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# THE LOGICAL FORM OF DETERMINERS

## INTRODUCTION

Since Frege and Russell, one of the key projects in the philosophy of language has been to elucidate the underlying logical form of various natural language constructions. Among the constructions of central interest have been quantified sentences. So, for example, sentences containing the determiners 'all', 'a', 'no', and 'the' have been argued to have the logical forms indicated in (1–4).

(1) All $A$s are $B$s: $\forall x(A(x) \rightarrow B(x))$

(2) An $A$ is $B$: $\exists x(A(x)\ \&\ B(x))$

(3) No $A$ is $B$: $\forall x(A(x) \rightarrow \sim B(x))$

(4) The $A$ is $B$: $\exists x(A(x)\ \&\ \forall y(A(y) \rightarrow x = y)\ \&\ B(x))$

Montague showed that it was possible to derive the logical forms for these sentences in a systematic way in the lambda calculus if one translated the determiners as follows.

(1′) All: $\lambda P \lambda Q \forall x(P(x) \rightarrow Q(x))$

(2′) An: $\lambda P \lambda Q \exists x(P(x)\ \&\ Q(x))$

(3′) No: $\lambda P \lambda Q \forall x(P(x) \rightarrow \sim Q(x))$

(4′) The: $\lambda P \lambda Q \exists x(P(x)\ \&\ \forall y(P(y) \rightarrow x = y)\ \&\ Q(x))$

Montague also showed, however, that a model-theoretic semantics could be provided directly for surface English form, and that one could dispence with translations like those in (1′–4′).

It was also observed (by many) that no first-order logical form could be provided for a number of natural language determiners, including 'most', 'more than half', 'few', 'infinitely many', etc. Subsequent research on the logic of determiners (e.g. by Wiggins (1980), Barwise

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and Cooper (1981), Higginbotham and May (1981), Keenan and Stavi (1986)) thus concentrated on the model theory of determiners, and either skirted the question of what the logical form of determiners may be, or assumed their logical forms to be trivial.

The upshot of this focus in research has been that a number of determiner properties have been explored, but that these properties have been studied as exclusively semantic phenomena. In this paper I shall argue that determiners and some of their properties can also be studied as features of the logical forms of determiners. Specifically, I shall (1) offer a language L* in which logical forms can be provided for determiners such as 'most', 'infinitely many', etc., (2) show that the logical forms provided allow us to syntactically characterize the property of directional entailingness, and consequently (3) give a syntactic account of the licensing of negative polarity items by determiners, and (4) provide syntactic accounts of certain logical inferences involving these determiners.

## 1. L* AND THE LOGICAL FORM OF DETERMINERS

While it can be proved that quantifiers like 'most' cannot be defined in first order logic,[1] this does not close the door on the question of whether they can be defined in other languages. In this section I shall briefly describe the treatment of determiners in a language L* developed in Law and Ludlow (1985).

We begin by enriching first order logic with the introduction of subscripted objectual quantifiers. For example: $(\exists_{\geqslant 2}x), (\exists_{\geqslant 3}x)$, etc. These may be informally understood as "there are two (or more) $x$'s, such that ..." and "there are three (or more) $x$'s such that ...". We also introduce subscripted universal quantifers of the form $(\forall_{\geqslant n}x)$ which are defined as $\sim (\exists_{\geqslant n}x) \sim$.

Next, we introduce substitutional quantifiers ($\Pi$ a universal substitutional quantifier over numerals, and $\Sigma$ the corresponding existential quantifier) which can quantify into the subscript positions. So, for example, we would have sentences like the following.

(5) $\Pi n \exists_{\geqslant n} x(\text{man}(x)\ \&\ \text{mortal}(x))$

This would be read as saying that for all numerals $n$, there are greater than or equal to $n$ $x$'s such that $x$ is a man and $x$ is mortal. In short, it expresses the proposition that infinitely many men are mortal. 'Finitely many men are mortal' may be expressed as in (6).

(6) $\sim \Pi n \exists_{\geqslant n} x(\text{man}(x) \ \& \ \text{mortal}(x))$

Finally, we can render 'Most men are mortal' as in (7).

(7) $\Pi n((\exists_{\geqslant n} x)\text{man}(x) \rightarrow (\exists_{\geqslant f(n)} x)(\text{man}(x) \ \& \ \text{mortal}(x)))$

$f$ is a primitive recursive operation on inscriptions which, in the case of the quantifier 'most' will yield $1/2n$ (less fractional remainder) plus 1. As shown in Law and Ludlow (1985), the account can be extended to determiners such as 'more than $1/3$', 'more than $1/33$', etc.[2]

In addition to 'Most $A$s are $B$s', we can also give an analysis of 'More $A$s than $B$s are $C$s'. Intuitively, the comparative form of the determiner 'more... than...' suggests that there is a comparison of one quantity with another.[3] This intuition can be cashed out in the following fashion.

(8) $\Sigma n((\exists_{\geqslant n} x)(A(x) \ \& \ C(x)) \ \& \sim (\exists_{\geqslant n} x)(B(x) \ \& \ C(x)))$

There is a choice in the analysis of 'More than $1/2$ the $A$s are $B$s'. One can render it immediately along the lines of (7), or alternatively, one can choose to be faithful to the comparative structure of the determiner 'more than $1/2$'. If we take the latter route, then an analysis like that in (9) suggests itself.

(9) $\Sigma n(\sim (\exists_{\geqslant n} x)A(x) \ \& \ (\exists_{\geqslant f(n)} x)(A(x) \ \& \ B(x)))$

Because L* is an extension of first order logic, all the first-order definable determiners can be rendered in the manner familiar from introductory logic texts. However, L* also allows us to define such determiners with subcripted quantifiers. For example, '$a(n)$' can be rendered as in (10), 'no' can be rendered as in (11).

(10) An $A$ is $B$: $\exists_{\geqslant 1} x(A(x) \ \& \ B(x))$

(11) No $A$ is $B$: $\sim \exists_{\geqslant 1} x(A(x) \ \& \ B(x))$

Are there standard first order determiners which cannot be rendered as subscripted quantifiers? Not if one takes certain liberties in the analysis of determiners like 'all', 'every', etc. They may be rendered as in (12).

(12) All $A$s are $B$s: $\sim \exists_{\geqslant 1} x(A(x) \ \& \sim B(x))$

The Russellian analysis of definite descriptions can also be accommodated using the subscripted notation, as indicated in (13).

(13) The $A$ is $B$: $\exists_{\geqslant 1}x(A(x))$ & $\sim \exists_{\geqslant 2}x(A(x))$ & $\sim \exists_{\geqslant 1}x(A(x)$ & $\sim B(x))$

Details for L* are provided in Appendix I.

Finally, we will find it useful to introduce the notion of L* *canonical form.* A formula of L* is in L* canonical form iff it is built up from elementary formulae using only the symbols $\Pi, \Sigma, \exists, \forall, n, \geqslant, f, (, ), \vee, \&, \sim$, and the formula is in *prenex normal form*, and the quantifier free portion of the formula is in *disjunctive normal form.*

## 2. DIRECTIONAL ENTAILINGNESS, AND THE NOTION OF POSITIVE AND NEGATIVE OCCURRENCE

A number of natural language phenomena are sensitive to the phenomenon of directional entailingness, where upward and downward entailingness can be understood as follows.

An environment $\alpha$ in a sentence $\phi$ is *upward entailing* iff $[_{\phi} \ldots [_{\alpha} \ldots A \ldots] \ldots]$ entails $[_{\phi} \ldots [_{\alpha} \ldots B \ldots] \ldots]$, where all $A$s are $B$s

An environment $\alpha$ in a sentence $\phi$ is *downward entailing* iff $[_{\phi} \ldots [_{\alpha} \ldots A \ldots] \ldots]$ entails $[_{\phi} \ldots [_{\alpha} \ldots B \ldots] \ldots]$, where all $B$s are $A$s

The phenomenon of directional entailingness suggests a taxonomy for natural language determiners. Let's define the *first position* of a determiner as that in which the nominal occurs, and the *second position* as that in which the predicate occurs. Some determiners, such as 'some', are upward entailing in both first and second position. This is clear from the fact that (14) entails (15–16)a but not (15–16)b.

| | | | |
|---|---|---|---|
| (14) | some men run | | |
| (15) | (a) | => | some things run |
| | (b) | $\neq>$ | some tall men run |
| (16) | (a) | => | some men move |
| | (b) | $\neq>$ | some men run fast |

'No' is downward entailing in both first and second position. This can be seen from the fact that (17) entails (18–19)b, but not (18–19)a.

| | | | |
|---|---|---|---|
| (17) | no man runs | | |
| (18) | (a) | $\not=>$ | no things run |
| | (b) | $=>$ | no tall men run |
| (19) | (a) | $\not=>$ | no men move |
| | (b) | $=>$ | no men run fast |

There will also be determiners such as 'every' which are downward entailing in the first position, but upward entailing in the second. Thus (20) will entail (21)b and (22)a, but not (21)a and (22)b.

| | | | |
|---|---|---|---|
| (20) | every man runs | | |
| (21) | (a) | $\not=>$ | every thing runs |
| | (b) | $=>$ | every tall man runs |
| (22) | (a) | $=>$ | every man moves |
| | (b) | $\not=>$ | every man runs fast |

It is important to note that there are also determiners (for example 'most') which are neither downward not upward entailing in the first position. (23) entails neither (24)a, (24)b, nor (25)b, but it does entail (25)a.

| | | | |
|---|---|---|---|
| (23) | most men run | | |
| (24) | (a) | $\not=>$ | most things run |
| | (b) | $\not=>$ | most tall men run |
| (25) | (a) | $=>$ | most men move |
| | (b) | $\not=>$ | most men run fast |

These relations can be characterized set-theoretically,[4] but we can also offer an account of directional entailingness which makes use of the logical form of these constructions within L*. This account of directional entailing environments for determiners can be stated in the form of T-1.

(T-1) For $S$, a sentence in L* canonical form, and $\alpha$ an elementary formula in $S$: If $\alpha$ has all positive occurrences in $S$, then $\alpha$ is in an upward entailing environment in $S$. If $\alpha$ has all negative occurrences in $S$ then $\alpha$ is in a downward entailing environment in $S$.[5]

We will also find it useful to speak of a *predicate* having positive and negative occurrences. We can say that a predicate $P$ in $S$, has all negative (positive) occurrences in $S$ iff all elementary formulae in which $P$ occurs in $S$ are formulae which occur only negatively (positively) in $S$.

Let's consider T-1 first in the light of standard first order quantifiers. Recall (1–3) from above, but placed in L* canonical form.

(1″) All $A$s are $B$s: $\forall x(\sim A(x) \vee B(x))$

(2) An $A$ is $B$: $\exists x(A(x)\ \&\ B(x))$

(3″) No $A$ is $B$: $\forall x(\sim A(x) \vee \sim B(x))$

Notice given (1″) as an analysis for 'every', T-1 correctly predicts that '$A(x)$' (having only a negative occurrence) will be in a downward entailing environment, and '$B(x)$' (having only a positive occurrence) will be in an upward entailing environment. Given (2) as an analysis of 'a(n)', T-1 correctly predicts that both positions should be upward entailing. Finally, given (3″) as an analysis of 'no', T-1 correctly predicts that both positions will be downward entailing.

Notice that these properties are preserved in the analyses which utilized the subscripted quantifiers.

(10′) An $A$ is $B$: $\exists_{\geqslant 1} x(A(x)\ \&\ B(x))$

(11′) No $A$ is $B$: $\forall_{\geqslant 1} x(\sim A(x) \vee \sim B(x))$

(12′) All $A$s are $B$s: $\forall_{\geqslant 1} x(\sim A(x) \vee B(x))$

What happens when we consider the more complex determiners under the analyses they received in part 1? Given the analysis proposed for 'infinitely' in (5),

(5) $\Pi n \exists_{\geqslant n} x(A(x)\ \&\ B(x))$

T-1 correctly predicts that it is upward entailing in both positions. Given the L* canonical form for 'finitely many $A$s are $B$s',

(6′) $\Sigma n \forall_{\geqslant n} x(\sim A(x) \vee \sim B(x))$

T-1 correctly predicts that it is downward entailing in both positions. What of 'most'? Consider its L* canonical form.

(7′) $\Pi n(\forall_{\geqslant n}x)(\exists_{\geqslant f(n)}y)\ (\sim A(x) \vee (A(y)\ \&\ B(y)))$

Here '$A$' has a positive occurrence in '$A(x)$' and a negative occurrence in '$A(y)$', so '$A$' itself will be in neither an upward nor downward entailing environment. '$B$' has only a positive occurrence so it is predicted to be in an upward entailing environment.[6] Examples like this point out one of the more interesting features of L* and the value of defining positive (negative) occurrences over formulae in L*. What is suggested in the case of (7′) is that it will be possible to substitute for '$A(x)$' and '$A(y)$', but not for '$A$'. Thus we correctly predict the following distribution of possible inferences.

(7*)
- a. Most people sing
- b. ≠> Most *women* sing
- c. ≠> Most *animals* sing
- d. => If there are $n$ *women* then at least $f(n)$ people sing
- e. => If there are $n$ people then at least $f(n)$ *animals* sing

We noted above that 'more that $1/2$′ is best rendered as in (9).

(9) $\Sigma n(\sim(\exists_{\geqslant n}x)A(x)\ \&\ (\exists_{\geqslant f(n)}x)(A(x)\ \&\ B(x)))$

which will in turn have the canonical form in (9′).

(9′) $\Sigma n(\forall_{\geqslant n}x)(\exists_{\geqslant f(n)}y)(\sim A(x)\ \&\ (A(y)\ \&\ B(y)))$

Once again, the determiner is correctly predicted to be neither upward nor downward entailing in its first position, and to be upward entailing in its second.

It is interesting to note that given the canonical form of 'More $A$s than $B$s are $C$s' in (8′), we correctly predict that the first ('$A$') position is upward entailing, the second ('$B$') position is downward entailing, and the third ('$C$') position is neither upward nor downward entailing (since '$C$' has both a positive and negative occurrence).

(8′) $$\Sigma n(\exists_{\geqslant n}x)(\forall_{\geqslant n}y)((A(x)\ \&\ C(x)\ \&\ \sim B(y)) \vee (A(x)\ \&\ C(x)\ \&\ \sim C(y)))$$

Notice again, however, that additional inferences are predicted since '$C(y)$' is in a downward entailing environment and '$C(x)$' is in an upward entailing environment. For example, if $(\forall y)(C^\star(y) \rightarrow C(y))$ it should be possible to substitute '$C^\star(y)$' for '$C(y)$' throughout (8) and (8′). Analogous reasoning applies to the positively occurring '$C(x)$'. This is demonstrated in (8*) where across-the-board substitution for '$C$' is blocked, but where (upward) substitution for '$C(x)$' is possible, and (downward) substitution for '$C(y)$' is possible.

(8*) More dogs than cats bite *people*

a. $\not=>$ More dogs than cats bite *men*

b. $\not=>$ More dogs than cats bite *animals*

c. => for some $n$, there are $n$ things that are dogs that bite people and there are fewer than $f(n)$ things that are cats that bite *men*

d. => for some $n$, there are $n$ things that are dogs that bite *animals* and there are fewer than $f(n)$ things that are cats that bite people

A proof of T-1 is provided in Appendix II.

## 3. DIRECTIONAL ENTAILINGNESS AND THE LICENSING OF NEGATIVE POLARITY ITEMS

Ladusaw (1980) argued that negative polarity items (expressions such as 'any' and 'ever') are licensed (or triggered) by downward entailing environments. For instance, the (a) examples are cases where the negative polarity items (negpols) are found in upward entailing environments. The corresponding (b) examples, in which the negpols occur in downward entailing environments are much more acceptable.[7]

(26) (a) *John saw anything/anyone

(b) John didn't see anything/anyone

(27) (a) *Max said that he had ever been there
(b) Max never said he had ever been there

As we saw in Section 2.1, there are many other possible downward entailing environments, including the first or second position of certain determiners. Thus we have the following distribution of facts.[8]

(28) (a) Every [person who has ever been to NY] [has returned to it]
(b) *Every [person who has been to NY] [has ever returned to it]

(29) (a) *Some [person who has ever been to NY] [has returned to it]
(b) *Some [person who has been to NY] [has ever returned to it]

(30) (a) No [person who has ever been to NY] [has returned to it]
(b) No [person who has been to NY] [has ever returned to it]

One of the consequences drawn by Ladusaw was that certain apparently syntactic well-formedness conditions must in fact appeal to the semantics of the expression.

> We have seen that the property of [unacceptable sentences with negpols] which renders them unacceptable is to be defined in terms of the entailments licensed by certain lexical items, rather than by simply marking certain morphemes with a semantic feature. It seems to follow directly that no grammar can in principle distinguish [between acceptable and unacceptable sentences with negpols] unless its semantic component aims higher than at simply disambiguating sentences by deriving 'logical forms' for them to the goal of providing a theory of entailment for the language it generates (1980; pp. 14–15).

Clearly, if $L^*$ translations of natural language sentences are construed as giving the logical forms of those sentences, then there is an immediate answer to Ladusaw – specifically, that the property which renders some sentences with negpols unacceptable can be defined syntactically in terms of negative and positive occurrences.[9] On the other hand, Ladusaw is correct in insisting that there is an interesting relation between entailment relations and the licensing of negpols.

Theoretical questions aside, it has been known for some time that, despite its appeal, there are a number of counter-examples to Ladusaw's generalization. Specifically, there are a number of cases in which negpols are licensed, though they do not appear in a downward entailing environment.

(31) Most [people who know anything about politics] [hate it]

(32) Most [dinosaurs that ever ate a mammal] [hated it]

(33) The [philosopher that knows anything about logic] [can snow anyone]

(34) More [cats] than [dogs] [have ever eaten a mouse]

Considered in their $L^*$ canonical form, one observation regarding these sentences stands out. These are all sentences which have translations into $L^*$ canonical form in which the predicate containing the negpol has at least one negative occurrence. It might be, then, that a better generalization is available. Specifically, it is not directional entailingness which is key, but rather whether the negpol has at least one negative occurrence when in $L^*$ canonical form.

This revised generalization, if correct, would also shed light on certain facts about conditionals discussed in Heim (1984). Heim notes that while the antecedents of conditionals in natural language routinely license negpols (consider (35) and (36)),

(35) If you ever ate a balut, you know what I'm talking about

(36) If anyone sees you eat a balut, they will never talk to you again

they are not downward entailing environments. So, for example, if the antecedents of (35) and (36) are "strengthened," truth is not necessarily preserved.

(35′) # If you ever ate a balut but don't remember doing it, you know what I'm talking about

(36′) # If anyone sees you eat a balut deep fried in batter (and doesn't know what it is), they will never talk to you again

This anomaly can be accounted for if, following several current theories (e.g. Lycan (1984), Kratzer (1989)), conditionals are thought of as having an implicit quantification over events or situations. Specifically, if conditionals like the above are thought of as holding generally (that is, for most cases of balut eating), then they will receive an analysis like that in (37) below. Here (37) receives the surface analysis (37′) and its analysis in L* canonical form is (37″).

(37) Usually, if a man enters he will turn on the TV

(37′) $\Pi n(\sim (\exists_{\geqslant n} e)\text{enters}(\text{a man}, e) \rightarrow$
$(\exists_{\geqslant f(n)} e' Re)(\text{enters}(\text{the man}, e')\ \&$
$\text{turns-on-TV}(\text{the man}, e')))$

(37″) $\Pi n(\forall_{\geqslant n} e)(\exists_{\geqslant f(n)} e' Re)$
$(\sim \text{enters}(\text{a man}, e) \vee (\text{enters}(\text{the man}, e')\ \&$
$\text{turns-on-TV}(\text{the man}, e')))$

Notice that predicates occurring in the antecedent of the conditional in (37) will have both positive and negative occurrences. So we predict that these predicates will be in neither upward nor downward entailing environments, but that the environment *will* license negpols.[10]

## 4. DIRECTIONAL ENTAILINGNESS AND FORMAL THEORIES OF LOGICAL CONSEQUENCES

The phenomenon of directional entailingness also has implications for formal (i.e. syntactic) theories of logical consequence. The idea that logical inferences might be characterized formally is not new, of course. The idea dates to Aristotle. Of particular interest to us, however, is an observation by Hoeksema (1986) that medieval logicians discussed two formal inference paradigms, *dictum de omni*, and *dictum de nullo*.

*dictum de omni*: $\ldots A \ldots$
*All As are Bs*
$\ldots B \ldots$

*dictum de nullo*: Neg. $\ldots A \ldots$
*All Bs are As*
Neg. $\ldots B \ldots$

The medievals speculated that these two paradigms might underlie much of syllogistic reasoning. For example, it was observed that *modus ponens*, was simply a special case of *dictum de omni*, and *modus tollens* was a special case of *dictum de nullo*. (To see this, consider the case where '...' is null.)

One of the exercises of medieval logic was to explore the domain in which these paradigms would hold. It was known, for example, that a single negation, though not two nested negations, could introduce the *de nullo* environment. It was also argued that *de nullo* could also be introduced by distributed terms (e.g. the first position of 'all'). We are now equipped with a way of generalizing the medievals' observation. A predicate in a sentence $S$ is in a *de nullo* environment if and only if it has all negative occurrences in the translation of $S$ into $L^\star$ canonical form. The *de omni* inference paradigm applies to those predicates which have only positive occurrences in $L^\star$ canonical form.

Finally, if we move from the philosophy of language to the philosophy of mind and take $L^\star$ to be a description of the logical form of the language of thought, there are potential consequences in cognitive science. In the philosophy of cognitive science, one can distinguish two broad approaches to the characterization of human inferential capacities. Fodor (1980) advocates the *formality condition*, that inferences must be syntactic in nature. On the other hand, Barwise (1989) advocates what we might call the *situated inference hypothesis*. On this view, inferences must appeal to the semantic contents of our mental representations (in a particular context).[11]

> *formality condition*: Cognitive science ought to characterize human inference from $S_1 \ldots S_n$ to $S$ by appeal to the forms of $S_1 \ldots S_n$ and $S$.
>
> *situated inference hypothesis*: Characterizing an inference from $S_1 \ldots S_n$ to $S$ requires appeal to the

semantic contents of $S_1 \ldots S_n$ and $S$ (in context $c$). Appeal to syntactic form is not enough.

One occasionally hears the claim that determiners like 'most', 'few', and 'infinitely many' pose difficulties for the formality condition because one must appeal to their semantic contents (in a contex) in order to adequately characterize their role in human inferential capacities. According to this line of thinking, attempts to treat inferences involving these determiners *and* respect the formality condition require one to hold that agents mentally represent the complex model-theoretic properties of determiners. It is then argued that it is not plausible to attribute such rich representations to human agents – to attribute such richly structured representations to an agent is simply to "overburden" the beliefs of the agent.[12]

However, the possibility of defining determiners in languages like $L^\star$ shows that such claims must be approached with extreme caution. Clearly, a number of inferences can be formally characterized in $L^\star$ – and hence (in these cases) appeal to the model theory of determiners can be avoided. Moreover, the cognitive demands of making such inferences in $L^\star$ would be minimal; all that would be required is the inspection of the string for negations. Of course the question of whether other inferences involving determiners can be characterized formally cannot be answered by pronouncement, but only by careful investigation.

## 5. CONCLUSION

I have argued that in $L^\star$ the properties of a number of natural language determiners can be studied as features of their logical forms. Directional entailingness is one such property. The ability to license negpols is another. The literature on generalized quantifiers is vast, of course, and there are a number of other properties which have not been explored here. While I have not discussed those properties, it should be clear that a number of them will carry over into $L^\star$ – in particular those properties which are related to cardinality. This is apparent from the isomorphism between the set-theoretic expression $|\{x\colon Ax\}| \geqslant n$ and the $L^\star$ expression $(\exists_{\geqslant n} x)Ax$. I also have not addressed the question of whether all natural language determiners are definable in L*. It is clear, for example, that a number of possible determiners cannot be defined in L* (at least as it stands): 'uncountably many', 'measure of zero', 'fewer than zero', etc. Whether these are genuine natural language determiners is another matter.

My fundamental concern, of course, is not the utility of L*, but rather with the general project of finding formal languages in which natural language determiners can be defined, and in which the properties of those determiners can be studied as features of their logical forms. My goal in this paper has been to show that such a project can be carried out (at least up to a point) and that it is valuable to study determiners in this way.

## APPENDIX I

L is a first-order language with variables $v_0, v_1 \ldots$ and a (finite) stock of descriptive symbols. The following is a description of an expansion L* of L.

(A) *Vocabulary.*

In addition to that of L:

1. $0, ', \geqslant$
2. $f_i$, for $i \geqslant 0$
3. $x_i$, for $i \geqslant 0$
4. $a_i$, for $i \geqslant 0$ (substitutional variables)
5. $\Pi$ (universal substitutional quantifier symbol).

(B) *Quantifier-subscript terms* (*QS-terms*).

1. 0 and $a_i$, for $i \geqslant 0$
2. If $t$ is a QS-term, so is $t'$
3. If $t_1, \ldots, t_r$ are QS-terms, so is $f_i(t_1, \ldots, t_r)$.

Among the QS-terms, the numerals $0, 0', 0'', \ldots$ are distinguished as the substituends of the variables $a_i$. ($0(n)$ will be abbreviated by the appropriate Arabic numeral.)

A QS-term is closed if it contains no occurrences of the variables $a_i$.

(C) *Quantifier expressions.*

In addition to those of L: $\Sigma a_i$ and $\Pi a_i$.

(D) *Formulae.*

1. L-formulae are L*-formulae
2. Any expression obtained by substituting a variable $x_i$ for every free occurrence of an L-variable in an L-formula is an L*-formula
3. If Q is a quantifier expression and $A$ is an L*-formula, then $QA$ is an L*-formula

To (1–3) add any rules for the formation of L-formulae – e.g., those for sentential connectives.)

An elementary formula (e-formula) is an L*-formula which contains no occurrences of a substitutional variable.

A pure substitutional formula (ps-formula) is an L*-formula in which no substitutional quantifier occurs within the scope of an objectual quantifier.

(E) *Some definitions*:

$$\Sigma a_i = \mathrm{df} \sim \Pi a_i \sim$$
$$\forall_{\geqslant t} x A = \mathrm{df} \sim \exists_{\geqslant t} x \sim A$$

(F) A relation $\vdash$ of reduction of QS-terms to numerals is defined as follows.

1. $n \vdash n, n$ a numeral
2. If $t \vdash n$, then $t' \vdash n'$
3. $f(m) \vdash n$ iff $\mathrm{g}(m) = n$

$$\text{where } \mathrm{g}(m) = \begin{cases} (m/2 - r) + 1 \\ \text{where for some integer } k,\ m = 2k + r \\ \text{and } 0 \leqslant r < 1 \end{cases}$$

(G) Truth for $e$-sentences and ps-sentences can be defined in terms of a primitive predicate T0 for truth in L.

1. If $S$ contains no Q$S$-terms other than numerals, obtain $S^\star$ as follows:
   starting with the innermost, replace each subformula of $S$ of the form $\exists_{\geqslant n} xA$ by $\exists v_1, \ldots, \exists v_n[(\&_{i \neq j} v_i \neq v_j) \ \& \ (\&_{1 \leqslant i \leqslant n} A(v_i))]$, where $v_1, \ldots, v_n$ are the first $n$ L-variables which do not occur in $A$, and $A(v_i)$ is the result of substituting $v_i$ for $x$ in $A$. Then,

   $$T(S) \quad \text{iff} \quad T_0(S^\star).$$

2. If $t_0, \ldots, t_k$ are all the Q$S$-terms occurring in $S$ and $t_i \vdash n_i$, let $S\#$ be the result of substituting $n_i$ for every occurrence of $t_i$ in $S$, $0 \leqslant i \leqslant k$. Then,

   $$T(S) \quad \text{iff} \quad T(S\#).$$

3. If $S$ is of the form $\Pi a S'$, then

   $T(S)$ iff for every numeral $n$, $T(S'(n))$

   (where $S'(n)$ is the result of substituting $n$ for $a$ in $S'$).

4. (Clauses for connectives, as usual.)

(H) *Satisfaction.*

Let $M$ be a model for L. (For simplicity, assume that the descriptive vocabulary of L consists of a sole unary predicate symbol $R$. Subscripts on '$\models$' are dropped.) Below, $A$ and $B$ contain no free substitutional variables unless otherwise noted.

1. $\sigma \models Rv_i$ iff $\sigma(i) \in R^M$
2. $\sigma \models \sim A$ iff $\sigma \not\models A$
3. $\sigma \models A \ \& \ B$ iff $\sigma \models A$ and $\sigma \models B$
4. $\sigma \models \exists v_i A$ iff for some $\sigma' \approx_i \sigma$, $\sigma' \models A$
5. $\sigma \models \Pi \alpha_i A$ iff for all numerals $n, \sigma \models A(n)$, where $A$ contains at most $\alpha_i$ free
6. $\sigma \models \exists_{\geqslant 0} xA$
7. $\sigma \models \exists_{\geqslant 1} xA$ iff $\sigma \models \exists v_i A(v_i)$, where $v_i$ is the first L-variable which does not occur in $A$
8. $\sigma \models \exists_{\geqslant n} xA$ iff $\sigma \models \exists v_i \ \exists_{\geqslant n-1} x(v_i \neq x \ \& \ A(v_i) \ \& \ A)$ where $v_i$ is the first L-variable which does not occur in $A$ and $n$ is other than 0 or 1
9. $\sigma \models \exists_{\geqslant t} xA$ iff for some $n$, $t \vdash n$, $\sigma \models \exists_{\geqslant n} xA$.

## APPENDIX II: PROOF OF T-1

For L* formulae in prenex normal form where the quantifier free portion of the formulae are in disjunctive normal form (DNF).

DEFINITION. A formula $\zeta$ has a *negative occurrence* in $\alpha$ iff $\zeta$ is in the scope of negation in $\alpha$. (Recall that when in DNF a formula will either be in the scope of one negation or no negations.)

DEFINITION. A formula $\zeta$ is in a *downward entailing* (DE) environment in $\alpha$ iff $(\forall x)((\xi x \rightarrow \zeta x)\ \&\ \beta = \alpha\ |^{\zeta}_{\xi}) \rightarrow (\alpha \rightarrow \beta)$.

Prove: if a formula $\zeta$ has only negative occurrences in $\alpha$ then $\zeta$ is in a downward entailing (DE) environment. (Proof for the upward entailing case is logically dual.)

Proof by induction on definition of formulae:

1. *Basis*

1.0. Let $\phi$ be any quantifier free propositional sentence in DNF. Suppose that a sentence letter $S$ occurs only negatively in $\phi$. Then $\phi$ may be rendered as a disjunction of two complex formulae, with one of the disjuncts containing all occurrences of $S$ and the other disjunct, call it $\chi$, containing no occurrences of $S$. The occurrences of $\sim S$ in the first disjunct may be factored by distribution to obtain a formula of the form:

$$(\sim S\ \&\ (\ldots \vee \ldots \vee \ldots)) \vee \chi.$$

Consider first the case where $\chi$ is null (i.e., where $\sim S$ occurs in each original disjunct):

$$\sim S\ \&\ (\ldots \vee \ldots \vee \ldots).$$

Assume $(\xi \rightarrow S)\ \&\ \beta = \phi\ |^{S}_{\xi}$ and show $\phi \rightarrow \beta$.
Since $\xi \rightarrow S$, by contraposition we have $\sim S \rightarrow \sim \xi$.
Therefore $\phi \rightarrow \beta$, since

$$\sim S\ \&\ (\ldots \vee \ldots \vee \ldots) \rightarrow \sim \xi\ \&\ (\ldots \vee \ldots \vee \ldots).$$

If $\chi$ is not null then since $\sim S\ \&\ (\ldots \vee \ldots \vee \ldots) \rightarrow \sim \xi\ \&\ (\ldots \vee \ldots \vee \ldots)$ as above, then $[\sim S\ \&\ (\ldots \vee \ldots \vee \ldots) \vee \chi] \rightarrow [\sim \xi\ \&\ (\ldots \vee \ldots \vee \ldots) \vee \chi]$ by propositional logic.

Suppose next that $\phi$ is a quantified formula.

2.1. Let $\phi = \exists x\Psi$. Let $c_i$ be constant names for each element of the domain $D$. Then $\phi$ is equivalent to $\bigvee_{c_i \in D} \Psi \mid^{x}_{c_i}$ by a truth-functional expansion of the quantifier. It then follows by the induction hypothesis for each disjunct that $\Psi c_i \rightarrow \Psi c_i \mid^{\beta}_{\alpha}$, so $\bigvee_{c_i \in D} \Psi c_i \mid^{\beta}_{\alpha}$.

But the latter is just a truth-functional expansion of the following:

$$\exists x \Psi x \mid^{\beta}_{\alpha} .$$

Hence $\exists x\Psi \rightarrow \exists x \Psi x \mid^{\beta}_{\alpha}$.

2.2. Let $\phi = \forall x\Psi$. Let $c_i$ be constant names for each element of the domain $D$. Then $\phi$ is equivalent to $\bigwedge_{c_i \in D} \Psi \mid^{x}_{c_i}$ by a truth-functional expansion of the quantifier. It then follows by the induction hypothesis for each conjunct that $\Psi c_i \rightarrow \Psi c_i \mid^{\beta}_{\alpha}$. Reasoning proceeds as in 2.1.

2.3. Let $\phi = \Pi n \exists_{\geqslant n} x\Psi$. For a proof by *RAA*, we assume the induction hypothesis fails for 2.3. Then there exists a smallest counterexample $\exists_{\geqslant n'} x\Psi$. But for any given $n', \exists_{\geqslant n'} x\Psi$ is logically equivalent to $\exists x_1 \ldots \exists x_{n'} [x_1 \neq x_2 \;\&\; x_1 \neq x_3 \;\&\ldots x_1 \neq x_{n'} \;\&\; x_2 \neq x_3 \;\&\ldots x_2 \neq x_{n'} \ldots \;\&\; x_{n'-1} \neq x_{n'} \;\&\; \Psi]$, and we have already shown that the induction hypothesis holds for formulae of standard first order logic.

2.4. Let $\phi = \Sigma n \exists_{\geqslant n} x\Psi$. For a proof by *RAA*, we assume the smallest $n'$ for which the induction hypothesis fails. Reasoning proceeds as in 2.3.

2.5. Let $\phi = \Pi n \forall_{\geqslant n} x\Psi$. For a proof by *RAA*, we assume the induction hypothesis fails for 2.5. Then there is a smallest counterexample $\forall_{\geqslant n'} x\Psi$. But $\forall_{\geqslant n'} x\Psi$ is equivalent to $\sim \exists_{\geqslant n'} x \sim \Psi$, which is in turn equivalent to a formula of first order logic, for which the induction hypothesis has been shown to hold.

2.6. Let $\phi = \Sigma n \forall_{\geqslant n} x\Psi$. For a proof by *RAA*, we assume the smallest $n'$ for which the induction hypothesis fails. Reasoning proceeds as in 2.5.

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## NOTES

1 See Rescher (1962). Barwise and Cooper (1981) extend the result to finite domains.

2 'Few' and 'Many' will have similar analyses, but will differ in that some contextual operator will determine what operation $f$ is to perform. For example, in a given context, many might be 9/10. Then $f$ will be a primitive recursive operation on inscriptions which yields 9/10 $n$ less fractional remainder. The formula for 'many' would otherwise be identical to (9). Likewise, in a given context, few might be 1/10. The analysis of 'Few men are mortal' would then be as in (i), with $f(n)$ yielding 1/10 $n$.

(i) $$\Pi n(\sim (\exists_{\geqslant n} x)\,\mathrm{man}(x) \rightarrow \sim (\exists_{\geqslant f(n)} x)(\mathrm{man}(x)\ \&\ \mathrm{mortal}(x))).$$

3 This point was brought to my attention by Richard Larson.

4 See Larson (1990) for an example.

5 We can define a *positive occurrence* as an occurrence within the scope of an even number of negations. A *negative occurrence* is an occurrence within the scope of an odd number of negations. Since formulae in $\mathrm{L}^*$ canonical form are in DNF, an occurrence of a formula will either be in the scope of a single negation or none, hence in $\mathrm{L}^*$ canonical form a negative occurrence will simply be an occurrence within the scope of negation.

It is also possible to prove the converse of T-1 (that is, to show that all downward entailing environments are environments with all negative occurrences) if we say that a formula $\alpha$ in $S$ has all negative occurrences iff all occurrences of $\alpha$ in $S'$ are in the scope of negation and $S'$ is logically equivalent to $S$. See Appendix II for a proof of T-1.

[6] Similar results are available for 'many' and 'few' if we adopt the analyses proposed in Note 2 above. 'Many' would have the L* canonical form given in (7′) and 'few' would have the following L* canonical form.

(i) $\Pi n(\forall_{\geqslant n}x)(\forall_{\geqslant f(n)}y)\ (A(x) \vee \sim A(y) \vee \sim B(y))$.

Note that this correctly predicts that 'few' will be neither upward nor downward entailing in the first position, but downward entailing in the second.

[7] Care is necessary to distinguish these instances of 'any' from so-called "free-choice 'any'," which need not appear within the scope of negation. Examples would be (i) and (ii).

(i) I might have said anything. I was furious.

(ii) I would have punched anyone who said that to me.

It's clear that 'any' in these examples has a different meaning than it does in (26) above. 'Any' in (26) means something or someone. Not so in (i) and (ii), where there is the suggestion that everything is sayable and that everyone is a possible target of my rath. A standard analysis is that free-choice 'any' is a universal quantifier with wide scope over the modal. Not only is there an apparent difference in meaning, but as a general rule, free-choise 'any' needs to be licensed by modals (see Carlson (1981) for a discussion of the distribution of free-choice 'any'). It is interesting to note that in certain natural languages, 'any' and free-choice 'any' are not homophones. Serbo-Croatian, for example, has 'iko' (any) and 'bilo' (free-choice any) (see Progavac (1990) for discussion).

[8] These specific examples are drawn from Larson (1990).

[9] This suggestion would also require that the general treatment of directional entailingness can also be extended to certain intuitively negative predicates. For example, predicates like 'doubts', 'forgets', and 'difficult'.

(i) I doubt that he ever speaks to her

(ii) John forgot to bring anything to dinner

(iii) It is difficult to find any squid at Safeway

Following Baker (1970) and Linebarger (1987), we might decompose these lexical items into a more primitive predicate and a negation. The suggestion is natural, given that all of the verbs which induce downward entailing environments appear to have a negative connotation to them. For example, in the pairs doubt/believe, forgot/remembered, difficult/easy any informant would judge that 'doubt', 'forgot', and 'difficult' were the pair members which have a negative element. The idea would be that (i)–(iii) might be rendered as in (i′)–(iii′).

(i′) I [not-believe] that he ever speaks to her

(ii′) John [not-remember-ed] to bring anything to dinner

(iii′) It is [not-easy] to find any squid at Safeway

Ladusaw (1980) objected to this proposal, observing that the analysis predicts that sentences like (i′)–(iii′), when conjoined with the negation of their corresponding sentence in (i)–(iii), should form a contradiction. In some cases, this prediction is born out (e.g. (iv)), but in others it is not (e.g. in (v)).

(iv) John didn't remenber to bring anything, but he didn't forget to either

(v) It isn't hard to find squid at Safeway, but it isn't easy either

It is interesting to note that the predicates which do not invoke contradictions admit a middle range of objects or events which have neither of the contrastive properties (e.g. which are neither hard nor easy). Furthermore, one need not define this middle range as the negation of two positive attributes. So, for example, we need not define an average sized flea as one which is neither large nor small for a flea. Indeed, there is no reason to suppose that the middle range cannot be a kind of primitive predicate. Accordingly, there is a natural extension of the original decomposition thesis which avoids the problem in these cases. Consider the following analysis.

(vi) hard = [not-easy and not-average]

(vii) sorry = [not-glad and not-neutral]

In each case there is an implicit predicate which covers the middle ground. If, for example, 'hard' is analyzed as in (vi), then it is clear why there is no contradiction in (vi) – in short, because there is nothing contradictory about (vi′).

(vi′) It's not [not-easy and not-average] to find squid at Safeway,
but it isn't easy.

[10] This revised generalization also opens the door to a possible explanation of why questions license negpols (e.g. 'Did you see anyone?', 'Who saw anything?'), when it is unclear whether the notion of directional entailingness is even applicable (but see Higginbotham (1993; appendix) for a suggestion that it is). If a question such as 'Is the Earth round?' is thought of as having the underlying form of a disjunction ('The Earth is round or it is not the case that the Earth is round') then we would have a natural explanation of why yes/no questions license negpols when they do not seem to support either downward entailing or upward entailing inferences. More generally, we can adopt a modified version of the analysis of questions in Higginbotham and May (1981) and particularly Higginbotham (1993), in which questions refer to abstract questions which in turn are partitions of the possible states of nature into mutually exclusive cells $P_i$ for $i \in I$. So, for example, the yes/no question 'Is the Earth round?' expresses an abstract question which in turn can be analyzed as a partition of two possible states of the world, represented as follows: $\{\{$The Earth is round$\} \mid \{$The Earth is not round$\}\}$. A wh-question such as 'What did you see?' in a world containing three objects $A, B$ and $C$, would express an abstract question having the following partition $\{\{$You see $A\}, \{$You see $B\}, \{$You see $C\}\} \mid \{\{$You see $A\}, \{$You see $B\}, \{\sim$ You see $C\}\} \mid \{\{$You see $A\}, \{\sim$ You see $B\}, \{\sim$ You see $C\}\} \mid \ldots$. The intuition is that wh-questions allow for partial answers like "Well, I didn't see $B$."

If we adopt this account, but regard the partition as consisting of sentences in L* canonical form, we can account for why questions license negpols – every partition contains at least one sentence in which the negpol has an occurrence within the scope of negation.

[11] It is important to see that 'semantic' is being used here in the sense of some kind of language/world relation. So Barwise has something much stronger in mind than simple appeal to model-theoretic semantics. Likewise, appeal to model-theoretic semantics does not necessarily flout Fodor's formality condition. Strictly speaking, model-theoretic representations of a determiner can satisfy the formality condition if it is plausible that humans can mentally represent the model theory of determiners. Such a possibility is not *prima facie* absurd. Computational models which essentially represent the model theory of Montague Grammar have been known for some time. (See Friedman, Moran, and Warren (1978a,b).)

[12] The terminology is drawn from Perry (1986). I should point out that Perry himself does not address the issue of determiners, but is concerned with theories of belief which attribute to agents extremely rich internal representations of the environment.

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