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Wittgenstein on Colour Exclusion: Not fatally mistaken

ABSTRACT: The problem of colour exclusion is not fatal to Ludwig Wittgenstein’s early philosophy, nor was it the catalyst for his later philosophy. The remarks in the *Tractatus* about the impossibility of the simultaneous occurrence of two colours at a point in the visual field sit comfortably with the remarks in the rest of the book, the discussion of mathematical physics above all. Furthermore Wittgenstein’s second thoughts about the impossibility were a consequence, not the cause, of the subsequent turn in his philosophy.

Ludwig Wittgenstein is generally held to have stumbled badly in 1916/1917 when discussing the impossibility of two colours occurring at the same place at the same time and to have moved towards his later philosophy in 1929 on realising he had stumbled. The solution to the problem of colour exclusion sketched in 6.3751 of *Tractatus Logico-Philosophicus* is deemed ruinous for the book, and Wittgenstein’s belated appreciation of the point reckoned responsible for his dismantling the philosophy he had promoted and his developing the philosophy of *Philosophical Investigations*. As a leading commentator has it: “The programme implicit in 6.3751 was to show that when ‘*A* is red’ is fully analysed into its constituents, its truth will perspicuously entail that *A* is not blue”, a programme later rejected on the grounds that “A is red and A is blue” is “not a simple logical *contradiction*, to be revealed as such by *analysis*, but a *nonsense*” (Hacker 1986, 108-109). I first consider what Wittgenstein says about colour exclusion in the *Tractatus*, then take up the question of what set him on the path to the *Philosophical Investigations*. Mainly I aim to show that Wittgenstein was initially on firm ground and the problem of colour exclusion was not what prompted him to rethink what he had earlier written but other considerations, considerations that required that he revised his explanation of colour exclusion.

 The received view of the treatment of colour exclusion in the *Tractatus* is not without basis in the text. In the third of the three paragraphs devoted to the topic, Wittgenstein observes that while elementary propositions may be true simultaneously, conjunctions of propositions attributing different colours to a point in the visual field are contradictions. He writes (within parentheses): “It is clear that the logical product [conjunction] of two elementary propositions [*Elementarsätze*] can neither be a tautology nor a contradiction. The assertion that a point in the visual field has two different colours at the same time, is a contradiction” (1922, 6.3751). From this, it follows – *Elementarsätze* being logically independent and contradictory propositions being invariably false – that propositions about the colours of points in the visual field are complex, not elementary. In other words, just as the conventional wisdom has it, Wittgenstein holds that assertions to the effect that a point in the visual field has two colours are not conjunctions of elementary propositions (presumably if either conjunct is elementary, so is the other). As Wittgenstein puts it in material he drew on when drafting 6.3751: “If the logical product of two propositions is a contradiction and the propositions appear to be elementary propositions, we can see that in this case the appearance is deceptive. (E.g.: A is red and A is green)” (1979, 91).

 This thought was not a last minute addition reasonably ignored. The discussion of colour exclusion in the *Tractatus* derives from remarks composed some time before the book was compiled, and there is every reason to think Wittgenstein was satisfied with what he says at 6.3751. He recycled remarks in the *Prototractatus* penned in 1917/1918 practically verbatim, remarks that in turn distilled entries in *Notebooks 1914-1916* dated 16 August 1916 and 8 January 1917 (1971, 6.3751-6.3752 and 1979, 81, 91/2000, MS 104, 88, 94). Had he come to doubt his treatment of the problem, he would surely have reworked it, not reproduced it with very small changes and minor omissions. (The only noteworthy difference between the versions is that in the final version Wittgenstein speaks of points in the visual field instead of points as having two colours.) Nor can the remarks on the topic be regarded as incidental. Wittgenstein could not duck the task of accounting for the contradictory character of “A is red and A is green” given his view that “[t]here is only *logical* necessity” (6.37) and the fact that the principle that elementary propositions are logically independent was central to his thinking as a whole, not just to the discussion of 6.3751 (compare 4.211 and 5.134).

 To clarify the difficulty alleged to bedevil the treatment of colour exclusion in the *Tractatus*, it helps to consider Wittgenstein’s treatment of the example of a point – or point in the visual field – that is simultaneously both red and green. On the one hand it is argued that the conclusion that “A is red” and “A is green” cannot be elementary is “unwelcome” since the attribution of a colour to a point is “close to our conception of what an elementary proposition ought to be like” (Black 1964, 367-368). On the other hand it is argued that “A is red” and “A is green” cannot be complex since an analysis of them in terms, say, of “surface reflectance potentials” would only shift the problem to one of explaining “the impossibility of a particle (photon) having different velocities at the same time” (Landini 2007, 87). Simply put, the contention is that once it is granted that the attribution of two colours to a single point in the visual field at the same time is contradictory, attributions of colours to points, whether regarded as elementary or as non-elementary, cannot be accommodated within the framework of the *Tractatus*, and it is only puzzling that Wittgenstein took so long to see the problem and come up with a solution.

 Though Wittgenstein is now widely taken to have erred badly regarding colour exclusion at 6.3751 of the *Tractatus*, this was not always so. One would expect Bertrand Russell, a philosopher with a nose for philosophical gaffes second to none, to have spotted Wittgenstein’s error and commented on it in the “Introduction” he wrote for the *Tractatus*. Were the discussion of the topic as flawed as regularly charged, it is hard to imagine Russell passing over it, as he does, in silence. He does not hesitate to point out what he takes to be defects in the *Tractatus*, some large, some small, and it is unlikely he would have refrained from noting so big a defect, if such it be, along with the others. This would have meant him pulling punches to an unusual degree, and it is a fairly safe assumption that in his eyes Wittgenstein had not blundered terribly. While his not saying anything about how the likes of “A is red and A is green” are to be understood is no guarantee that he agreed with Wittgenstein, his saying nothing provides some reason for thinking he did not reject Wittgenstein’s thinking out of hand. It is even reasonable, given the tenor of his “Introduction”, to conclude that he found Wittgenstein’s explanation of colour exclusion to be, if not entirely adequate, very nearly so.

 The plausibility of the charge that the discussion of colour exclusion in the *Tractatus* falls dreadfully short is also compromised by the fact that Wittgenstein seems to have been unfazed by the objection that an analysis of “A is red and A is green” would get us no further ahead. He would have known that Frank Ramsey had argued in his 1923 review of the *Tractatus* that “Mr Wittgenstein is only reducing the difficulty to that of the *necessary* properties of space, time, and matter or the ether”, properties “hardly capable of further reduction of this kind” (Ramsey 1923, 473). Ramsey visited him in Austria some months after writing the review and it is scarcely credible that the supposed difficulty was not mentioned when the two of them went through the *Tractatus* remark by remark. In the event, however, it seems to have been Ramsey, not Wittgenstein, who ended up persuaded he had things wrong. He never subsequently pressed the objection, and when he hazarded his own explanation of colour exclusion in 1927, he allowed that “formal logic ... presupposes that the truth-possibilities of atomic sentences are possible”, declared that the self-contradictoriness of “This is both blue and red” is “concealed by a defective analysis” and compared the contradiction with a “mechanically impossible” situation (1990, 48).

 Moreover there is reason to believe that, when working on the ideas of the *Tractatus*,Wittgenstein’s remarks about colour exclusion made good sense and the objection that they do not anything but conclusive. As regards the complaint there is no better example of an elementary proposition than an attribution of colour to a point in the visual field, Wittgenstein expressly challenges the claim. He would have been well aware of the plausibility of the suggestion if only because he says in the *Notebooks 1914-1916*: “As examples of simples I always think of points in the visual field (just as parts of the visual field always come before my mind as typical composite objects)” (1979, 45, dated 6 May 1915). However even when he wrote this he had his doubts about the supposition that the likes of “A is red” are unanalysable. For right after confessing that he thinks of points in the visual field as simples, he asks: “But what is a uniformly coloured part of my visual field composed of? Of *minima sensibilia*?” (1979, 45, dated 7 May 1915; also compare 1979, 64, dated 18 June 1915, and 1979, 67, dated 20 June 1915). And between writing these remarks and compiling the *Tractatus* he clearly came to think that, however “unwelcome”, the fact that “A is red and A is green” is a contradiction shows that the attribution of a colour to a point cannot be elementary. It is even likely that he set down the material that became the third paragraph of 6.3751 in January 1917, some months after drafting the material that became the other two paragraphs, for the express purpose of signalling that “A is red” and “A is green” should be regarded as non-elementary because their conjunction is “a contradiction”.

 And it is just as unlikely that Wittgenstein would have been swayed by the objection, urged by Ramsey in his review and repeatedly echoed ever since, that it only defers the problem to regard attributions of colours to points as non-elementary. Had Wittgenstein believed “A is red” is analysable in terms of positions, velocities, wavelengths, surface reflectance potentials or the like, he would, no two ways about it, have left himself open to the charge of putting off the evil day. But it is difficult to credit that he missed that an analysis of “A is red” in more basic terms would be beset by a similar, if not an identical, problem. Nor is he properly read, as he is habitually read, as intimating in the second paragraph of 6.3751 – oblivious of just how little this would achieve – that attributions of colours to points are analysable in spatio-temporal terms. In this paragraph he confines himself to noting a comparable case. “Let us”, he says, “consider how this contradiction presents itself in physics: Somewhat as follows: That a particle cannot at the same time have two velocities, *i.e.* that at the same time it cannot be in two places, *i.e.* that particles in different places at the same time cannot be identical”. This is not to say “A is red and A is green” is analysable in physical terms, only to observe that there are similar contradictions in physics, “X is at two positions” and “X has two velocities”, for instance.

 The burden of Wittgenstein’s remarks in the third paragraph of 6.3751 is not that attributions of colours to points in the visual field are analysable in spatio-temporal terms (or terms of the same ilk) but that they are analysable in a manner consistent with the fact that attributing two colours to a point is a contradiction (and the conception of elementary propositions as logically independent). In the *Tractatus* non-reductive analysis is extolled (compare 4.0031, 5.532-5.5321), and Wittgenstein would not, I conjecture, have balked at the idea that assertions about the joint occurrence of colours are subject to such analysis. While reductive analyses in terms of physical or other more basic notions only relocate the problem, nonreductive analyses, which remain at the same level as the original assertion, may resolve it. “Area A in the visual field has at least one blemish and is without blemish” is analysable as “There is a blemish that A has, and it is false that there is a blemish that A has”, and it is not out of the question that contradictions of the sort referred to at 6.3751 are subject to comparable – if more complex – analysis. There is no guarantee that “A is red and A is green” can be re-expressed as a formal contradiction stating that something both is and is not the case but neither is it impossible that Wittgenstein had something of the kind in mind.

 The assumption that Wittgenstein believed that attributions of colours to points could be re-expressed as formal contradictions without falling foul of central claims of the *Tractatus* deserves sympathetic consideration especially as the discussion of colour exclusion in the *Tractatus* echoes the discussion of the topic in Russell’s *Principles of Mathematics*, a work Wittgenstein had studied closely (Landini 2007, 86-88 and Lugg forthcoming). Wittgenstein reprises – one could say appropriates – Russell’s view that the impossibility is to be traced to the essential nature of colour, the sole difference between his treatment of the problem and Russell’s being that he traces the impossibility to “the logical structure of colour” (1922, 6.3751, first paragraph), Russell to a “fundamental characteristic of matter” (1903, 467). While Wittgenstein does not refer to “impenetrability”, the characteristic of matter that Russell takes to show that “no two colours can be in the same place at once”, he agrees that two colours occurring together is logically excluded. And it is also striking that Wittgenstein follows Russell in the second paragraph of 6.3751 when he draws attention to the occurrence of similar contradictions in physics, Russell having already mentioned in the *Principles* that colour exclusion is similar to mechanical impossibility (1903, 473). No wonder, then, that Wittgenstein does not go into detail and Russell says nothing about 6.3751. Both of them would have taken the problem to have been already taken care of.

 Wittgenstein’s solution of the problem occurs in the first paragraph of 6.3751, a paragraph much less examined than the other two. It is, I believe, insufficiently recognised that in this paragraph Wittgenstein, following in Russell’s footsteps, provides an explanation of colour exclusion that is immune to criticism of the sort generally taken to put paid to the argument of the *Tractatus*. ~~Far fr~~om simply ignoring the question of what prevents two colours occurring simultaneous at the same point in the visual field at the same time, he observes that this is impossible because of the nature of colour itself. Summarising his thinking, he writes: “For two colours, *e.g.* to be at one place in the visual field, is impossible, logically impossible, for it is excluded by the logical structure of colour [*logische Struktur der Farbe*]”. By this he means the occurrence of two colours at a point is logically ruled out by our concept of colour or, what for him is the same, by how colour is thought and spoken about. Nothing can be simultaneously red and green all over, he would have us recognise, because it is essential to colour – i.e. integral to the logic of colour concepts – that occurrences of red exclude occurrences of green and *vice versa*.

 When the first paragraph of 6.3751 is seen as providing Wittgenstein’s leading idea, the second paragraph poses no special problem. There is no hiatus between his claiming that two colours cannot be at the same point in the visual field because of the logical structure of colour and his observing that “this contradiction presents itself in physics”. He is simply noting that colour exclusion is matched by position and velocity exclusion and points in the visual field can no more have two colours given the logical structure of colour than particles can have two positions or two velocities given the logical structure of matter and motion, i.e. in the one case as in the other the impossibility is logical, not empirical. In fact in *Notebooks 1914-1916* Wittgenstein compares the impossibility of a point of more than one colour with the impossibility of a particle at more than one place, which is excluded by “the structure of space and of particles” (1979: 81). While “a particle cannot be in two places at the same time does look more like a logical impossibility”, he writes, the impossibility of a point having two colours at the same time is ruled out by how we conceive the world, not by how the world happens to be, the reason being that “the very language of physics reduces it to a kinetic [i.e. mathematical] impossibility”.

 The account of colour exclusion in 6.3751 that emerges when the spotlight is shone on the first paragraph of the number rather than the other two is thus very different from the standard account. It is no mystery why Wittgenstein would have believed he had answered the question and found Ramsey’s criticism unpersuasive, perhaps even why Ramsey did not repeat his objection after visiting Wittgenstein in 1923. In the *Tractatus* colour is regarded, like matter and motion, as logically structured, and colour exclusion taken, along with position and velocity exclusion, to be logically instead of empirically or metaphysically impossible. For Wittgenstein colour attributions, position attributions and velocity attributions can all be analysed in a way that brings out the contradictory character of the “impossibilities”. Moreover he gives the impression that he took this sort of analysis to comport with ~~such analyses to conform to~~ the argument of the rest of the book, not least to the account of logic he provides. But if so, how exactly did he understand the all-important notion of “the logical structure of colour” and how did he imagine – given how he construes logic – that he could show the joint occurrence of red and green is logically excluded by appealing to this notion?

 Wittgenstein does not tackle this question explicitly in the *Tractatus*. There are in the book, however, hints as to how he would have answered it had he been challenged. At 2.0131 he implies that red, yellow, green, blue and the rest are logically interrelated inasmuch as the relationships of entailment and inconsistency among colours mirror the relationships of entailment and inconsistency among points in “colour space”. He writes: “A speck in a visual field need not be red but it must have a colour; it has, so to speak, a colour space [*Farbenraum*] around it” (also 1971, 2.0142). It cannot be by chance that he placed the discussion of colour exclusion in the part of the *Tractatus* devoted to mathematical physics, material in which the notion of a space of possibilities looms large. He is best understood as going along with Russell and taking his account of mathematical physics to extend – with trivial revisions and qualifications – to colour, i.e. as holding that colour is no less mathematically representable than matter and motion. Most charitably read, he was of the opinion that our system of colour concepts, like mechanics, determines “a form of description”, one akin to a grid or network (“*Netz*”) (1922, 6.341). In his view it too “is *purely* geometrical, and all its properties can be given a priori” (6.35; also 1979, 38).

 The conception of mathematical representation was deep in Wittgenstein’s thinking. Possibly as a result of his training in science, possibly as a result of his own early reading, he viewed qualities and quantities in the manner favoured by mathematical physicists and by all appearances regarded colour along with position and velocity as representable by points in abstract spaces of possibilities. Ludwig Boltzmann and Heinrich Hertz are the first two thinkers on the list of influences Wittgenstein drew up in 1930 (1998, 16), and their way of representing phenomena underlies his reflections on language and the world. It is unmistakable that in his discussion of Newtonian Mechanics (1979, 35; 1922, 6.341) he is working with what Hertz variously calls “representations of the principles of mechanics”, “modes of expression” and “modes of representation” (1899, 4, 9, 24). And it cannot be fortuitous either that in *Notebooks 1914-1916* he says – just before referring to Hertz’s “invisible masses” – he has “felt for a *long* time” that mechanics is a form of description (1979, 36, dated 6 December 1914). Moreover in the *Tractatus* itself he alludes to “Hertz’s Mechanics, on Dynamic Models” (4.04) and expresses his thinking at one point in “the terminology of Hertz” (6.361). (Also I fancy Wittgenstein would have been familiar with Russell’s account of Hertz’s dynamics in Chapter LIX of the *Principles*.)

 When Wittgenstein is understood as construing representation mathematically and equating the logical structure of colour with the geometry of colour space, his treatment of colour exclusion is readily understood. To his way of thinking, it is no more puzzling than position exclusion, specks in the visual field being associated with one and only one colour in colour space just as particles are associated with one and only one position in Euclidean space (and the squares of a net being either empty or filled). Given that different colours are represented by different points in colour space, it immediately follows that specks can have no more than one colour and ‘A is red’ and ‘A is green’ cannot be true together. Just as the logical structure of matter and motion (and “[t]he form of description” associated with Newtonian mechanics) entails that particles are located at one and only one position, so the logical structure of colour (and the form of description associated with our system of colour concepts) entails that specks in the visual field have one and only one colour. A “form of description”, whether for colour or matter and motion, defines what can and cannot be sensibly said and hence defines what is logically “excluded [*ausgeschlosse*n]” (6.3751).

 The suggestion that Wittgenstein embraces the Hertz/Boltzmann mathematical conception of representation when explaining colour exclusion is further bolstered by the fact that he takes phenomena, without any apparent reluctance, to be representable by their coordinates (compare locating a city by means of its latitude and longitude). In a remark composed early on he writes: “We might conceive two co-ordinates aP and bP as a proposition stating that the material point P is to be found in the place (ab)” (1979, 20, dated 29 October 1914), and it is beyond belief that he regarded the attribution of colour to a point any differently. On this conception there is no asserting that a point can be red and green simultaneously since there is no coordinate corresponding to both “A is red” and “A is green”. The assertion that a point in visual space is correlated with two distinct coordinates in colour space is as absurd as the assertion that a material point is correlated with both (a1b1) and (a2b2) in geometrical space (compare stating that a city has more than one latitude and longitude). As Wittgenstein says: “It is no more possible to present something ‘contradicting logic’ in language than to present a figure contradicting the laws of space in geometry by means of its co-ordinates; or, say, to give the co-ordinates of a point that does not exist” (1979, 40, dated 16 May 1915; also 1922, 3.032).

 If qualities and quantities are representable by points in a multi-dimensional space, attributions of colours to points are representable in functional terms. “[T]he general concept of colour can”, as has been noted, “be represented in language, not by a class of colour-predicates, but by a function *c* which maps points in visual space into a colour space” (Hintikka and Hintikka 1986, 123). (Compare the “height”-function which associates people with their heights.) Adopting this conception of representation, “A is red” is symbolised as “c(A) = r” (where “r” designates red) rather than as “Ra” (where “R” holds a place for “is red”). And likewise “A is green” is symbolised as “c(A) = g” (where “g” designates green) rather than as “Ga” (where “G” holds a place for “is green”). Whence since “a function cannot have two different values for the same argument”, it can be inferred without further ado that “the two propositions are *logically* incompatible”. In other words “c(a) = r & c(a) = g” is logically excluded for the simple reason that the colour function,c, always returns a single value in colour space for each argument in visual (or speck) space. The only remaining question is whether Wittgenstein had this explanation of the impossibility in mind when he wrote the *Tractatus*.

 This may be doubted as “the historical truth seems to be that Wittgenstein never says he is construing colour functionally” and “it may very well turn out that he never assented to it verbally” (Hintikka and Hintikka 1986, 124). But even granting Wittgenstein neither said nor assented to it, it seems wrong to conclude that “the most that we can claim here is that the construal of colour as a function mapping points in visual space into colour space is in keeping with the spirit of [his] thinking”. Leaving aside the possibility that he deployed the construal without a second thought (or considered it too commonplace to belabour), he is most plausibly read as committed to the construal in the *Tractatus*. At 2.0131 (and 1971, 2.01411), he not only observes that that specks in the visual field are surrounded by “a colour space”, he also writes: “A spatial object must lie in infinite space. (A point in space is an argument place)”. Given this view – that objects are functionally related to positions – it would have been remarkable had he not also regarded specks as functionally related to colours, the one as “arguments”, the other as “values”. Moreover if it is “forcefully asserted” at 2.0131 that “attributions of different perceptual qualities are ... represented logically speaking by genuine functions”, why regard mapping construal of colour as merely a “thought experiment” (Hintikka and Hintikka 1986, 123)?

 To argue that Wittgenstein’s discussion of colour exclusion in the *Tractatus* accords well with his remarks about colour space and mathematical physics does not, however, show he is out of the woods. In the first paragraph of 6.3751 he refers to “the logical structure of colour”, not to its mathematical structure. In the second paragraph he speaks of “this contradiction” rather than this mathematical impossibility as arising in physics. And in the third paragraph he states that the assertion that a point in the visual field has two different colours is “a contradiction”, not that it is mathematically inconsistent. One might even be forgiven for thinking that I have unwittingly demonstrated that Wittgenstein is in a deeper hole than generally supposed, his discussion of mathematics and mathematical physics being equally at variance with what he says in the rest of the book. If his account of colour exclusion goes hand in hand with his account of mathematics and mathematical physics, is not the major difficulty alleged to plague the *Tractatus* exacerbated rather than put to rest? Wittgenstein cannot, it may well be argued, have been committed to treating representation like Hertz and Boltzmann without cutting himself off from treating representation, as he certainly did, like Frege and Russell.

 The trouble that I see with this is that it wrongly presumes that Wittgenstein believed mathematical representation is irreducible to logical representation whereas in fact he regarded the two conceptions of representation as going together and took mathematical impossibility to be of a piece with logically impossibility. The most reasonable hypothesis is that he believed Russell had shown that mathematics is in essence logic and following on from this had concluded that it is logically impossible – since mathematically impossible – for a point to be two colours simultaneously. By all indications, he accepted that by specifying “the kinematical conditions for a system of material particles, generalized and expressed in terms of logical constants”, Russell had provided an “abstract logical statement of what rational Dynamics requires its matter to be” (1903, 468). Wittgenstein does not, as commonly believed, challenge Russell on the relationship of mathematics to logic but accepts that he had shown that “[t]he *à priori* truths involved in Dynamics are only those of logic” (488), indeed managed to establish “the logical nature of mathematics ... throughout” (498). As he saw it, any residual errors in Russell’s discussionare remediable without materially compromising his own treatment of mathematics, mathematical physics and colour exclusion.

 The main snag with this defence of Wittgenstein’s argument in the *Tractatus* would seem to be that it fails to take into account that he construes logical propositions very differently from mathematical propositions. It is taken as gospel that he differs from Russell regarding the relationship of mathematics to logic since he explicitly says: “The propositions of logic are tautologies” (1922, 6.1). This does not, however, mean he remains on the hook, there being ample reason to think he did not regard tautologies as narrowly as they are nowadays regarded. The present-day notion of tautology was not in place in the early decades of the Twentieth Century, and Wittgenstein mostly worked with the traditional notion of tautologies as propositions that convey no information (compare 4.461, 5.142, 6.11; also Dreben and Floyd 1991 and Lugg 2003 and 2013). In this regard it bears underlining that Russell did not hesitate to write some years after writing his “Introduction” that “[t]he propositions which form part of logic, or can be proved by logic, are all *tautologie*s” and “all pure mathematics consists of tautologies in [this] sense” (1927, 171). And still more tellingly Ramsey maintained around the same time that “mathematics consists of tautologies” (1990, 176) and referred to the Multiplicative Axiom (i.e. the axiom of choice) as “an obvious tautology” (1990, 222).

 These historical observations notwithstanding, it may be argued that in the *Tractatus* Wittgenstein identifies logic with what is nowadays referred to as predicate logic (if not just sentence logic), a system of logic much too weak to encompass mathematics. There is, I imagine it being objected, no escaping the fact that mathematical propositions cannot be obtained, as Wittgenstein claims all propositions can be obtained, from a base of elementary propositions by successive application of the “N-operator”, the operator that yields the conjunction of the negations of a given set of propositions (5.5 and 6). In response, I would point out that Wittgenstein actually had a conception of logic close, if not equal, in expressive power to logic as Russell conceived it. At 5.501 he mentions three ways of forming propositions, the first covering sentence logic, the second predicate logic and the third higher-order logic, a system strong enough to retrieve at least a fair amount of mathematics (compare Ricketts 2013, 129-139). While 5.501 and other remarks that suggest that Wittgenstein had a rich conception of logic (e.g. 3.331 and 4.1273) require interpretation and it is debatable how much mathematics he could justifiably be said to recover, there can be no denying that he took logic to comprise more than the tautologies of sentence and predicate logic.

 All this notwithstanding, Wittgenstein may be thought at odds with Russell since he holds that mathematics is categorically different from logic. The most important obstacle to what I am urging would seem to be that in the *Tractatus* Wittgenstein eschews Russell’s view of mathematics as an extension of logic when he writes: “The propositions of logic are tautologies. ... The propositions of mathematics are equations [*Gleichungen*]” (6.1 and 6.2). None of this can be gainsaid but neither is it conclusive. Wittgenstein does not state straight-out that mathematics falls in a different category from logic but rather comments on the two endeavours. For him mathematics is “a logical method [*eine logische Methode*]” (1922, 6.2), “a method of logic [*eine Methode der Logik*]” (6.234), and “[t]he equation characterises the standpoint from which I consider the two expressions, that is to say the standpoint of their equality of meaning [*Bedeutungsgleicheit*]” (6.2323). While these remarks require interpretation and scarcely prove that Wittgenstein did not repudiate Russell’s view that mathematics is in essence logic, they show that the easy assumption that they differed substantially over the question of whether mathematics is reducible to logic leaves something to be desired (also compare Friedman 1999, 182-183).

 In favour of the present way of understanding Wittgenstein on mathematics, it is also worth noting that in the *Tractatus* he is concerned with the general question of the nature of mathematics, not with the technical question of its reducibility to logic. In 6.02-6.031, his first series of remarks on mathematics, he suggests that numbers can be generated by repeated application of the successor operation starting from zero in much the same way as propositions can be generated by repeated application of the N-operator. And in 6.2-6.241, his other series on the subject, he observes that mathematical propositions are like logical propositions in that they are “*Scheinsätze*” (6.2), say nothing (6.21), show “the logic of the world” (6.22), do not require anything by way of “intuition” for their proof beyond what “language itself ... supplies” (6.233) and “must be self-evident” (6.2341). None of this runs counter to Russell’s claim that mathematics is at root logic, a view that Russell himself seems to have agreed with. In his “Introduction”, he notes that “[t]here are some respects, in which ... Mr Wittgenstein’s theory stands in need of greater technical development”, especially “his theory of numbers”, he does not find “anything in Mr Wittgenstein’s system that makes it impossible for him to fill this lacuna” (Wittgenstein 1922, 21).

 Consideration of the background to Wittgenstein’s remarks also suggests that his view of mathematics and logic is close to Russell’s. Like most interested parties at the time, he was impressed by the argument that Russell (and Alfred North Whitehead) mounted in *Principia Mathematica* for regarding mathematics as essentially logic. What bothered him was not Russell’s technical argument but his understanding and presentation of it. When compiling the *Tractatus*, Wittgenstein’s chief interest remained – as David Pinsent, a close friend, described it in his diary on 25 August 1913 – “the very fundamental part of the subject”, not Russell’s “purely Mathematical work – for instance most of his ‘Principia’” (1990, 59). As a matter of fact Wittgenstein drafted the remarks on mathematics late in the day (2000, MS 104, 70 and 100-101, 117-118; 1971, 200 and 216-218) and one has the sense that he believed they could be smoothly grafted onto what he had written about logic in *Notebooks 1914-1916*. He seems to have had no compunction about recycling what he had earlier written about mathematics (and mathematical physics) pretty much word for word in the *Prototractatus* and the *Tractatus* after he had introduced the notion of an operation, sketched his “theory of number” and discussed the nature of mathematics. Nor should it be overlooked that in the *Prototractatus* he restates that the likes of “A is red and A is green” are contradictory after giving “[t]he general form of an integer” (2000, MS 104, 70 and 94; 1971, 229 and 203).

 Turning now to the question of why Wittgenstein came to repudiate the discussion of colour exclusion in the *Tractatus*, this too calls for a different answer from the one typically offered. There is no indication that Wittgenstein came to believe he had erred at 6.3751 about the topic prior to 1929, when after some ten years engaged in other business, he started writing again on philosophy. During the period he was only minimally concerned with philosophy and there is no sign of his having recanted what he says in 6.3751, never mind his regarding it as dooming the philosophical vision ~~promot~~ed in the book. Nothing in the material that has come down to us indicates that he did not remain committed to the view that the joint occurrence of two colours at a point in the visual field is excluded by “the logical structure of colour” (and as comparable to position and velocity exclusion). But if he continued to go along, with minor reservations, with Russell’s account of the phenomenon in the *Principles* and to hew to a Hertzian view of mathematical physics, what prompted him to revise his thinking about colour exclusion in the months following his return to philosophy? Nobody denies that he began in 1929 to dismantle the philosophy in the *Tractatus*. What is debatable is only how his thinking about colour exclusion contributed to bringing the house down.

 Wittgenstein is standardly taken to have criticised the discussion of colour exclusion in the *Tractatus* in “Some Remarks on Logical Form”, a paper he ~~initiall~~y planned to deliver at the (British) Aristotelian Society in July 1929 (1993, pp. 29-35). In this paper he is supposed to have targeted the thought advanced in the third paragraph of 6.3751 about the joint attribution of colours to a point in the visual field being contradictory. The contention is that in “Some Remarks on Logical Form” “he stated clearly the inadequacy of the solution [to the problem of colour exclusion] in the *Tractatus*” and “tackled the issue head on” (Hacker 1986, p. 109). This is a simple and seemingly explanatory account of the development of Wittgenstein’s thought. The only hitch is that it labours under considerable difficulty, and there is an alternative account, one in keeping with what I have suggested Wittgenstein says in the *Tractatus* and what he wrote in MS 105, the first surviving document from the period. As I understand the situation, his about-face was induced by considerations other than what he had written at 6.3751, considerations that convinced him that the attribution of a colour to a point in the visual field is elementary, not complex, and colour exclusion required a new explanation.

 As often noted, Wittgenstein treats the impossibility of two colours at a point differently in “Some Remarks on Logical Form” from how he treats it in the *Tractatus*. Rather than claiming analysis reveals the assertion of two colours at a point to be formally contradictory, Wittgenstein writes in the fifth of the six paragraphs of the paper: “Our symbolism, which allows us to form the sign of the logical product of ‘RPT’ and ‘BPT’ [i.e. ‘Colour R and colour B are at place P at time T’] gives no correct picture of reality” (1993, 34). He now regards the invariable falsity of “RPT & BPT” as traceable to the fact that “‘( )PT’ leaves room only for one entity”. Taking “RPT” and “BPT” to be “in a certain sense *complete*” (33) and “RPT & BPT” to lead to “collision” (34), he concludes that it is “a deficiency of our notation that it does not prevent the formation of such nonsensical constructions” (35). What is at stake regarding the question of the role of Wittgenstein’s thinking about colour exclusion in the development of his philosophy, however, is not whether Wittgenstein believed “a perfect notation will exclude such structures by definite rules of syntax” but why he believed it. The crucial question is what persuaded him to regard “RPT” and “BPT” as “*complete*” and conclude that there have to be rules of syntax that “tell us that in the case of certain kinds of atomic propositions ... certain combinations of [truth possibilities] must be left out”.

 It is poorly appreciated, if appreciated at all, that there is no argument in “Some Remarks on Logical Form” for the conclusion that attributions of colours to points are elementary/atomic beyond the bare statement that “RPT” and “BPT” collide. In the fourth paragraph he notes in support of the view that “the statement which attributes a degree to a quality cannot be further analysed” (1993, 33) that “Entity E has two units of brightness” cannot be analysed either as “E has one unit of brightness and E has one unit of brightness” or as “E has *b’* and *b”* units of brightness”, the first analysis being logically equivalent to “E has one unit of brightness”, the second leaving open the “obviously absurd” question of whether “E has one unit of brightness” is the same as “E has *b’* units of brightness” or “E has *b”* units of brightness” (32-33). But this by itself hardly proves that statements of degree, let alone attributions of colours to points, are elementary. And how likely is it anyway that Wittgenstein believed “E has two units of brightness” is beyond the scope of the *Tractatus*? He knew as well as anyone that mechanics deals with degrees of qualities and would have regarded “two units of brightness” as no more problematic than “two units of momentum”.

 Since Wittgenstein says next to nothing in “Some Remarks on Logical Form” in support of treating “RPT” and “BPT” as elementary, it is highly unlikely he was out to establish the point. The least risky assumption is thus that in the fourth paragraph of the paper he is motivating the claim that atomic/elementary propositions may involve numbers, more specifically the claim, stated in the third paragraph, that “numbers will have to enter [the forms of elementary propositions] when – as we should say in ordinary language – we are dealing with properties which admit of gradation” (1993, 32). On this reading of Wittgenstein’s remarks, it is a premise of his discussion, not a conclusion, that “the occurrence of numbers in the forms of atomic propositions is ... not merely a feature of a special symbolism, but an essential and, consequently, unavoidable feature of [our way of representing phenomena]”. He is presupposing rather than arguing for the view that when “we try to get at an actual analysis, we find logical forms that have very little similarity with the norms of ordinary language”, in fact “meet with the forms of space and time with the whole manifold of spatial and temporal objects, as colours, sounds, etc., etc., with their gradations, continuous transitions, and combinations in various proportions” (31).

 “Some Remarks on Logical Form” falls into two parts. In the first three paragraphs Wittgenstein announces that “[e]very proposition has a form and a content” (1993, 29), considers what analysis involves and the type of proposition it reveals (29-31), and floats the idea that an analysis of propositions uncovers the forms of space and time, colours and the like (31-32). Only then, in the remaining three paragraphs, does Wittgenstein discuss the problem of colour exclusion. What he says here rests on the unproven assumption that “numbers (rational and irrational) must enter into the structure of the atomic propositions themselves” (31). This crucial premise – Wittgenstein bills it as his “first definite remark on the logical analysis of actual phenomena” – is the pivot around which his discussion turns. Without it, his examination of the analysability of “RPT” and “BPT”, an examination that is usually regarded as showing he had erred in the *Tractatus* about colour exclusion, is totally lacking in force. (It is, I am inclined to think, not a bad bet that it was Wittgenstein’s “first definite remark” that inspired him to write “Some Remarks on Logical Form” and, given he does not defend the remark, why he attempted to withdraw the paper and spoke on another topic at the Aristotelian Society.)

 So there was, as I see it, no failure of awareness on Wittgenstein’s part in the *Tractatus* concerning the attribution of colours to points in the visual field and no acknowledgement in “Some Remarks on Logical Form” that there is something wrong with what he wrote about it in the book. He should not be criticised for missing something no thoughtful philosopher should have missed and for noticing that he had slipped up only very much later. The explanation of colour exclusion in the first paragraph of 6.3751 of the *Tractatus* is not easily rebutted, and Wittgenstein does not palm off “the most famous counter-example [to taking all necessity to be logical necessity] as an *illustration* of the thesis”, explicable “only with reference to [his] vaunting confidence that the truth of his thoughts was ‘unassailable and definitive’” (Fogelin 1987, 91-92). On the contrary, when he is read as going along with Russell regarding colour exclusion and accepting that mathematics is an extension of logic, he is seen to be fully justified in sticking with his earlier view of colour exclusion. More generously understood, he came to believe something else in the *Tractatus* is unsatisfactory, and it was this, not what he says at 6.3751, that led to his adopting the view of elementary propositions set out in “Some Remarks on Logical Form”.

 Granting that the common reading of Wittgenstein’s 1929 paper gets things back to front and Wittgenstein developed a new account of the logical impossibility of the joint occurrence of colours after – rather than before – coming to think that rational and irrational numbers can figure in elementary propositions, what sent him back to the drawing board? No doubt one important factor was that in the early months of 1929 he “ceas[ed] to believe in the possibility of a truth-functional account of generality and the quantifiers” (White 2006, 140), this being something that Wittgenstein later referred to as a shortcoming of the *Tractatus* (e.g. 1979, 38-41). There also seems, however, to have been more to it (and not only because it is not immediately obvious why such a ~~short~~coming would have caused him to turn his back on the explanation of colour exclusion offered in the *Tractatus*). More likely, it seems to me, his reflections on the nature of mathematics and the geometry of the visual field tipped the balance. While it is hard to say exactly what Wittgenstein was thinking during the months prior to his writing “Some Remarks on Logical Form”, there is good circumstantial evidence that these reflections converted him to the view that numbers enter the form of elementary propositions, or at least were an important part of the mix.

 On returning to Cambridge in January 1929 Wittgenstein was in regular conversation with Ramsey about mathematics, conversation that could not but have encouraged him (along with his own deliberations) to reconsider his earlier thinking about mathematical and logical representation. It is a reasonable conjecture that Ramsey told Wittgenstein that he had concluded, after working long and hard on the philosophy of mathematics adumbrated in the *Tractatus*, that it is “faced with insuperable difficulties” (1990, 180; published in 1925) and he was currently of the belief that “not merely some [of the terms of our language] but even all may be best symbolized by numbers”, indeed that “colours have a structure, in which any given colour can be assigned a place by three numbers” (1990, 113; drafted in 1929). Such observations, assuming they were aired, would have made Wittgenstein think again about mathematics and colour attribution, not least whether he had – as he is later reported to have conceded – mistakenly treated “mathematics as a part of logic” (Waismann 1979, 218). In addition there is the striking fact that in what seem to be preparatory notes for his talk at the Aristotelian Society Wittgenstein takes an “intuition [*Anschauung*]” about mathematical infinity to entail that “there are no elementary propositions” (2010, 177).

 It is also likely that Wittgenstein’s thinking about the logical character of the visual field added fuel to the fire. As has been pointed out, “[t]he central theme of his reflections [in MS 105] is the logical analysis of the structure of the visual field” (Hacker 1986, 109), and “the numerical representation of the visual field is at the centre of [his writing on the topic]” (Engelmann 2103, 280, ftn 40). During the first months of 1929 Wittgenstein retains his earlier conception of an elementary proposition but sees that the geometry of visual space does not coincide in any straightforward way with the geometry of physical space, the only sort of geometry he had considered in the *Tractatus* (2000, MS 105, 1ff). This could not but have fostered the idea that the geometry of the visual field is in a class of its own and arithmetic belongs to the base of language, not only to its superstructure. There is, I picture Wittgenstein thinking, no avoiding the conclusion that numbers enter into the structure of elementary propositions, a conclusion that virtually duplicates the “first definite remark on the logical analysis of actual phenomena” in “Some Remarks on Logical Form”.

 Several remarks in MS 105 further support the hypothesis that Wittgenstein’s consideration of the geometry of the visual field was instrumental to the development of his philosophy (and possibly what set him on the path to writing “Some Remarks on Logical Form”). One is that he asks: “How can the shape of a fleck in the visual field be described? Can coordinate geometry be done in the visual field [*Kann man im Gesichtsfeld Coordinatengeometrie treiben*]?” (2000, MS 105, 9), a second that he writes some pages later: “I am apparently thrown back against my will on arithmetic. The number is a means of representation” (19), and a third that he avers that the geometry of visual space is different from Euclidean geometry (45). While writing these remarks – all of which were drafted before the remarks in MS 105 that served as a basis for “Some Remarks on Logical Form” – Wittgenstein remained of the view that “[s]igns must have the multiplicity and qualities of the spaces” (55), from which it is no more than a short step to the conclusion that the mathematics of visual space is exceptional. Whence it directly follows that the logic detailed in the *Tractatus* is not the final word, signs referring to the phenomena of visual space failing to exhibit “the multiplicity and qualities” of Euclidean space.

 On the present account of the shift in Wittgenstein’s thinking, then, it was his reflections on the character of the visual field and mathematics (along perhaps with other considerations) that made the difference. When compiling the *Tractatus* in 1917/1918 he was committed to regarding the mathematical physicist’s conception of representation as standing and falling with the logician’s conception and he only came to repudiate this assumption in 1929. At this juncture he could retain his conception of colour as mathematically representable but not his conception of elementary propositions as number-free. He found he had to retract his earlier explanation of the attribution of more than one colour to a point as logically impossible (or, what amounts to the same thing, to accept that such attributions are logically impossible in a broader sense of logic). This ~~in tu~~rn resulted in the idea of mathematical representation he had embraced very early on moving to centre stage and the idea of logical representation he had regarded as fundamental when writing the *Tractatus* fading into the background. In fact even as late as 1950 Wittgenstein was still wedded to the conception of a “mathematics” or “geometry” of colour and to explaining colour exclusion as a simple consequence of the logical structure of colour (1977, III-3 and III-86).

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Andrew Lugg

Montreal

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