

# Syntactic Cut-Elimination for Intuitionistic Fuzzy Logic via Linear Nested Sequents<sup>\*</sup>

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**Abstract.** This paper employs the *linear nested sequent* framework to design a new cut-free calculus (LNIF) for intuitionistic fuzzy logic—the first-order Gödel logic characterized by linear relational frames with constant domains. Linear nested sequents—which are nested sequents restricted to linear structures—prove to be a well-suited proof-theoretic formalism for intuitionistic fuzzy logic. We show that the calculus LNIF possesses highly desirable proof-theoretic properties such as invertibility of all rules, admissibility of structural rules, and syntactic cut-elimination.

**Keywords:** Cut-elimination · Fuzzy logic · Gödel logic · Intermediate logic · Intuitionistic logic · Linear nested sequents · Proof theory

## 1 Introduction

Intuitionistic fuzzy logic (IF) has attracted considerable attention due to its unique nature as a logic blending fuzzy reasoning and constructive reasoning [1,3,4,13,24]. The logic, which was initially defined by Takeuti and Titani in [24], has its roots in the work of Kurt Gödel. Gödel introduced extensions of propositional intuitionistic logic (now called, “Gödel logics”) in order to prove that propositional intuitionistic logic does not possess a finite characteristic matrix [11]. These logics were later studied by Dummett who extended Gödel’s finite-valued semantics to include an infinite number of truth-values [7]. Dummett additionally provided an axiomatization for the propositional fragment of IF [7]. The first-order logic IF also admits a finite axiomatization, obtained by extending an axiomatization of first-order intuitionistic logic with the *linearity axiom*  $(A \supset B) \vee (B \supset A)$  and the *quantifier shift axiom*  $(\forall x)(A(x) \vee C) \supset \forall xA(x) \vee C$  (where  $x$  does not occur free in  $C$ ) [14].

Over the last few decades, propositional and first-order Gödel logics (including the prominent logic IF) have been applied in various areas of logic and computer science [3,4,6,13,17,18,26]. For example, Visser [26] applied the propositional fragment of IF while analyzing the provability logic of Heyting arithmetic, Lifschitz et al. [18] employed a Gödel logic to model the strong equivalence of logic programs, and Borgwardt et al. [6] studied standard reasoning problems of first-order Gödel logics in the context of fuzzy description logics.

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Additionally—and quite significantly—the logic IF has been recognized as one of the fundamental formalizations of fuzzy logic [13].

The question of whether or not a logic possesses an *analytic* proof calculus—that is, a calculus which stepwise (de)composes the formula to be proved—is of critical importance. Such calculi are effective tools for designing automated reasoning procedures and for proving certain (meta-)logical properties of a logic. For example, analytic calculi have been leveraged to provide decidability procedures for logics [10], to prove that logics interpolate [17], for counter-model extraction [20], and to understand the computational content of proofs [22].

In his seminal work [10], Gentzen proposed the *sequent calculus* framework for classical and intuitionistic logic, and subsequently, proved his celebrated *Hauptsatz* (i.e. cut-elimination theorem), which ultimately provided analytic calculi for the two logics. Gentzen’s sequent calculus formalism has become one of the preferred proof-theoretic frameworks for providing analytic calculi, and indeed, many logics of interest have been equipped with such calculi. Nevertheless, one of the alluring features of the formalism—namely, its simplicity—has also proven to be one of the formalism’s drawbacks; there remain many logics for which no cut-free, or analytic, sequent calculus (à la Gentzen) is known [12,23]. In response to this, the sequent calculus formalism has been extended in various ways over the last 30 years to include additional structure, allowing for numerous logics to be supplied with cut-free, analytic calculi. Some of the most prominent extensions of Gentzen’s formalism include *display calculi* [5], *labelled calculi* [20,25], *hypersequent calculi* [23], and *nested calculi* [8,12].

In this paper, we employ the *linear nested sequent* formalism, introduced by Lellmann in [15]. Linear nested sequents fall within the *nested calculus paradigm*, but where sequents are restricted to linear, instead of treelike, structures. Linear nested sequents are based off of Masini’s 2-sequent framework [21,22] that was used to provide cut-free calculi for the modal logic KD as well as various other constructive logics. The linear nested formalism proves to be highly compatible with the well-known first-order Gödel logic IF (i.e. intuitionistic fuzzy logic), due to the fact that IF can be semantically characterized by *linear* relational frames (see Sect. 2). The present work provides the linear nested calculus LNIF for IF, which enjoys a variety of fruitful properties, such as:<sup>1</sup>

- ▶ *Separation*: Each logical rule exhibits no other logical connectives than the one to be introduced.
- ▶ *Symmetry*: Each logical connective has a left and right introduction rule.
- ▶ *Internality*: Each sequent translates into a formula of the logical language.
- ▶ *Cut-eliminability*: There exists an algorithm allowing the permutation of a (*cut*) rule (encoding reasoning with lemmata) upwards in a derivation until the rule is completely eliminated from the derivation.
- ▶ *Subformula property*: Every formula occurring in a derivation is a subformula of some formula in the end sequent.
- ▶ *Admissibility of structural rules*: Everything derivable with a structural rule (cf. (*iw*) and (*mrg*) in Sect. 4) is derivable without the structural rule.

<sup>1</sup> We refer to [27] for a detailed discussion of fundamental proof-theoretic properties.

- *Invertibility of rules*: If the conclusion of an inference rule is derivable, then so is the premise.

In [4], a cut-free hypersequent calculus **HIF** for **IF** was introduced to overcome the shortcomings of previously introduced systems [14,24] that violated fundamental proof-theoretic properties such as cut-elimination. In contrast to **HIF**, the current approach of exploiting linear nested sequents has two main benefits. First, the admissibility of structural rules has not been shown in **HIF**, and as such, the calculus does not offer a purely formula-driven approach to proof search. Therefore, the calculus **LNIF** serves as a better basis for automated reasoning in **IF**—bottom-up applications of the rules in **LNIF** simply decompose or propagate formulae, and so, the question of if/when structural rules need to be applied does not arise. Second, the calculus **HIF** cannot be leveraged to prove interpolation for the logic **IF** (see [17]) via the so-called *proof-theoretic method* (cf. [17,19]) due to the presence of the communication structural rule [1]. In [17], it was shown that the propositional fragment of **LNIF** can be harnessed to prove Lyndon interpolation for the propositional fragment of **IF**. This result suggests that **LNIF**, in conjunction with the aforementioned proof-theoretic method, may potentially be harnessed to study and determine interpolable fragments of **IF**, or to solve the longstanding open problem of if the entire logic **IF** interpolates or not.

The contributions and organization of this paper can be summarized as follows: In Sect. 2, we introduce the semantics and axiomatization for intuitionistic fuzzy logic (**IF**). Sect. 3 introduces linear nested sequents and the calculus **LNIF**, as well as proves the calculus sound and complete relative to **IF**. In Sect. 4, we provide invertibility, structural rule admissibility, and cut-elimination results. Last, Sect. 5 concludes and discusses future work.

## 2 Logical Preliminaries

Our language consists of denumerably many *variables*  $\{x, y, z, \dots\}$ , denumerably many *n*-ary *predicates*  $\{p, q, r, \dots\}$  (with  $n \in \mathbb{N}$ ), the *connectives*  $\perp, \wedge, \vee, \supset$ , the *quantifiers*  $\forall, \exists$ , and *parentheses* ‘(’ and ‘)’. We define the language  $\mathcal{L}$  via the BNF grammar below, and will use  $A, B, C$ , etc. to represent formulae from  $\mathcal{L}$ .

$$A ::= p(x_1, \dots, x_n) \mid \perp \mid (A \vee A) \mid (A \wedge A) \mid (A \supset A) \mid (\forall x)A \mid (\exists x)A$$

In the above grammar,  $p$  is any *n*-ary predicate symbol and  $x_1, \dots, x_n, x$  are variables. We refer to formulae of the form  $p(x_1, \dots, x_n)$  as *atomic formulae*, and (more specifically) refer to formulae of the form  $p$  as *propositional variables* (i.e. a 0-ary predicate  $p$  is a propositional variable). The *free variables* of a formula  $A$  are defined in the usual way as variables unbound by a quantifier, and *bound variables* as those bounded by a quantifier.

We opt for the relational semantics of **IF**—as opposed to the fuzzy semantics (cf. [4])—since the structure of linear nested sequents is well-suited for interpretation via linear relational frames.

**Definition 1 (Relational Frames, Models [9]).** A relational frame is a triple  $F = (W, R, D)$  such that: (i)  $W$  is a non-empty set of worlds  $w, u, v, \dots$ , (ii)  $R$  is a reflexive, transitive, antisymmetric, and connected binary relation on  $W$ , and (iii)  $D$  is a function that maps a world  $w \in W$  to a non-empty set of parameters  $D_w$  called the domain of  $w$  such that the following condition is met:

(CD) If  $Rwu$ , then  $D_w = D_u$ .

A model  $M$  is a tuple  $(F, V)$  where  $F$  is a relational frame and  $V$  is a valuation function such that  $V(p, w) \subseteq (D_w)^n$  for each  $n$ -ary predicate  $p$  and

(TP) If  $Rwu$ , then  $V(p, w) \subseteq V(p, u)$  (if  $p$  is of arity  $n > 0$ );  
If  $Rwu$  and  $w \in V(p, w)$ , then  $u \in V(p, v)$  (if  $p$  is of arity 0).

We uphold the convention in [9] and assume that for each world  $w \in W$ ,  $(D_w)^0 = \{w\}$ , so  $V(p, w) = \{w\}$  or  $V(p, w) = \emptyset$ , for a propositional variable  $p$ .

The distinctive feature of relational frames for IF is the *connected* property, which states that for any  $w, u, v \in W$  of a frame  $F = (W, R, D)$ , if  $Rwu$  and  $Rwv$ , then either  $Ruv$  or  $Rvu$ . Imposing this property on reflexive, transitive, and antisymmetric (i.e. intuitionistic) frames causes the set of worlds to become linearly ordered, thus validating the linearity axiom  $(A \supset B) \vee (B \supset A)$  (shown in Fig. 1). Furthermore, the constant domain condition (CD) validates the quantifier shift axiom  $\forall x(A(x) \vee B) \supset \forall xA(x) \vee B$  (also shown in Fig. 1).

Rather than interpret formulae from  $\mathcal{L}$  in relational models, we follow [9] and introduce  $D_w$ -sentences to be interpreted in relational models. This gives rise to a notion of validity for formulae in  $\mathcal{L}$  (see Def. 3). The definition of validity also depends on the *universal closure* of a formula: if a formula  $A$  contains only  $x_0, \dots, x_m$  as free variables, then the universal closure  $\bar{\forall}A$  is taken to be the formula  $\forall x_0 \dots \forall x_m A$ .

**Definition 2 ( $D_w$ -Sentence).** Let  $M$  be a relational model with  $w \in W$  of  $M$ . We define  $\mathcal{L}_{D_w}$  to be the language  $\mathcal{L}$  expanded with parameters from the set  $D_w$ . We define a  $D_w$ -formula to be a formula in  $\mathcal{L}_{D_w}$ , and we define a  $D_w$ -sentence to be a  $D_w$ -formula that does not contain any free variables. Last, we use  $a, b, c, \dots$  to denote parameters in a set  $D_w$ .

**Definition 3 (Semantic Clauses [9]).** Let  $M = (W, R, D, V)$  be a relational model with  $w \in W$  and  $R(w) := \{v \in W \mid (w, v) \in R\}$ . The satisfaction relation  $M, w \Vdash A$  between  $w \in W$  and a  $D_w$ -sentence  $A$  is inductively defined as follows:

- $M, w \not\Vdash \perp$
- If  $p$  is a propositional variable, then  $M, w \Vdash p$  iff  $w \in V(p, w)$ ;
- If  $p$  is an  $n$ -ary predicate symbol (with  $n > 0$ ), then  $M, w \Vdash p(a_1, \dots, a_n)$  iff  $(a_1, \dots, a_n) \in V(p, w)$ ;
- $M, w \Vdash A \vee B$  iff  $M, w \Vdash A$  or  $M, w \Vdash B$ ;
- $M, w \Vdash A \wedge B$  iff  $M, w \Vdash A$  and  $M, w \Vdash B$ ;
- $M, w \Vdash A \supset B$  iff for all  $u \in R(w)$ , if  $M, u \Vdash A$ , then  $M, u \Vdash B$ ;
- $M, w \Vdash \forall xA(x)$  iff for all  $u \in R(w)$  and all  $a \in D_u$ ,  $M, u \Vdash A(a)$ ;

–  $M, w \Vdash \exists x A(x)$  iff there exists an  $a \in D_w$  such that  $M, w \Vdash A(a)$ .

We say that a formula  $A$  is globally true on  $M$ , written  $M \Vdash A$ , iff  $M, u \Vdash \bar{\forall} A$  for all worlds  $u \in W$ . A formula  $A$  is valid, written  $\Vdash A$ , iff it is globally true on all relational models.

**Lemma 1 (Persistence).** *Let  $M$  be a relational model with  $w, u \in W$  of  $M$ . For any  $D_w$ -sentence  $A$ , if  $M, w \Vdash A$  and  $Rwu$ , then  $M, u \Vdash A$ .*

*Proof.* See [9, Lem. 3.2.16] for details.  $\square$

A sound and complete axiomatization for the logic IF is provided in Fig. 1. We define the *substitution*  $[y/x]$  of the variable  $y$  for the free variable  $x$  on a formula  $A$  in the standard way as the replacement of all free occurrences of  $x$  in  $A$  with  $y$ . The substitution  $[a/x]$  of the parameter  $a$  for the free variable  $x$  is defined similarly. Last, the side condition  *$y$  is free for  $x$*  (see Fig. 1) is taken to mean that  $y$  does not become bound by a quantifier if substituted for  $x$ .

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$$\begin{array}{l}
 A \supset (B \supset A) \quad (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)) \quad A \supset (B \supset (A \wedge B)) \\
 (A \wedge B) \supset A \quad (A \wedge B) \supset B \quad A \supset (A \vee B) \quad B \supset (A \vee B) \quad \frac{A \quad A \supset B}{B} \text{ mp} \\
 (A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C)) \quad \perp \supset A \quad (A \supset B) \vee (B \supset A) \quad \frac{A}{\forall x A} \text{ gen} \\
 \forall x A \supset A[y/x] \text{ } y \text{ free for } x \quad A[y/x] \supset \exists x A \text{ } y \text{ free for } x \quad \forall x (B \supset A(x)) \supset (B \supset \forall x A(x)) \\
 \forall x (A(x) \supset B) \supset (\exists x A(x) \supset B) \quad \forall x (A(x) \vee B) \supset \forall x A(x) \vee B \text{ with } x \notin B
 \end{array}$$


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**Fig. 1.** Axiomatization for the logic IF [9]. The logic IF is the smallest set of formulae from  $\mathcal{L}$  closed under substitutions of the axioms and applications of the inference rules *mp* and *gen*. We write  $\vdash_{\text{IF}} A$  to denote that  $A$  is an element, or *theorem*, of IF.

**Theorem 1 (Adequacy of IF).** *For any  $A \in \mathcal{L}$ ,  $\Vdash A$  iff  $\vdash_{\text{IF}} A$ .*

*Proof.* The forward direction follows from [9, Prop. 7.2.9] and [9, Prop. 7.3.6], and the backwards direction follows from [9, Lem. 3.2.31].  $\square$

### 3 Soundness and Completeness of LNIF

Let us define *linear nested sequents* (which we will refer to as *sequents*) to be syntactic objects  $\mathcal{G}$  given by the BNF grammar shown below:

$$\mathcal{G} ::= \Gamma \vdash \Gamma \mid \mathcal{G} // \mathcal{G} \text{ where } \Gamma ::= A \mid \Gamma, \Gamma \text{ with } A \in \mathcal{L}.$$

Each sequent  $\mathcal{G}$  is of the form  $\Gamma_1 \vdash \Delta_1 // \cdots // \Gamma_n \vdash \Delta_n$  with  $n \in \mathbb{N}$ . We refer to each  $\Gamma_i \vdash \Delta_i$  (for  $1 \leq i \leq n$ ) as a *component* of  $\mathcal{G}$  and use  $\|\mathcal{G}\|$  to denote the number of components in  $\mathcal{G}$ .

We often use  $\mathcal{G}$ ,  $\mathcal{H}$ ,  $\mathcal{F}$ , and  $\mathcal{K}$  to denote sequents, and will use  $\Gamma$  and  $\Delta$  to denote antecedents and consequents of components. Last, we take the comma operator to be commutative and associative; for example, we identify the sequent  $p(x) \vdash q(x), r(y), p(x)$  with  $p(x) \vdash r(y), p(x), q(x)$ . This interpretation lets us view an antecedent  $\Gamma$  or consequent  $\Delta$  as a multiset of formulae.

To ease the proof of cut-elimination (Thm. 4), we follow [?, Sect. 2.3.1] and make a syntactic distinction between *bound variables*  $\{x, y, z, \dots\}$  and *parameters*  $\{a, b, c, \dots\}$ , which will take the place of free variables occurring in formulae. Thus, our sequents make use of formulae from  $\mathcal{L}$  where each free variable has been replaced by a unique parameter. For example, we would use the sequent  $p(a) \vdash \forall x q(x, b) // \perp \vdash r(a)$  instead of the sequent  $p(x) \vdash \forall x q(x, y) // \perp \vdash r(x)$  in a derivation (where the parameter  $a$  has been substituted for the free variable  $x$  and  $b$  has been substituted for  $y$ ). We also use the notation  $A(a_0, \dots, a_n)$  to denote that the parameters  $a_0, \dots, a_n$  occur in the formula  $A$ , and write  $A(\vec{a})$  as shorthand for  $A(a_0, \dots, a_n)$ . This notation extends straightforwardly to sequents as well.

The linear nested calculus LNIF for IF is given in Fig. 2. (NB. The linear nested calculus LNG introduced in [17] is the propositional fragment of LNIF, i.e. LNG is the calculus LNIF without the quantifier rules and where propositional variables are used in place of atomic formulae.) The  $(\supset_{r2})$  and  $(\forall_{r2})$  rules in LNIF are particularly noteworthy; as will be seen in the next section, the rules play a vital role in ensuring the invertibility and admissibility of certain rules, ultimately permitting the elimination of (*cut*) (see Thm. 4).

To obtain soundness, we interpret each sequent as a formula in  $\mathcal{L}$  and utilize the notion of validity in Def. 3. The following definition specifies how each sequent is interpreted.

**Definition 4 (Interpretation  $\iota$ ).** *The interpretation of a sequent is defined inductively as follows:*

$$\iota(\Gamma \vdash \Delta) := \bigwedge \Gamma \supset \bigvee \Delta \quad \iota(\Gamma \vdash \Delta // \mathcal{G}) := \bigwedge \Gamma \supset \left( \bigvee \Delta \vee \iota(\mathcal{G}) \right)$$

We interpret a sequent  $\mathcal{G}$  as a formula in  $\mathcal{L}$  by taking the universal closure  $\bar{\forall}\iota(\mathcal{G})$  of  $\iota(\mathcal{G})$  and we say that  $\mathcal{G}$  is valid if and only if  $\Vdash \bar{\forall}\iota(\mathcal{G})$ .

**Theorem 2 (Soundness of LNIF).** *For any linear nested sequent  $\mathcal{G}$ , if  $\mathcal{G}$  is provable in LNIF, then  $\Vdash \bar{\forall}\iota(\mathcal{G})$ .*

*Proof.* We prove the result by induction on the height of the derivation of

$$\mathcal{G} = \Gamma_1 \vdash \Delta_1 // \cdots // \Gamma_n \vdash \Delta_n // \Gamma_{n+1} \vdash \Delta_{n+1} // \cdots // \Gamma_m \vdash \Delta_m$$

and only present the more interesting  $\forall$  quantifier cases in the inductive step. All remaining cases can be found in App. A. Each inference rule considered is of one of the following two forms.

$$\begin{array}{c}
 \frac{}{\mathcal{G} \parallel \Gamma, p(\vec{a}) \vdash p(\vec{a}), \Delta \parallel \mathcal{H}} (id_1) \quad \frac{}{\mathcal{G} \parallel \Gamma_1, p(\vec{a}) \vdash \Delta_1 \parallel \mathcal{H} \parallel \Gamma_2 \vdash p(\vec{a}), \Delta_2 \parallel \mathcal{F}} (id_2) \\
 \\
 \frac{}{\mathcal{G} \parallel \Gamma, \perp \vdash \Delta \parallel \mathcal{H}} (\perp_l) \quad \frac{\mathcal{G} \parallel \Gamma, A, B \vdash \Delta \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma, A \wedge B \vdash \Delta \parallel \mathcal{H}} (\wedge_l) \quad \frac{\mathcal{G} \parallel \Gamma \vdash \Delta, A, B \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma \vdash \Delta, A \vee B \parallel \mathcal{H}} (\vee_r) \\
 \frac{\mathcal{G} \parallel \Gamma \vdash \Delta, A \parallel \mathcal{H} \quad \mathcal{G} \parallel \Gamma \vdash \Delta, B \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma \vdash \Delta, A \wedge B \parallel \mathcal{H}} (\wedge_r) \quad \frac{\mathcal{G} \parallel \Gamma, A \vdash \Delta \parallel \mathcal{H} \quad \mathcal{G} \parallel \Gamma, B \vdash \Delta \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma, A \vee B \vdash \Delta \parallel \mathcal{H}} (\vee_l) \\
 \frac{\mathcal{G} \parallel \Gamma \vdash \Delta \parallel A \vdash B}{\mathcal{G} \parallel \Gamma \vdash \Delta, A \supset B} (\supset_{r1}) \quad \frac{\mathcal{G} \parallel \Gamma, B \vdash \Delta \parallel \mathcal{H} \quad \mathcal{G} \parallel \Gamma, A \supset B \vdash A, \Delta \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma, A \supset B \vdash \Delta \parallel \mathcal{H}} (\supset_l) \\
 \frac{\mathcal{G} \parallel \Gamma_1, A \vdash \Delta_1 \parallel \Gamma_2, A \vdash \Delta_2 \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma_1, A \vdash \Delta_1 \parallel \Gamma_2 \vdash \Delta_2 \parallel \mathcal{H}} (lift) \quad \frac{\mathcal{G} \parallel \Gamma, A[a/x], \forall x A \vdash \Delta \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma, \forall x A \vdash \Delta \parallel \mathcal{H}} (\forall_l) \\
 \frac{\mathcal{G} \parallel \Gamma \vdash \Delta \parallel \vdash A[a/x]}{\mathcal{G} \parallel \Gamma \vdash \Delta, \forall x A} (\forall_{r1})^\dagger \quad \frac{\mathcal{G} \parallel \Gamma, A[a/x] \vdash \Delta \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma, \exists x A \vdash \Delta \parallel \mathcal{H}} (\exists_l)^\dagger \quad \frac{\mathcal{G} \parallel \Gamma \vdash A[a/x], \Delta \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma \vdash \exists x A, \Delta \parallel \mathcal{H}} (\exists_r) \\
 \frac{\mathcal{G} \parallel \Gamma_1 \vdash \Delta_1 \parallel A \vdash B \parallel \Gamma_2 \vdash \Delta_2 \parallel \mathcal{H} \quad \mathcal{G} \parallel \Gamma_1 \vdash \Delta_1 \parallel \Gamma_2 \vdash \Delta_2, A \supset B \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma_1 \vdash \Delta_1, A \supset B \parallel \Gamma_2 \vdash \Delta_2 \parallel \mathcal{H}} (\supset_{r2}) \\
 \frac{\mathcal{G} \parallel \Gamma_1 \vdash \Delta_1 \parallel \vdash A[a/x] \parallel \Gamma_2 \vdash \Delta_2 \parallel \mathcal{H} \quad \mathcal{G} \parallel \Gamma_1 \vdash \Delta_1 \parallel \Gamma_2 \vdash \Delta_2, \forall x A \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma_1 \vdash \Delta_1, \forall x A \parallel \Gamma_2 \vdash \Delta_2 \parallel \mathcal{H}} (\forall_{r2})^\dagger
 \end{array}$$

**Fig. 2.** The Calculus LNIF. The side condition  $\dagger$  stipulates that the parameter  $a$  is an *eigenvariable*, i.e. it does not occur in the conclusion. Occasionally, we write  $\vdash_{\text{LNIF}} \mathcal{G}$  to mean that the sequent  $\mathcal{G}$  is derivable in LNIF.

$$\frac{\mathcal{G}'}{\mathcal{G}} (r_1) \quad \frac{\mathcal{G}_1 \quad \mathcal{G}_2}{\mathcal{G}} (r_2)$$

We argue by contraposition and prove that if  $\mathcal{G}$  is invalid, then at least one premise is invalid. Assuming  $\mathcal{G}$  is invalid (i.e.  $\not\models \bar{\forall} \iota(\mathcal{G})$ ) implies the existence of a model  $M = (W, R, D, V)$  with world  $v \in W$  such that  $Rvw_0$ ,  $\vec{a} \in D_{w_0}$ , and  $M, w_0 \not\models \iota(\mathcal{G})(\vec{a})$ , where  $\vec{a}$  represents all parameters in  $\iota(\mathcal{G})$ . Hence, there is a sequence of worlds  $w_1, \dots, w_m \in W$  such that  $Rw_j w_{j+1}$  (for  $0 \leq j \leq m-1$ ),  $M, w_i \Vdash \bigwedge \Gamma_i$ , and  $M, w_i \not\models \bigvee \Delta_i$ , for each  $1 \leq i \leq m$ . We assume all parameters in  $\bigwedge \Gamma_i$  and  $\bigvee \Delta_i$  are interpreted as elements of the associated domain  $D_{w_i}$  (for  $1 \leq i \leq m$ ).

**( $\forall_{r1}$ )-rule:** By our assumption  $M, w_m \Vdash \bigwedge \Gamma_m$  and  $M, w_m \not\models \bigvee \Delta_m \vee \forall x A$ . The latter implies that  $M, w_m \not\models \forall x A$ , meaning there exists a world  $w_{m+1} \in W$  such that  $Rw_m w_{m+1}$  and  $M, w_{m+1} \not\models A[b/x]$  for some  $b \in D_{w_{m+1}}$ . If we interpret the eigenvariable of the premise as  $b$ , then the premise is shown invalid.

**( $\forall_{r2}$ )-rule:** It follows from our assumption that  $M, w_n \Vdash \bigwedge \Gamma_n$ ,  $M, w_n \not\models \bigvee \Delta_n \vee \forall x A$ ,  $M, w_{n+1} \Vdash \bigwedge \Gamma_{n+1}$ , and  $M, w_{n+1} \not\models \bigvee \Delta_{n+1}$ . The fact that  $M, w_n \not\models \bigvee \Delta_n \vee \forall x A$  implies that there exists a world  $w \in W$  such that  $Rw_n w$  and for some  $b \in D_w$ ,  $M, w \not\models A[b/x]$ . Since our frames are connected, there are two cases to consider: (i)  $Rw w_{n+1}$ , or (ii)  $Rw_{n+1} w$ . Case (i) falsifies the left premise, and case (ii) falsifies the right premise.

**( $\forall_I$ )-rule:** We know that  $M, w_n \Vdash \bigwedge \Gamma_n \wedge \forall x A$  and  $M, w_n \not\Vdash \bigvee \Delta_n$ . Hence, for any world  $w \in W$ , if  $Rw_n w$ , then  $M, w \Vdash A[b/x]$  for all  $b \in D_w$ . Since  $Rw_n w_n$ , it follows that  $M, w_n \Vdash A[b/x]$  for any  $b \in D_{w_n}$ . If  $a$  occurs in the conclusion  $\mathcal{G}$ , then by the constant domain condition (**CD**), we know that  $a \in D_{w_n}$ , so we may falsify the premise of the rule. If  $a$  does not occur in  $\mathcal{G}$ , then it is an eigenvariable, and assigning  $a$  to any element of  $D_{w_n}$  will falsify the premise.  $\square$

**Theorem 3 (Completeness of LNIF).** *If  $\vdash_{\text{IF}} A$ , then  $A$  is provable in LNIF.*

*Proof.* It is not difficult to show that LNIF can derive each axiom of IF and can simulate each inference rule. We refer the reader to App. A for details.  $\square$

## 4 Proof-Theoretic Properties of LNIF

In this section, we present the fundamental proof-theoretic properties of LNIF, thus extending the results in [17] from the propositional setting to the first-order setting. (NB. We often leverage results from [17] to simplify our proofs.) Most results are proved by induction on the *height* of a given derivation  $\Pi$ , i.e. on the length (number of sequents) of the longest branch from the end sequent to an initial sequent in  $\Pi$ . Proofs of Lem. 14, Lem. 16, and Thm. 4 are given by induction on the lexicographic ordering of pairs  $(|A|, h)$ , where  $|A|$  is the *complexity* of a certain formula  $A$  (defined in the usual way as the number of logical operators in  $A$ ) and  $h$  is the height of the derivation. Lemmata whose proofs are omitted can be found in App. A.

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$$\begin{array}{c}
\frac{\mathcal{G} \parallel \Gamma_1 \vdash \Delta_1 \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \parallel \mathcal{H}} \text{ (iw)} \quad \frac{\mathcal{G} \parallel \Gamma, A, A \vdash \Delta \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma, A \vdash \Delta \parallel \mathcal{H}} \text{ (ic}_l\text{)} \quad \frac{\mathcal{G} \parallel \mathcal{H}}{\mathcal{G} \parallel \vdash \parallel \mathcal{H}} \text{ (ew)} \\
\frac{\mathcal{G}}{\mathcal{G}[b/a]} \text{ (sub)} \quad \frac{\mathcal{G} \parallel \Gamma_1 \vdash A, \Delta_1 \parallel \Gamma_2 \vdash \Delta_2 \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma_1 \vdash \Delta_1 \parallel \Gamma_2 \vdash A, \Delta_2 \parallel \mathcal{H}} \text{ (lwr)} \quad \frac{\mathcal{G} \parallel \Gamma \vdash \Delta, \perp \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma \vdash \Delta \parallel \mathcal{H}} \text{ ( $\perp_r$ )} \\
\frac{\mathcal{G} \parallel \Gamma \vdash A, A, \Delta \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma, \Gamma \vdash A, \Delta \parallel \mathcal{H}} \text{ (ic}_r\text{)} \quad \frac{\mathcal{G} \parallel \Gamma_1 \vdash \Delta_1 \parallel \Gamma_2 \vdash \Delta_2 \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \parallel \mathcal{H}} \text{ (mrg)}
\end{array}$$


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**Fig. 3.** Admissible rules in LNIF.

We say that a rule is *admissible* in LNIF iff derivability of the premise(s) implies derivability of the conclusion in LNIF. Additionally, a rule is *height preserving (hp-)admissible* in LNIF iff if the premise of the rule has a derivation of a certain height in LNIF, then the conclusion of the rule has a derivation of the same height or less in LNIF. Last, a rule is *invertible (hp-invertible)* iff derivability of the conclusion implies derivability of the premise(s) (with a derivation of the same height or less). Admissible rules of LNIF are given in Fig. 3.



**Lemma 2.** For any  $A, \Gamma, \Delta, \mathcal{G}$ , and  $\mathcal{H}$ ,  $\vdash_{\text{LNIF}} \mathcal{G} // \Gamma, A \vdash A, \Delta // \mathcal{H}$ .

**Lemma 3.** The  $(\perp_r)$  rule is hp-admissible in LNIF.

*Proof.* By induction on the height of the given derivation. In the base case, applying  $(\perp_r)$  to  $(id_1)$ ,  $(id_2)$ , or  $(\perp_l)$  gives an initial sequent, and for each case of the inductive step we apply IH followed by the corresponding rule.  $\square$

**Lemma 4.** The  $(sub)$  rule is hp-admissible in LNIF.

**Lemma 5.** The  $(iw)$  rule is hp-admissible in LNIF.

**Lemma 6.** The  $(ew)$  rule is admissible in LNIF.

*Proof.* By [17, Lem. 5.6] we know that  $(ew)$  is admissible in LNG, thus leaving us to prove the  $(\forall_{r1})$ ,  $(\exists_l)$ ,  $(\forall_{r2})$ ,  $(\forall_l)$ , and  $(\exists_r)$  cases. The  $(\exists_l)$ ,  $(\forall_l)$ , and  $(\exists_r)$  cases are easily shown by applying IH and then the rule. We therefore prove the  $(\forall_{r1})$  and  $(\forall_{r2})$  cases, beginning with the former, which is split into the two subcases, shown below:

$$\frac{\frac{\mathcal{G} // \Gamma \vdash \Delta // \vdash A[a/x]}{\mathcal{G} // \Gamma \vdash \Delta, \forall x A} (\forall_{r1})}{\mathcal{G}' // \Gamma \vdash \Delta, \forall x A} (ew) \quad \frac{\frac{\mathcal{G} // \Gamma \vdash \Delta // \vdash A[a/x]}{\mathcal{G} // \Gamma \vdash \Delta, \forall x A} (\forall_{r1})}{\mathcal{G} // \Gamma \vdash \Delta, \forall x A // \vdash} (ew)}$$

In the top left case, where we weaken in a component prior to the component  $\Gamma \vdash \Delta, \forall x A$ , we may freely permute the two rule instances. The top right case is resolved as shown below.

$$\frac{\frac{\mathcal{G} // \Gamma \vdash \Delta // \vdash A[a/x] // \vdash}{\mathcal{G} // \Gamma \vdash \Delta, \forall x A // \vdash} \text{IH} \quad \frac{\frac{\mathcal{G} // \Gamma \vdash \Delta // \vdash A[a/x]}{\mathcal{G} // \Gamma \vdash \Delta // \vdash \forall x A} (\forall_{r1})}{\mathcal{G} // \Gamma \vdash \Delta, \forall x A // \vdash} (\forall_{r2})}{\mathcal{G} // \Gamma \vdash \Delta, \forall x A // \vdash}$$

Suppose now that we have an  $(\forall_{r2})$  inference (as in Fig. 2) followed by an  $(ew)$  inference. The only nontrivial case (which is resolved as shown below) occurs when a component is weakened in directly after the component  $\Gamma_1 \vdash \Delta_1, \forall x A$ . All other cases follow by an application of IH followed by an application of the  $(\forall_{r2})$  rule.

$$\frac{\frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \vdash A[a/x] // \vdash // \Gamma_2 \vdash \Delta_2 // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x A // \vdash // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{IH}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x A // \vdash // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \Pi (\forall_{r2})$$

$$\Pi = \left\{ \frac{\frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \vdash A[a/x] // \Gamma_2 \vdash \Delta_2 // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \vdash \forall x A // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{IH} \quad \frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \vdash // \Gamma_2 \vdash \Delta_2, \forall x A // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \vdash // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{IH}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \vdash // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} (\forall_{r2}) \right. \square$$

**Lemma 7.** The rule  $(lwr)$  is hp-admissible in LNIF.

*Proof.* By [17, Lem. 5.7] we know that  $(lwr)$  is admissible in LNG, and so, we may prove the claim by extending it to include the quantifier rules. We have two cases to consider: either (i) the lower-formula is a side formula in the quantifier inference, or (ii) it is principal. In case (i), the  $(\forall_{r1})$ ,  $(\forall_l)$ ,  $(\exists_l)$ , and  $(\exists_r)$  cases can be resolved by applying IH followed by an application of the rule. Concerning the  $(\forall_{r2})$  rule, all cases follow by applying IH and then the rule, with the exception of the following:

$$\frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, A // \vdash B[a/x] // \Gamma_2 \vdash \Delta_2 // \mathcal{H} \quad \mathcal{G} // \Gamma_1 \vdash \Delta_1, A // \Gamma_2 \vdash \Delta_2, \forall x B // \mathcal{H}}{\frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, A, \forall x B // \Gamma_2 \vdash \Delta_2 // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x B // \Gamma_2 \vdash \Delta_2, A // \mathcal{H}}} (\forall_{r2})} (lwr)$$

In the above case, we apply IH twice to the top left premise and apply IH once to the top right premise. A single application of  $(\forall_{r2})$  gives the desired result.

Let us now consider case (ii). Observe that the principal formulae in  $(\forall_{r1})$ ,  $(\forall_l)$ , and  $(\exists_l)$  cannot be principal in the use of  $(lwr)$ , so we need only consider the  $(\exists_r)$  and  $(\forall_{r2})$  cases. The  $(\exists_r)$  case is shown below top-left and the case is resolved as shown below top-right. In the  $(\forall_{r2})$  case (shown below bottom), we take the derivation of the top right premise as the proof of the desired conclusion.

$$\frac{\frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, A[a/x], \exists x A // \Gamma_2 \vdash \Delta_2 // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \exists x A // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} (\exists_r) \quad \frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \Gamma_2 \vdash \Delta_2, A[a/x], \exists x A // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \Gamma_2 \vdash \Delta_2, \exists x A // \mathcal{H}} \text{IH} \times 2 (\exists_r)}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \Gamma_2 \vdash \Delta_2, \exists x A // \mathcal{H}} (lwr)}$$

$$\frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \vdash A[a/x] // \Gamma_2 \vdash \Delta_2 // \mathcal{H} \quad \mathcal{G} // \Gamma_1 \vdash \Delta_1 // \Gamma_2 \vdash \Delta_2, \forall x A // \mathcal{H}}{\frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x A // \Gamma_2 \vdash \Delta_2 // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \Gamma_2 \vdash \Delta_2, \forall x A // \mathcal{H}}} (\forall_{r2})} (lwr)$$

□

Our version of the  $(lift)$  rule necessitates a stronger form of invertibility, called  $m$ -invertibility, for the  $(\wedge_l)$ ,  $(\forall_l)$ ,  $(\supset_l)$ ,  $(\forall_l)$ , and  $(\exists_l)$  rules (cf. [17]). We use  $A^{k_i}$  to represent  $k_i$  copies of a formula  $A$ , with  $i \in \mathbb{N}$ .

**Lemma 8.** *If  $\sum_{i=1}^n k_n \geq 1$ , then*

- (i) (1) implies (2)                      (iii) (6) implies (7) and (8)    (v) (11) implies (12)  
(ii) (3) implies (4) and (5)    (iv) (9) implies (10)

$$\vdash_{\text{LNIF}} \Gamma_1, (A \wedge B)^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, (A \wedge B)^{k_n} \vdash \Delta_n \quad (1)$$

$$\vdash_{\text{LNIF}} \Gamma_1, A^{k_1}, B^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, A^{k_n}, B^{k_n} \vdash \Delta_n \quad (2)$$

$$\vdash_{\text{LNIF}} \Gamma_1, (A \vee B)^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, (A \vee B)^{k_n} \vdash \Delta_n \quad (3)$$

$$\vdash_{\text{LNIF}} \Gamma_1, A^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, A^{k_n} \vdash \Delta_n \quad (4)$$

$$\vdash_{\text{LNIF}} \Gamma_1, B^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, B^{k_n} \vdash \Delta_n \quad (5)$$

$$\vdash_{\text{LNIF}} \Gamma_1, (A \supset B)^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, (A \supset B)^{k_n} \vdash \Delta_n \quad (6)$$

$$\vdash_{\text{LNIF}} \Gamma_1, B^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, B^{k_n} \vdash \Delta_n \quad (7)$$

$$\vdash_{\text{LNIF}} \Gamma_1, (A \supset B)^{k_1} \vdash \Delta_1, A^{k_1} // \cdots // \Gamma_n, (A \supset B)^{k_n} \vdash \Delta_n, A^{k_n} \quad (8)$$

$$\vdash_{\text{LNIF}} \Gamma_1, (\forall xA)^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, (\forall xA)^{k_n} \vdash \Delta_n \quad (9)$$

$$\vdash_{\text{LNIF}} \Gamma_1, A[a/x]^{k_1}, (\forall xA)^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, A[a/x]^{k_n}, (\forall xA)^{k_n} \vdash \Delta_n \quad (10)$$

$$\vdash_{\text{LNIF}} \Gamma_1, (\exists xA)^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, (\exists xA)^{k_n} \vdash \Delta_n \quad (11)$$

$$\vdash_{\text{LNIF}} \Gamma_1, A[a/x]^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, A[a/x]^{k_n} \vdash \Delta_n \quad (12)$$

**Lemma 9.** *The  $(\wedge_r)$ ,  $(\vee_r)$ , and  $(\exists_r)$  rules are hp-invertible in LNIF.*

*Proof.* By [17, Lem. 5.8] we know that the claim holds for the  $(\wedge_r)$  and  $(\vee_r)$  rules relative to LNG. The proof may be extended to LNIF by considering the quantifier rules in the inductive step; however, it is quick to verify the claim for the quantifier rules by applying IH and then the corresponding rule. Proving invertibility of the  $(\exists_r)$  rule is straightforward, and is shown by induction on the height of the given derivation.  $\square$

**Lemma 10.** *The  $(\supset_{r2})$  rule is invertible in LNIF.*

*Proof.* We extend the proof of [17, Lem. 5.10] to include the quantifier rules, and prove the result by induction on the height of the given derivation of  $\mathcal{G} // \Gamma_1 \vdash \Delta_1, A \supset B // \Gamma_2 \vdash \Delta_2 // \mathcal{H}$ . Derivability of the right premise  $\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \Gamma_2 \vdash \Delta_2, A \supset B // \mathcal{H}$  follows from Lem. 7, so we focus on showing that the left premise  $\mathcal{G} // \Gamma_1 \vdash \Delta_1 // A \vdash B // \Gamma_2 \vdash \Delta_2 // \mathcal{H}$  is derivable. For the  $(\forall_{r1})$ ,  $(\forall_l)$ ,  $(\exists_l)$ , and  $(\exists_r)$  rules the desired conclusion is obtained by applying IH, followed by an application of the corresponding rule. The nontrivial  $(\forall_{r2})$  case is shown below top and is resolved as shown below bottom. In all other  $(\forall_{r2})$  cases, we apply IH followed by the  $(\forall_{r2})$  rule.

$$\frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, A \supset B // \vdash C[a/x] // \Gamma_2 \vdash \Delta_2 // \mathcal{H} \quad \mathcal{G} // \Gamma_1 \vdash \Delta_1, A \supset B // \Gamma_2 \vdash \Delta_2, \forall xC // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall xC, A \supset B // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} (\forall_{r2})$$

$$\frac{\Pi_1 \quad \Pi_2}{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall xC // A \vdash B // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} (\forall_{r2}) \quad \Pi_1 = \left\{ \begin{array}{l} \frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, A \supset B // \vdash C[a/x] // \Gamma_2 \vdash \Delta_2 // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \vdash C[a/x], A \supset B // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{Lem. 7} \\ \frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \vdash C[a/x], A \supset B // \Gamma_2 \vdash \Delta_2 // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \vdash C[a/x] // A \vdash B // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{IH} \end{array} \right.$$

$$\Pi_2 = \left\{ \begin{array}{l} \frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, A \supset B // \vdash C[a/x] // \Gamma_2 \vdash \Delta_2 // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // A \vdash B // \vdash C[a/x], \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{IH} \quad \frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, A \supset B // \Gamma_2 \vdash \Delta_2, \forall xC // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // A \vdash B // \Gamma_2 \vdash \Delta_2, \forall xC // \mathcal{H}} \text{IH} \\ \frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // A \vdash B // \vdash C[a/x], \Gamma_2 \vdash \Delta_2 // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // A \vdash B, \forall xC // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} (\forall_{r2}) \end{array} \right. \quad \square$$

**Lemma 11.** *The  $(\forall_{r2})$  rule is invertible in LNIF.*

*Proof.* Let the sequent  $\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall xA // \Gamma_2 \vdash \Delta_2 // \mathcal{H}$  be derivable in LNIF. Derivability of the right premise  $\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \Gamma_2 \vdash \Delta_2, \forall xA // \mathcal{H}$  follows from the hp-admissibility of  $(lwr)$  (Lem. 7). We prove that the left premise  $\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \vdash A[a/x] // \Gamma_2 \vdash \Delta_2 // \mathcal{H}$  is derivable by induction on the height of the given derivation.

*Base case.* Regardless of if  $\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall xA // \Gamma_2 \vdash \Delta_2 // \mathcal{H}$  is derived by an application of  $(id_1)$ ,  $(id_2)$ , or  $(\perp_l)$ ,  $\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \vdash A[a/x] // \Gamma_2 \vdash \Delta_2 // \mathcal{H}$  is an initial sequent as well.

*Inductive step.* For all rules, with the exception of (*lift*), ( $\supset_{r2}$ ), ( $\forall_{r1}$ ), ( $\exists_l$ ), and ( $\forall_{r2}$ ), we apply IH to the premise(s) followed by the corresponding rule. We consider the aforementioned nontrivial cases below.

If the (*lift*) rule is applied as shown below left, then the desired conclusion may be derived as shown below right. In all other cases, we apply IH and then (*lift*) to achieve the desired result.

$$\frac{\mathcal{G} // \Gamma_1, B \vdash \Delta_1, \forall x A // \Gamma_2, B \vdash \Delta_2 // \mathcal{H}}{\mathcal{G} // \Gamma_1, B \vdash \Delta_1, \forall x A // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{ (lift)} \quad \frac{\frac{\mathcal{G} // \Gamma_1, B \vdash \Delta_1, \forall x A // \Gamma_2, B \vdash \Delta_2 // \mathcal{H}}{\mathcal{G} // \Gamma_1, B \vdash \Delta_1 // B \vdash A[a/x] // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{ IH}}{\mathcal{G} // \Gamma_1, B \vdash \Delta_1 // B \vdash A[a/x] // \Gamma_2, B \vdash \Delta_2 // \mathcal{H}} \text{ Lem. 5}}{\mathcal{G} // \Gamma_1, B \vdash \Delta_1 // B \vdash A[a/x] // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{ (lift)}$$

If the ( $\supset_{r2}$ ) rule is applied as shown below top, then the desired conclusion may be derived as shown below bottom. In all other cases, we apply IH and then the ( $\supset_{r2}$ ) rule to obtain the desired result.

$$\frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x A // B \vdash C // \Gamma_2 \vdash \Delta_2 // \mathcal{H} \quad \mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x A // \Gamma_2 \vdash \Delta_2, B \supset C // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x A, B \supset C // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{ } (\supset_{r2})$$

$$\frac{\Pi_1 \quad \Pi_2}{\mathcal{G} // \Gamma_1 \vdash \Delta_1, B \supset C // B \vdash A[a/x] // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{ } (\supset_{r2}) \quad \Pi_1 = \left\{ \begin{array}{l} \frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x A // B \vdash C // \Gamma_2 \vdash \Delta_2 // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // B \vdash C // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{ Lem. 7} \\ \frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x A // \Gamma_2 \vdash \Delta_2, B \supset C // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // B \vdash C // B \vdash A[a/x] // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{ IH} \end{array} \right.$$

$$\Pi_2 = \left\{ \begin{array}{l} \frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x A // B \vdash C // \Gamma_2 \vdash \Delta_2 // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // B \vdash A[a/x] // B \vdash C // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{ IH} \\ \frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x A // \Gamma_2 \vdash \Delta_2, B \supset C // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // B \vdash A[a/x] // \Gamma_2 \vdash \Delta_2, B \supset C // \mathcal{H}} \text{ IH} \end{array} \right. \text{ } (\supset_{r2})$$

In the ( $\forall_{r1}$ ) and ( $\exists_l$ ) cases, we must ensure that the eigenvariable of the inference is not identical to the parameter  $a$  in  $A[a/x]$  introduced by IH. However, this can always be ensured by Lem. 4. Therefore, we move onto the last nontrivial case, which concerns the ( $\forall_{r2}$ ) rule. The only nontrivial case occurs as shown below top and is resolved as shown below bottom. In all other cases, we apply IH followed by the ( $\forall_{r2}$ ) rule (evoking Lem. 4 if necessary).

$$\frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x A // B[b/y] // \Gamma_2 \vdash \Delta_2 // \mathcal{H} \quad \mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x A // \Gamma_2 \vdash \Delta_2, \forall y B // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x A, \forall y B // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{ } (\forall_{r2})$$

$$\frac{\Pi_1 \quad \Pi_2}{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall y B // B[b/y] // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{ } (\forall_{r2}) \quad \Pi_1 = \left\{ \begin{array}{l} \frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x A // B[b/y] // \Gamma_2 \vdash \Delta_2 // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // B[b/y] // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{ Lem. 4} \\ \frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x A // \Gamma_2 \vdash \Delta_2, \forall y B // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // B[b/y] // B[b/y] // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{ IH} \end{array} \right.$$

$$\Pi_2 = \left\{ \begin{array}{l} \frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x A // B[b/y] // \Gamma_2 \vdash \Delta_2 // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // B[b/y] // B[b/y] // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{ IH} \\ \frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x A // \Gamma_2 \vdash \Delta_2, \forall y B // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // B[b/y] // B[b/y] // \Gamma_2 \vdash \Delta_2, \forall y B // \mathcal{H}} \text{ IH} \end{array} \right. \text{ } (\forall_{r2})$$

□

**Lemma 12.** *The ( $\supset_{r1}$ ) rule is invertible in LNIF.*

*Proof.* We extend the proof of [17, Lem. 5.11] to include the quantifier cases. The claim is shown by induction on the height of the given derivation. When the last rule of the derivation is  $(\forall_l)$ ,  $(\exists_l)$ ,  $(\exists_r)$ , or  $(\forall_{r2})$  in the inductive step, we apply IH to the premise(s) of the inference followed by an application of the corresponding rule. If the last inference of the derivation is an application of the  $(\forall_{r1})$  rule (as shown below left), then the case is resolved as shown below right.

$$\frac{\frac{\mathcal{G} \parallel \Gamma \vdash \Delta, A \supset B \parallel \vdash C[a/x]}{\mathcal{G} \parallel \Gamma \vdash \Delta, A \supset B, \forall x C} (\forall_{r1})}{\frac{\frac{\mathcal{G} \parallel \Gamma \vdash \Delta, A \supset B \parallel \vdash C[a/x]}{\mathcal{G} \parallel \Gamma \vdash \Delta \parallel \vdash C[a/x], A \supset B} \text{Lem. 7} \quad \frac{\mathcal{G} \parallel \Gamma \vdash \Delta, A \supset B \parallel \vdash C[a/x]}{\mathcal{G} \parallel \Gamma \vdash \Delta \parallel A \vdash B \parallel \vdash C[a/x]} \text{Lem. 10}}{\mathcal{G} \parallel \Gamma \vdash \Delta \parallel \vdash C[a/x] \parallel A \vdash B} \text{IH}}{\mathcal{G} \parallel \Gamma \vdash \Delta, \forall x C \parallel A \vdash B} (\forall_{r2})} \quad \square$$

**Lemma 13.** *The  $(\forall_{r1})$  rule is invertible in LNIF.*

*Proof.* We prove the result by induction on the height of the given derivation of  $\mathcal{G} \parallel \Gamma \vdash \Delta, \forall x A$  and show that  $\mathcal{G} \parallel \Gamma \vdash \Delta \parallel \vdash A[a/x]$  is derivable.

*Base case.* If  $\mathcal{G} \parallel \Gamma \vdash \Delta, \forall x A$  is obtained via  $(id_1)$ ,  $(id_2)$ , or  $(\perp_l)$ , then  $\mathcal{G} \parallel \Gamma \vdash \Delta \parallel \vdash A[a/x]$  is an instance of the corresponding rule as well.

*Inductive step.* All cases, with the exception of the  $(\supset_{r1})$ ,  $(\forall_{r1})$ ,  $(\exists_l)$ , and  $(\forall_{r2})$  rules, are resolved by applying IH to the premise(s) and then applying the relevant rule. Let us consider each of the additional cases in turn.

The  $(\supset_{r1})$  case is shown below left and is resolved as shown below right.

$$\frac{\frac{\mathcal{G} \parallel \Gamma \vdash \Delta, \forall x A \parallel B \vdash C}{\mathcal{G} \parallel \Gamma \vdash \Delta, \forall x A, B \supset C} (\supset_{r1})}{\frac{\frac{\mathcal{G} \parallel \Gamma \vdash \Delta, \forall x A \parallel B \vdash C}{\mathcal{G} \parallel \Gamma \vdash \Delta \parallel B \vdash C, \forall x A} \text{Lem. 7} \quad \frac{\mathcal{G} \parallel \Gamma \vdash \Delta, \forall x A \parallel B \vdash C}{\mathcal{G} \parallel \Gamma \vdash \Delta \parallel \vdash A[a/x] \parallel B \vdash C} \text{Lem. 11}}{\mathcal{G} \parallel \Gamma \vdash \Delta \parallel B \vdash C \parallel \vdash A[a/x]} \text{IH}}{\mathcal{G} \parallel \Gamma \vdash \Delta, B \supset C \parallel \vdash A[a/x]} (\supset_{r2})} \quad \square$$

In the  $(\forall_{r1})$  case where the relevant formula  $\forall x A$  is principal, the premise of the inference is the desired conclusion. If the relevant formula  $\forall x A$  is not principal, then the  $(\forall_{r1})$  inference is of the form shown below left and is resolved as shown below right.

$$\frac{\frac{\mathcal{G} \parallel \Gamma \vdash \Delta, \forall x A \parallel B[b/y]}{\mathcal{G} \parallel \Gamma \vdash \Delta, \forall x A, \forall y B} (\forall_{r1})}{\frac{\frac{\mathcal{G} \parallel \Gamma \vdash \Delta, \forall x A \parallel B[b/y]}{\mathcal{G} \parallel \Gamma \vdash \Delta \parallel B[b/y], \forall x A} \text{Lem. 7} \quad \frac{\mathcal{G} \parallel \Gamma \vdash \Delta, \forall x A \parallel B[b/y]}{\mathcal{G} \parallel \Gamma \vdash \Delta \parallel \vdash A[a/x] \parallel B[b/y]} \text{Lem. 11}}{\mathcal{G} \parallel \Gamma \vdash \Delta \parallel B[b/y] \parallel \vdash A[a/x]} \text{IH}}{\mathcal{G} \parallel \Gamma \vdash \Delta, \forall y B \parallel \vdash A[a/x]} (\forall_{r2})} \quad \square$$

If the last inference is an instance of the  $(\exists_l)$  or  $(\forall_{r2})$  rule, then we must ensure that the eigenvariable of the inference is not identical to the parameter  $a$  in  $A[a/x]$  introduced by IH, but this can always be ensured due to Lem. 4.  $\square$

**Lemma 14.** *The  $(ic_l)$  rule is admissible in LNIF.*

*Proof.* We extend the proof of [17, Lem. 5.12] and prove the result by induction on the lexicographic ordering of pairs  $(|A|, h)$ , where  $|A|$  is the complexity of the contraction formula  $A$  and  $h$  is the height of the derivation. We know the result holds for LNG, and so, we argue the inductive step for the quantifier rules.

With the exception of the  $(\exists_l)$  case shown below left, all quantifier cases are settled by applying IH followed by an application of the corresponding rule.

The only nontrivial case occurs when a contraction is performed on a formula  $\exists xA$  with one of the contraction formulae principal in the  $(ic_l)$  inference. The situation is resolved as shown below right.

$$\frac{\frac{\mathcal{G} \parallel \Gamma, A[a/x], \exists xA \vdash \Delta \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma, \exists xA, \exists xA \vdash \Delta \parallel \mathcal{H}} (\exists_l) \quad \frac{\mathcal{G} \parallel \Gamma, A[a/x], \exists xA \vdash \Delta \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma, \exists xA, \exists xA \vdash \Delta \parallel \mathcal{H}} (ic_l)}{\mathcal{G} \parallel \Gamma, \exists xA \vdash \Delta \parallel \mathcal{H}} \quad \frac{\frac{\mathcal{G} \parallel \Gamma, A[a/x], \exists xA \vdash \Delta \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma, A[a/x], A[a/x] \vdash \Delta \parallel \mathcal{H}} \text{Lem. 8} \quad \frac{\mathcal{G} \parallel \Gamma, A[a/x], A[a/x] \vdash \Delta \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma, A[a/x] \vdash \Delta \parallel \mathcal{H}} \text{IH}}{\mathcal{G} \parallel \Gamma, \exists xA \vdash \Delta \parallel \mathcal{H}} (\exists_l)$$

Notice that IH is applicable since we are contracting on a formula of smaller complexity.  $\square$

**Lemma 15.** *The  $(mrg)$  rule is admissible in LNIF.*

*Proof.* We extend the proof of [17, Lem. 5.13], which proves that  $(mrg)$  is admissible in LNG, and prove the admissibility of  $(mrg)$  in LNIF by induction on the height of the given derivation. We need only consider the quantifier rules due to [17, Lem. 5.13]. The  $(\forall_{r1})$ ,  $(\forall_l)$ ,  $(\exists_l)$ , and  $(\exists_r)$  cases are all resolved by applying IH to the premise of the rule followed by an application of the rule. If  $(mrg)$  is applied to the principal components of the  $(\forall_{r2})$  rule as follows:

$$\frac{\frac{\mathcal{G} \parallel \Gamma_1 \vdash \Delta_1 \parallel \vdash A[a/x] \parallel \Gamma_2 \vdash \Delta_2 \parallel \mathcal{H} \quad \mathcal{G} \parallel \Gamma_1 \vdash \Delta_1 \parallel \Gamma_2 \vdash \Delta_2, \forall xA \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma_1 \vdash \Delta_1, \forall xA \parallel \Gamma_2 \vdash \Delta_2 \parallel \mathcal{H}} (\forall_{r2})}{\frac{\mathcal{G} \parallel \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \forall xA \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, \forall xA \parallel \mathcal{H}} (mrg)}$$

then the desired conclusion is obtained by applying IH to the top right premise. In all other cases, we apply IH to the premises of  $(\forall_{r2})$  followed by an application of the rule.  $\square$

**Lemma 16.** *The  $(ic_r)$  rule is admissible in LNIF.*

*Proof.* We extend the proof of [17, Lem. 5.14] to include the quantifier rules and argue the claim by induction on the lexicographic ordering of pairs  $(|A|, h)$ , where  $|A|$  is the complexity of the contraction formula  $A$  and  $h$  is the height of the derivation. The  $(\forall_l)$  and  $(\exists_l)$  cases are settled by applying IH to the premise of the inference followed by an application of the rule. For the  $(\exists_r)$  case, we evoke Lem. 9, apply IH, and then apply the corresponding rule. The nontrivial case (occurring when the principal formula is contracted) for the  $(\forall_{r1})$  rule is shown below left, and the desired conclusion is derived as shown below right (where IH is applicable due to the decreased complexity of the contraction formula).

$$\frac{\frac{\mathcal{G} \parallel \Gamma \vdash \Delta, \forall xA \parallel \vdash A[a/x]}{\mathcal{G} \parallel \Gamma \vdash \Delta, \forall xA, \forall xA} (\forall_{r1}) \quad \frac{\mathcal{G} \parallel \Gamma \vdash \Delta, \forall xA \parallel \vdash A[a/x]}{\mathcal{G} \parallel \Gamma \vdash \Delta \parallel \vdash A[a/x] \parallel \vdash A[a/x]} \text{Lem. 11}}{\frac{\mathcal{G} \parallel \Gamma \vdash \Delta, \forall xA, \forall xA}{\mathcal{G} \parallel \Gamma \vdash \Delta, \forall xA} (ic_r)} \quad \frac{\frac{\mathcal{G} \parallel \Gamma \vdash \Delta \parallel \vdash A[a/x] \parallel \vdash A[a/x]}{\mathcal{G} \parallel \Gamma \vdash \Delta \parallel \vdash A[a/x], A[a/x]} \text{Lem. 15} \quad \frac{\mathcal{G} \parallel \Gamma \vdash \Delta \parallel \vdash A[a/x]}{\mathcal{G} \parallel \Gamma \vdash \Delta \parallel \vdash A[a/x]} \text{IH}}{\frac{\mathcal{G} \parallel \Gamma \vdash \Delta, \forall xA}{\mathcal{G} \parallel \Gamma \vdash \Delta, \forall xA} (\forall_{r1})}$$

When the contracted formulae are both non-principal in an  $(\forall_{r1})$  inference, we apply IH to the premise followed by an application of the  $(\forall_{r1})$  rule. If the contracted formulae are both non-principal in an  $(\forall_{r2})$  inference, then we apply IH to the premises followed by an application of the rule. If one of the contracted formulae is principal in an  $(\forall_{r2})$  inference (as shown below top), then the case is settled as shown below bottom.

$$\frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall xA // \vdash A[a/x] // \Gamma_2 \vdash \Delta_2 // \mathcal{H} \quad \mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall xA // \Gamma_2 \vdash \Delta_2, \forall xA // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall xA, \forall xA // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} (\forall_{r2})$$

$$\frac{\frac{\frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall xA // \vdash A[a/x] // \Gamma_2 \vdash \Delta_2 // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \vdash A[a/x] // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{Lem. 11} \quad \frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall xA // \Gamma_2 \vdash \Delta_2, \forall xA // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \Gamma_2 \vdash \Delta_2, \forall xA, \forall xA // \mathcal{H}} \text{Lem. 7}}{\frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \vdash A[a/x], A[a/x] // \Gamma_2 \vdash \Delta_2 // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \text{IH}} \text{IH}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall xA // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} (\forall_{r2})$$

Note that we may apply IH in the left branch of the derivation since the complexity of the contraction formula is less than  $\forall xA$ , and we may apply IH in the right branch since the height of the derivation is less than the original.  $\square$

Before moving on to the cut-elimination theorem, we present the definition of the *splice* operation [17,21]. The operation is used to formulate the (*cut*) rule.

**Definition 5 (Splice [17]).** *The splice  $\mathcal{G} \oplus \mathcal{H}$  of two linear nested sequents  $\mathcal{G}$  and  $\mathcal{H}$  is defined as follows:*

$$\begin{aligned} (\Gamma_1 \vdash \Delta_1) \oplus (\Gamma_2 \vdash \Delta_2) &:= \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 \\ (\Gamma_1 \vdash \Delta_1) \oplus (\Gamma_2 \vdash \Delta_2 // \mathcal{F}) &:= \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 // \mathcal{F} \\ (\Gamma_1 \vdash \Delta_1 // \mathcal{F}) \oplus (\Gamma_2 \vdash \Delta_2) &:= \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 // \mathcal{F} \\ (\Gamma_1 \vdash \Delta_1 // \mathcal{F}) \oplus (\Gamma_2 \vdash \Delta_2 // \mathcal{K}) &:= \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2 // (\mathcal{F} \oplus \mathcal{K}) \end{aligned}$$

**Theorem 4 (Cut-Elimination).** *The rule*

$$\frac{\mathcal{G} // \Gamma \vdash \Delta, A // \mathcal{H} \quad \mathcal{F} // A^{k_1}, \Gamma_1 \vdash \Delta_1 // \cdots // A^{k_n}, \Gamma_n \vdash \Delta_n}{(\mathcal{G} \oplus \mathcal{F}) // \Gamma, \Gamma_1 \vdash \Delta, \Delta_1 // (\mathcal{H} \oplus (\Gamma_2 \vdash \Delta_2 // \cdots // \Gamma_n \vdash \Delta_n))} (\text{cut})$$

where  $\|\mathcal{G}\| = \|\mathcal{F}\|$ ,  $\|\mathcal{H}\| = n - 1$ , and  $\sum_{i=1}^n k_i \geq 1$ , is eliminable in LNIF.

*Proof.* We extend the proof of [17, Thm. 5.16] and prove the result by induction on the lexicographic ordering of pairs  $(|A|, h)$ , where  $|A|$  is the complexity of the cut formula  $A$  and  $h$  is the height of the derivation of the right premise of the (*cut*) rule. Moreover, we assume w.l.o.g. that (*cut*) is used once as the last inference of the derivation (given a derivation with multiple applications of (*cut*), we may repeatedly apply the elimination algorithm described here to the topmost occurrence of (*cut*), ultimately resulting in a cut-free derivation). By [17, Thm. 5.16], we know that (*cut*) is eliminable from any derivation in LNG, and therefore, we need only consider cases which incorporate quantifier rules.

If  $h = 0$ , then the right premise of (*cut*) is an instance of  $(id_1)$ ,  $(id_2)$ , or  $(\perp_l)$ . If none of the cut formulae  $A$  are principal in the right premise, then the conclusion of (*cut*) is an instance of  $(id_1)$ ,  $(id_2)$ , or  $(\perp_l)$ . If, however, one of the

cut formulae  $A$  is principal in the right premise and is an atomic formula  $p(\vec{a})$ , then the top right premise of  $(cut)$  is of the form

$$\mathcal{F} \parallel p(\vec{a})^{k_1}, \Gamma_1 \vdash \Delta_1 \parallel \cdots \parallel p(\vec{a})^{k_i}, \Gamma_i \vdash p(\vec{a}), \Delta'_i \parallel \cdots \parallel p(\vec{a})^{k_n}, \Gamma_n \vdash \Delta_n$$

where  $\Delta_i = p(\vec{a}), \Delta'_i$ . Observe that since  $\Delta_i$  occurs in the conclusion of  $(cut)$ , so does  $p(\vec{a})$ . To construct a cut-free derivation of the conclusion of  $(cut)$ , we apply  $(lwr)$  to the left premise  $\mathcal{G} \parallel \Gamma \vdash \Delta, p(\vec{a}) \parallel \mathcal{H}$  until  $p(\vec{a})$  is in the  $i^{th}$  component, and then apply hp-admissibility of  $(iw)$  (Lem. 5) to add in the missing formulae. Last, if the cut formula  $A$  is principal in the right premise and is equal to  $\perp$ , then the left premise of  $(cut)$  is of the form  $\mathcal{G} \parallel \Gamma \vdash \Delta, \perp \parallel \mathcal{H}$ . We obtain a cut-free derivation of the conclusion of  $(cut)$  by first applying hp-admissibility of  $(\perp_r)$  (Lem. 3), followed by hp-admissibility of  $(iw)$  (Lem. 5) to add in the missing formulae.

Suppose that  $h > 0$ . If none of the cut formulae  $A$  are principal in the inference  $(r)$  of the right premise of  $(cut)$ , then for all cases (with the exception of the  $(\forall_{r1}), (\supset_{r1}), (\exists_l), (\forall_{r2}),$  and  $(\supset_{r2})$  cases) we apply IH to the premise(s) of  $(r)$ , followed by an application of  $(r)$ . Let us now consider the  $(\forall_{r1}), (\exists_l), (\forall_{r2}), (\supset_{r1}),$  and  $(\supset_{r2})$  cases when none of the cut formulae  $A$  are principal. First, assume that  $(\forall_{r1})$  is the rule used to derive the right premise of  $(cut)$ :

$$\frac{\mathcal{G} \parallel \Gamma \vdash \Delta, A \parallel \mathcal{H} \quad \frac{\mathcal{F} \parallel A^{k_1}, \Gamma_1 \vdash \Delta_1 \parallel \cdots \parallel A^{k_n}, \Gamma_n \vdash \Delta_n \parallel \vdash B[a/x] \quad (\forall_{r1})}{\mathcal{F} \parallel A^{k_1}, \Gamma_1 \vdash \Delta_1 \parallel \cdots \parallel A^{k_n}, \Gamma_n \vdash \Delta_n, \forall x B} \quad (cut)}{(\mathcal{G} \oplus \mathcal{F}) \parallel \Gamma, \Gamma_1 \vdash \Delta, \Delta_1 \parallel (\mathcal{H} \oplus (\Gamma_2 \vdash \Delta_2 \parallel \cdots \parallel \Gamma_n \vdash \Delta_n, \forall x B))} \quad (cut)$$

We evoke hp-admissibility of  $(sub)$  (Lem. 4) to substitute the eigenvariable  $a$  of  $(\forall_{r1})$  with a fresh variable  $b$  that does not occur in either premise of  $(cut)$ . We then apply admissibility of  $(ew)$  (Lem. 6) to the left premise of  $(cut)$ , apply IH to the resulting derivations, and last apply the  $(\forall_{r1})$  rule, as shown below:

$$\frac{\frac{\mathcal{G} \parallel \Gamma \vdash \Delta, A \parallel \mathcal{H}}{\mathcal{G} \parallel \Gamma \vdash \Delta, A \parallel \mathcal{H} \parallel \vdash} \text{Lem. 6} \quad \frac{\mathcal{F} \parallel A^{k_1}, \Gamma_1 \vdash \Delta_1 \parallel \cdots \parallel A^{k_n}, \Gamma_n \vdash \Delta_n \parallel \vdash B[a/x] \quad \text{Lem. 4}}{\mathcal{F} \parallel A^{k_1}, \Gamma_1 \vdash \Delta_1 \parallel \cdots \parallel A^{k_n}, \Gamma_n \vdash \Delta_n \parallel \vdash B[b/x]} \text{IH}}{\frac{(\mathcal{G} \oplus \mathcal{F}) \parallel \Gamma, \Gamma_1 \vdash \Delta, \Delta_1 \parallel (\mathcal{H} \oplus (\Gamma_2 \vdash \Delta_2 \parallel \cdots \parallel \Gamma_n \vdash \Delta_n)) \parallel \vdash B[b/x] \quad (\forall_{r1})}{(\mathcal{G} \oplus \mathcal{F}) \parallel \Gamma, \Gamma_1 \vdash \Delta, \Delta_1 \parallel (\mathcal{H} \oplus (\Gamma_2 \vdash \Delta_2 \parallel \cdots \parallel \Gamma_n \vdash \Delta_n, \forall x B))} \quad (\forall_{r1})}$$

In the  $(\exists_l)$  case below

$$\frac{\mathcal{G} \parallel \Gamma \vdash \Delta, A \parallel \mathcal{H} \quad \frac{\mathcal{F} \parallel A^{k_1}, \Gamma_1 \vdash \Delta_1 \parallel \cdots \parallel A^{k_i}, B[a/x], \Gamma_i \vdash \Delta_i \parallel \cdots \parallel A^{k_n}, \Gamma_n \vdash \Delta_n \quad (\exists_l)}{\mathcal{F} \parallel A^{k_1}, \Gamma_1 \vdash \Delta_1 \parallel \cdots \parallel A^{k_i}, \exists x B, \Gamma_i \vdash \Delta_i \parallel \cdots \parallel A^{k_n}, \Gamma_n \vdash \Delta_n} \quad (cut)}{(\mathcal{G} \oplus \mathcal{F}) \parallel \Gamma, \Gamma_1 \vdash \Delta, \Delta_1 \parallel (\mathcal{H} \oplus (\Gamma_2 \vdash \Delta_2 \parallel \cdots \parallel \exists x B, \Gamma_i \vdash \Delta_i \parallel \cdots \parallel \Gamma_n \vdash \Delta_n))} \quad (cut)$$

we also make use of the hp-admissibility of  $(sub)$  to ensure that the  $(\exists_l)$  rule can be applied after evoking the inductive hypothesis:

$$\frac{\frac{\mathcal{G} \parallel \Gamma \vdash \Delta, A \parallel \mathcal{H}}{(\mathcal{G} \oplus \mathcal{F}) \parallel \Gamma, \Gamma_1 \vdash \Delta, \Delta_1 \parallel (\mathcal{H} \oplus (\Gamma_2 \vdash \Delta_2 \parallel \cdots \parallel \exists x B, \Gamma_i \vdash \Delta_i \parallel \cdots \parallel \Gamma_n \vdash \Delta_n))} \text{IH}}{\frac{\mathcal{F} \parallel A^{k_1}, \Gamma_1 \vdash \Delta_1 \parallel \cdots \parallel A^{k_i}, B[a/x], \Gamma_i \vdash \Delta_i \parallel \cdots \parallel A^{k_n}, \Gamma_n \vdash \Delta_n \quad \text{Lem. 4}}{\mathcal{F} \parallel A^{k_1}, \Gamma_1 \vdash \Delta_1 \parallel \cdots \parallel A^{k_i}, B[b/x], \Gamma_i \vdash \Delta_i \parallel \cdots \parallel A^{k_n}, \Gamma_n \vdash \Delta_n} \text{IH}}{(\mathcal{G} \oplus \mathcal{F}) \parallel \Gamma, \Gamma_1 \vdash \Delta, \Delta_1 \parallel (\mathcal{H} \oplus (\Gamma_2 \vdash \Delta_2 \parallel \cdots \parallel \exists x B, \Gamma_i \vdash \Delta_i \parallel \cdots \parallel \Gamma_n \vdash \Delta_n))} \quad (\exists_l)$$



Let us consider the  $(\forall_{r2})$  case

$$\begin{array}{c}
 \frac{\mathcal{F} \parallel A^{k_1}, \Gamma_1 \vdash \Delta_1 \parallel \cdots \parallel A^{k_i}, \Gamma_i \vdash \Delta_i, \forall xB \parallel A^{k_{i+1}}, \Gamma_{i+1} \vdash \Delta_{i+1} \parallel \cdots \parallel A^{k_n}, \Gamma_n \vdash \Delta_n}{(\mathcal{G} \oplus \mathcal{F}) \parallel \Gamma, \Gamma_1 \vdash \Delta, \Delta_1 \parallel (\mathcal{H} \oplus (\Gamma_2 \vdash \Delta_2 \parallel \cdots \parallel \Gamma_i \vdash \Delta_i, \forall xB \parallel \Gamma_{i+1} \vdash \Delta_{i+1} \parallel \cdots \parallel \Gamma_n \vdash \Delta_n))} \text{(\forall}_{r2}) \\
 \text{(1)} \quad \frac{\mathcal{G} \parallel \Gamma \vdash \Delta, A \parallel \mathcal{H}_1 \parallel \Gamma'_i \vdash \Delta'_i \parallel \Gamma'_{i+1} \vdash \Delta'_{i+1} \parallel \mathcal{H}_2}{\mathcal{G} \parallel \Gamma \vdash \Delta, A \parallel \mathcal{H}_1 \parallel \Gamma'_i \vdash \Delta'_i \parallel \Gamma'_{i+1} \vdash \Delta'_{i+1} \parallel \mathcal{H}_2} \\
 \text{(2)} \quad \mathcal{F} \parallel A^{k_1}, \Gamma_1 \vdash \Delta_1 \parallel \cdots \parallel A^{k_i}, \Gamma_i \vdash \Delta_i \parallel B[a/x] \parallel A^{k_{i+1}}, \Gamma_{i+1} \vdash \Delta_{i+1} \parallel \cdots \parallel A^{k_n}, \Gamma_n \vdash \Delta_n \\
 \text{(3)} \quad \mathcal{F} \parallel A^{k_1}, \Gamma_1 \vdash \Delta_1 \parallel \cdots \parallel A^{k_i}, \Gamma_i \vdash \Delta_i \parallel A^{k_{i+1}}, \Gamma_{i+1} \vdash \Delta_{i+1}, \forall xB \parallel \cdots \parallel A^{k_n}, \Gamma_n \vdash \Delta_n
 \end{array}$$

where  $\mathcal{H} = \mathcal{H}_1 \parallel \Gamma'_i \vdash \Delta'_i \parallel \Gamma'_{i+1} \vdash \Delta'_{i+1} \parallel \mathcal{H}_2$ . To resolve the case we evoke admissibility of  $(ew)$  (Lem. 6) on (1) to obtain a derivation of

$$(1)' \quad \mathcal{G} \parallel \Gamma \vdash \Delta, A \parallel \mathcal{H}_1 \parallel \Gamma'_i \vdash \Delta'_i \parallel \Gamma'_{i+1} \vdash \Delta'_{i+1} \parallel \mathcal{H}_2$$

Moreover, to ensure that the eigenvariable  $a$  in (2) does not occur in (1), we apply hp-admissibility of  $(sub)$  (Lem. 4) to obtain (2)' where  $a$  has been replaced by a fresh parameter  $b$ . Applying IH between (1)' and (2)', and (1) and (3), followed by an application of  $(\forall_{r2})$ , gives the desired result. Last, note that the  $(\supset_{r1})$  and  $(\supset_{r2})$  cases are resolved as explained in the proof of [17, Thm. 5.16].

We assume now that one of the cut formulae  $A$  is principal in the inference yielding the right premise of  $(cut)$ . The cases where  $A$  is an atomic formula  $p(\vec{a})$  or is identical to  $\perp$  are resolved as explained above (when  $h = 0$ ). For the case when  $A$  is principal in an application of  $(lift)$ , we simply apply IH between the left premise of  $(cut)$  and the premise of the  $(lift)$  rule. Also, if  $A$  is of the form  $B \wedge C$ ,  $B \vee C$ , or  $B \supset C$ , then all such cases can be resolved as explained in the proof of [17, Thm. 5.16]. Thus, we only consider the cases where  $A$  is of the form  $\exists xB$  and  $\forall xB$ ; we begin with the former and assume our derivation ends with:

$$\frac{\mathcal{G} \parallel \Gamma \vdash \Delta, \exists xB \parallel \mathcal{H} \quad \frac{\mathcal{F} \parallel \exists xB^{k_1}, \Gamma_1 \vdash \Delta_1 \parallel \cdots \parallel \exists xB^{k_i}, B[a/x], \Gamma_i \vdash \Delta_i \parallel \cdots \parallel \exists xB^{k_n}, \Gamma_n \vdash \Delta_n}{\mathcal{F} \parallel \exists xB^{k_1}, \Gamma_1 \vdash \Delta_1 \parallel \cdots \parallel \exists xB^{k_{i+1}}, \Gamma_{i+1} \vdash \Delta_{i+1} \parallel \cdots \parallel \exists xB^{k_n}, \Gamma_n \vdash \Delta_n} (\exists_l)}{(\mathcal{G} \oplus \mathcal{F}) \parallel \Gamma, \Gamma_1 \vdash \Delta, \Delta_1 \parallel (\mathcal{H} \oplus (\Gamma_2 \vdash \Delta_2 \parallel \cdots \parallel \Gamma_i \vdash \Delta_i \parallel \cdots \parallel \Gamma_n \vdash \Delta_n))} \text{(cut)}$$

Evoking IH with the left premise of  $(cut)$  and the premise of  $(\exists_l)$  gives a cut-free derivation of:

$$(\mathcal{G} \oplus \mathcal{F}) \parallel \Gamma, \Gamma_1 \vdash \Delta, \Delta_1 \parallel (\mathcal{H} \oplus (\Gamma_2 \vdash \Delta_2 \parallel \cdots \parallel B[a/x], \Gamma_i \vdash \Delta_i \parallel \cdots \parallel \Gamma_n \vdash \Delta_n))$$

By invertibility of  $(\exists_r)$  (Lem. 9), there exists a cut-free derivation of  $\mathcal{G} \parallel \Gamma \vdash \Delta, B[a/x] \parallel \mathcal{H}$ . Since  $|B[a/x]| < |\exists xB|$ , we can apply IH to this sequent as well as the sequent above to obtain a cut-free derivation of:

$$(\mathcal{G} \oplus \mathcal{G} \oplus \mathcal{F}) \parallel \Gamma, \Gamma, \Gamma_1 \vdash \Delta, \Delta, \Delta_1 \parallel (\mathcal{H} \oplus \mathcal{H} \oplus (\Gamma_2 \vdash \Delta_2 \parallel \cdots \parallel \Gamma_i \vdash \Delta_i \parallel \cdots \parallel \Gamma_n \vdash \Delta_n))$$

Applying admissibility of  $(ic_l)$  and  $(ic_r)$  (Lem. 14 and 16), we obtain the desired conclusion.

Last, let us consider the case where  $A$  is of the form  $\forall xB$ :

$$\frac{\mathcal{G} \parallel \Gamma \vdash \Delta, \forall xB \parallel \mathcal{H} \quad \frac{\mathcal{F} \parallel \forall xB^{k_1}, \Gamma_1 \vdash \Delta_1 \parallel \cdots \parallel \forall xB^{k_i}, B[a/x], \Gamma_i \vdash \Delta_i \parallel \cdots \parallel \forall xB^{k_n}, \Gamma_n \vdash \Delta_n}{\mathcal{F} \parallel \forall xB^{k_1}, \Gamma_1 \vdash \Delta_1 \parallel \cdots \parallel \forall xB^{k_i}, \Gamma_i \vdash \Delta_i \parallel \cdots \parallel \forall xB^{k_n}, \Gamma_n \vdash \Delta_n} (\forall_l)}{(\mathcal{G} \oplus \mathcal{F}) \parallel \Gamma, \Gamma_1 \vdash \Delta, \Delta_1 \parallel (\mathcal{H} \oplus (\Gamma_2 \vdash \Delta_2 \parallel \cdots \parallel \Gamma_i \vdash \Delta_i \parallel \cdots \parallel \Gamma_n \vdash \Delta_n))} \text{(cut)}$$

Applying IH between the left premise of (*cut*) and the premise of the ( $\forall_l$ ) rule, we obtain

$$(\mathcal{G} \oplus \mathcal{F}) // \Gamma, \Gamma_1 \vdash \Delta, \Delta_1 // (\mathcal{H} \oplus (\Gamma_2 \vdash \Delta_2 // \cdots // B[a/x], \Gamma_i \vdash \Delta_i // \cdots // \Gamma_n \vdash \Delta_n))$$

Depending on if  $\mathcal{H}$  is empty or not, we evoke the invertibility of ( $\forall_{r1}$ ) or ( $\forall_{r2}$ ) (Lem. 13 and 11), admissibility of (*mrg*) (Lem. 15), and hp-admissibility of (*sub*) (Lem. 4) to obtain a derivation of the sequent  $\mathcal{G} // \Gamma \vdash \Delta, B[a/x] // \mathcal{H}$ . Since  $|B[a/x]| < |\forall x B|$  we can apply IH between this sequent and the one above to obtain a cut-free derivation of:

$$(\mathcal{G} \oplus \mathcal{G} \oplus \mathcal{F}) // \Gamma, \Gamma, \Gamma_1 \vdash \Delta, \Delta, \Delta_1 // (\mathcal{H} \oplus \mathcal{H} \oplus (\Gamma_2 \vdash \Delta_2 // \cdots // \Gamma_i \vdash \Delta_i // \cdots // \Gamma_n \vdash \Delta_n))$$

Admissibility of (*ic<sub>l</sub>*) and (*ic<sub>r</sub>*) (Lem. 14 and 16) give the desired conclusion.  $\square$

## 5 Conclusion

This paper presented the cut-free calculus LNIF for intuitionistic fuzzy logic within the relatively new paradigm of *linear nested sequents*. The calculus possesses highly fundamental proof-theoretic properties such as (m-)invertibility of all logical rules, admissibility of structural rules, and syntactic cut-elimination.

In future work the author aims to investigate corollaries of the cut-elimination theorem, such as a *midsequent theorem* [4]. In our context, such a theorem states that every derivable sequent containing only prenex formulae is derivable with a proof containing quantifier-free sequents, called *midsequents*, which have only propositional inferences (and potentially (*lift*)) above them in the derivation, and only quantifier inferences (and potentially (*lift*)) below them. Moreover, the present formalism could offer insight regarding which fragments interpolate (or if all of IF interpolates) by applying the so-called *proof-theoretic method* of interpolation [17,19]. Additionally, it could be fruitful to adapt linear nested sequents to other first-order Gödel logics and to investigate decidable fragments [2] by providing proof-search algorithms with implementations (e.g. [16] provides an implementation of proof-search in PROLOG for a class of modal logics within the linear nested sequent framework).

Last, [8] introduced both a nested calculus for first-order intuitionistic logic with *constant domains*, and a nested calculus for first-order intuitionistic logic with *non-constant domains*. The fundamental difference between the two calculi involves the imposition of a side condition on the left  $\forall$  and right  $\exists$  rules. The author aims to investigate whether such a condition can be imposed on quantifier rules in LNIF in order to readily convert the calculus into a sound and cut-free complete calculus for first-order Gödel logic with *non-constant domains*. This would be a further strength of LNIF since switching between the calculi for the constant domain and non-constant domain versions of first-order Gödel logic would result by simply imposing a side condition on a subset of the quantifier rules.

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## A Proofs

**Theorem 2. (Soundness of LNIF)** *For any linear nested sequent  $\mathcal{G}$ , if  $\mathcal{G}$  is provable in LNIF, then  $\Vdash \bar{\forall} \iota(\mathcal{G})$ .*

*Proof.* We prove the result by induction on the height of the derivation of  $\mathcal{G}$ .

*Base case.* We argue by contradiction that  $(r) \in \{(id_1), (id_2), (\perp_1)\}$  always produces a valid linear nested sequent under the universal closure of  $\iota$ . Each rule is of the form shown below left with the sequent  $\mathcal{G}$  of the form shown below right:

$$\overline{\mathcal{G}}(r) \quad \mathcal{G} = \Gamma_1 \vdash \Delta_1 // \cdots // \Gamma_n \vdash \Delta_n // \cdots // \Gamma_m \vdash \Delta_m$$

We therefore assume that  $\mathcal{G}$  is invalid. This implies that there exists a model  $M = (W, R, D, V)$  with world  $v$  such that  $Rvw_0$ ,  $\vec{a} \in D_{w_0}$ , and  $M, w_0 \not\models \iota(\mathcal{G})(\vec{a})$ . It follows that there is a sequence of worlds  $w_1, \dots, w_m \in W$  such that  $Rw_j w_{j+1}$  (for  $0 \leq j \leq m-1$ ),  $M, w_i \Vdash \bigwedge \Gamma_i$ , and  $M, w_i \not\models \bigvee \Delta_i$ , for each  $1 \leq i \leq m$ . We assume all parameters in  $\bigwedge \Gamma_i$  and  $\bigvee \Delta_i$  (for  $1 \leq i \leq m$ ) are interpreted as elements of the associated domain.

**$(id_1)$ -rule:** Let  $\mathcal{G}$  be in the form above and assume that  $\Gamma_n = \Gamma'_n, p(\vec{b})$  and  $\Delta_n = p(\vec{b}), \Delta'_n$ . If this happens to be the case, then  $M, w_n \Vdash \bigwedge \Gamma'_n \wedge p(\vec{b})$  and

$M, w_n \not\vdash \bigvee \Delta'_n \vee p(\vec{b})$ . The former implies that  $M, w_n \Vdash p(\vec{b})$  and the latter implies that  $M, w_n \not\vdash p(\vec{b})$ , which is a contradiction.

**(id<sub>2</sub>)-rule:** Similar to the case above, but uses Lem. 1.

**( $\perp$ )<sub>l</sub>-rule:** Let  $\mathcal{G}$  be as above and assume that  $\Gamma_n = \Gamma'_n, \perp$ . If this is the case, then it follows that  $M, w_n \Vdash \bigwedge \Gamma_i \wedge \perp$ , from which the contradiction  $M, w_n \Vdash \perp$  follows.

*Inductive step.* Each inference rule considered is of one of the following two forms:

$$\frac{\mathcal{G}'}{\mathcal{G}} (r_1) \qquad \frac{\mathcal{G}_1 \quad \mathcal{G}_2}{\mathcal{G}} (r_2)$$

where  $\mathcal{G} = \Gamma_1 \vdash \Delta_1 // \cdots // \Gamma_n \vdash \Delta_n // \Gamma_{n+1} \vdash \Delta_{n+1} // \cdots // \Gamma_m \vdash \Delta_m$ .

Assuming that  $\mathcal{G}$  is invalid implies the existence of a model  $M = (W, R, D, V)$  with world  $v$  such that  $Rvw_0$ ,  $\vec{a} \in D_{w_0}$ , and  $M, w_0 \not\vdash \iota(\mathcal{G})\sigma$ . Hence, there is a sequence of worlds  $w_1, \dots, w_m \in W$  such that  $Rw_j w_{j+1}$  (for  $0 \leq j \leq m-1$ ),  $M, w_i \Vdash \bigwedge \Gamma_i$ , and  $M, w_i \not\vdash \bigvee \Delta_i$ , for each  $1 \leq i \leq m$ . We assume all parameters in  $\bigwedge \Gamma_i$  and  $\bigvee \Delta_i$  (for  $1 \leq i \leq m$ ) are interpreted as elements of the associated domain.

**(lift)-rule:** By our assumption that the conclusion  $\mathcal{G}$  is falsifiable, it follows that  $M, w_n \Vdash \bigwedge \Gamma_n \wedge A$ ,  $M, w_{n+1} \Vdash \bigwedge \Gamma_{n+1}$ ,  $M, w_n \not\vdash \bigvee \Delta_n$ , and  $M, w_{n+1} \not\vdash \bigvee \Delta_{n+1}$ . By the fact that  $Rw_n w_{n+1}$  and Lem. 1,  $M, w_{n+1} \Vdash A$ , which falsifies the premise.

**( $\wedge$ ), ( $\wedge_r$ ), ( $\vee_r$ ), ( $\vee_l$ )-rules:** It is not difficult to show that for each of these rules the premise, or at least one of the premises, is invalid if the conclusion is assumed invalid.

**( $\supset_{r1}$ )-rule:** It follows from our assumption that  $M, w_m \Vdash \bigwedge \Gamma_m$  and  $M, w_m \not\vdash \bigvee \Delta_m \vee A \supset B$ . The latter statement implies that  $M, w_m \not\vdash A \supset B$ , from which it follows that there exists a world  $w_{m+1} \in W$  such that  $Rw_m w_{m+1}$  and  $M, w_{m+1} \Vdash A$ , but  $M, w_{m+1} \not\vdash B$ , letting us falsify the premise.

**( $\supset_{r2}$ )-rule:** Our assumption implies that  $M, w_n \Vdash \bigwedge \Gamma_n$ ,  $M, w_n \not\vdash \bigvee \Delta_n \vee A \supset B$ ,  $M, w_{n+1} \Vdash \bigwedge \Gamma_{n+1}$ , and  $M, w_{n+1} \not\vdash \bigvee \Delta_{n+1}$ . The fact that  $M, w_n \not\vdash \bigvee \Delta_n \vee A \supset B$  holds implies that there exists a world  $w \in W$  such that  $Rw_n w$  and  $M, w \Vdash A$  and  $M, w \not\vdash B$ . Since our frames are connected, we have two cases to consider: (i)  $Rw w_{n+1}$ , or (ii)  $Rw_{n+1} w$ . In case (i), the left premise is falsified, and in case (ii) the right premise is falsified.

**( $\supset_l$ )-rule:** Our assumption implies that  $M, w_n \Vdash \bigwedge \Gamma_n \wedge A \supset B$  and  $M, w_n \not\vdash \bigvee \Delta_n$ . Since  $R$  is reflexive, we know that  $Rw_n w_n$ ; this fact, in conjunction with the fact that  $M, w_n \Vdash A \supset B$ , entails that  $M, w_n \not\vdash A$  or  $M, w_n \Vdash B$ , which confirms that one of the premises of the rule is falsified in  $M$ .

**( $\exists_l$ )-rule:** Our assumption implies that  $M, w_n \Vdash \bigwedge \Gamma_n \wedge \exists x A$  and  $M, w_n \not\vdash \bigvee \Delta_n$ . Therefore,  $M, w_n \Vdash A[b/x]$  for some  $b \in D_{w_n}$ . Since  $a$  is an eigenvariable, this implies that the premise is falsified when we interpret  $a$  as  $b$  at the world  $w_n$ .

**( $\exists_r$ )-rule:** Similar to ( $\forall_l$ ) case. □

**Theorem 3. (Completeness of LNIF)** *If  $\vdash_{\text{IF}} A$ , then  $A$  is provable in LNIF.*



□

**Lemma 2.** For any  $A, \Gamma, \Delta, \mathcal{G}$ , and  $\mathcal{H}$ ,  $\vdash_{\text{LNIF}} \mathcal{G} // \Gamma, A \vdash A, \Delta // \mathcal{H}$ .

*Proof.* We prove the result by induction on the complexity of  $A$ .

*Base case.* When  $A$  is atomic or  $\perp$ , the desired result is an instance of  $(id_1)$  and  $(\perp_l)$ , respectively.

*Inductive step.* We consider each case below and use IH to denote the proof given by the inductive hypothesis.

The cases where  $A$  is of the form  $B \wedge C$ ,  $B \vee C$ , or  $\exists xB$  are simple, and are shown below.

$$\frac{\frac{\overline{\mathcal{G}} // \Gamma, \overline{B}, \overline{C} \vdash \overline{B}, \overline{\Delta} // \overline{\mathcal{H}} \quad \text{IH} \quad \overline{\mathcal{G}} // \Gamma, \overline{B}, \overline{C} \vdash \overline{C}, \overline{\Delta} // \overline{\mathcal{H}} \quad \text{IH}}{\overline{\mathcal{G}} // \Gamma, \overline{B}, \overline{C} \vdash \overline{B} \wedge \overline{C}, \overline{\Delta} // \overline{\mathcal{H}}}}{\overline{\mathcal{G}} // \Gamma, \overline{B} \wedge \overline{C} \vdash \overline{B} \wedge \overline{C}, \overline{\Delta} // \overline{\mathcal{H}}}}$$

$$\frac{\frac{\overline{\mathcal{G}} // \Gamma, \overline{B} \vdash \overline{B}, \overline{C}, \overline{\Delta} // \overline{\mathcal{H}} \quad \text{IH} \quad \overline{\mathcal{G}} // \Gamma, \overline{C} \vdash \overline{B}, \overline{C}, \overline{\Delta} // \overline{\mathcal{H}} \quad \text{IH}}{\overline{\mathcal{G}} // \Gamma, \overline{B} \vee \overline{C} \vdash \overline{B}, \overline{C}, \overline{\Delta} // \overline{\mathcal{H}}}}{\overline{\mathcal{G}} // \Gamma, \overline{B} \vee \overline{C} \vdash \overline{B} \vee \overline{C}, \overline{\Delta} // \overline{\mathcal{H}}}}$$

$$\frac{\overline{\mathcal{G}} // \Gamma, \overline{B[a/x]} \vdash \overline{B[a/x]}, \overline{\exists xB}, \overline{\Delta} // \overline{\mathcal{H}} \quad \text{IH}}{\frac{\overline{\mathcal{G}} // \Gamma, \overline{B[a/x]} \vdash \overline{\exists xB}, \overline{\Delta} // \overline{\mathcal{H}}}{\overline{\mathcal{G}} // \Gamma, \overline{\exists xB} \vdash \overline{\exists xB}, \overline{\Delta} // \overline{\mathcal{H}}}}$$

The cases where  $A$  is of the form  $B \supset C$  or  $\forall xB$  are a bit more cumbersome, and are explained below. We first define the linear nested sequents  $\mathcal{G}_i$  (for  $0 \leq i \leq n$ ) and  $\mathcal{H}_j$ , where  $\mathcal{G}_0 = \mathcal{G}$ .

$$\mathcal{G}_i = \mathcal{G} // \Gamma_1, B \supset C \vdash \Delta_1 // \cdots // \Gamma_i, B \supset C \vdash \Delta_i \quad \mathcal{H}_j = \Gamma_j \vdash \Delta_j // \cdots // \Gamma_n \vdash \Delta_n$$

$$\frac{\frac{\overline{\mathcal{G}} // \Gamma_1, \overline{B} \supset \overline{C} \vdash \overline{\Delta}_1 // \overline{B}, \overline{C} \vdash \overline{C} // \overline{\mathcal{H}}_2 \quad \text{IH} \quad \overline{\mathcal{G}} // \Gamma_1, \overline{B} \supset \overline{C} \vdash \overline{\Delta}_1 // \overline{B}, \overline{B} \supset \overline{C} \vdash \overline{B}, \overline{C} // \overline{\mathcal{H}}_2 \quad \text{IH}}{\frac{\overline{\mathcal{G}} // \Gamma_1, \overline{B} \supset \overline{C} \vdash \overline{\Delta}_1 // \overline{B}, \overline{B} \supset \overline{C} \vdash \overline{C} // \overline{\mathcal{H}}_2}{\overline{\mathcal{G}} // \Gamma_1, \overline{B} \supset \overline{C} \vdash \overline{\Delta}_1 // \overline{B} \vdash \overline{C} // \overline{\mathcal{H}}_2}}}{\overline{\mathcal{G}} // \Gamma_1, \overline{B} \supset \overline{C} \vdash \overline{B} \supset \overline{C}, \overline{\Delta}_1 // \overline{\mathcal{H}}_2} \quad \Pi_0$$

The derivations  $\Pi_i$ , with  $0 \leq i \leq n-3$ , are as follows:

$$\frac{\frac{\frac{\overline{\mathcal{G}}_{i+1} // \Gamma_{i+2}, \overline{B} \supset \overline{C} \vdash \overline{\Delta}_{i+2} // \overline{B}, \overline{C} \vdash \overline{C} // \overline{\mathcal{H}}_{i+3} \quad \text{IH} \quad \overline{\mathcal{G}}_{i+1} // \Gamma_{i+2}, \overline{B} \supset \overline{C} \vdash \overline{\Delta}_{i+2} // \overline{B}, \overline{B} \supset \overline{C} \vdash \overline{B}, \overline{C} // \overline{\mathcal{H}}_{i+3} \quad \text{IH}}{\overline{\mathcal{G}}_{i+1} // \Gamma_{i+2}, \overline{B} \supset \overline{C} \vdash \overline{\Delta}_{i+2} // \overline{B}, \overline{B} \supset \overline{C} \vdash \overline{C} // \overline{\mathcal{H}}_{i+3}}}{\overline{\mathcal{G}}_{i+1} // \Gamma_{i+2}, \overline{B} \supset \overline{C} \vdash \overline{\Delta}_{i+2} // \overline{B} \vdash \overline{C} // \overline{\mathcal{H}}_{i+3}}}{\overline{\mathcal{G}}_{i+1} // \Gamma_{i+2}, \overline{B} \supset \overline{C} \vdash \overline{B} \supset \overline{C}, \overline{\Delta}_{i+2} // \overline{\mathcal{H}}_{i+3}} \quad \Pi_{i+1} =$$

$$\frac{\overline{\mathcal{G}}_{i+1} // \Gamma_{i+2}, \overline{B} \supset \overline{C} \vdash \overline{\Delta}_{i+2} // \overline{B} \vdash \overline{C} // \overline{\mathcal{H}}_{i+3}}{\overline{\mathcal{G}}_i // \Gamma_{i+1}, \overline{B} \supset \overline{C} \vdash \overline{\Delta}_{i+1} // \Gamma_{i+2}, \overline{B} \supset \overline{C} \vdash \overline{B} \supset \overline{C}, \overline{\Delta}_{i+2} // \overline{\mathcal{H}}_{i+3}} =$$

$$\frac{\overline{\mathcal{G}}_i // \Gamma_{i+1}, \overline{B} \supset \overline{C} \vdash \overline{\Delta}_{i+1} // \Gamma_{i+2}, \overline{B} \supset \overline{C} \vdash \overline{B} \supset \overline{C}, \overline{\Delta}_{i+2} // \overline{\mathcal{H}}_{i+3}}{\overline{\mathcal{G}}_i // \Gamma_{i+1}, \overline{B} \supset \overline{C} \vdash \overline{\Delta}_{i+1} // \Gamma_{i+2}, \overline{B} \supset \overline{C}, \overline{\Delta}_{i+2} // \overline{\mathcal{H}}_{i+3}}$$

Last, the derivation  $\Pi_{n-2}$  is as follows:

$$\begin{array}{c}
\frac{\overline{\overline{\mathcal{G}_n \parallel B, C \vdash C}} \text{ IH} \quad \overline{\overline{\mathcal{G}_n \parallel B, B \supset C \vdash B, C}} \text{ IH}}{\overline{\overline{\mathcal{G}_n \parallel B, B \supset C \vdash C}}} \\
\frac{\dots \overline{\overline{\mathcal{G}_{n-2} \parallel \Gamma_{n-1}, B \supset C \vdash \Delta_{n-1} \parallel \Gamma_n, B \supset C \vdash \Delta_n \parallel B, B \supset C \vdash C}} =}{\overline{\overline{\mathcal{G}_{n-2} \parallel \Gamma_{n-1}, B \supset C \vdash \Delta_{n-1} \parallel \Gamma_n, B \supset C \vdash \Delta_n \parallel B \vdash C}}} \\
\frac{\overline{\overline{\mathcal{G}_{n-2} \parallel \Gamma_{n-1}, B \supset C \vdash \Delta_{n-1} \parallel \Gamma_n \vdash \Delta_n \parallel B \vdash C}}}{\overline{\overline{\mathcal{G}_{n-2} \parallel \Gamma_{n-1}, B \supset C \vdash \Delta_{n-1} \parallel \Gamma_n \vdash B \supset C, \Delta_n}}}
\end{array}$$

Let us consider the case where  $A$  is of the form  $\forall xB$ . We first define the linear nested sequents  $\mathcal{G}_i$  (for  $0 \leq i \leq n$ ) and  $\mathcal{H}_j$ , where  $\mathcal{G}_0 = \mathcal{G}$ .

$$\mathcal{G}_i = \mathcal{G} \parallel \Gamma_1, \forall xB \vdash \Delta_1 \parallel \dots \parallel \Gamma_i, \forall xB \vdash \Delta_i \quad \mathcal{H}_j = \Gamma_j \vdash \Delta_j \parallel \dots \parallel \Gamma_n \vdash \Delta_n$$

$$\begin{array}{c}
\overline{\overline{\overline{\overline{\mathcal{G} \parallel \Gamma_1, \forall xB \vdash \Delta_1 \parallel A[y/x], \forall xA \vdash A[y/x] \parallel \mathcal{H}_2}} \text{ IH}}} \\
\frac{\overline{\overline{\overline{\mathcal{G} \parallel \Gamma_1, \forall xB \vdash \Delta_1 \parallel \forall xA \vdash A[y/x] \parallel \mathcal{H}_2}}}}{\overline{\overline{\overline{\mathcal{G} \parallel \Gamma_1, \forall xB \vdash \Delta_1 \parallel A[y/x] \parallel \mathcal{H}_2}}} \quad \Pi_0} \\
\frac{\overline{\overline{\overline{\mathcal{G} \parallel \Gamma_1, \forall xB \vdash \forall xB, \Delta_1 \parallel \mathcal{H}_2}}}
\end{array}$$

The derivations  $\Pi_i$ , with  $0 \leq i \leq n-3$ , are as follows:

$$\begin{array}{c}
\overline{\overline{\overline{\overline{\mathcal{G}_{i+2} \parallel \forall xB, B[y_i/x] \vdash B[y_i/x] \parallel \mathcal{H}_{i+3}}} \text{ IH}}} \\
\frac{\overline{\overline{\overline{\mathcal{G}_{i+2} \parallel \forall xB \vdash B[y_i/x] \parallel \mathcal{H}_{i+3}}}}}{\overline{\overline{\overline{\mathcal{G}_i \parallel \Gamma_{i+1}, \forall xB \vdash \Delta_{i+1} \parallel \Gamma_{i+2}, \forall xB \vdash \Delta_{i+2} \parallel \forall xB \vdash B[y_i/x] \parallel \mathcal{H}_{i+3}}} = \quad \Pi_{i+1}} \\
\frac{\overline{\overline{\overline{\mathcal{G}_i \parallel \Gamma_{i+1}, \forall xB \vdash \Delta_{i+1} \parallel \Gamma_{i+2}, \forall xB \vdash \Delta_{i+2} \parallel \vdash B[y_i/x] \parallel \mathcal{H}_{i+3}}}}}{\overline{\overline{\overline{\mathcal{G}_i \parallel \Gamma_{i+1}, \forall xB \vdash \Delta_{i+1} \parallel \Gamma_{i+2}, \forall xB \vdash \forall xB, \Delta_{i+2} \parallel \mathcal{H}_{i+3}}}}} \\
\frac{\overline{\overline{\overline{\mathcal{G}_i \parallel \Gamma_{i+1}, \forall xB \vdash \Delta_{i+1} \parallel \Gamma_{i+2} \vdash \forall xB, \Delta_{i+2} \parallel \mathcal{H}_{i+3}}}}}
\end{array}$$

The last component of the derivation,  $\Pi_{n-2}$  is given below:

$$\begin{array}{c}
\overline{\overline{\overline{\overline{\overline{\overline{\overline{\mathcal{G}_{n-2} \parallel \Gamma_{n-1}, \forall xB \vdash \Delta_{n-1} \parallel \Gamma_n, \forall xB \vdash \Delta_n \parallel \forall xB, B[y_n/x] \vdash B[y_n/x]}} \text{ IH}}}}}}}} \\
\frac{\overline{\overline{\overline{\overline{\overline{\mathcal{G}_{n-2} \parallel \Gamma_{n-1}, \forall xB \vdash \Delta_{n-1} \parallel \Gamma_n, \forall xB \vdash \Delta_n \parallel \forall xB \vdash B[y_n/x]}}}}}}}{\overline{\overline{\overline{\overline{\overline{\mathcal{G}_{n-2} \parallel \Gamma_{n-1}, \forall xB \vdash \Delta_{n-1} \parallel \Gamma_n, \forall xB \vdash \Delta_n \parallel \vdash B[y_n/x]}}}}}} \\
\frac{\overline{\overline{\overline{\overline{\overline{\mathcal{G}_{n-2} \parallel \Gamma_{n-1}, \forall xB \vdash \Delta_{n-1} \parallel \Gamma_n \vdash \Delta_n \parallel \vdash B[y_n/x]}}}}}}}{\overline{\overline{\overline{\overline{\overline{\mathcal{G}_{n-2} \parallel \Gamma_{n-1}, \forall xB \vdash \Delta_{n-1} \parallel \Gamma_n \vdash \forall xB, \Delta_n}}}}}}
\end{array}$$

□

**Lemma 4.** *The (sub) rule is hp-admissible in LNIF.*

*Proof.* We prove the result by induction on the height of the given derivation of  $\mathcal{G}$ .

*Base case.* Any instance of the rule ( $id_1$ ), ( $id_2$ ), or ( $\perp_l$ ) is still an instance of the rule under the variable substitution  $[b/a]$ .

*Inductive step.* For all rules, with the exception of the ( $\forall_{r1}$ ), ( $\exists_l$ ), and ( $\forall_{r2}$ ) rules, the claim follows straightforwardly by applying IH followed by the corresponding rule. The nontrivial cases occur when the last rule applied is an instance of ( $\forall_{r1}$ ), ( $\exists_l$ ), or ( $\forall_{r2}$ ), and the variable substituted into the conclusion of the inference is also the eigenvariable of the inference:



$$\begin{array}{c}
 \frac{\mathcal{G} // \Gamma \vdash \Delta // \vdash A[a/x]}{\mathcal{G} // \Gamma \vdash \Delta, \forall x A} \quad \frac{\mathcal{G} // \Gamma, A[a/x] \vdash \Delta // \mathcal{H}}{\mathcal{G} // \Gamma, \exists x A \vdash \Delta // \mathcal{H}} \\
 \hline
 (\mathcal{G} // \Gamma \vdash \Delta, \forall x A)[a/b] \quad (\mathcal{G} // \Gamma, \exists x A \vdash \Delta // \mathcal{H})[a/b] \\
 \\
 \frac{\mathcal{G} // \Gamma_1 \vdash \Delta_1 // \vdash A[a/x] // \Gamma_2 \vdash \Delta_2 // \mathcal{H} \quad \mathcal{G} // \Gamma_1 \vdash \Delta_1 // \Gamma_2 \vdash \Delta_2, \forall x A // \mathcal{H}}{\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x A // \Gamma_2 \vdash \Delta_2 // \mathcal{H}} \\
 \hline
 (\mathcal{G} // \Gamma_1 \vdash \Delta_1, \forall x A // \Gamma_2 \vdash \Delta_2 // \mathcal{H})[a/b]
 \end{array}$$

In such cases we evoke the inductive hypothesis twice: first, we apply the substitution  $[c/a]$  to the premise, where  $c$  is a fresh parameter, and then we evoke the inductive hypothesis again and apply the substitution  $[a/b]$ . The desired result follows by a single application of each rule.  $\square$

**Lemma 5.** *The (iw) rule is hp-admissible in LNIF.*

*Proof.* We know by [17, Lem. 5.5] that (iw) is admissible in LNG. We extend the argument to LNIF and consider the cases of permuting (iw) past the  $(\forall_{r1})$ ,  $(\exists_l)$ , and  $(\forall_{r2})$  rules; the  $(\forall_l)$  and  $(\exists_r)$  cases are trivial.

Suppose we have a  $(\forall_{r1})$  inference followed by an instance of (iw):

$$\frac{\frac{\mathcal{G} // \Gamma \vdash \Delta // \vdash A[a/x]}{\mathcal{G} // \Gamma \vdash \Delta, \forall x A} (\forall_{r1})}{\mathcal{G}' // \Gamma' \vdash \Delta', \forall x A} (iw)$$

The nontrivial case occurs when the (iw) rule weakens in a formula containing the parameter  $a$ . If this happens to be the case, then we evoke Lem. 4 and apply a substitution  $[b/a]$  where  $b$  is a fresh parameter not occurring in the derivation above. After performing this operation, we may complete the derivation as shown below:

$$\begin{array}{c}
 \frac{\mathcal{G} // \Gamma \vdash \Delta // \vdash A[a/x]}{\mathcal{G}[b/a] // \Gamma[b/a] \vdash \Delta[b/a] // \vdash A[a/x][b/a]} \text{ Lem. 4} \\
 \hline
 \frac{\mathcal{G} // \Gamma \vdash \Delta // \vdash A[b/x]}{\mathcal{G}' // \Gamma' \vdash \Delta' // \vdash A[b/x]} (iw) \\
 \hline
 \frac{\mathcal{G}' // \Gamma' \vdash \Delta' // \vdash A[b/x]}{\mathcal{G}' // \Gamma' \vdash \Delta', \forall x A} (\forall_{r1})
 \end{array}$$

The second and third line are equal because  $a$  does not occur in  $\mathcal{G}$ ,  $\Gamma$ , or  $\Delta$  (i.e.,  $a$  is an eigenvariable) and also because  $A[a/x][b/a] = A[b/x]$  (we may assume w.l.o.g. that  $A$  does not contain any occurrences of  $a$ ). Last, we may apply the  $(\forall_{r1})$  rule since we are guaranteed that  $b$  is an eigenvariable by choice.

The cases for  $(\exists_l)$  and  $(\forall_{r2})$  are shown similarly.  $\square$

**Lemma 8.** *If  $\sum_{i=1}^n k_n \geq 1$ , then*

- (i) (1) implies (2)                      (iii) (6) implies (7) and (8)      (v) (11) implies (12)  
(ii) (3) implies (4) and (5)      (iv) (9) implies (10)

$$\vdash_{\text{LNIF}} \Gamma_1, (A \wedge B)^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, (A \wedge B)^{k_n} \vdash \Delta_n \quad (1)$$

$$\vdash_{\text{LNIF}} \Gamma_1, A^{k_1}, B^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, A^{k_n}, B^{k_n} \vdash \Delta_n \quad (2)$$

$$\vdash_{\text{LNIF}} \Gamma_1, (A \vee B)^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, (A \vee B)^{k_n} \vdash \Delta_n \quad (3)$$

$$\vdash_{\text{LNIF}} \Gamma_1, A^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, A^{k_n} \vdash \Delta_n \quad (4)$$

$$\vdash_{\text{LNIF}} \Gamma_1, B^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, B^{k_n} \vdash \Delta_n \quad (5)$$

$$\vdash_{\text{LNIF}} \Gamma_1, (A \supset B)^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, (A \supset B)^{k_n} \vdash \Delta_n \quad (6)$$

$$\vdash_{\text{LNIF}} \Gamma_1, B^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, B^{k_n} \vdash \Delta_n \quad (7)$$

$$\vdash_{\text{LNIF}} \Gamma_1, (A \supset B)^{k_1} \vdash \Delta_1, A^{k_1} // \cdots // \Gamma_n, (A \supset B)^{k_n} \vdash \Delta_n, A^{k_n} \quad (8)$$

$$\vdash_{\text{LNIF}} \Gamma_1, (\forall x A)^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, (\forall x A)^{k_n} \vdash \Delta_n \quad (9)$$

$$\vdash_{\text{LNIF}} \Gamma_1, A[a/x]^{k_1}, (\forall x A)^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, A[a/x]^{k_n}, (\forall x A)^{k_n} \vdash \Delta_n \quad (10)$$

$$\vdash_{\text{LNIF}} \Gamma_1, (\exists x A)^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, (\exists x A)^{k_n} \vdash \Delta_n \quad (11)$$

$$\vdash_{\text{LNIF}} \Gamma_1, A[a/x]^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, A[a/x]^{k_n} \vdash \Delta_n \quad (12)$$

*Proof.* By [17, Lem. 5.9] we know that claims (i)-(iii) hold for LNG. It is easy to show that the proof of [17, Lem. 5.9] can be extended to the quantifier rules for claims (i) and (ii) by applying the IH and then the rule. We therefore argue that claim (iii) continues to hold in the presence of the quantifier rules, and also argue that claims (iv) and (v) hold.

*Claim (iii).* We know that (6) implies (8) by Lem. 5. One can prove that (6) implies (7) by induction on the height of the given derivation. By [17, Lem. 5.9] all cases with the exception of the quantifier rules hold. All of the quantifier cases are handled by applying IH followed by an application of the corresponding rule.

*Claim (iv).* Statement (9) implies (10) by Lem. 5.

*Claim (v).* We argue that (11) implies (12) by induction on the height of the given derivation of  $\mathcal{G} = \Gamma_1, (\exists x A)^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, (\exists x A)^{k_n} \vdash \Delta_n$ .

*Base case.* If  $\mathcal{G}$  is the result of  $(id_1)$ ,  $(id_2)$ , or  $(\perp_l)$ , then  $\Gamma_1, A[a/x]^{k_1} \vdash \Delta_1 // \cdots // \Gamma_n, A[a/x]^{k_n} \vdash \Delta_n$  is an instance of the corresponding rule as well.

*Inductive step.* For all rules, with the exception of  $(\forall_{r1})$ ,  $(\exists_l)$ , and  $(\forall_{r2})$ , we apply IH to the premise(s) followed by the rule. In the  $(\forall_{r1})$ ,  $(\exists_l)$ , and  $(\forall_{r2})$  cases we must ensure that the eigenvariable of the inference is not identical to the parameter  $a$  occurring in  $A[a/x]$ ; however, due to Lem. 4, this can always be ensured.  $\square$