# THE CASE AGAINST AXIOMATIZATION

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ABSTRACT. This paper provides a philosophical proof for the case against axiomatization.

"We hold these truths to be self-evident, that all men are created equal, that they are endowed by their Creator with certain unalienable Rights."

— *Declaration of Independence, 1776*

## 1. MOTIVATION

With what conviction can we assert mathematical truths? Could a description of mathematical behavior constitutes a definition thereof? Mathematical induction asserts that if $P(0)$ holds and $\forall n \in \mathbb{N}(P(n) \implies P(n+1))$, then one may assert $\forall n P(n)$. Yet the Sorites paradox demonstrates that the property $P$ could definitely hold for $P(0)$ and is vaguely conserved between each increment of $n$, yet an observer could state the existence of some $k \in \mathbb{N}$ such that $\neg P(k)$ holds without necessarily being able to assert what $k$ is.

As an abstraction, let $\mathcal{M}$ be a well-defined object whose conceptual existence is necessarily contingent upon a number of distinct, primitive notions in $\mathcal{N}_{\mathcal{M}}$. Let $\Omega$ be another well-defined object whose existence is equivalently contingent on notions in $\mathcal{N}_\Omega$. If there exists a bijective function $I$ such that $I : \mathcal{N}_{\mathcal{M}} \to \mathcal{N}_\Omega$, then we say that $\mathcal{M}$ is trivially isomorphic to $\Omega$. To arrive at a more meaningful philosophical comparison, we consider objects that share the same base theory $\mathcal{T}$. In other words, there exists some subset of notions in both sets such that $\mathcal{N}_{\mathcal{M}} \cap \mathcal{N}_\Omega = \mathcal{T}$. A contradiction between the objects is expressed as the derivation of the following

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statement from $\mathcal{T}$: $(\exists x)[\mathcal{M}(x) = \Omega(x)] \implies \bot$. There are two logical responses to this, either we accept $\mathcal{T}$ and admit $(\forall x)[\mathcal{M}(x) \neq \Omega(x)]$, or we reject the theory and insist that the world is such that $\neg\mathcal{T} + (\exists x)[\mathcal{M}(x) = \Omega(x)]$. It is not apt to compare the ardency of logicians for some particular existential assertion, such as the axiom of choice, to that with which the founding fathers declared America's independence. But this comparison is indeed appropriate for the conviction in the logical conditionals $\left[\mathcal{T} \implies (\forall x)[\mathcal{M}(x) \neq \Omega(x)]\right] \vee \left[\neg\mathcal{T} \implies (\exists x)[\mathcal{M}(x) = \Omega(x)]\right]$. As an analogy, the logician is indifferent between a world where all people are created equal or where at least one person is created unequal: only that the world must be one case and not the other, nor both at the same time.

To the majority of logicians, the acceptability of an axiom is intrinsically extrinsic. The axiom of choice, for instance, is desirable because it accords with the way we intuitively think about how sets should behave on a fundamental level. In the context of Reverse Mathematics (RM), it is *mathematically natural* to view set existence principles as closure conditions (Eastaugh, 2019). But these approaches are soft justifications that fall short of what is usually expected of a logically vindicated *conviction*. What distinguishes the conviction in the necessary existence of an object $x$ from the justification for its hypothetical existence is the axiomatic existence of its underlying notions $\mathcal{N}_x$; more specifically, to believe in something axiomatically is to believe it without cause. By *ex falso quodlibet*, from contradiction, everything follows. Yet the acceptance of this principle is ultimately predicated on a myopic view of what a practitioner of logic is aiming to conserve. The practitioner is not necessarily aiming to conserve truth across the usual logical connectives but across locally held beliefs.

It is common to argue that axioms should be accepted because they accord with our intuition. We must therefore take it as self-evident that there necessarily exists a totality of all things that are within the realm of our mental faculties, as they can be expressed.

## 2. Proof

Let $\mathfrak{T}$ denote the totality of all things, objects, processes, axioms, observations, physical or metaphysical entities, etc., that are mentally conceivable:

$$\mathfrak{T} = \{r_1, r_2, r_3, r_4, \dots\}.$$

The sole criterion for what is mentally conceivable is the mind's ability to articulate its existence in any language $\mathcal{L}$.

We pigeonhole each member of $\mathfrak{T}$ into three categories: an atomic axiom, a consequence, or a deductive axiom. A deductive axiom is defined as a member $\delta_0$ of the relation

$$\vdash_{\mathcal{L}} \subseteq \mathfrak{T}^{<\omega}$$

such that it can be characterized as an $n$-tuple $\langle s_0, \dots, s_n \rangle \in \vdash_{\mathcal{L}}$ where $s_0$ is defined as an atomic axiom and $s_m$ is defined as a consequence for all $n \geq m > 0$ in $\omega$.

Let $f_{[\mathrm{OR}]}$ (Ontological Reduction) be a mapping

$$f_{[\mathrm{OR}]} : \mathcal{P}(\mathfrak{T}) \to \mathcal{P}(\mathfrak{T}).$$

We define the function $f_{[\mathrm{OR}]}$ as follows:

$$f_{[\mathrm{OR}]}(\Gamma) = \{\chi \in \Gamma \mid \forall i \in (0, n]\ \forall \langle s_0, \dots, s_n \rangle \in \Gamma\ [s_0 \in \Gamma \implies \chi \neq s_i]\}.$$

Suppose that $\Gamma$ were ontologically reduced to $\Gamma^*$ yet consisted of a subset of atomic axioms of the form $\{\varphi_0, \neg\varphi_0\}$ or deductive axioms of the form $\{\neg\delta_0, \delta_0\}$ where $\neg\delta_0 := \langle \psi, \dots, \neg\varphi_0 \rangle$ and $\delta_0 := \langle \psi, \dots, \varphi_0 \rangle$. Let $\mathbb{P}_0^{\Gamma^*}$ be the initial union of every such subset.

Consider the function $M$ which performs the following:

$$M_0(\Gamma^*) = \begin{cases} \left(\Gamma^* \setminus \mathbb{P}_0^{\Gamma^*}\right) \cup \{\varphi_0\} & \text{if } \exists\langle \varphi_0, \dots, s_k\rangle \in \Gamma \wedge \neg\exists\langle \neg\varphi_0, \dots, s_l\rangle \in \Gamma, \\ \left(\Gamma^* \setminus \mathbb{P}_0^{\Gamma^*}\right) \cup \{\neg\varphi_0\} & \text{if } \exists\langle \neg\varphi_0, \dots, s_k\rangle \in \Gamma \wedge \neg\exists\langle \varphi_0, \dots, s_l\rangle \in \Gamma, \\ \left(\Gamma^* \setminus \mathbb{P}_0^{\Gamma^*}\right) \cup \{\langle \psi, \dots, \varphi_0\rangle\} & \text{if } (\exists\psi \in \Gamma \wedge \exists\varphi_0 \in \Gamma) \wedge \neg\exists\neg\varphi_0 \in \Gamma, \\ \left(\Gamma^* \setminus \mathbb{P}_0^{\Gamma^*}\right) \cup \{\langle \psi, \dots, \neg\varphi_0\rangle\} & \text{if } (\exists\psi \in \Gamma \wedge \exists\neg\varphi_0 \in \Gamma) \wedge \neg\exists\varphi_0 \in \Gamma, \\ \Gamma^* \setminus \mathbb{P}_0^{\Gamma^*} & \text{otherwise.} \end{cases}$$

Moreover,

$$M_{n+1}(\Gamma^*) = \begin{cases} \left(M_n(\Gamma^*) \setminus \mathbb{P}_n^{\Gamma^*}\right) \cup \{\varphi_n\} & \text{if } \exists\langle \varphi_n, \dots, s_k\rangle \in \Gamma \wedge \neg\exists\langle \neg\varphi_n, \dots, s_l\rangle \in \Gamma, \\ \left(M_n(\Gamma^*) \setminus \mathbb{P}_n^{\Gamma^*}\right) \cup \{\neg\varphi_n\} & \text{if } \exists\langle \neg\varphi_n, \dots, s_k\rangle \in \Gamma \wedge \neg\exists\langle \varphi_n, \dots, s_l\rangle \in \Gamma, \\ \left(M_n(\Gamma^*) \setminus \mathbb{P}_n^{\Gamma^*}\right) \cup \{\langle \psi, \dots, \varphi_n\rangle\} & \text{if } (\exists\psi \in \Gamma \wedge \exists\varphi_n \in \Gamma) \wedge \neg\exists\neg\varphi_n \in \Gamma, \\ \left(M_n(\Gamma^*) \setminus \mathbb{P}_n^{\Gamma^*}\right) \cup \{\langle \psi, \dots, \neg\varphi_n\rangle\} & \text{if } (\exists\psi \in \Gamma \wedge \exists\neg\varphi_n \in \Gamma) \wedge \neg\exists\varphi_n \in \Gamma, \\ M_n(\Gamma^*) \setminus \mathbb{P}_n^{\Gamma^*} & \text{otherwise.} \end{cases}$$

where

$$\mathbb{P}_n^{\Gamma^*} = \mathbb{P}_{n-1}^{\Gamma^*} \setminus \{A_{n-1}, \neg A_{n-1}\}.$$

We now consider $f_{[\mathrm{OR}]}(\mathfrak{T})$, or $\mathfrak{T}^*$. Since $\mathfrak{T}$ consists of everything that is mentally conceivable or expressible in any language $\mathcal{L}$, it is necessarily and maximally inconsistent. For every axiom $p, \langle \phi, \dots, \psi\rangle \in \mathfrak{T}^*$, we can conceive of there being some $\neg p, \langle \phi, \dots, \neg\psi\rangle \in \mathfrak{T}^*$ as well. It follows that

$$\mathfrak{T}^* \setminus \mathbb{P}_0^{\mathfrak{T}^*} = \varnothing.$$

Hence, for any axiom $A$, we are indifferent between

$$M_0(\mathfrak{T}^*) = \left(\mathfrak{T}^* \setminus \mathbb{P}_0^{\mathfrak{T}^*}\right) \cup \{\neg A\} \quad \text{or} \quad M_0(\mathfrak{T}^*) = \left(\mathfrak{T}^* \setminus \mathbb{P}_0^{\mathfrak{T}^*}\right) \cup \{A\}.$$

If $A$ and $\neg A$ were atomic axioms, then there does not exist any consistent extrinsic justification to extend it to some conclusion $s_k$. Similarly, if they were deductive axioms, there does not exist any consistent intrinsic justification $\psi$ such that $\psi$ accounts for some consequence $s_k \in \mathfrak{T}$ through $A$ or $\neg A$.

The selection of an axiom represents that that axiom, unlike any other, is perceived as being strictly superior in every respect to every other axiom that could have been selected. Since we are indifferent, we cannot make any meaningful *a priori* decision on how to perform this axiomatization. It follows that we must reject this mode of inquiry altogether. □

## References

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