SEMANTICS FOR SECOND-ORDER RELEVANT LOGICS

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ABSTRACT. In this paper we extend Kit Fine's semantics for first-order relevant logic so as to treat dyadic second-order logic. Along the way we discuss how to understand the semantics, problems one encounters in extending it to account for various sorts of comprehension, and partial solutions to those problems.

NOTE: THIS IS A PREPRINT OF A CHAPTER TO APPEAR IN THE FORTHCOMING COLLECTION *NEW DIRECTIONS IN RELEVANT LOGIC*. PLEASE CITE THE PUBLISHED VERSION!

1. PREAMBLE

Here's the thing: when you look at it from just the right angle, it's entirely obvious how semantics for second-order relevant logics ought to go. Or at least, if you've understood how semantics for *first*-order relevant logics ought to go (those seeking a gentle introduction might find the discussions in [10] or [13] helpful), there are perspectives like this. What's more is that from any such angle, the metatheory that needs doing can be summed up in one line: everything is just as in the first-order case, but with more indices.

Of course, it's no small matter finding the magical angle from which everything becomes obvious. And even having found this perspective, one cannot assume one's audience will find things as obvious as oneself. All that to say this: if the results in the paper below strike you as obvious, pay attention to the perspective that makes that possible. And if they don't, feel free to ignore this preamble in its entirety.

2. THE LANGUAGE

We will work in a somewhat idiosyncratic setting: dyadic second-order logic. I've chosen this setting not for technical reasons but for reasons of pedagogical expediency. I've explained this material to a number of people and have found the following:

- Monadic second-order logic, while easiest to understand, doesn't leave everyone clear on what to do when it comes to extending yet higher.
- Third-(or higher-)order logic has too much machinery for any but the devout to make it through. Those that do, though, are left able to see their way anywhere they want.
- Dyadic second-order logic is a middle ground—even the apostate are usually able to tolerate working through it if they decide they really care. And seeing one's way from dyadic second-order logic to third-order logic and higher is usually doable in a matter of days.

That said, we explicitly define the language $\mathcal{L}$ as follows:

Vocabulary: The set of symbols of $\mathcal{L}$ consists of

- Countably many individual constants $(c_1, c_2, \dots)$, the set of which we denote by Con_0;
- Countably many unary predicate constants $(P_1, P_2, \dots)$, the set of which we denote by Con_1;
- Countably many binary predicate constants $(Q_1, Q_2, \dots)$, the set of which we denote by Con_2;
- Countably many individual variables $(x_1, x_2, \dots)$, the set of which we denote by Var_0;
- Countably many unary predicate variables $(X_1, X_2, \dots)$, the set of which we denote by Var_1;
- Countably many binary predicate variables $(Y_1, Y_2, \dots)$, the set of which we denote by Var_2;
- The connectives $\neg$, $\wedge$, $\vee$, and $\to$; and
- The quantifiers $\forall$ and $\exists$.

Grammar: $\mathcal{L}$, understood as its set of formulas, is then defined as follows:

- If $T \in \mathsf{Con}_1 \cup \mathsf{Var}_1$ and $t \in \mathsf{Con}_0 \cup \mathsf{Var}_0$, then Tt is a(n atomic) formula;
- If $T \in \mathsf{Con}_2 \cup \mathsf{Var}_2$ and $t_i \in \mathsf{Con}_0 \cup \mathsf{Var}_0$, then Tt_1t_2 is an atomic formula;
- If A and B are formulas, then so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, and $(A \to B)$;
- If A is a formula and x is an individual variable, then $\forall x A$ and $\exists x A$ are formulas;
- If A is a formula and X is a unary predicate variable, then $\forall X A$ and $\exists X A$ are formulas; and
- If A is a formula and Y is a binary predicate variable, then $\forall Y A$ and $\exists Y A$ are formulas.

I will usually stick to the metavariable conventions implicitly specified in the above definition. We'll also occasionally use $\leftrightarrow$, defined in its usual way. Free and bound occurrences of a variable are also defined in the expected ways; for substitution, we write $A(x/t)$ (resp. $A(X/T)$; $A(Y/S)$) for the formula that results from replacing each free occurrence of x (resp. X; Y) in A with an occurrence of t (resp. T; S). With respect to such substitutions, we define what it means for t (resp. T; S) to be *free for* x (resp. X; Y) in A in the expected ways. Finally, where A is a formula and x_1 is an individual variable (resp. x_1 and x_2 are individual variables), we write $B(X/A(x_1))$ (resp. $B(Y/A(x_1, x_2))$) for the formula that results from replacing, for each individual term t (resp. for each pair of individual terms t_1 and t_2) each occurrence of Xt (resp. Yt_1t_2) in B in which the 'X' (resp. 'Y') is occurring freely with an occurrence of $A(x/t)$ (resp. $A(x_1/t_1, x_2/t_2)$). We extend the notion of 'free for' from variable-substitutions to formula-substitutions in the obvious way.

We write $\mathcal{L}$ for the language so-defined. Where F_0, F_1, and F_2 are pairwise disjoint sets of symbols not found in our language and $\overline{F} = \langle F_0, F_1, F_2 \rangle$, we take $\mathcal{L}(\overline{F})$ to be the extension of $\mathcal{L}$ that adds the members of F_0 as additional individual constants, adds the members of F_1 as additional unary predicate constants, and adds the members of F_2 as additional binary predicate constants. An $\mathcal{L}(\overline{F})$-sentence is an $\mathcal{L}(\overline{F})$-formula in which no variables occur free.

3. The Logic

The propositional (or zero-order) fragment of each of the logics we discuss in this paper is the weak relevant logic **B** discussed in (among many other places) §4.3 of [15]. This is also the base logic for which Kit Fine first defined stratified semantics in [8].[1] The reader interested in extending the results of this paper to logics with stronger propositional fragments will find that the tools for doing so introduced by Fine are sufficient. On the other hand, the reader interested in extending my results by including further types of quantification will have to piece together how to do so on their own; I nonetheless think that anyone who takes the time to properly understand what I've done here will find doing so entirely straightforward.

Now to business. For each language $\mathcal{L}(\overline{F})$ (including $\mathcal{L} = \mathcal{L}(\langle\emptyset,\emptyset,\emptyset\rangle)$), the dyadic second-order logic $\mathbf{B2Q}(\overline{F})$ is defined to be the logic axiomatized by the $\mathcal{L}(\overline{F})$-instances of the following axioms and rules:

A1 $A \to A$

A2 $(A \wedge B) \to A$

$(A \wedge B) \to B$

A3 $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

A4 $A \to (A \vee B)$

$B \to (A \vee B)$

A5 $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

A6 $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$

A7 $\neg\neg A \to A$

A8 $\forall x A \to A(x/t)$

$\forall X A \to A(X/T)$

$\forall Y A \to A(Y/S)$

A9 $\forall x(A \to B) \to (A \to \forall x B)$

$\forall X(A \to B) \to (A \to \forall X B)$

$\forall Y(A \to B) \to (A \to \forall Y B)$

A10 $\forall x(A \vee B) \to (A \vee \forall x B)$

$\forall X(A \vee B) \to (A \vee \forall X B)$

$\forall Y(A \vee B) \to (A \vee \forall Y B)$

A11 $A(t/x) \to \exists x A$

$A(T/X) \to \exists X A$

$A(S/Y) \to \exists Y A$

A12 $(A \wedge \exists x B) \to \exists x(A \wedge B)$

$(A \wedge \exists X B) \to \exists X(A \wedge B)$

$(A \wedge \exists Y B) \to \exists Y(A \wedge B)$

R1 $\dfrac{A \qquad A \to B}{B}$

R2 $\dfrac{A \qquad B}{A \wedge B}$

R3 $\dfrac{A \to B \qquad C \to D}{(B \to C) \to (A \to D)}$

R4 $\dfrac{A \to \neg B}{B \to \neg A}$

R5 $\dfrac{A}{\forall x A}$

R6 $\dfrac{A}{\forall X A}$

R7 $\dfrac{A}{\forall Y A}$

R8 $\dfrac{\forall x(B \to A)}{\exists x B \to A}$

R9 $\dfrac{\forall X(B \to A)}{\exists X B \to A}$

R10 $\dfrac{\forall Y(B \to A)}{\exists Y B \to A}$

Note that we do *not* take $\exists$ to be a defined connective here, which partially explains some of the apparent redundancies in this presentation. Also note that

[1] This is not, strictly speaking, correct, as a referee has pointed out. Fine's system included the law of excluded middle, which is not typical for **B**. These days, **B** extended to include excluded middle is usually called **BX**.

in A8 and A11, we require that t (resp. T; S) be free for x (resp. X; Y), and in A9, A10, A12, and in R8-R10, we require that x (resp. X; Y) not occur free in A. Finally, note that there is an asymmetry in our treatment of intensional confinement for universals in A9 and our treatment of intensional confinement for existentials in R8-R10. This is done to preserve the usual negation duality between the existential and the universal in the presence of rule contraposition (rule R4) rather than axiomatic contraposition.

Recall that a comprehension axiom for A is a formula of one of the following forms:

$$\exists X \forall x (Xx \leftrightarrow A)$$
$$\exists Y \forall x_1 \forall x_2 (Yx_1x_2 \leftrightarrow A)$$

We implicitly assume when discussing comprehension axioms that X (resp. Y) does not occur freely in A. Given a set $\mathbb{A}$ of formulas, the logic $\mathbf{B2Q}^{\mathbb{A}}(\overline{F})$ augments $\mathbf{B2Q}(\overline{F})$ with comprehension axioms for each $A \in \mathbb{A}$.

For any of the above logics L, we write $\vdash_L A$ to mean that there is a sequence of formulas $B_1, \ldots, B_n$ with $B_n = A$ so that for $1 \leq i \leq n$, either B_i is an instance of an L-axiom or B_i follows from previous members of the sequence using one of the L-rules. We write $X \vdash_L A$ to mean that there is a sequence of formulas $B_1, \ldots, B_n$ with $B_n = A$ so that for $1 \leq i \leq n$, either B_i is a member of X or there are $j < i$ and $k < i$ so that $B_i = B_j \wedge B_k$ or there is $j < i$ such that $\vdash B_j \to B_i$.

Given a set of formulas Γ, we say that Γ is disjunctively closed when $A \in \Gamma$ and $B \in \Gamma$ only if $A \vee B \in \Gamma$. We say that Γ is prime when $A \vee B \in \Gamma$ only if either $A \in \Gamma$ or $B \in \Gamma$. We say that Γ is an L-theory when $\Gamma \vdash_L A$ only if $A \in \Gamma$.

4. Semantics

We define an $\overline{F}$-premodel to be a tuple $\langle T, P, \ell, \sqsubseteq, \cdot, \star, v \rangle$ where $\ell \in T \supseteq P$, $\sqsubseteq$ is a partial ordering of T, $\cdot$ is a binary operation on T, $\star$ is a unary operation on P, and v is a function mapping $t \in T$ to a set of atomic $\mathcal{L}(\overline{F})$ sentences. Note that for a more traditional presentation of things, one can recover from v functions v^1 and v^2 that (respectively) map each unary $\overline{F}$-predicate constant to a function from theories to sets of names and map each binary $\overline{F}$-predicate constant to a function from theories to sets of pairs of names. Explicitly, we have

$$v^1(P, t) = \{c : Pc \in v(t)\}$$
$$v^2(Q, t) = \{\langle c_1, c_2 \rangle : Qc_1c_2 \in v(t)\}$$

We define $t \sqsubseteq_P q$ to mean $t \sqsubseteq q$ and $q \in P$. An $\overline{F}$-model is an $\overline{F}$-premodel that satisfies the following conditions:

- Covariance: If $s \sqsubseteq t$, then
  - $u \cdot s \sqsubseteq u \cdot t$,
  - $s \cdot u \sqsubseteq t \cdot u$, and
  - $v(s) \subseteq v(t)$.
- Minimality: If $s \cdot t \sqsubseteq_P p$, then
  - There is $s \sqsubseteq_P q$ so that $q \cdot t \sqsubseteq p$ and
  - There is $t \sqsubseteq_P r$ so that $s \cdot r \sqsubseteq p$.
- If $A \in v(p)$ for all $t \sqsubseteq_P p$, then $A \in v(t)$.
- $\ell \cdot t = t$
- $p^{\star\star} = p$

- If $P \ni p \sqsubseteq_P q$, then $q^\star \sqsubseteq p^\star$.
- For all t there is p so that $t \sqsubseteq_P p$.

As the reader can verify, aside from the fact that we are working in a language with additional structure, an $\overline{F}$-premodel is exactly a model in the sense of Fine's [7], with two modifications: first, we've dropped the requirement that primes above the logic be above their duals (that is, if $\ell \sqsubseteq_P p$, then $p^\star \sqsubseteq p$). Second, we've added (in the final bullet above) the requirement that all theories be extended by some prime. The first change reflects our rejection of $A \vee \neg A$ as an axiom. The second change makes no difference at all in the logic. This isn't hard to see: since it amounts to a restriction on the class of models, the soundness proof still goes through, and since the language $\mathcal{L}(\overline{F})$, thought of as a theory, is obviously prime, the canonical model verifies the condition and the completeness proof will still go through as well.

We intuitively interpret the members of T as theories, the members of P as prime theories, ℓ as the logic, $\sqsubseteq$ as the containment relation among the theories, $\cdot$ as the application operation, $\star$ as the dual of a theory, and v as the function mapping each theory to its set of atomic members. For more detail on these interpretations, see e.g. [7], [11], or [13].

4.1. **The Base System.** As in [8], so also here a *stratified model* is essentially a poset of $\overline{F}$-models. Fine, in constructing stratified models for first-order quantification, gave models that were 'fibered along' a poset $\mathcal{D}$ of what he called domains. But there was nothing all that 'domain-y' about $\mathcal{D}$—its underlying class was just a class of sets satisfying the following:

- Extendibility: For all $\alpha \in \mathcal{D}$, there is $\beta \in \mathcal{D}$ so that $\beta \supsetneq \alpha$.
- Upper Bound: For all $\alpha \in \mathcal{D}$ and $\beta \in \mathcal{D}$, there is $\gamma \in \mathcal{D}$ so that $\alpha \cup \beta \subseteq \gamma$.
- Reversibility: if $\alpha \in \mathcal{D}$, $\beta \in \mathcal{D}$, $\gamma \in \mathcal{D}$, and $\alpha \subseteq \beta \subseteq \gamma$, then $\alpha \cup (\gamma - \beta) \in \mathcal{D}$ as well.

Thinking of the members of $\mathcal{D}$ as sets of names, we can justify these conditions as follows:

- Extendibility: language is indefinitely extensible; no matter how many names we've added to our vocabulary, we can always add more.
- Upper Bound: No two ways of adding names to our language are incompatible with each other.
- Reversibility: Given any two sets of names we might add to our language, we can add them in either order without consequence.

We can generalize these to the case at hand by 'fibering along' not just a poset of sets of names, but along a poset of *vocabularies*, now understood to include not just *names*, but also predicate symbols and relation symbols. Of course, we'll have to require that such a poset satisfy the obvious analogues of the conditions Fine gave. This does not introduce as much new complexity as one would expect, especially once we take the step of extending set-theoretic notions from sets to triples of sets in the expected ways—that is, by defining e.g. $\overline{F} \subseteq \overline{F'}$ to mean that $F_0 \subseteq F'_0$, $F_1 \subseteq F'_1$, and $F_2 \subseteq F'_2$ and, for $\overline{F} \subseteq \overline{F'}$, by defining $\overline{F} - \overline{F'}$ to be the triple $\langle F_0 - F'_0, F_1 - F'_1, F_2 - F'_2 \rangle$. With this in hand, we define the (oh so) technical term 'appropriate class of vocabularies' by saying that an appropriate class of vocabularies is any class V of triples $\overline{F} = \langle F_0, F_1, F_2 \rangle$ of sets such that

- Extendibility: For each $\overline{F} \in V$, there are (not necessarily distinct) $\overline{F} \subseteq \overline{G} \in V$, $\overline{F} \subseteq \overline{H} \in V$ and $\overline{F} \subseteq \overline{K} \in V$ so that $F_0 \subsetneq G_0$, $F_1 \subsetneq H_1$, and $F_2 \subsetneq K_2$.
- Upper Bound: For all $\overline{F} \in V$ and $\overline{G} \in V$, there is $\overline{H} \in V$ with $\overline{F} \cup \overline{G} \subseteq \overline{H}$.
- Reversibility: if $\overline{F} \in V$, $\overline{G} \in V$, $\overline{H} \in V$, and $\overline{F} \subseteq \overline{G} \subseteq \overline{H}$, then $\overline{F} \cup (\overline{H} - \overline{G}) \in V$ as well.

To define stratified models we begin by settling on a particular appropriate class of vocabularies V such that no member of any component of any member of V occurs as a symbol in $\mathcal{L}$. We then define a V-stratified premodel to be a tuple $\langle M, \Uparrow, \Downarrow, [\![-]\!] \rangle$ where

- M is a function that maps each $\overline{F} \in V$ to a $\overline{F}$-model $M(\overline{F}) = \langle T_{\overline{F}}, P_{\overline{F}}, \ell_{\overline{F}}, \sqsubseteq_{\overline{F}}, \cdot_{\overline{F}}, \star_{\overline{F}}, v_{\overline{F}} \rangle$;
- $\Uparrow$ is a set containing one function $\uparrow^{\overline{F'}}_{\overline{F}} \colon T_{\overline{F}} \longrightarrow T_{\overline{F'}}$ for each pair $\langle \overline{F}, \overline{F'} \rangle$ with $\overline{F} \subseteq \overline{F'}$.
- $\Downarrow$ is a set containing one function $\downarrow^{\overline{F'}}_{\overline{F}} \colon T_{\overline{F'}} \longrightarrow T_{\overline{F}}$ for each pair $\langle \overline{F}, \overline{F'} \rangle$ with $\overline{F} \subseteq \overline{F'}$.
- $[\![-]\!]$ is a set containing one function $[-]^{\sigma_1,\sigma_2}_{\overline{F}} : T_{\overline{F}} \longrightarrow T_{\overline{F}}$ for each triple $\langle \overline{F}, \sigma_1, \sigma_2 \rangle$ with $\langle \sigma_1, \sigma_2 \rangle \in (\mathsf{Con}_0 \cup F_0)^2 \cup (\mathsf{Con}_1 \cup F_1)^2 \cup (\mathsf{Con}_2 \cup F_2)^2$.

Intuitively, each model $M(\overline{F})$ is a space of theories in the language $\mathcal{L}(\overline{F})$, each function $\uparrow^{\overline{F'}}_{\overline{F}}$ maps the $\mathcal{L}(\overline{F})$-theory t to the $\mathcal{L}(\overline{F'})$-theory $t\uparrow^{\overline{F'}}_{\overline{F}}$ that t generates under $\ell_{\overline{F'}}$, each function $\downarrow^{\overline{F'}}_{\overline{F}}$ maps the $\mathcal{L}(\overline{F'})$-theory t to the $\mathcal{L}(\overline{F})$-theory $t \cap \mathcal{L}(\overline{F})$, and each function $[-]^{\sigma_1,\sigma_2}_{\overline{F}}$ maps the $\mathcal{L}(\overline{F})$-theory t to the $\mathcal{L}(\overline{F})$-theory $[t]^{\sigma_1,\sigma_2}_{\overline{F}}$ that we get by extending t so as to make σ_1 and σ_2 indistinguishable. We extend all of these functions from functions from theories to theories to functions from sets of theories to sets of theories in the obvious ways.

Given $\langle \sigma_1, \sigma_2 \rangle \in (\mathsf{Con}_0 \cup F_0)^2 \cup (\mathsf{Con}_1 \cup F_1)^2 \cup (\mathsf{Con}_2 \cup F_2)^2$ and a formula $A \in \mathcal{L}(V)$, a $\langle \sigma_1, \sigma_2 \rangle$-variant of A is a formula that results from replacing zero or more occurrences of σ_i in A with σ_j, where $i \neq j \in \{1, 2\}$. Now consider the function $\Sigma_{\langle \sigma_1, \sigma_2 \rangle}$ from subsets of $\mathcal{L}(V)$ to subsets of $\mathcal{L}(V)$ defined by

$$\Sigma_{\langle \sigma_1, \sigma_2 \rangle}(\Gamma) = \bigcup_{A \in \Gamma} \{B : B \text{ is a } \langle \sigma_1, \sigma_2 \rangle\text{-variant of } A\}$$

$\Sigma_{\langle \sigma_1, \sigma_2 \rangle}(\Gamma)$ is the *symmetrization* of Γ at $\langle \sigma_1, \sigma_2 \rangle$. We say that Γ is symmetric when $\Sigma_{\langle \sigma_1, \sigma_2 \rangle}(\Gamma) = \Gamma$.

A V-stratified *model* is a V-stratified premodel that meets the following conditions:

(1) Covariance: If $s \sqsubseteq_{\overline{F}} t$, then
   - $s\uparrow^{\overline{F'}}_{\overline{F}} \sqsubseteq_{\overline{F'}} t\uparrow^{\overline{F'}}_{\overline{F}}$, and
   - $s\downarrow^{\overline{F}}_{\overline{F'}} \sqsubseteq_{\overline{F'}} t\downarrow^{\overline{F}}_{\overline{F'}}$.

(2) Identity: $t\uparrow^{\overline{F}}_{\overline{F}} = t\downarrow^{\overline{F}}_{\overline{F}} = t$.

(3) Transitivity: $t\uparrow^{\overline{F'}}_{\overline{F}}\uparrow^{\overline{F''}}_{\overline{F'}} = t\uparrow^{\overline{F''}}_{\overline{F}}$ and $t\downarrow^{\overline{F}}_{\overline{F'}}\downarrow^{\overline{F'}}_{\overline{F''}} = t\downarrow^{\overline{F}}_{\overline{F''}}$.

(4) Extension-Restriction: $t\uparrow^{\overline{F'}}_{\overline{F}}\downarrow^{\overline{F'}}_{\overline{F}} = t$, but $t\downarrow^{\overline{F}}_{\overline{F'}}\uparrow^{\overline{F}}_{\overline{F'}} \sqsubseteq t$.

(5) Vertical Atomic Heredity: $v_{\overline{F'}}(t\downarrow^{\overline{F}}_{\overline{F'}}) = v_{\overline{F}}(t) \cap \mathcal{L}(\overline{F'})$.

(6) Primes Down: $P_{\overline{F}}\downarrow^{\overline{F}}_{\overline{F'}} \subseteq P_{\overline{F'}}$.

(7) Prime Restriction Down: If $q \in P_{\overline{F}}$ and $p \sqsubseteq q{\downarrow}^{\overline{F}}_{\overline{F'}}$, then there is $r \in P_{\overline{F}}$, with $r{\downarrow}^{\overline{F}}_{\overline{F'}} = p$ and $r \sqsubseteq q$.
(8) Prime Extension Down: If $t{\downarrow}^{\overline{F}}_{\overline{F'}} \sqsubseteq_{P_{\overline{F'}}} p$, then there is $t \sqsubseteq_{P_{\overline{F}}} q$ with $q{\downarrow}^{\overline{F}}_{\overline{F'}} = p$.
(9) Duality Down: $p^{\star_{\overline{F}}}{\downarrow}^{\overline{F}}_{\overline{F'}} = (p{\downarrow}^{\overline{F}}_{\overline{F'}})^{\star_{\overline{F'}}}$.
(10) Distribution Up: $(t \cdot_{\overline{F}} u){\uparrow}^{\overline{F'}}_{\overline{F}} = t{\uparrow}^{\overline{F'}}_{\overline{F}} \cdot_{\overline{F'}} u{\uparrow}^{\overline{F'}}_{\overline{F}}$.
(11) Distribution Down: $(t \cdot_{\overline{F}} u{\uparrow}^{\overline{F}}_{\overline{F'}}){\downarrow}^{\overline{F}}_{\overline{F'}} = t{\downarrow}^{\overline{F}}_{\overline{F'}} \cdot_{\overline{F'}} u$.
(12) Logics Up: $\ell_{\overline{F}}{\uparrow}^{\overline{F'}}_{\overline{F}} = \ell_{\overline{F'}}$.
(13) Bracket is a Closure Operator:
  - $t \sqsubseteq_{\overline{F}} [t]^{\sigma_1,\sigma_2}_{\overline{F}}$;
  - If $s \sqsubseteq_{\overline{F}} t$, then $[t]^{\sigma_1,\sigma_2}_{\overline{F}} \sqsubseteq_{\overline{F}} [s]^{\sigma_1,\sigma_2}_{\overline{F}}$; and
  - $[[t]^{\sigma_1,\sigma_2}_{\overline{F}}]^{\sigma_1,\sigma_2}_{\overline{F}} = [t]^{\sigma_1,\sigma_2}_{\overline{F}}$.
(14) Bracket Duality: $[([t]^{\sigma_1,\sigma_2}_{\overline{F}})^{\star_{\overline{F}}}]^{\sigma_1,\sigma_2}_{\overline{F}} = ([t]^{\sigma_1,\sigma_2}_{\overline{F}})^{\star_{\overline{F}}}$
(15) Bracket Application: $[s \cdot_{\overline{F}} t]^{\sigma_1,\sigma_2}_{\overline{F}} \sqsubseteq_{\overline{F}} [s]^{\sigma_1,\sigma_2}_{\overline{F}} \cdot_{\overline{F}} [t]^{\sigma_1,\sigma_2}_{\overline{F}} \sqsubseteq_{\overline{F}} [s]^{\sigma_1,\sigma_2}_{\overline{F}} \cdot_{\overline{F}} t$
(16) Bracket Up: $[t{\uparrow}^{\overline{F}}_{\overline{F'}}]^{\sigma_1,\sigma_2}_{\overline{F}} = [t]^{\sigma_1,\sigma_2}_{\overline{F'}}{\uparrow}^{\overline{F}}_{\overline{F'}}$
(17) Bracket Down: If $\sigma_1 \in F_i - F'_i$ and $\sigma_2 \in F'_i$, then $[t{\uparrow}^{\overline{F}}_{\overline{F'}}]^{\sigma_1,\sigma_2}_{\overline{F}}{\downarrow}^{\overline{F}}_{\overline{F'}} \sqsubseteq_{\overline{F'}} t$
(18) Bracket Symmetry: $[t]^{\sigma_1,\sigma_2}_{\overline{F}}$ is symmetric in σ_1 and σ_2.
(19) Symmetric Prime Extension: If $v(t)$ is symmetric in σ_1 and σ_2 and $t \sqsubseteq_{P_{\overline{F}}} p$, then there is a q so that $v(q)$ is symmetric in σ_1 and σ_2 and $t \sqsubseteq_{P_{\overline{F}}} q \sqsubseteq_{P_{\overline{F}}} p$.

The forcing relation, which holds between triples $\langle S, \overline{F}, t\rangle$—where S is a V-stratified model, $\overline{F} \in V$, and $t \in M_S(\overline{F})$—and $\mathcal{L}(\overline{F})$-*sentences* (not *formulas*!) is then defined as follows:

- $S, \overline{F}, t \vDash Pt$ iff $Pt \in v_{\overline{F}}(t)$.
- $S, \overline{F}, t \vDash Qt_1t_2$ iff $Qt_1t_2 \in v_{\overline{F}}(t)$.
- $S, \overline{F}, t \vDash A \wedge B$ iff $S, \overline{F}, t \vDash A$ and $S, \overline{F}, t \vDash B$
- $S, \overline{F}, t \vDash A \vee B$ iff for all $t \sqsubseteq_{P_{\overline{F}}} p$, $S, \overline{F}, p \vDash A$ or $S, \overline{F}, p \vDash B$
- $S, \overline{F}, t \vDash \neg A$ iff for all $t \sqsubseteq_{P_{\overline{F}}} p$, $S, \overline{F}, p^{\star}_{\overline{F}} \nvDash A$.
- $S, \overline{F}, t \vDash A \to B$ iff for all $u \in T_{\overline{F}}$, if $S, \overline{F}, u \vDash A$, then $S, \overline{F}, t \cdot_{\overline{F}} u \vDash B$.
- $S, \overline{F}, t \vDash \forall x A$ iff for some $\overline{G} \supsetneq \overline{F}$ and $g \in G_0 - F_0$, $S, \overline{G}, t{\uparrow}^{\overline{G}}_{\overline{F}} \vDash A(x/g)$.
- $S, \overline{F}, t \vDash \exists x A$ iff for all $t \sqsubseteq_{P_{\overline{F}}} p$ there are $\overline{G} \supseteq \overline{F}$, $q \in P_{\overline{G}}$, and $g \in G_0$ so that $q{\downarrow}^{\overline{G}}_{\overline{F}} = p$ and $S, \overline{G}, q \vDash A(x/g)$.
- $S, \overline{F}, t \vDash \forall X A$ iff for some $\overline{G} \supsetneq \overline{F}$ and $G \in G_1 - F_1$, $S, \overline{G}, t{\uparrow}^{\overline{G}}_{\overline{F}} \vDash A(X/G)$.
- $S, \overline{F}, t \vDash \exists X A$ iff for all $t \sqsubseteq_{P_{\overline{F}}} p$ there are $\overline{G} \supseteq \overline{F}$, $q \in P_{\overline{G}}$, and $G \in G_1$ so that $q{\downarrow}^{\overline{G}}_{\overline{F}} = p$ and $S, \overline{G}, q \vDash A(X/G)$.
- $S, \overline{F}, t \vDash \forall Y A$ iff for some $\overline{G} \supsetneq \overline{F}$ and $G \in G_2 - F_2$, $S, \overline{G}, t{\uparrow}^{\overline{G}}_{\overline{F}} \vDash A(Y/G)$.
- $S, \overline{F}, t \vDash \exists Y A$ iff for all $t \sqsubseteq_{P_{\overline{F}}} p$ there are $\overline{G} \supseteq \overline{F}$, $q \in P_{\overline{G}}$, and $G \in G_2$ so that $q{\downarrow}^{\overline{G}}_{\overline{F}} = p$ and $S, \overline{G}, q \vDash A(Y/G)$.

Letting $\overline{\Omega} = \langle \Omega_0, \Omega_1, \Omega_2\rangle$, with $\Omega_i = \cup_{\overline{F} \in V} F_i$, we say that an $\mathcal{L}(\overline{\Omega})$-sentence A is valid in S when $S, \overline{F}, \ell_{\overline{F}} \vDash A$ whenever $A \in \mathcal{L}(\overline{F})$. We say that a formula $A(\Lambda_1, \ldots, \Lambda_n)$ in which the Λ_i occur free is valid in S when all its substitution instances are valid in S. We say that A is V-valid when A is valid in every V-stratified model S and that A is valid when A is V-valid for every appropriate class of vocabularies V. Finally, where $t \in T_{\overline{F}}$ for some $\overline{F} \in V$, we write $\underline{t}$ for $\{A : S, \overline{F}, t \vDash A\}$. In the remainder, when they are abundantly clear from context, we will drop some of the subscripts to enhance readability.

We now record a few important facts about the semantics:

Lemma 1.

- *If $s \sqsubseteq_{\overline{F}} t$ and $S, \overline{F}, s \vDash A$, then $S, \overline{F}, t \vDash A$.*
- *If $S, \overline{F}, p \vDash A$ for all $t \sqsubseteq_{P_{\overline{F}}} p$, then $S, \overline{F}, t \vDash A$ as well.*
- *If $v(t)$ is symmetric in σ_1 and σ_2, then $\underline{t}$ is symmetric in σ_1 and σ_2.*
- *If $\overline{F} \subseteq \overline{G}$ and $A \in \mathcal{L}(\overline{F})$, then $S, \overline{G}, t \vDash A$ iff $S, \overline{F}, t{\downarrow}^{\overline{G}}_{\overline{F}}$.*
- *If $\overline{F} \subseteq \overline{G}$ and $A \in \mathcal{L}(\overline{F})$, then $S, \overline{F}, t \vDash A$ iff $S, \overline{G}, t{\uparrow}^{\overline{G}}_{\overline{F}}$.*
- *If $S, \overline{F}, t \vDash \forall x A$ (resp. $S, \overline{F}, t \vDash \forall X A$; $S, \overline{F}, t \vDash \forall Y A$) and $f \in F_0$ (resp. $F \in F_1$; $F \in F_2$), then $S, \overline{F}, t \vDash A(x/t)$.*

Proof. By induction on A starting at the top of the list and working to the bottom of the list. In each case, the proof is exactly as in [8] except in the case of the existentials and in each the proof for existentials is essentially immediate. □

Theorem 2. **B2Q** *is sound for the semantics.*

Proof. By induction on the complexity of the proof. Again, almost everything goes as it did in [8]. We'll look only at one instance each of A11 and A12, since those are newish, and also at R10, just to round things out.

For A11, we examine the individual existential. So let $A(t/x) \to \exists x A \in \mathcal{L}(\overline{F})$ and $S, \overline{F}, s \vDash A(t/x)$. Then for all $s \sqsubseteq_{P_{\overline{F}}} p$, $S, \overline{F}, p \vDash A(t/x)$. Thus, letting $\overline{G} = \overline{F}$, $q = p$ and $g = t$ and recalling that ${\downarrow}^{\overline{F}}_{\overline{F}}$ is the identity function, we have that for all $t \sqsubseteq_{P_{\overline{F}}} p$ there is $\overline{G} \supseteq \overline{F}$, $q \in P_{\overline{G}}$ and $g \in G_0$ so that $q{\downarrow}^{\overline{F}}_{\overline{G}} = p$ and $S, \overline{G}, q \vDash A(g/x)$. So $S, \overline{F}, s \vDash \exists x A$. If follows that $S, \overline{F}, \ell_{\overline{F}} \vDash A(t/x) \to \exists x A$.

For A12 we examine the relation existential. So, let $(A \wedge \exists Y B) \to \exists Y (A \wedge B) \in \mathcal{L}(\overline{F})$ and $S, \overline{F}, t \vDash A \wedge \exists Y B$. Choose $t \sqsubseteq_{P_{\overline{F}}} p$. Clearly $p \vDash A$ and $p \vDash \exists Y B$. So there is $\overline{G} \supseteq \overline{F}$, $G \in G_2$, and $q \in P_{\overline{G}}$ with $q{\downarrow}^{\overline{G}}_{\overline{F}} = p$ and $q \vDash B(Y/G)$. Since $q{\downarrow}^{\overline{G}}_{\overline{F}} = p$, $q \supseteq q{\downarrow}^{\overline{G}}_{\overline{F}}{\uparrow}^{\overline{G}}_{\overline{F}} = p{\uparrow}^{\overline{G}}_{\overline{F}}$. Thus since $A \in \mathcal{L}(\overline{F})$, $q \vDash A$. So $q \vDash A \wedge B(Y/G) = (A \wedge B)(Y/G)$. So $t \vDash \exists Y (A \wedge B)$.

For R10, we examine the predicate existential. So, let $\forall X (B \to A) \in \mathcal{L}(\overline{F})$ and suppose $S, \overline{F}, \ell_{\overline{F}} \vDash \forall X (B \to A)$. To see that $S, \overline{F}, \ell_{\overline{F}} \vDash \exists X B \to A$, let $S, \overline{F}, t \vDash \exists X t$. Choose $t \sqsubseteq_{P_{\overline{F}}} p$. Then there is $\overline{G} \supseteq \overline{F}$, $q \in P_{\overline{G}}$, and $G \in G_1$ so that $q{\downarrow}^{\overline{G}}_{\overline{F}}$ and $S, \overline{G}, q \vDash B(X/G)$. Since $S, \overline{F}, \ell_{\overline{F}} \vDash \exists X B \to A$ we also have that $S, \overline{G}, \ell_{\overline{G}} \vDash \exists X B \to A$. So $S, \overline{G}, \ell_{\overline{G}} \vDash B(X/G) \to A$ Thus $S, \overline{G}, q \vDash A$. And since $A \in \mathcal{L}(\overline{F})$, and $q{\downarrow}^{\overline{G}}_{\overline{F}} = p$, it then follows that $S, \overline{F}, p \vDash A$. So all prime extensions of t verify A and thus $S, \overline{F}, t \vDash A$. □

Lemma 3 (Deduction)**.** *If $A \vdash B$, and $A \to B \in \mathcal{L}(\overline{F})$, then $A \to B \in$* **B2Q**$(\overline{F})$.

Proof. By induction along $\vdash$; see e.g. [8]. □

Lemma 4 (Lindenbaum)**.** *Let t be an $\mathcal{L}(\overline{F})$-theory, $\Delta \subseteq \mathcal{L}(\overline{F})$ be closed under disjunction, and $t \cap \Delta = \emptyset$. Then there is a prime $\mathcal{L}(\overline{F})$-theory $p \supseteq t$ with $p \cap \Delta = \emptyset$*

Proof. In the usual way; see e.g. [8]. □

Corollary 5. *If t is an $\mathcal{L}(\overline{F})$-theory and $Pr_{\overline{F}}$ is the set of prime $\mathcal{L}(\overline{F})$-theories, then* $t = \bigcap_{t \subseteq p \in Pr_{\overline{F}}} p$.

Theorem 6. B2Q *is complete for the semantics.*

Proof. By a canonical model construction. In case it's not clear, the key is to use the appropriate class of vocabularies given by the finite triples of sets of variables and then—surprise surprise—to do what Fine did, one more time.

The only interesting thing worth pausing to verify is that in the key lemma (which says that $t \vDash A$ iff $A \in t$) the induction still goes through in the existential cases. We'll prove this for the individual existential since the other cases are exactly parallel.

So, let C be the canonical model, t be an $\mathcal{L}(\overline{F})$-theory, and $t \vDash \exists x A$. Choose $t \subseteq p \in Pr_{\overline{F}}$. Then since $t \vDash \exists x A$, there is $\overline{G} \supseteq \overline{F}$, $g \in G_0$, and $q \in Pr_{\overline{G}}$ so that $q \cap \mathcal{L}(\overline{F}) = p$ and $q \vDash A(x/g)$. But then by the inductive hypothesis, $A(x/g) \in q$. So since $A(x/g) \to \exists x A$ is a theorem, $\exists x A \in q$. But also $\exists x A \in \mathcal{L}(\overline{F})$, so $\exists x A \in q \cap \mathcal{L}(\overline{F}) = p$. Thus, $\exists x A$ is in every prime extension of t and thus in t.

Now suppose that $\exists x A \in t$. Choose $t \subseteq p \in Pr_{\overline{F}}$ and let $p^{c_{\overline{F}}} = \mathcal{L}_{\overline{F}} - p$. Choose $g \in \Omega_0 - F_0$ and let $\overline{G} = \overline{F} \cup \{g\}$. I claim that

$$q = \{B \in \mathcal{L}(\overline{G}) : p \cup \{A(x/g)\} \vdash B\} \cap p^{c_{\overline{F}}} = \emptyset.$$

To see this, suppose to the contrary that for some B, $p \cup \{A(x/g)\} \vdash B$ and $B \in p^{c_{\overline{F}}}$. Then there will be some $C \in p$ so that $C \wedge A(x/g) \vdash B$. Thus $(C \wedge A(x/g)) \to B \in \mathbf{B2Q}(\overline{G})$. It follows that $\forall g(\neg B \to \neg(C \wedge A(x/g))) \in \mathbf{B2Q}(\overline{G})$. But then since A and B are in $\mathcal{L}(\overline{F})$, confining the universal twice (first intensionally, then extensionally) we get that $\neg B \to (\neg C \vee \forall g \neg A(x/g)) \in \mathbf{B2Q}(\overline{G})$. But then we also get that $(C \wedge \exists x A) \to B \in \mathbf{B2Q}(\overline{F})$, and thus since $C \in p$ and $\exists x A \in t \subseteq p$, $B \in p$, which is a contradiction.

Thus, since it is obvious that $p^{c_{\overline{F}}}$ is closed under disjunctions and q is a theory that contains $A(x/g)$ (and thus that, by the inductive hypothesis, $q \vDash A(x/g)$) there is, by the Lindenbaum Lemma, a prime $q' \in Pr_{\overline{G}}$ with $q' \cap \mathcal{L}(\overline{F}) = p$ and $q' \vDash A(x/g)$. So $t \vDash \exists x A$. □

4.2. **Adding Comprehension. B2Q** is a second-order logic in only the most technical sense. Lacking comprehension axioms, it's really just a many-sorted first-order logic with a complicated vocabulary. In order to add comprehension axioms, we of course have to restrict the class of models we allow.

There's also a complication to consider that arises in logics that don't admit axiomatic transitivity; that is in which the following are not theorems:

$$(A \to B) \to ((B \to C) \to (A \to C))$$
$$(B \to C) \to ((A \to B) \to (A \to C))$$

The problem is this: one of the key uses to which one puts comprehension axioms is in proving that second-order universals are well-behaved. As a particular case of the general phenomenon, it is typically the case in second-order systems that once one has added comprehension axioms for B, one can then derive all instances of

$\forall XA \to A(X/B(y))$ in which $B(y)$ is free for X.[2] This is a fairly natural thing to want of one's second-order universals.

The natural way to go about deriving all such formulas is by first doing something like this:

$$\cfrac{\cfrac{\cfrac{\forall XA \to A \qquad A(X/B) \to A(X/B)}{(A \to A(X/B)) \to (\forall XA \to A(X/B))}}{\forall X[(A \to A(X/B)) \to (\forall XA \to A(X/B))]}}{\exists X(A \to A(X/B)) \to (\forall XA \to A(X/B))}$$

All one has to do then is derive the antecedent; viz. $\exists X(A \to A(X/B))$. Now, if A is atomic, there's no problem here: either X occurs in A (so A has the form Xa for some a) or it doesn't. In the latter case, $A = A(X/B)$, so by a degenerate instance of A11 and the fact that $A \to A$ is an axiom, $\exists X(A \to A(X/B))$ is provable. In the former case, $\exists X(A \to A(X/B))$ is (essentially) one half of an instance of comprehension.

For complex A, the case where X doesn't occur in A goes through as before. But for the other case things are trickier. After poking at the problem a bit, one sees that the right approach is to prove something slightly stronger; viz. that all instances of $\exists X(A \leftrightarrow A(X/B))$ are theorems. And since comprehension gives us this for atoms, the natural way to proceed is by induction on the complexity of A. The inductive hypothesis then gives that all formulas of the form $\exists X(A \leftrightarrow A(X/B))$ are theorems when A has at most k connectives and quantifiers. There are then induction cases for each connective and each quantifier. For most of these, elbow grease is sufficient to get things done. For the conditional, it's not.

To say more, let's make a few assumptions. For simplicity, we'll consider a conditional $A_1 \to A_2$ in which X doesn't occur free in A_1. Our goal is to show that we can derive $\exists X[(A_1 \to A_2) \leftrightarrow (A_1 \to A_2)(X/B)]$. Given our assumption, though, this is just $\exists X[(A_1 \to A_2) \leftrightarrow (A_1 \to A_2(X/B))]$.

Now, as we'll leave the reader to check, if we have axiomatic transitivity on board, we can then prove that the following is a theorem:[3]

$$(A_2 \leftrightarrow A_2(X/B)) \to ((A_1 \to A_2) \leftrightarrow (A_1 \to A_2(X/B)))$$

And from here it's not hard at all to get to the following:

$$\exists X(A_2 \leftrightarrow A_2(X/B)) \to \exists X((A_1 \to A_2) \leftrightarrow (A_1 \to A_2(X/B)))$$

And induction can do the rest of the job from there.

The problem, now returning to the thread, is what to do if (as in the case at hand) we *don't* have the transitivity axioms in the logic we're working with. It's clear (in a highly suggestive comment that we'll be returning to below) that if we were working not with existentials but *instances*, the induction could still be done. That is, if rather than having $\exists X(A_2 \leftrightarrow A_2(X/B))$ as a theorem, we had (e.g.) $A_2(X/P) \leftrightarrow A_2(X/B)(X/P)$ as a theorem, then there would be no problem.[4] Indeed, were this the case, the derivation would be straightforward: just apply R3 to $A_1 \to A_1$ and the two halves of $A_2(X/P) \leftrightarrow A_2(X/B)(X/P)$ to get the two

[2]See e.g. [14] for a discussion.

[3]Actually, you only need one of the two forms of axiomatic transitivity to do this. But the argument corresponding to the one I'm giving but assuming that X isn't free in A_2 relies on the other form of axiomatic transitivity so the point is moot.

[4]It's worth pointing out that $A_2(X/B)(X/P)$ needn't reduce to $A_2(X/B)$ since X might occur free in B.

halves of $(A_1 \to A_2(X/P)) \leftrightarrow (A_1 \to A_2(X/B)(X/P))$, which is an instance of what we're looking for. Note that this argument will work given absolutely any *particular* instance, but it doesn't work if all we know is that *there is* an instance—that is if all we know is the existential $\exists X(A_2 \leftrightarrow A_2(X/B))$. The problem, in short is that absent transitivity axioms we just lack any resources that would let us 'pass' this fact about A_2 outside the scope of the existential.

We'll be returning below to the strange and suggestive bits of the above discussion. For now we'll note that what we can conclude is the following: if we indeed want a decently well-behaved universal on hand—one for which we can prove $\forall X(A \to A(X/B))$, then we need fuller-than-full comprehension; not just comprehension *for* every formula in the language, but also comprehension *within* every formula of the language. More helpfully, what's needed to ensure a well-behaved universal is *something* that ensures all of the following comprehension-esque sentences are theorems:

$$\exists X \forall y_1 \dots \forall y_n (B \leftrightarrow B(X/A(z)))$$

Where the y_i are the variables that occur free in B. Call such a sentence a comprehension axiom *for* A *along* B. Note that in the case where $B = Xy$, this just is ordinary comprehension for A, so ordinary comprehension is just comprehension along atoms.

It remains unclear to me exactly how to model, semantically, a system incorporating comprehension not only *for* all formulas but also *along* all formulas. So, for the time being, we'll focus on what needs to be done to semantically model comprehension along atoms. And since there's nothing particularly interesting that's different between how things go in the unary case and how things go in the binary case, we'll present only the former.

To that end, let $\mathbb{A}$ be the set of formulas one wishes to adopt comprehension axioms for. Choose an appropriate class of vocabularies V and for each $A \in \mathbb{A}$, each variable y, and each $\overline{F} \in V$, let $in_{\overline{F}}(A, y)$ be the set of formulas that result from replacing free variables in A with appropriate members of (components of) $\overline{F}$ until at most y remains free in A.

The semantic condition we then require is this:

- For all $A \in \mathbb{A}$ and $B \in in_{\overline{F}}(A, y)$, if $\overline{F} \subsetneq \overline{G}$ and $F_1 \subsetneq G_1$, then there is $G \in G_1$ and $\overline{H} \supsetneq \overline{G}$ with $h \in H_0 - G_0$ so that $Gh \in v_{\overline{H}}(t)$ iff $S, \overline{H}, t \vDash B(y/h)$.

The loose idea of the condition is this: at every level $\overline{G}$, and for every instance B of a formula we want comprehension over that is defined at one of the levels $\overline{F}$ preceding $\overline{G}$, we require that there be a predicate $G \in G_1$ and a fresh constant h so that Gh is equivalent to $B(y/h)$.

Call a V-stratified model satisfying the above condition an $\mathbb{A}, V$-stratified model

Theorem 7. *Every theorem of* $\mathbf{B2Q}^{\mathbb{A}}$ *is valid in every* $\mathbb{A}, V$*-stratified model.*

Proof. Clearly the only thing to check is whether the comprehension axioms are valid. To that end, suppose $A \in \mathbb{A}$ and S is an $\mathbb{A}, V$-stratified model. Then $\exists X \forall y(Xy \leftrightarrow A)$ is valid in S just if all of its substitution instances are valid. In turn, this happens just if, for all $\overline{F} \in V$ and $B \in in_{\overline{F}}(A)$, $S, \overline{F}, \ell_{\overline{F}} \vDash \exists X \forall y(Xy \leftrightarrow B)$.

Note by Extendibility there is $\overline{G} \in V$ with $\overline{F} \subsetneq \overline{G}$ and $F_1 \subsetneq G_1$. Thus, by the new semantic condition, there is $G \in G_1$ and $\overline{H} \supsetneq \overline{G}$ with $h \in H_0 - G_0$ so that $Gh \in v_{\overline{H}}(t)$ iff $t \vDash B(y/h)$. Choose $\ell_{\overline{F}} \sqsubseteq_{P_{\overline{F}}} p$. By Prime Extension Down, there is then $\ell_{\overline{G}} \sqsubseteq q$ with $q{\downarrow}^{\overline{G}}_{\overline{F}} = p$. But since $\ell_{\overline{G}} \sqsubseteq q$, $\ell_{\overline{H}} \sqsubseteq q{\uparrow}^{\overline{H}}_{\overline{G}}$.

Now, since $Gh \in v_{\overline{H}}(t)$ iff $S, \overline{H}, t \vDash B(y/h)$, $S, \overline{H}, \ell_{\overline{H}} \vDash Gh \leftrightarrow B(y/h)$. So since $\ell_{\overline{H}} \sqsubseteq q{\uparrow}\frac{\overline{H}}{\overline{G}}$, $S, \overline{H}, q{\uparrow}\frac{\overline{H}}{\overline{G}} \vDash Gh \leftrightarrow B(y/h)$. Thus $S, \overline{G}, q \vDash \forall y(Gy \leftrightarrow B)$. It follows that $S, \overline{F}, \ell_{\overline{F}} \vDash \exists X(Xy \leftrightarrow B)$. □

Theorem 8. *If C is a nontheorem of* $\mathbf{B2Q}^{\mathbb{A}}$*, then there is an $\mathbb{A}, V$-stratified model in which C isn't valid.*

Proof. Extend the usual canonical model by adding to the language designated predicates that do what we need. □

4.3. **The 'Full' Semantics.** Call a function $T_{\overline{F}} \longrightarrow 2^{F_0}$ a generalized formula. Loosely, a generalized formula represents a way an atomic formula might have been interpreted. In the full semantics, at every level, every generalized formula in fact represents an atom; more to the point, we require that the following be satisfied:

- If $\phi : T_{\overline{F}} \longrightarrow 2^{F_0}$ is a function, then there is $F \in F_1$ so that $Fh \in v_{\overline{F}}(t)$ iff $h \in \phi(t)$.

Clearly full models are $\mathbb{A}$,V-stratified models for all $\mathbb{A}$. Equally clear is that, qua models of language, they're quite odd: each language at each level has uncountably many predicates, and if $F_0 \subsetneq G_0$, then $\overline{G}$ is uncountably enriched over $\overline{F}$. But such are the wages of sin.

Let's write $\mathbf{B2Q}^{full}$ for the set of formulas that are valid in all full models. Whether $\mathbf{B2Q}^{full}$ admits a recursive axiomatization at all is not clear to me. There are well-known reasons for suspecting it isn't that I won't rehearse. But there is also surprising evidence on the other side; e.g. in [9] it's shown that the second-order version of FDE (which many take to be a relevant logic) *is* recursively axiomatizable.

In fact, [9] proves a range of interesting things about second-order FDE (and about second-order LP) and it would be a fun project to see exactly which of them remain true about second-order **B**—or other second-order relevant logics, for that matter. But those are jobs for the future.

5. Adding Metarules

Let's return to the funny business involving comprehension and look at a concrete example of what goes wrong. So, suppose we wanted to prove the following was a theorem of the system we get by adding $\exists X \forall y(Xy \leftrightarrow Ray)$ as our lone comprehension axiom:

$$\exists X \forall y_1 \forall y_2((Xy_1 \to Xy_2) \leftrightarrow (Ray_1 \to Ray_2))$$

A natural thought is that we would prove this by showing it follows from the only thing it really could follow from (namely the comprehension axiom we've added) and that we'd accomplish this goal by proving the following:

$$\exists X \forall y(Xy \leftrightarrow Ray) \to \exists X \forall y_1 \forall y_2((Xy_1 \to Xy_2) \leftrightarrow (Ray_1 \to Ray_2))$$

This, in turn would most naturally be proved by using R9 applied to the following:

$$\forall X[\forall y(Xy \leftrightarrow Ray) \to \exists X \forall y_1 \forall y_2((Xy_1 \to Xy_2) \leftrightarrow (Ray_1 \to Ray_2))]$$

Of course, the right way to prove a universal is by universalization, so we expect to first prove the following:

$$\forall y(Xy \leftrightarrow Ray) \to \exists X \forall y_1 \forall y_2((Xy_1 \to Xy_2) \leftrightarrow (Ray_1 \to Ray_2))$$

And, since the existential consequent here follows from an instance, we expect to prove *that* by proving *this*:

$$\forall y(Xy \leftrightarrow Ray) \rightarrow \forall y_1 \forall y_2((Xy_1 \rightarrow Xy_2) \leftrightarrow (Ray_1 \rightarrow Ray_2))$$

But this is really just a universalized version of axiomatic transitivity. Thus, one obvious way to get from ordinary comprehension to all the fancy comprehension we really want is by adding axiomatic transitivity to our logic.

But this isn't the only way to go; an alternative way to do things is to add Bradian metarules.

Before doing so, a confession: I've always been a bit befuddled by Brady's metarules. So what I'd like to do here is describe how I've come to understand what they mean. And since we're at the confessing game, I'll also go ahead and tell you that since what I'm about to say is a bit complicated and what Brady usually says (see e.g. [1] or [2] or [3]) isn't, I worry that maybe I'm getting things wrong. I suppose if that *is* the case, then what we need isn't Bradian metarules, but pseudoBradian metarules.

Whichever way it is, we need some definitions. We define the terms 'derivation of A' and 'metaderivation of R in which $x/X/Y$ has/hasn't been universalized' by simultaneous recursion as follows:

- Each instance of one of A1-A12 is a derivation of itself.
- Each instance of one of R1-R4 and R8-R10 is a metaderivation of itself in which nothing has been universalized.
- Each instance of R5 is a metaderivation of itself in which x has been universalized.
- Each instance of R6 is a metaderivation of itself in which X has been universalized.
- Each instance of R7 is a metaderivation of itself in which Y has been universalized.
- If Δ is a metaderivation of $\dfrac{A_1 \ldots A_n}{B}$ and $\delta_1, \ldots, \delta_n$ are derivations of $A_1, \ldots, A_n$, then $\Delta[\delta_1, \ldots, \delta_n] := \dfrac{\delta_1 \cdots \delta_n}{B}$ is a derivation of B.
- If Δ_1 is a metaderivation of $\dfrac{A_1 \cdots A_n}{B}$ and Δ_2 is a metaderivation of $\dfrac{B \qquad C_1 \cdots C_m}{D}$ then $\Delta_3 := \left(\dfrac{\dfrac{A_1 \cdots A_n}{B} \qquad C_1 \cdots C_m}{D}\right)$ is a metaderivation of $\dfrac{A_1 \cdots A_n \qquad C_1 \cdots C_m}{D}$. x (resp. X; Y) has been universalized in Δ_3 iff it has been universalized in Δ_1 or in Δ_2.
- If δ is a derivation of A and Δ_1 is a metaderivation of $\dfrac{A \qquad B_1 \cdots B_n}{C}$ then $\Delta_2 := \left(\dfrac{\delta \qquad B_1 \cdots B_n}{C}\right)$ is a metaderivation of $\dfrac{B_1 \cdots B_n}{C}$. x (resp. X; Y) has been universalized in Δ_2 iff it has been universalized in Δ_1.[5]

[5]Note that using the generalization rules in the derivation δ does *not* make it the case that one has used generalization in Δ_2.

- If Δ is a metaderivation of $\dfrac{A}{B}$ in which x (resp. X; Y) has not been universalized, then $\dfrac{\exists x A}{\exists x B}$ (resp. $\dfrac{\exists X A}{\exists X B}$; $\dfrac{\exists Y A}{\exists Y B}$) is a metaderivation of itself in which x (resp. X; Y) has not been universalized.

We say that A is a theorem just if there exists a derivation of A.

The 'metarule' bit is in the last bullet in the above list. The idea is that we close the set of rules under a metarule that roughly looks like this:

$$\text{If } A \Rightarrow B, \text{ then } \exists x A \Rightarrow \exists x B$$

Recall now, if you will, the funny business about comprehension from the previous section. The issue was this: to get a well-behaved universal, we need our logic to contain as theorems not only all instances of comprehension *for* all formulas, but also *along* all formulas. We also saw that, provided we had axiomatic transitivity along, giving ourselves access to comprehension for all formulas along atoms—which is to say, giving ourselves what's usually called 'full' comprehension—sufficed for ensuring all instances of comprehension both for and along all formulas.

The argument establishing this broke if we didn't have transitivity. But as we noted, there was something a bit fishy going on—a slightly different argument *did* work given any *particular* instance, but not if all we knew was that *there was* an instance.

Examining the explanation of the metarule above, one suspects that it directly gets around this issue. Indeed, what the metarule seems to say is that anytime you have an argument that works when given an instance, you're allowed to conclude that its existential analogue works.

Happily, this suspicion turns out to be correct: $\mathbf{B2Q}(\overline{F})$ contains a metaderivation of the following in which no predicate universalization at all occurs:

$$\frac{\forall y(Xy \leftrightarrow A(y))}{\forall y_1 \forall y_2((Xy_1 \to Xy_2) \leftrightarrow (A(y_1) \to A(y_2)))}$$

Thus, the metarule on board, the following is a metaderivation of itself:

$$\frac{\exists X \forall y(Xy \leftrightarrow A(y))}{\exists X \forall y_1 \forall y_2((Xy_1 \to Xy_2) \leftrightarrow (A(y_1) \to A(y_2)))}$$

So if we write **B2QE** for **B2Q** plus the existential metarule, then if $A \in \mathbf{A}$, then $\exists X \forall y_1 \forall y_2((Xy_1 \to Xy_2) \leftrightarrow (A(y_1) \to A(y_2)))$ is a theorem of $\mathbf{B2QE}^{\mathbb{A}}$. And from here, we can very easily finish the job we started in §4.2 of showing that $\forall X(A \to A(X/B))$ is a theorem.

In short, adding the metarule lets us bootstrap our way from comprehension along atoms to comprehension along formulas of arbitrary complexity, and thus to a well-behaved universal.

Here are two more fun facts about Bradian metarules. Fun fact the first: I don't know how to modify my semantic story to accommodate this metarule. Fun fact the second: it's entirely clear how to modify the semantics to accommodate the other Bradian metarule; viz. the metarule 'if $A \Rightarrow B$, then $C \vee A \Rightarrow C \vee B$'. I'll leave the proof to the reader, but all it takes to model this is adding the condition $\ell \in P$ to the semantics. So I suspect that there is a natural way to accommodate the above metarule in the semantics, I just haven't figured out what it is yet.

6. Conclusion

Some natural options for next steps include adding λ terms, exploring second-order analogues of various relevant theories—arithmetic comes to mind, given [6]—and a rethinking of identity using higher-order resources. Of course, the most glaring thing to do as a next step is to figure out exactly what to do about comprehension.

An alternative next step would be to attempt to work *backwards* from an appropriate second (or higher) order relevant logic to a relevant set theory of some sort. It's been shown (see [5]) that such theories aren't going to work in exactly the way one might have hoped. But there's also good reason (see e.g. [4]) to think such theories have some promise.

All of this points to a general thing worth keeping in mind: there are a range of interesting *expressive* extensions of the usual languages that relevance logicians concern themselves with that are massively, embarrassingly underexplored. The problem seems to be essentially the one we encountered with comprehension above: in many cases, the classical logicians have set the syllabus and decided not just how things ought to be explored, but what there is to explore in the first place. But we're not in school anymore, and thus there's no real reason to stick to the syllabus. So go on; explore the strange rich expressive potential we've been gifted. And when you come back, tell us how the semantics goes.

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