially implies $M^*(\psi) = \mathsf{T}$.
MV7: $M^*(\phi \to \psi) = \mathsf{T}$ iff $M(\phi) = \mathsf{T}$ materially implies $M^*(\psi) = \mathsf{T}$.
MV8: $M(\forall x\phi(x)) = \mathsf{T}$ iff $M(\phi(\tau)) = \mathsf{T}$ for all terms τ that are free for x in $\phi(x)$.

MV9: $M^*(\forall x\phi(x)) = \mathsf{T}$ iff $M^*(\phi(\tau)) = \mathsf{T}$ for all terms τ that are free for x in $\phi(x)$.

We now prove two lemmas that show these conditions force M and M^* to agree with our intuitive picture. For all of these, it helps to first notice that if $\vdash_\emptyset \phi$, then by the definition of **RWQ**, $\phi \in$ **RWQ**, so each of Facts 1-12 has an analogue in terms of elements of **RWQ**. In what follows we will often cite the fact when, technically speaking, we should be citing its analogue in this sense.

Lemma 20 $M(\phi) = \mathsf{T}$ *materially implies* $\phi \in$ **RWQ**.

Proof By induction on the complexity of ϕ. The base case holds vacuously; the cases for the connectives are straightforward. For the quantifier, suppose $M(\forall x\phi(x)) = \mathsf{T}$. Then $M(\phi(t)) = \mathsf{T}$ for all terms t that are free for x in ϕ. Letting y be a variable that is free for x in ϕ, $\phi(y)$ is in **RWQ**. But then by **QR1**, so is $\forall y\phi(y)$. Thus, since $\forall y\phi(y) \to \forall x\phi(x) \in$ **RWQ**, $\forall x\phi(x) \in$ **RWQ** as required. □

Lemma 21 $M^*(\phi) = \mathsf{F}$ *materially implies* $\neg\phi \in$ **RWQ**.

Proof By a slightly more subtle induction than in the previous lemma. The base case is still vacuously true. Now suppose $M^*(\phi \wedge \psi) = \mathsf{F}$. Then either $M^*(\phi) = \mathsf{F}$ or $M^*(\psi) = \mathsf{F}$. So, by the inductive hypothesis, either $\neg\phi \in$ **RWQ** or $\neg\psi \in$ **RWQ**. Thus, by either Fact 5 or Fact 6, $\neg(\phi \wedge \psi) \in$ **RWQ**.

Suppose $M^*(\neg\phi) = \mathsf{F}$. Then $M(\phi) = \mathsf{T}$. So by Lemma 20, $\phi \in$ **RWQ**. Thus, since $\phi \to \neg\neg\phi \in$ **RWQ** by Fact 1, $\neg\neg\phi \in$ **RWQ**.

Suppose $M^*(\phi \to \psi) = \mathsf{F}$. Then $M(\phi) = \mathsf{T}$ and $M^*(\psi) = \mathsf{F}$. By Lemma 20, from the first conjunct we can conclude that $\phi \in$ **RWQ**. Since $\phi \to ((\phi \to \psi) \to \psi)$ is an instance of **A9**, it's in **RWQ**. Thus so is $(\phi \to \psi) \to \psi$. By Fact 1 and Fact 3, it then follows that $(\phi \to \psi) \to \neg\neg\psi \in$ **RWQ**, whence by an instance of **A7**, so is $\neg\psi \to \neg(\phi \to \psi)$. But since $M^*(\psi) = \mathsf{F}$, by the inductive hypothesis, $\neg\psi \in$ **RWQ**, and thus $\neg(\phi \to \psi)$ is as well.

Finally, suppose $M^*(\forall x\phi(x)) = \mathsf{F}$. Then for some term t, $M^*(\phi(t)) = \mathsf{F}$. Thus by the inductive hypothesis, $\neg\phi(t) \in$ **RWQ**. But by **QA1**, $\forall x\phi(x) \to \phi(t) \in$ **RWQ**. So by a similar argument to the one in the previous case, $\neg\phi(t) \to \neg\forall x\phi(x) \in$ **RWQ** as well. Thus so is $\neg\forall x\phi(x)$. □

Before proving the next lemma, we need a tool:

Lemma 22 *Suppose* x *occurs freely in* $\phi(x)$*. Then (a) if* $M(\phi(x)) = \mathsf{T}$*, then* $M(\phi(\tau)) = \mathsf{T}$ *for all terms* τ *that are free for* x *in* $\phi(x)$ *and (b) if* $M^*(\phi(x)) = \mathsf{F}$*, then* $M^*(\phi(\tau)) = \mathsf{F}$ *for all terms* τ *that are free for* x *in* $\phi(x)$.

Proof By a simultaneous induction (on the complexity of ϕ) in both (a) and (b). The atomic cases are vacuously true and the conjunction case is straightforward. If ϕ has the form $\neg\psi(x)$, then if $M(\phi(x)) = \mathsf{T}$, then $\neg\psi(x) \in$ **RWQ** and $M^*(\psi(x)) = \mathsf{F}$. Since $\neg\psi(x) \in$ **RWQ**, by an instance of **QR1**, $\forall x\neg\psi(x) \in$ **RWQ**, whence by **QA1**, for all terms τ that are free for x in $\psi(x)$, $\neg\psi(\tau) \in$ **RWQ**. And since $M^*(\psi(x)) = \mathsf{F}$, the inductive hypothesis gives that for all such terms τ, $M^*(\psi(\tau)) = \mathsf{F}$. Together these show that for all terms τ that are free for x in $\psi(x)$, $M(\phi(\tau)) = \mathsf{T}$. Aside from the need to keep track of more details, the conditional and quantifier cases are essentially the same. □

Lemma 23 $\phi \in$ **RWQ** *materially implies* $M(\phi) = \mathsf{T}$.

Proof We will first show that if ϕ is an axiom of **RWQ**, then $M(\phi) = \mathsf{T}$. We only examine **A4** and **QA2**, leaving the remaining cases to the reader.

For this case, notice that $M(((\alpha \to \beta) \wedge (\alpha \to \gamma)) \to (\alpha \to (\beta \wedge \gamma))) = \mathsf{T}$ iff (a) $((\alpha \to \beta) \wedge (\alpha \to \gamma)) \to (\alpha \to (\beta \wedge \gamma)) \in$ **RWQ**; (b) $M((\alpha \to \beta) \wedge (\alpha \to \gamma)) = \mathsf{T}$ materially implies $M(\alpha \to (\beta \wedge \gamma)) = \mathsf{T}$; and (c) $M^*((\alpha \to \beta) \wedge (\alpha \to \gamma)) = \mathsf{T}$ materially implies $M^*(\alpha \to (\beta \wedge \gamma)) = \mathsf{T}$. (a) is obviously true. For (b), we reason as follows (using '$\sqsupset$' for our metalinguistic material conditional); (c) then follows by a similarly tedious 'metaderivation':

1	$M((\alpha \to \beta) \wedge (\alpha \to \gamma)) = \mathsf{T}$	Assumption
2	(a) $M(\alpha \to \beta) = \mathsf{T}$ and (b) $M(\alpha \to \gamma) = \mathsf{T}$	From 1 by **MV2**
3	(i) $\alpha \to \beta \in$ **RWQ**, (ii) $M(\alpha) = \mathsf{T} \sqsupset M(\beta) = \mathsf{T}$, and (iii) $M^*(\alpha) = \mathsf{T} \sqsupset M^*(\beta) = \mathsf{T}$	From 2(a) by **MV6**
4	(i) $\alpha \to \gamma \in$ **RWQ**, (ii) $M(\alpha) = \mathsf{T} \sqsupset M(\gamma) = \mathsf{T}$, and (iii) $M^*(\alpha) = \mathsf{T} \sqsupset M^*(\gamma) = \mathsf{T}$	From 2(b) by **MV6**
5	$(\alpha \to \beta) \wedge (\alpha \to \gamma) \in$ **RWQ**	**R2** applied to 3(i) and 4(i)
6	$\alpha \to (\beta \wedge \gamma) \in$ **RWQ**	**R1** applied to **A4**, 5
7	$M(\alpha) = \mathsf{T}$	Assumption
8	$M(\beta) = \mathsf{T}$ and $M(\gamma) = \mathsf{T}$	Via 3(ii) and 7; 4(ii) and 7
9	$M(\beta \wedge \gamma) = \mathsf{T}$	**MV2** applied to 8
10	$M(\alpha) = \mathsf{T} \sqsupset M(\beta \wedge \gamma) = \mathsf{T}$	7-9, discharging 7
11	$M^*(\alpha) = \mathsf{T} \sqsupset M^*(\beta \wedge \gamma) = \mathsf{T}$	7-10, mutatis mutandis
12	$M(\alpha \to (\beta \wedge \gamma)) = \mathsf{T}$	**MV6** applied to 6, 10, 11
13	$M((\alpha \to \beta) \wedge (\alpha \to \gamma)) = \mathsf{T} \sqsupset M(\alpha \to (\beta \wedge \gamma)) = \mathsf{T}$	1-12, discharging 1.

For **QA2**, suppose ν is not free in ϕ. $M(\forall\nu(\phi \to \psi(\nu)) \to (\phi \to \forall\nu\psi(\nu))) = \mathsf{T}$ iff (a) $\forall\nu(\phi \to \psi(\nu)) \to (\phi \to \forall\nu\psi(\nu)) \in$ **RWQ**; (b) $M(\forall\nu(\phi \to \psi(\nu))) = \mathsf{T}$ materially implies $M(\phi \to \forall\nu\psi(\nu)) = \mathsf{T}$; and (c) $M^*(\forall\nu(\phi \to \psi(\nu))) = \mathsf{T}$ materially implies $M^*(\phi \to \forall\nu\psi(\nu)) = \mathsf{T}$. (a) is obviously true. For (b), we reason as follows; (c) then follows by a similarly tedious 'metaderivation':

1	$M(\forall\nu(\phi \to \psi(\nu))) = \mathsf{T}$	Assumption
2	$M(\phi \to \psi(t)) = \mathsf{T}$	From 1 by **MV8**
3	(i) $\phi \to \psi(t) \in$ **RWQ**, (ii) $M(\phi) = \mathsf{T} \sqsupset M(\psi(t)) = \mathsf{T}$, and (iii) $M^*(\phi) = \mathsf{T} \sqsupset M^*(\psi(t)) = \mathsf{T}$	From 2 by **MV6**
4	$M(\phi) = \mathsf{T}$	Assumption
5	$M(\psi(t)) = \mathsf{T}$	By 4 and 3(ii)
6	$M(\psi(\tau)) = \mathsf{T}$ for all appropriate τ	From 5
7	$M(\forall\nu\psi(\nu)) = \mathsf{T}$	From 6 by **MV8**
8	$M(\phi) = \mathsf{T} \sqsupset M(\forall\nu\psi(\nu)) = \mathsf{T}$	4-7, discharging 4.
9	$M^*(\phi) = \mathsf{T} \sqsupset M^*(\forall\nu\psi(\nu)) = \mathsf{T}$	4-8, mutatis mutandis
10	$M(\phi \to \forall\nu\phi(\nu)) = \mathsf{T}$	From 3(i), 8, and 9 by **MV6**

In lines 2, 3, and 5 we take t to be an arbitrary term that is free for the appropriate variable. We complete the proof by showing that each of the rules preserves T. This is immediately obvious for **R1** and **R2**. For **QR1**, notice that if ν occurs free in $\psi(\nu)$ and $M(\psi(\nu)) = \mathsf{T}$, then by Lemma 22, $M(\psi(\tau)) = \mathsf{T}$ for all terms τ. So by **MV8**, $M(\forall\nu\psi(\nu)) = \mathsf{T}$ as well. □

With our metavaluation in hand, the following result is nearly immediate:

Lemma 24 *For each X,* **RWQ**$_X$ *is prime.*

Proof We prove that **RWQ** itself is prime, since the result immediately follows from this. To that end, let $\alpha \vee \beta \in$ **RWQ**. Then by Lemma 23, $M(\alpha \vee \beta) = \mathsf{T}$. So $M(\neg(\neg\alpha \wedge \neg\beta)) = \mathsf{T}$. Thus $M^*(\neg\alpha \wedge \neg\beta) = \mathsf{F}$. So $M^*(\neg\alpha) = \mathsf{F}$ or $M^*(\neg\beta) = \mathsf{F}$. Thus by Lemma 21, $\neg\neg\alpha \in$ **RWQ** or $\neg\neg\beta \in$ **RWQ**. So by **A6**, either $\alpha \in$ **RWQ** or $\beta \in$ **RWQ**. □

D.5 Modelhood

With that out of the way, we can now prove that the canonical model is a premodel. This requires showing the models M_X satisfy the ordering, monotonicity, closure, rearranging, and heredity constraints. But these are proved exactly as in Appendix B; we note here the following two Lemmas:

Lemma 25 $a \leq b$ *iff* $a \subseteq b$

Lemma 26 *The canonical model is an* **RWQ**-*premodel*

Now we turn to modelhood proper.

Lemma 27 *The canonical model is an* **RWQ**-*model.*

We've already seen that the canonical model is an **RWQ**-premodel. It remains to show that it satisfies the vertical atomic heredity, normality, lifting, homomorphism, extension, and symmetry conditions. Atomic heredity and homomorphism are straightforward; for the remainder we argue as follows:

Normality: It's clear that if $a \in N_X$, then $a{\downarrow}^X_Y \in N_Y$. Now suppose $a \notin N_X$. Let $\phi \in \mathbf{RWQ}_X - a$. Let $x_1, \dots, x_n$ be the variables that occur in ϕ that are not in $\{v_i\}_{i \in Y}$. Since $\phi \in \mathbf{RWQ}$, repeated application of **QR1** gives that $\forall x_1 \dots \forall x_n \phi \in \mathbf{RWQ}$ as well. Notice that $\forall x_1 \dots \forall x_n \phi \in \mathrm{LQ}_Y$. Thus $\phi \in \mathbf{RWQ}_Y$.

Now suppose $\forall x_1 \dots \forall x_n \phi \in a$. Then clearly by repeated application of **QA1**, $\phi \in a$ as well. But $\phi \notin a$. So $\forall x_1 \dots \forall x_n \phi \notin a$. So $\forall x_1 \dots \forall x_n \phi \notin a{\downarrow}^X_Y \subseteq a$. But then since $\forall x_1 \dots \forall x_n \phi \in \mathbf{RWQ}_Y$, $a{\downarrow}^X_Y \notin N_Y$.

Lifting: Let $\overline{a} = \mathrm{LQ}_X - a$ and let $c' = \{\phi \in \mathrm{LQ}_{X \cup Y} : b \Vdash \phi\}$. Clearly c' is a theory and $c' \supseteq b$. Suppose $\phi \in c' \cap \overline{a}$. Let $\{x_1, \dots, x_n\}$ be the free variables of ϕ that are not in Y. Then since $b \Vdash \phi$, $b \Vdash \forall x_1 \dots \forall x_n \phi$. But $\forall x_1 \dots \forall x_n \phi \in b \cap \mathrm{LQ}_{X \cap Y} = a \cap \mathrm{LQ}_{X \cap Y}$. So $a \Vdash \forall x_1 \dots \forall x_n \phi$, and thus by instantiation, $a \Vdash \phi$, which is impossible since $\phi \in \overline{a}$. It follows by the Lindenbaum Lemma that there is a $c \supseteq c'$ such that $c \cap \overline{a} = \emptyset$. Thus, $c \downarrow^{X \cup Y}_X \leq a$ and $b \leq c \downarrow^{X \cup Y}_Y$.

Extension: For each part, the proof is essentially a modification of the proof of one of the extension lemmas from Appendix B. We give only one of them, leaving the others to the reader.

Suppose $R_Y abc$ and $d{\downarrow}^X_Y = c$. Let $\overline{d} = \mathrm{LQ}_X - d$, $e' = \{\phi \in \mathrm{LQ}_X : a \Vdash \phi\}$ and let $\Theta = \left\{\alpha \in \mathrm{LQ}_X : \alpha \to \beta \in e' \text{ for some } \beta \in \overline{d}\right\}$. By familiar arguments, we see that Θ is closed under disjunction.

Suppose $\phi \in b \cap \Theta$. Then for some $\beta \in \overline{d}$, $a \Vdash \phi \to \beta$. Let $x_1, \dots, x_n$ be the variables that are free in $\phi \to \beta$ that are not in Y. Since a is a Y-theory and $a \Vdash \phi \to \beta$, $a \Vdash \forall x_1 \dots \forall x_n (\phi \to \beta)$. Since $\phi \in b \subseteq \mathrm{LQ}_Y$, it follows that $a \Vdash \phi \to \forall x_1 \dots \forall x_n \beta$. But then since a is a Y-theory $\phi \to \forall x_1 \dots \forall x_n \beta \in a$. And since $Rabc$ and $\phi \in b$, it follows from this that $\forall x_1 \dots \forall x_n \beta \in c \subseteq d$. It then quickly follows that $\beta \in d$, contradicting $\beta \in \overline{d}$. So $b \cap \Theta$ is empty.

Applying the Lindenbaum Lemma restricted to LQ_X, we then get that there is a prime X-theory f with $b \subseteq f$ and $f \cap \Theta = \emptyset$. It follows from definitions that $R_X e' f d$. Applying an analogue of Lemma 7 then gives the result.

Symmetry: Following the discussion near Lemma 15 in [13], for any set of first-order wffs Σ, and any first-order wff ϕ, we write $\Sigma \Vdash^{v,w} \phi$ when there is a sequence $\phi_1, \phi_2, \dots, \phi_n$ such that $\phi_n = \phi$ and for all $1 \leq i \leq n$, either

- $\phi_i \in \Sigma$, or
- For some $j < i$ and $k < i$, $\phi_i = \phi_j \wedge \phi_k$, or
- For some $j < i$, v not free in Σ and free for x in $\phi_j(x)$, $\phi_i = \forall v \phi_j(x/v)$, or
- For some $j < i$, ϕ_j is a v, w-variant of ϕ_i – that is, ϕ_i is ϕ_j with some occurrences of v replaced by w or with some occurrences of w replaced by v.
- For some $j < i$, $\phi_j \to \phi_i \in \mathbf{RWQ}$.

The proof of the Lindenbaum Lemma generalizes to the use of $\Vdash^{v,w}$. So if the hypotheses are met and we let $\Delta = \{\alpha : a \nVdash \alpha\}$, then the Lindenbaum Lemma gives a prime theory that satisfies the conclusion.

We are nearly done. What remains is to prove the truth-and-containment result.

Lemma 28 *Let $\mathfrak{C}$ be the canonical model. Then if $a \in S_X$, $\mathtt{va}$ is X-coherent, and $\phi \in \mathrm{LQ}_X$, then $1 \in \mathfrak{C}^a_X(\mathtt{va}, \phi)$ iff $\phi \in a$ and $0 \in \mathfrak{C}^a_X(\mathtt{va}, \phi)$ iff $\neg\phi \in a$.*

Proof By induction on complexity as usual. The base case and the cases involving the connectives are as they were in Appendix B. For the quantified cases, we argue as follows.

Suppose $1 \in \mathfrak{C}^a_X(\mathtt{va}, \forall v_k \psi(v_k))$. Then for some $Y \supsetneq X$ and $i \in Y - X$, for all $b \in S_Y$, if $b{\downarrow}^Y_X = a$, then $1 \in \mathfrak{C}^b_Y(\mathtt{va}^{v_k}_{v_i}, \psi(v_k))$.

Now let c be a prime theory containing a. Since $c \supseteq a$, $c_X \supseteq a$. Thus, by Lemma 25, $a \leq c_X$. So for some $n \in N_X$, $Rnac_X$. Suppose $b{\downarrow}^Y_X = a$. It follows that $1 \in \mathfrak{C}^b_Y(\mathtt{va}^{v_k}_{v_i}, \psi(v_k))$. Also, by the Extension Condition, there is an $m \in N_Y$ and $d \in S_Y$ such that $Rmbd$ and $d{\downarrow}^Y_X = c_X$. Since $m \in N_Y$ and $Rmbd$, $b \leq d$. Thus, by Horizontal Heredity, $1 \in \mathfrak{C}^d_Y(\mathtt{va}^{v_k}_{v_i}, \psi(v_k))$. It follows by the Evaluation Lemma that $1 \in \mathfrak{C}^d_Y(\mathtt{va}, \psi(v_i))$. Thus, by Vertical Heredity, $1 \in \mathfrak{C}^{c_X}_X(\mathtt{va}, \psi(v_i))$. So by the inductive hypothesis, $\psi(v_i) \in c_X$. Thus $\psi(v_i) \in c$. Since c was arbitrary, by Lemma 19, $\psi(v_i) \in a$. Thus $\forall v_k \psi(v_k) \in a$, as required.

On the other hand, suppose $\forall v_k \psi(v_k) \in a$. Let $Y \supsetneq X$ and $i \in Y - X$. If $b{\downarrow}^Y_X = a$, then $a \subseteq b$. So $\forall v_k \psi(v_k) \in b$. Thus, by application of **QA1**, $\psi(v_i) \in b$. So by the inductive hypothesis, $1 \in \mathfrak{C}^b_Y(\mathtt{va}, \psi(v_i))$. So by the Evaluation Lemma, $1 \in \mathfrak{C}^b_Y(\mathtt{va}^{v_k}_{v_i}, \psi(v_k))$. Since b was arbitrary, $1 \in \mathfrak{C}^a_X(\mathtt{va}, \forall v_k \psi(v_k))$.

For the zero case, suppose $0 \in \mathfrak{C}^a_X(\mathtt{va}, \forall v_k \psi(v_k))$. Then for every $Y \supsetneq X$ and $i \in Y - X$ there is a $b \in S_Y$ so that $b{\downarrow}^Y_X = a$ and $0 \in \mathfrak{C}^b_Y(\mathtt{va}^{v_k}_{v_i}, \psi(v_k))$. Thus, by the Evaluation Lemma, $0 \in \mathfrak{C}^b_Y(\mathtt{va}, \psi(v_i))$. So by the inductive hypothesis, $\neg\psi(v_i) \in b$. But by **QA1**, $\neg\psi(v_i) \to \neg\forall v_k \psi(v_k) \in$ **RWQ**. So since b is a theory, $\neg\forall v_k \psi(v_k) \in b$. And since $\neg\forall v_k \psi(v_k) \in \mathrm{LQ}_X$ and $b{\downarrow}^Y_X = a$, it follows that $\neg\forall v_k \psi(v_k) \in a$ as required.

On the other hand, suppose $\neg\forall v_k \psi(v_k) \in a$. Let $Y \supsetneq X$, $i \in Y - X$, $b' = \{\beta \in \mathrm{LQ}_Y : a, \neg\psi(v_i) \Vdash \beta\}$, and $\overline{a} = \mathrm{LQ}_X - a$. I claim that $b' \cap \overline{a} = \emptyset$; that is, that if $\beta \in b' \cap \mathrm{LQ}_X$, then $\beta \in a$.

To see this, note that if $a, \neg\psi(v_i) \Vdash \beta$, then there is $\alpha \in a$ so that $\alpha \wedge \neg\psi(v_i) \Vdash \beta$. Thus by the Deduction Lemma, $(\alpha \wedge \neg\psi(v_i)) \to \beta \in$ **RWQ**. An instance of Fact 4 then gives that $\neg\beta \to \neg(\alpha \wedge \neg\psi(v_i)) \in$ **RWQ**, from which **A6** and Facts 2 and 7 give that $\neg\beta \to (\neg\alpha \vee \psi(v_i)) \in$ **RWQ**. Thus by **QR1**, so is $\forall v_i(\neg\beta \to (\neg\alpha \vee \psi(v_i)))$. On the assumption that $\beta \in \mathrm{LQ}_X$, it follows by instances of **QA2**, **QA3**, and Fact 2 that $\neg\beta \to (\neg\alpha \vee \forall v_i \psi(v_i)) \in$ **RWQ**. Applying **A6** and Facts 2 and 4, we then see that $\neg(\neg\alpha \vee \forall v_i \psi(v_i)) \to \beta \in$ **RWQ**. The definition of '$\vee$' and instances of Fact 1 and **A6** then give $(\alpha \wedge \neg\forall v_i \psi(v_i)) \to \beta \in$ **RWQ**. Thus, $a, \neg\forall v_i \psi(v_i) \Vdash \beta$. But $\neg\forall v_i \psi(v_i) \in a$, so it follows that $a \Vdash \beta$. Thus since a is an X-theory and $\beta \in \mathrm{LQ}_X$, it follows that $\beta \in a$.

Since $\beta \in b' \cap \mathrm{LQ}_X$ only if $\beta \in a$, $\beta \cap \overline{a} = \emptyset$, the (restricted) Lindenbaum Lemma guarantees a prime Y-theory $b \supseteq b'$ with $b \cap \overline{a} = \emptyset$. Since $b \supset a$ by construction, $b\downarrow^X_Y = a$. And since $\neg\psi(v_i) \in b$ as well, the inductive hypothesis gives that $0 \in \mathfrak{C}^b_Y(\mathtt{va}^{v_k}_{v_i}\psi(v_k))$. Thus $0 \in \mathfrak{C}^a_X(\mathtt{va}, \forall v_k \psi(v_k))$ as required. □

Theorem 4 *If* ϕ *is* **RWQ**-*valid, then* $\phi \in$ **RWQ**.

Proof Suppose $\phi \notin$ **RWQ**. Then since **RWQ** is normal and prime, there are normal prime X-theories that do not contain ϕ. So ϕ is not true in the canonical model and is thus not valid. □

D.5.1 Constant Domain Models

It's straightforward to construct, from a varying-domain **RWQ**-model $C = \langle D, \Omega, \delta, \mathcal{M}, \Downarrow\rangle$, an Ω-model $C^{cd} = \langle D \cup \Omega, \delta, \mathcal{M}', \Downarrow'\rangle$. $\mathcal{M}'$ and $\Downarrow'$ are, essentially, just as they were before. The only difference is that $\mathcal{M}'$ now maps each X to the model M'_X that is just like M_X except that $D_X = D \cup \Omega$. Because of the way the M_X are defined in the varying-domain case, each of the M'_X automatically satisfies the featurelessness condition.

Going the other direction is just as easy. So to prove that the constant-domain semantics is adequate (sound and complete) it suffices to prove that for all ϕ and C, $C \vDash \phi$ iff $C^{cd} \vDash \phi$. But this result, in turn, follows by an incredibly straightforward induction. So the constant domain semantics is adequate.

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