Infinity, Choice, and Hume's Principle

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August 8, 2024

penultimate draft; please cite published version

1 Introduction

Neo-Fregean philosophers of mathematics have for some time been interested in the implications of *Hume's Principle* (HP). This principle says: "the number of Fs is equal to the number of Gs just in case there is a bijection between the Fs and the Gs." In second-order logic, HP is expressed in the following way:

$$\#F = \#G \leftrightarrow F \approx G.$$

Here F and G are monadic second-order variables. The operator # (read: "the number of") is a type-lowering operator that combines with monadic second-order variables to yield terms of object type. Finally, $F \approx G$ abbreviates the second-order statement that there is a bijection between the Fs and the Gs.

Hume's Principle is surprisingly strong in the context of axiomatic second-order logic (SOL).[1] In fact SOL + HP interprets second-order arithmetic, a beautiful result known as *Frege's Theorem*.[2] What is most surprising about Frege's Theorem is the fact that SOL + HP proves the

[1]By *axiomatic second-order logic*, I mean essentially Shapiro's (1991, pp. 66–67) system D2 minus the axiom scheme of choice. This system includes full second-order comprehension. See section 2.1 for details.

[2]*Second-order arithmetic* is a powerful theory of arithmetic that seems capable of formalizing almost any ordinary mathematical theorem only involving countable objects and structures.

existence of *infinitely many* cardinal numbers. This is not at all obvious just by inspecting HP; for HP seems merely to give us the criterion of identity of cardinal numbers. Yet in fact HP has the strength of an axiom of infinity, though it doesn't wear it on its sleeve. SOL + HP interprets the natural numbers: it can define an infinite progression of objects and a successor function on those objects. Hence, SOL + HP proves that the universe is Dedekind infinite (DI): "there is an injection from the universe into itself that is not a surjection." This is a version of the axiom of infinity expressed in the language of pure second-order logic.

How strong is HP, exactly? One way of approaching this question is to measure strength by interpretability. It has long been known that the following theories are mutually interpretable:

1. SOL + DI
2. SOL + HP
3. second-order arithmetic.

That (2) interprets (3) is essentially due to Frege (1884, 1893). That (1) interprets (3) is essentially due to Dedekind (1888). That (3) interprets (2) is due to Boolos (1987). Finally, it is easy to see that (3) interprets (1). So, the interpretability strength of HP is well understood. Furthermore, this characterization of the strength of HP reveals an interesting fact: in a certain sense, the strength of HP is already fully captured by one of its purely logical consequences, namely DI.

In this note, we give a more fine-grained characterization of the logical strength of HP, as measured by deductive implications rather than interpretability.[3] Our main result is that HP is not deductively conservative over SOL + DI (Theorem 7). In other words, HP proves additional theorems in the language of pure second-order logic that are not provable from SOL + DI alone. Arguably, then, HP is not just a pure axiom of infinity, but rather it carries additional logical content.

Next, we establish some limits on the non-conservativeness of HP by giving a partial conservativity result: HP is Π^1_1 conservative over SOL + DI (Theorem 12). A sentence in the

[3]This is a more fine-grained characterization, because everything that is deductively implied by a theory is interpretable in that theory, but not vice versa.

language of pure second-order logic is said to be Π^1_1 just in case it has the form $\forall X_1 \cdots \forall X_k\, \varphi$, where φ contains no second-order quantifiers.[4] The partial conservativity result says that any Π^1_1 sentence in the language of pure second-order logic that is provable from SOL + HP is already provable from SOL + DI. In the course of proving this result, we also give a full conservativity result with respect to a stronger base theory: HP is conservative over SOL + DI + "the universe is well-ordered" (WO) (Lemma 13). The principle WO is a strong form of the axiom of choice expressed in the language of pure second-order logic.

The latter result motivates us to ask whether HP might prove some form of the axiom of choice. This is a natural question, because HP itself feels like a choice principle: it lets us choose a representative object for each equivalence class of equinumerous concepts (i.e., sets or monadic relations).[5] We do not settle this question completely. However, we show that HP does not prove some particularly simple and natural forms of the axiom of choice (Theorem 15).

Since we compare HP with different versions of the axiom of choice, it is natural and interesting to compare HP with different versions of the axiom of infinity as well.[6] We close this paper with a brief discussion of this question. We consider eleven natural versions of the axiom of infinity in second-order logic, which form an almost linearly ordered hierarchy in terms of deductive strength. We show that DI is the strongest axiom in the hierarchy that is provable from SOL + HP (Theorem 31). This is a pleasing result, for a couple of reasons. Firstly, it suggests that we were right to focus on the relationship between HP and DI, rather than some other axiom of infinity. Secondly, the axioms of infinity stronger than DI can be viewed as smuggling in some consequences of the axiom of choice; they express claims that would normally be proved using choice, and are not provable in Zermelo-Fraenkel set theory without choice. So, the fact that HP does not prove these stronger axioms of infinity bolsters our claim that HP does not prove any

[4]See (Väänänen 2021, §4) for discussion of prenex classes of formulas in second-order logic. In general, one needs a version of the axiom of choice (*Hilbert-Ackermann choice*, or HAC) to show that every second-order formula is equivalent to a prenex formula. Since we do not wish to assume HAC, we will speak of prenex complexity only for formulas that are actually in prenex form or which are provably equivalent over SOL (without choice) to a prenex formula. Thus, for example, DI is a Σ^1_1 sentence, while HP is equivalent over SOL to a Π^1_2 sentence.

[5]The analogy between HP and certain forms of the axiom of choice has been noted by Siskind et al. (2023).

[6]I am grateful to an anonymous reviewer for raising this question.

natural form of the axiom of choice.

In summary, we can characterize the logical implications of HP as follows: SOL + HP proves a strong axiom of infinity (namely DI), and even a bit more than that. But it does not seem to prove any simple, natural form of the axiom of choice, nor any stronger axiom of infinity.

Our plan is as follows. In section 2, we present some background material on second-order logic and permutation models. In section 3, we study deductive relationships between HP and DI. In section 4, we examine deductive relationships between HP and various forms of the axiom of choice. Lastly, in section 5, we consider deductive relationships between HP and other axioms of infinity.

2 Background

2.1 Second-order logic

Our terminology and notation for second-order logic are fairly standard. But it is still good to review these definitions briefly. For a comprehensive reference, see (Shapiro 1991).

The *language of pure second-order logic*, $\mathscr{L}$, contains first-order variables $x, y, z, \ldots$ and n-adic relation variables $X, Y, Z, \ldots$ for all $n \geq 1$. Atomic formulas of $\mathscr{L}$ have the form $x = y$ or $Xy_1 \cdots y_n$. If φ, ψ are formulas and x, X are variables, then $\varphi \to \psi$, $\neg\varphi$, $\exists x\varphi$, $\exists X\varphi$ are also formulas.

The *deductive system* for $\mathscr{L}$ is Shapiro's (1991, pp. 66–67) system D2 minus the axiom scheme of choice.[7] This deductive system contains the usual axioms and rules for propositional logic, quantification, and identity, together with the axiom scheme of comprehension:

$$\exists X \forall y_1, \cdots, y_n (Xy_1 \cdots y_n \leftrightarrow \varphi(y_1, \cdots, y_n)),$$

where φ is any $\mathscr{L}$-formula not containing free occurrences of X.

[7] However, Shapiro (1991) allows function variables in the language of pure second-order logic, whereas we only allow relation variables.

An $\mathscr{L}$*-prestructure* $\mathscr{M} = (M, M_1, M_2, M_3, \cdots)$ is a sequence of nonempty sets such that $M_n \subseteq \mathscr{P}(M^n)$ for every natural number $n \geq 1$.[8] The set M is called the *object domain* or *universe* of $\mathscr{M}$, over which first-order variables range. For each $n \geq 1$, the set M_n is called the n-adic *relation domain*, over which n-adic relation variables range. Following Fregean tradition, we refer to monadic relations as *concepts*. Satisfaction and truth are defined for $\mathscr{L}$-prestructures in the usual way.

A *general* $\mathscr{L}$*-structure* is an $\mathscr{L}$-prestructure in which the comprehension scheme is satisfied. Our deductive system is sound and complete with respect to general $\mathscr{L}$-structures. A *full* $\mathscr{L}$*-structure* $\mathscr{M}$ is a general $\mathscr{L}$-structure in which $M_n = \mathscr{P}(M^n)$ for all $n \geq 1$. Thus, a full $\mathscr{L}$-structure is completely specified by its object domain M. Our deductive system is sound but not complete with respect to full $\mathscr{L}$-structures.

The language $\mathscr{L}^+$ is obtained by adding to $\mathscr{L}$ the operator #, which combines with a monadic relation variable to yield a term of object type. Alternatively, we can represent # by means of a dyadic "third-order" relation symbol $\#(X, y)$. The formation rules, deductive system, and semantics for $\mathscr{L}^+$ are based on those for $\mathscr{L}$ in the obvious way. Note that the deductive system for $\mathscr{L}^+$ includes comprehension axioms for formulas containing #. An $\mathscr{L}^+$-prestructure has the form $\mathscr{M} = (M, M_1, M_2, M_3, \cdots, f)$, where f is a function from M_1 into M.

We write "SOL" ambiguously for the logical axioms of $\mathscr{L}$ and for those of $\mathscr{L}^+$. For example, in saying that HP is not conservative over SOL + DI, we mean there is an $\mathscr{L}$-sentence φ such that HP $\vdash_{\mathscr{L}^+} \varphi$ but DI $\nvdash_{\mathscr{L}} \varphi$. The reason for the ambiguity is that when we are reasoning in an $\mathscr{L}$-theory, we should not make use of any logical axioms involving additional vocabulary not found in $\mathscr{L}$. This is particularly important when full comprehension is included among the logical axioms, since adding more comprehension can vastly increase the deductive strength of a theory.[9] We allow ourselves the use of full comprehension *for the language under consideration*

[8] As usual, M^n means $M \times \cdots \times M$ (n times).

[9] E.g., pure axiomatic third-order logic (TOL) with full comprehension is not conservative over pure axiomatic second-order logic (SOL) with full comprehension. (Proof sketch: Recall that SOL + DI is mutually interpretable with second-order arithmetic (Z_2). A consistency statement for Z_2 can be formulated in $\mathscr{L}$ in a reasonable way; call it Con(Z_2). By a strong form of the second incompleteness theorem, SOL + DI does not prove Con(Z_2). But TOL + DI proves Con(Z_2), thanks to the additional layer of comprehension. Hence TOL proves DI $\rightarrow$ Con(Z_2), but SOL does

at any given time—but no more.

2.2 Permutation models

Permutation models were originally devised by Fraenkel and Mostowski in the context of set theory. We will only need a particularly simple kind of permutation model, which we describe below. Our presentation is based on (Mackereth and Avigad 2023, §6). For reference, see (Väänänen 2021, §9.1), (Asser 1981, §7), and (Jech 1973, ch. 4).

Let M be a nonempty set, and let G be a group of permutations of M. Let $A \subseteq M^n$ and $E \subseteq M$. We say that E is a *support* of A iff every permutation $\pi \in G$ that fixes E pointwise fixes A setwise:

$$(\forall e \in E)(\pi(e) = e) \implies \forall x_1, \cdots, x_n((x_1, \cdots, x_n) \in A \leftrightarrow (\pi(x_1), \cdots, \pi(x_n)) \in A).$$

Using the notation $\pi(A) = \{(\pi(x_1), \cdots, \pi(x_n)) \in M^n : (x_1, \cdots, x_n) \in A\}$, we can restate this property as follows: for every permutation $\pi \in G$,

$$(\forall e \in E)(\pi(e) = e) \implies \pi(A) = A.$$

A set $A \subseteq M^n$ is *symmetric* iff it has a finite support $E \subseteq M$. Let us consider a couple of examples. (1) If every permutation $\pi \in G$ fixes $A \subseteq M^n$ setwise, then A is symmetric. Hence $\varnothing$ and M^n (for any $n \geq 1$) are symmetric, no matter what group G we choose. (2) If $E \subseteq M$ is any finite set, then E and $M \setminus E$ are both symmetric, no matter what group G we choose.

In the special case where A is a function, there is an equivalent characterization which we shall often use.[10] Let $f : M^n \to M$ be an n-ary function, and let $E \subseteq M$. Then E is a *support* of f

not.) I do not know whether the logical axioms of $\mathscr{L}^+$ are conservative over those of $\mathscr{L}$.

[10]Officially, we will identify functions with their graphs, representing n-ary functions as $(n+1)$-adic relations. However, we will sometimes use function notation as an informal aid.

iff every permutation $\pi \in G$ that fixes E pointwise commutes with f:

$$(\forall e \in E)(\pi(e) = e) \implies \forall x_1, \cdots, x_n(f(\pi(x_1), \cdots, \pi(x_n)) = \pi(f(x_1, \cdots, x_n))).$$

Furthermore, $f : M^n \to M$ is *symmetric* iff it has a finite support $E \subseteq M$. It is easy to check that this characterization agrees with our previous definition.

Definition. Let M be any nonempty set, and let G be any group of permutations of M. The *permutation model* $\mathscr{M}_G$ is the $\mathscr{L}$-prestructure $(M, M_1, M_2, M_3, \cdots)$ where for all $n \geq 1$,

$$M_n = \{A \subseteq M^n : A \text{ is symmetric}\}.$$

Proposition 1. *$\mathscr{M}_G$ is a general $\mathscr{L}$-structure.*

This result is proved in (Asser 1981, pp. 112–114). Unfortunately, this text is out of print and difficult to obtain. In (Mackereth and Avigad 2023, Lemmas 6.36 and 6.38), we presented Asser's proof as applied to the special case where $M := \mathbb{N}$ and G is the group of all permutations of $\mathbb{N}$. We now present the same proof in greater generality in Lemmas 2–5, taking the opportunity to give Lemma 5 in more detail.

Lemma 2. *Let M be any nonempty set, and let G be any group of permutations of M. Take any symmetric set $A \subseteq M^n$ and any $\sigma \in G$. Then $\sigma(A) \subseteq M^n$ is also symmetric.*

Proof. Let E be a finite support for A. We show that $\sigma(E)$ is a finite support for $\sigma(A)$.[11] It is clear that $\sigma(E)$ is finite. To see that $\sigma(E)$ is a support for $\sigma(A)$, take any permutation $\pi \in G$ that fixes $\sigma(E)$ pointwise. Then $\sigma^{-1}\pi\sigma \in G$ is a permutation that fixes E pointwise:

$$e \in E \implies \sigma^{-1}(\pi(\sigma(e))) = \sigma^{-1}(\sigma(e)) = e.$$

So, $(\sigma^{-1}\pi\sigma)(A) = A$, and hence $\pi(\sigma(A)) = \sigma(A)$. ∎

[11] In (Mackereth and Avigad 2023, Lemma 6.36) we mistakenly wrote "$\sigma^{-1}(E)$".

Corollary 3. *Let $\mathscr{M}_G$ be defined as above. Then each relation domain M_n is closed under the action of G on M^n.*

Lemma 4. *$\mathscr{M}_G$ satisfies the axiom scheme of comprehension.*

Proof. Take any $\mathscr{L}$-formula $\varphi(\vec{x},\vec{b},\vec{B})$, with free variables $\vec{x}=(x_1,\cdots,x_n)$ and parameters $\vec{b}=(b_1,\cdots,b_j)$ and $\vec{B}=(B_1,\cdots,B_k)$ drawn from $\mathscr{M}_G$. Let $A=\{\vec{a}\in M^n : \mathscr{M}_G \vDash \varphi(\vec{a},\vec{b},\vec{B})\}$. We show that $A\in M_n$.

Since the relation parameters $\vec{B}$ are drawn from $\mathscr{M}_G$, each set B_i has a finite support E_i ($i=1,\cdots,k$). Let $E=\{b_1,\cdots,b_j\}\cup E_1\cup\cdots\cup E_k$. Clearly, E is finite. We show that E is a support for A.

Take any permutation $\pi\in G$ that fixes E pointwise, and take any $\vec{a}=(a_1,\cdots,a_n)\in M^n$. We check that $\vec{a}\in A \iff \pi(\vec{a})=(\pi(a_1),\cdots,\pi(a_n))\in A$. Indeed,

$$\begin{aligned}
\vec{a}\in A &\iff \mathscr{M}_G \vDash \varphi(\vec{a},\vec{b},\vec{B})\\
&\iff \mathscr{M}_G \vDash \varphi(\pi(\vec{a}),\pi(\vec{b}),\pi(\vec{B}))\\
&\iff \mathscr{M}_G \vDash \varphi(\pi(\vec{a}),\vec{b},\vec{B})\\
&\iff \pi(\vec{a})\in A.
\end{aligned}$$

(Notation: $\pi(\vec{b})=(\pi(b_1),\cdots,\pi(b_j))$ and $\pi(\vec{B})=(\pi(B_1),\cdots,\pi(B_k))$. By Lemma 2, each $\pi(B_i)$ is symmetric, and hence is available for use as a parameter from $\mathscr{M}_G$.) The first and fourth steps follow from the definition of A. The second step works because for any variable assignment s, which assigns values both to object variables and to relation variables, we have

$$\mathscr{M}_G[s]\vDash \varphi(\vec{x},\vec{y},\vec{Y}) \iff \mathscr{M}_G[\pi\circ s]\vDash \varphi(\vec{x},\vec{y},\vec{Y}).$$

We prove this in the next lemma. Intuitively, the idea is that π moves objects and relations in tandem, so it does not change any incidence relations. The third step works because π fixes E

pointwise, hence fixes all the parameters. $\square$

Lemma 5. *Let $\mathscr{M}_G$ be defined as above, and fix a permutation $\pi \in G$. For every $\mathscr{L}$-formula φ and every variable assignment s,*

$$\mathscr{M}_G[s] \vDash \varphi \iff \mathscr{M}_G[\pi \circ s] \vDash \varphi.$$

Proof. By induction on formulas φ.

Base cases. Suppose φ has the form $x = y$. Take any variable assignment s. Then we have

$$\begin{aligned}\mathscr{M}_G[s] \vDash x = y &\iff s(x) = s(y) \\ &\iff \pi(s(x)) = \pi(s(y)) \\ &\iff \mathscr{M}_G[\pi \circ s] \vDash x = y,\end{aligned}$$

since π is an injective function.

Suppose φ has the form $Xx_1 \cdots x_n$. Take any variable assignment s. Then we have

$$\begin{aligned}\mathscr{M}_G[s] \vDash Xx_1 \cdots x_n &\iff (s(x_1), \cdots, s(x_n)) \in s(X) \\ &\iff (\pi(s(x_1)), \cdots, \pi(s(x_n))) \in \pi(s(X)) \\ &\iff \mathscr{M}_G[\pi \circ s] \vDash Xx_1 \cdots x_n,\end{aligned}$$

again just using the fact that π is an injective function (and unpacking definitions).

Induction step. The cases where φ has the form $\psi \to \theta$ or $\neg\psi$ pose no difficulty. Suppose

φ has the form $\exists x \psi$. Take any variable assignment s. Then we have

$$
\begin{aligned}
\mathscr{M}_G[s] \vDash \exists x \varphi(x) &\iff (\exists a \in M) \mathscr{M}_G[s[x \mapsto a]] \vDash \varphi(x) \\
&\iff (\exists a \in M) \mathscr{M}_G[\pi \circ s[x \mapsto a]] \vDash \varphi(x) \\
&\iff (\exists a \in M) \mathscr{M}_G[(\pi \circ s)[x \mapsto \pi(a)]] \vDash \varphi(x) \\
&\iff (\exists b \in M) \mathscr{M}_G[(\pi \circ s)[x \mapsto b]] \vDash \varphi(x) \\
&\iff \mathscr{M}_G[\pi \circ s] \vDash \exists x \varphi(x).
\end{aligned}
$$

The second step holds by the inductive hypothesis. The fourth step uses the fact that π is surjective onto M.

Finally, suppose φ has the form $\exists X \psi$, where X is an n-adic relation variable. Take any variable assignment s. Then we have

$$
\begin{aligned}
\mathscr{M}_G[s] \vDash \exists X \varphi(X) &\iff (\exists A \in M_n) \mathscr{M}_G[s[X \mapsto A]] \vDash \varphi(X) \\
&\iff (\exists A \in M_n) \mathscr{M}_G[\pi \circ s[X \mapsto A]] \vDash \varphi(X) \\
&\iff (\exists A \in M_n) \mathscr{M}_G[(\pi \circ s)[X \mapsto \pi(A)]] \vDash \varphi(X) \\
&\iff (\exists B \in M_n) \mathscr{M}_G[(\pi \circ s)[X \mapsto B]] \vDash \varphi(X) \\
&\iff \mathscr{M}_G[\pi \circ s] \vDash \exists X \varphi(X).
\end{aligned}
$$

The second step holds by the inductive hypothesis. The fourth step uses the fact that M_n is closed under the action of π and π^{-1}, which follows from Lemma 2. □

This completes the proof of Proposition 1.

Here are three simple examples of permutation models, analogous to the permutation models of set theory described in (Jech 1973, pp. 48–51). These examples will be useful to us in section 4. An interesting property shared by these three permutation models is that of being infinite but not Dedekind-infinite: i.e., they are infinite models of SOL + ¬ DI.

Example. The *first Fraenkel model* is obtained by setting $M := \omega$ and taking G to be the group of all permutations of ω. The only concepts that occur in the first Fraenkel model are the finite and cofinite subsets of ω. More generally, the relations occurring in the first Fraenkel model are precisely those that are definable by boolean combinations of equalities (with object parameters). It follows that the first Fraenkel model is the minimal infinite model of pure second-order logic, in the sense that it is a substructure of any countably infinite general $\mathscr{L}$-structure.[12]

Example. The *second Fraenkel model* is obtained by setting $M := \omega \times \{0,1\}$ and taking G to be the group of all permutations that preserve the first coordinate of each member of M:

$$G := \{\pi \in \mathit{Perms}(M) : \pi(i,j) = (i,k) \text{ for some } k\}.$$

In other words, each $\pi \in G$ consists of an infinite sequence of independent permutations of $\{0,1\}$. Let us write $B_i := \{(i,0),(i,1)\}$ for each $i < \omega$. The concepts and relations occurring in the second Fraenkel model are precisely those which do not discriminate between members of the pair B_i when i is sufficiently large. (So, the second Fraenkel model contains concepts that are both infinite and co-infinite.) The set-theoretic analogue of the second Fraenkel model is a paradigm of the failure of the axiom of choice: it contains a sequence of nonempty sets for which there is no choice function. This does not quite work in the present context, because we have no good way of indexing the sequence. In section 4, we will define a modified version of the second Fraenkel model that avoids this difficulty.

Example. The *ordered Mostowski model* is obtained by setting $M := \mathbb{Q}$ and taking G to be the group of all order-preserving permutations of $\mathbb{Q}$. The concepts occurring in the Mostowski model are precisely the finite unions of intervals with endpoints in $\mathbb{Q} \cup \{-\infty, \infty\}$. An interesting property of this model is that it contains a linear ordering of the universe (namely $<_{\mathbb{Q}}$), but no well-ordering of the universe.

[12] For proofs of these claims, see (Mackereth and Avigad 2023, §6).

We close this section by generalizing the definition of a permutation model to $\mathscr{L}^+$. Let M, G be as before, and let $M_1 = \{A \subseteq M : A \text{ is symmetric}\}$. Let $B \subseteq M_1 \times M$ and $E \subseteq M$. We say that E is a *support* of B iff every permutation $\pi \in G$ that fixes E pointwise fixes B setwise:

$$(\forall e \in E)(\pi(e) = e) \implies (\forall (X,x) \in M_1 \times M)((X,x) \in B \leftrightarrow (\pi(X), \pi(x)) \in B).$$

In the special case where B is a function $f : M_1 \to M$, this is equivalent to saying that every permutation $\pi \in G$ that fixes E pointwise commutes with f:

$$(\forall e \in E)(\pi(e) = e) \implies (\forall X \in M_1)(f(\pi(X) = \pi(F(X))).$$

A set $B \subseteq M_1 \times M$ is *symmetric* if it has a finite support $E \subseteq M$.

Definition. Let M be a nonempty set, and let G be a group of permutations of M. A *permutation model* $\mathscr{M}_G[f]$ is an $\mathscr{L}^+$-prestructure $(M, M_1, M_2, M_3, \cdots, f)$ where for all $n \geq 1$,

$$M_n = \{A \subseteq M^n : A \text{ is symmetric}\}$$

and where f is symmetric.

Proposition 6. *Any permutation model $\mathscr{M}_G[f]$ is a general $\mathscr{L}^+$-structure.*

Proof. Essentially the same as that of Proposition 1. Lemma 2 goes through unchanged. In Lemma 4, we now have formulas containing #. Let D be a finite support for f. Then replace E with $E \cup D$, and the rest goes through unchanged. In Lemma 5, we just need to add one more base case to the induction:

$$\begin{aligned}
\mathscr{M}_G[f][s] \vDash \#F = x &\iff f(s(F)) = s(x) \\
&\iff \pi(f(s(F))) = \pi(s(x)) \\
&\iff f(\pi(s(F))) = \pi(s(x)) \\
&\iff \mathscr{M}_G[f][\pi \circ s] \vDash \#F = x.
\end{aligned}$$

The third step uses the fact that f is symmetric. $\square$

3 HP and DI

Without further ado we proceed to our main theorem.

Theorem 7. *HP is not deductively conservative over SOL + DI.*

The proof of this theorem will be given in the following definitions and Lemmas 8–11.

If $R(i,x)$ is a dyadic relation, we write R_i to denote the concept $\{x : R(i,x)\}$, called the *section* of R at i.

Definition. Let θ be the following principle, expressed as an $\mathscr{L}$-sentence: "For every dyadic relation $F(i,x)$, there is a function f from objects to objects such that $f(i) = f(j)$ iff there is a bijection between F_i and F_j."[13]

Clearly HP $\vdash_{\mathscr{L}^+} \theta$, since we can define $f(i) := \#F_i$. We show that DI $\nvdash_{\mathscr{L}} \theta$ by exhibiting a permutation model of DI + $\neg\theta$.

Definition. We define a permutation model $\mathscr{M}$ as follows. The domain of objects is the union of two disjoint parts, namely $A = \omega$ and $B = \mathbb{R} \times \mathbb{R}$. Thus we have $M := A \cup B$. For the group of permutations, G, we take all permutations $\pi : A \cup B \to A \cup B$ of the following form:

[13]Note that the unary function f is represented by a dyadic relation. The idea of considering this sentence θ was suggested by Jeremy Avigad.

1. For all $x \in A$, we have $\pi(x) = x$.

2. For all $x = (r,s) \in B$, we have $\pi(x) = (\pi_1(r), \pi_{2,r}(s))$, where π_1 and $\pi_{2,r}$ are order-preserving permutations of $\mathbb{R}$.

For convenience we designate such a permutation by $\pi = (\mathrm{id}_A, \pi_1, \pi_{2,r})_{r\in\mathbb{R}}$. Note that $\pi_{2,r}$ may depend on r, but π_1 may not. In other words, for points in B, we first perform an order-preserving permutation on the first coordinate, and then we perform another order-preserving permutation on the second coordinate, where the choice of the second permutation is allowed to vary. For example, we can have

$$\pi_1(r) = 2r$$
$$\pi_{2,r}(s) = r + s.$$

It is easy to check that these permutations form a group.

Lemma 8. *$\mathscr{M}$ is a general model of DI.*

Proof. We have that $\mathscr{M}$ is a general $\mathscr{L}$-structure because it is a permutation model. Furthermore, $\mathscr{M}$ is full over A, in the sense that $\mathscr{P}(A^n) \subseteq M_n$ for all $n \geq 1$. (This is because every relation over A is symmetric with respect to id_A.) So in particular, the successor relation succ_A is in $\mathscr{M}$. Hence $\mathscr{M}$ is a general model of DI, as desired. □

Define a binary relation $\ll$ on B as follows: $(r,s) \ll (r',s')$ iff $r < r'$ in the usual ordering on $\mathbb{R}$.

Lemma 9. *The relation $\ll$ is in $\mathscr{M}$.*

Proof. We show that $\ll$ is symmetric with support $\varnothing$. Take any $\pi = (\mathrm{id}_A, \pi_1, \pi_{2,r})_{r\in\mathbb{R}} \in G$, and take any $x, y \in M$. We check that $x \ll y$ iff $\pi(x) \ll \pi(y)$.

Suppose $x \in A$. Then $\neg x \ll y$, since $\ll$ is a relation on B (i.e., $\ll \subseteq B \times B$). Furthermore, $\pi(x) = x$, so we have $\neg\pi(x) \ll \pi(y)$ as desired. Similarly if $y \in A$. So, the only remaining case is when $x, y \in B$. Writing $x = (r, s)$, $y = (r', s')$, we have:

$$\begin{aligned} x \ll y &\iff r < r' \\ &\iff \pi_1(r) < \pi_1(r') \\ &\iff \pi(x) \ll \pi(y), \end{aligned}$$

as desired. $\square$

Set $F(i, x) := x \ll i$. By the previous lemma, $\ll$ is in $\mathscr{M}$, so it is available for use as a parameter. Furthermore, for any $i \in M$, the section F_i is definable from the parameters $\ll$ and i. So these sections are also in $\mathscr{M}$, and are also available as parameters.

Lemma 10. *Fix parameters $i = (r, s)$ and $j = (r', s')$ in M. Then $\mathscr{M} \vDash F_i \approx F_j$ iff $r = r'$.*

Proof. For the "if" direction, suppose $r = r'$. Then $F_i = F_j$, and the identity function is a bijection between them. The identity function is in $\mathscr{M}$. Hence, the restriction of the identity function to F_i is also in $\mathscr{M}$, since it is definable from the parameter F_i.

For the "only if" direction, suppose without loss of generality that $r < r'$. Take any bijection $g : F_i \to F_j$. We show that g is not symmetric. Take any finite set $E \subseteq A \cup B$. We want to define a permutation $\pi = (\mathrm{id}_A, \pi_1, \pi_{2,t})_{t \in \mathbb{R}} \in G$ that fixes E pointwise but nevertheless ruins g, in the sense that for some $x \in F_i$,

$$\pi(g(x)) \neq g(\pi(x)).$$

Set $\pi_{2,t}$ to be the identity function on $\mathbb{R}$, for all $t \in \mathbb{R}$. As for π_1, it is enough if we can choose π_1 so as to satisfy the following constraints:

- $\pi_1(u) = u$ for all $u < r$
- $\pi_1(e) = e$ for all $(e, e') \in E$
- $\pi_1(v) > v$ for some $r \leq v < r'$

- π_1 is an order-preserving bijection from $\mathbb{R}$ to $\mathbb{R}$.

To see that this is enough, suppose that $\pi_1(v) > v$ for some $r \leq v < r'$. Let $y = (v,0)$, and let $x = g^{-1}(y) = (w,t)$ for some $w,t \in \mathbb{R}$. Note that $x \in F_i$, so $w < r$ and hence $\pi_1(w) = w$. Thus we obtain

$$\begin{aligned}
\pi(g(x)) &= \pi(y) \\
&= (\pi_1(v), \pi_{2,v}(0)) \\
&= (\pi_1(v), 0), \\
g(\pi(x)) &= g((\pi_1(w), \pi_{2,\pi_1(w)}(t))) \\
&= g((w,t)) \\
&= g(x) \\
&= y \\
&= (v,0).
\end{aligned}$$

Since $\pi_1(v) > v$, it follows that $\pi(g(x)) \neq g(\pi(x))$, as desired.

We now check that it is possible to satisfy the four constraints listed above. Define e as follows:

$$e = \begin{cases} \min\{d \in \mathbb{R} : r < d < r' \wedge \exists d'((d,d') \in E)\}, & \text{if this exists} \\ r' & \text{otherwise.} \end{cases}$$

That is, e is the smallest first coordinate of a member of $E \cap B$ such that $r < e < r'$, if there is one; otherwise $e = r'$. Let $v = \frac{r+e}{2}$, and let $v' = \frac{r+3e}{4}$. Now we can satisfy all four constraints by

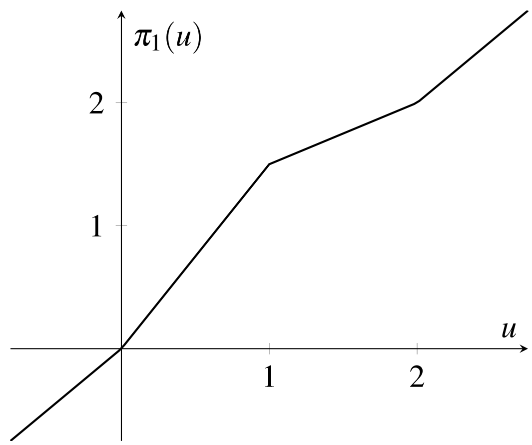

Figure 1: Depiction of the graph of π_1. For illustrative purposes, we chose the values $r = 0$, $v = 1$, $v' = 1.5$, $e = 2$.

choosing π_1 to be the following piecewise linear function:

$$\pi_1(u) = \begin{cases} u & \text{if } u \leq r \\ r + \frac{3}{2}(u - r) & \text{if } r < u < v \\ v' + \frac{1}{2}(u - v) & \text{if } v \leq u < e \\ u & \text{if } u \geq e. \end{cases}$$

This function is depicted in Figure 1. In particular, $\pi_1(v) = v' > v$, and the other constraints are satisfied. The proof is complete. $\square$

Lemma 11. *$\mathscr{M}$ is not a model of θ.*

Proof. Suppose for sake of contradiction that $\mathscr{M} \vDash \theta$. By Lemma 9, we have that F is in $\mathscr{M}$. So, $\mathscr{M}$ contains a function $f : B \to A \cup B$ such that $f(i) = f(j)$ iff $\mathscr{M} \vDash F_i \approx F_j$. By Lemma 10, this is equivalent to saying that $f(i) = f(j)$ iff i and j have the same first coordinate. Since there are uncountably many real numbers, but A is countable, it follows that range$(f) \cap B$ is infinite.

We show that f is not symmetric. Take any finite set $E \subseteq A \cup B$. We want to define a permutation $\pi = (\mathrm{id}_A, \pi_1, \pi_{2,t})_{t \in \mathbb{R}} \in G$ that fixes E pointwise but ruins f. Since range$(f) \cap B$ is infinite, we can choose i so that $f(i) = (r,s) \in B \setminus E$. Set π_1 to be the identity function on $\mathbb{R}$. For all $t \in \mathbb{R}$ such that $t \neq r$, set $\pi_{2,t}$ to be the identity function on $\mathbb{R}$. As for $\pi_{2,r}$, it is enough if we can choose $\pi_{2,r}$ so as to satisfy the following constraints:

- $\pi_{2,r}(u) = u$ whenever $(r,u) \in E$
- $\pi_{2,r}(s) > s$
- $\pi_{2,r}$ is an order-preserving bijection from $\mathbb{R}$ to $\mathbb{R}$.

We can easily construct such a function by the same method as in the proof of Lemma 9. Now, $f(\pi(i)) = f(i) = (r,s)$, because $\pi(i)$ and i have the same first coordinate; but $\pi(f(i)) = (r,s')$ for some $s' > s$. So, we have $f(\pi(i)) \neq \pi(f(i))$, and hence f is not symmetric. Contradiction. $\square$

The proof of Theorem 7 is complete.

Remark. What happens if we replace θ with an axiom scheme θ^* saying, for each *definable* dyadic relation $F(i,x)$, that there is a function f from objects to objects such that $f(i) = f(j)$ iff $F_i \approx F_j$? Here "definable" should be understood in the sense of "definable with object parameters but no relation parameters."[14] The answer is that SOL + DI proves every instance of θ^*. Indeed, let $\mathscr{M} = (M, M_1, M_2, \cdots)$ be any general model of SOL + DI, and let $F(i,x)$ be any dyadic relation definable with object parameters from M. We show that $\mathscr{M}$ satisfies the relevant instance of θ^*. First of all, since $\mathscr{M} \vDash$ SOL + DI, we can find a copy of the natural numbers $(\omega, 0, \mathrm{succ})$ in $\mathscr{M}$. Since $F(i,x)$ is definable, each section F_i is a definable subset of M. But the definable subsets of M are just the finite and cofinite ones (see below for proof); furthermore, $\mathscr{M} \vDash F_i \approx V$ whenever F_i is cofinite (since we can map $V \setminus F_i$ onto an initial segment of ω). So, we can define

$$f(i) = \begin{cases} 0 & \text{if } F_i \approx V \\ n+1 & \text{if } F_i \approx \{0,1,2,\cdots,n-1\}. \end{cases}$$

This gives $\mathscr{M} \vDash f(i) = f(j) \leftrightarrow F_i \approx F_j$, as desired.

[14] I am grateful to an anonymous reviewer for raising this interesting question.

To see that the definable subsets of M are just the finite and cofinite ones, let $\mathscr{N} = (M, N_1, N_2, \cdots)$ be a permutation model analogous to the first Fraenkel model, but with object domain M instead of ω. By the natural analogue of (Mackereth and Avigad 2023, Lemma 6.40), the concepts and relations occurring in $\mathscr{N}$ are precisely those definable by Boolean combinations of equations with object parameters. In the case of concepts, these are just the finite and cofinite ones. Now, since $\mathscr{N}$ is a general $\mathscr{L}$-structure, every definable subset of M^n is a member of N_n. Conversely, every member of N_n is a definable subset of M^n (definable by a Boolean combination of equations). This gives the desired result.

Next, we prove a partial conservativity result:

Theorem 12. *HP is Π^1_1 conservative over SOL + DI.*

Our proof of this theorem is divided into the following two lemmas.

Lemma 13. *HP is conservative over SOL + DI + WO.*

Proof. The argument is based on the same trick used to show that SOL + HP is interpretable in second-order arithmetic. We reason in SOL + DI + WO. Suppose that $<$ is a well-ordering of the universe. Denote the $<$-least element of the universe by 0, and denote successor by $+1$. For any concept F that is not equinumerous with the universe, we define

$$\mu(F) := \text{ the least } z \text{ such that } F \approx \{x : x < z\}.$$

Then we define the operator # as follows:

$$\#F := \begin{cases} 0 & \text{if } F \approx V, \\ \mu(F)+1 & \text{if } F \text{ is finite,} \\ \mu(F) & \text{otherwise.} \end{cases}$$

It is straightforward to prove that # satisfies HP. □

Lemma 14. *WO is Π^1_1 conservative over SOL + DI.*

Proof. Let φ be any Π^1_1 sentence of $\mathscr{L}$ (i.e., any sentence of the form $\forall X_1 \cdots \forall X_k\, \psi$, where ψ contains no second-order quantifiers). We want to show that if DI + WO $\vdash_{\mathscr{L}} \varphi$, then DI $\vdash_{\mathscr{L}} \varphi$. We prove the contrapositive. Suppose that DI does not prove φ. Then there is a general model $\mathscr{M} = (M, M_1, M_2, \cdots) \vDash$ DI + $\neg\varphi$. We show that DI + WO does not prove φ by exhibiting a general model of DI + WO + $\neg\varphi$.

Let $<$ be a well-ordering of M according to the metatheory (namely, ZFC). Let $\mathscr{N} = (M, N_1, N_2, \cdots)$ be the general $\mathscr{L}$-structure obtained from $\mathscr{M}$ by adding the relation $<$ together with all the concepts and relations definable from $<$. Thus $\mathscr{N}$ has the same objects as $\mathscr{M}$, but possibly more concepts and relations. We show that $\mathscr{N} \vDash$ DI + WO + $\neg\varphi$.

First we check that $\mathscr{N} \vDash$ WO. Indeed, since $(M, <)$ is a well-ordering in the metatheory, every nonempty set $A \in \mathscr{P}(M)$ has a $<$-least element. So, a fortiori, every nonempty set $A \in N_1$ has a $<$-least element. But this is just to say that $\mathscr{N} \vDash$ "$<$ is well founded". Moreover, we also have that $\mathscr{N} \vDash$ "$<$ is a linear order". Hence $\mathscr{N} \vDash$ WO.

Next we check that $\mathscr{N} \vDash \neg\varphi$. Observe that $\neg\varphi$ is equivalent to a Σ^1_1 sentence, that is, a sentence of the form $\exists X_1 \cdots \exists X_k\, \psi(X_1, \cdots, X_k)$, where ψ contains no second-order quantifiers. Since $\mathscr{M} \vDash \neg\varphi$, there exist $A_1, \cdots, A_k \in \mathscr{M}$ such that $\mathscr{M} \vDash \psi(A_1, \cdots, A_k)$. Any relation that occurs in $\mathscr{M}$ also occurs in $\mathscr{N}$, so we have $A_1, \cdots, A_k \in \mathscr{N}$ as well. Furthermore we have $\mathscr{N} \vDash \psi(A_1, \cdots, A_k)$, because (unpacking the definition of satisfaction) this statement does not depend on $N_1, N_2, \cdots$ at all, and is equivalent to $\mathscr{M} \vDash \psi(A_1, \cdots, A_k)$. Hence $\mathscr{N} \vDash \neg\varphi$.

Lastly, we have $\mathscr{N} \vDash$ DI. Indeed, DI is a Σ^1_1 sentence that is true in $\mathscr{M}$, so we can apply the same reasoning as before. Thus we have $\mathscr{N} \vDash$ DI + WO + $\neg\varphi$, as desired. □

Remark. Our argument also shows that WO is Π^1_1 conservative over SOL, and over any theory obtained by adjoining finitely many Σ^1_1 sentences to SOL.

4 HP and choice

In the previous section, we showed that HP is not deductively conservative over SOL + DI, but is conservative over SOL + DI + WO. Now, the principle WO is a strong form of the axiom of choice. So, it is tempting to ask whether HP actually proves some form of the axiom of choice, and whether this might account for the extra strength of HP as compared with DI. Unfortunately, matters are not so simple.

There are many forms of the axiom of choice in second-order logic. For reference, see (Gaßner 1994), (Siskind et al. 2023), (Asser 1981). There is also an enormous literature on different forms of the axiom of choice in set theory; see especially (Jech 1973), (Howard and Rubin 1998), and the works cited there. We will not consider all of these variations. However, we will show that HP does not prove some of the simplest and most natural versions of the axiom of choice.[15]

Theorem 15. *SOL + HP does not prove any of the following versions of the axiom of choice:*

1. *well ordering principle (WO^1): the universe is well ordered,*
2. *linear ordering principle (LO^1): the universe is linearly ordered,*
3. *the axiom of choice ($AC^{1,1}$): for any indexed family of nonempty concepts, there is a choice function:*

$$\forall R \exists S \forall x (\exists y Rxy \to \exists ! y (Rxy \wedge Sxy)),$$

4. *the axiom of choice for families of disjoint sets ($AC_*^{1;1}$): for any indexed family of nonempty pairwise disjoint concepts, there is a choice function,*
5. *trichotomy (TR^1): for all concepts F, G, either there is an injection from F into G or there is an injection from G into F,*

[15] I am not aware of any interesting interpretability relationships between SOL + HP and SOL + φ, where φ is any version of the axiom of choice. We do have that SOL + DI + WO is mutually interpretable with SOL + DI, and hence with SOL + HP, but this does not seem to be a big win.

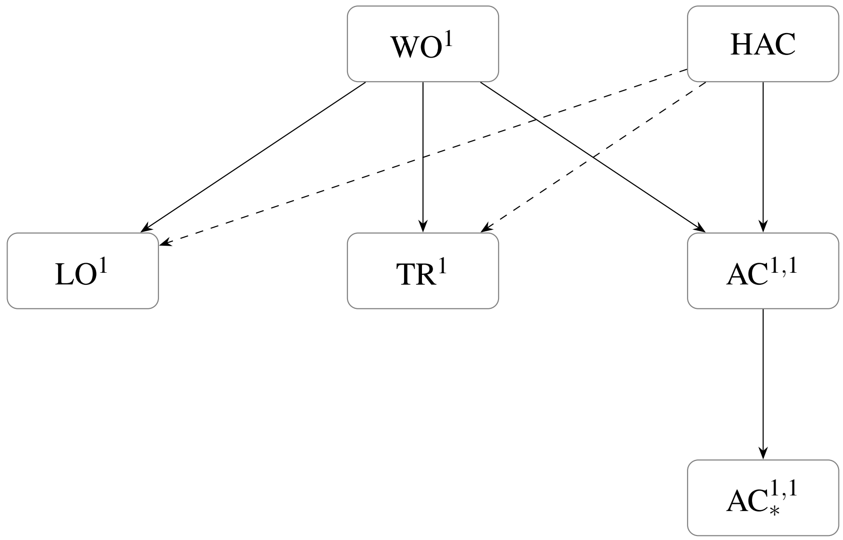

Figure 2: Versions of the axiom of choice in second-order logic. A solid arrow $P \longrightarrow Q$ means that $P \vdash_{\mathscr{L}} Q$ and $Q \nvdash_{\mathscr{L}} P$. A dashed arrow $P \dashrightarrow Q$ means that $Q \nvdash_{\mathscr{L}} P$, but it is unknown whether $P \vdash_{\mathscr{L}} Q$. If there is no arrow (or path of arrows) between P and Q, this means that $P \nvdash_{\mathscr{L}} Q$ and $Q \nvdash_{\mathscr{L}} P$. For proofs, see (Gaßner 1994), (Siskind et al. 2023), (Asser 1981).

6. *Hilbert-Ackermann choice (HAC): for any second-order formula $\varphi(x, X)$,*

$$\forall x \exists X \varphi(x, X) \rightarrow \exists R \forall x \varphi(x, R_x),$$

where X is a monadic relation variable and R is a dyadic relation variable.

The parenthetical abbreviations (WO^1, LO^1, etc.) are mostly drawn from (Gaßner 1994), where further details are given and the meaning of the superscripts is explained. Note that WO^1 is exactly the same as WO, from earlier. Trichotomy (TR^1) is also known as cardinal comparability. The deductive relationships between these versions of choice are depicted in Figure 2.

We will now prove Theorem 15 by constructing a permutation model $\mathscr{M}[f]$ such that $\mathscr{M}[f] \vDash \text{HP} + \neg\, \text{LO}^1 + \neg\, \text{TR}^1 + \neg\, \text{AC}_*^{1,1}$. In light of Figure 2, this is enough to establish the theorem. Intuitively, $\mathscr{M}[f]$ is just the second Fraenkel model with cardinalities added.

Our argument proceeds in three stages. First, we define a general $\mathscr{L}$-structure $\mathscr{M}$ and characterize the concepts of $\mathscr{M}$. Next, we define a function f to serve as the interpretation of #, and we check that f is symmetric. This gives us a general $\mathscr{L}^+$-structure $\mathscr{M}[f]$. Lastly, we check that $\mathscr{M}[f] \vDash \mathrm{HP} + \neg\, \mathrm{LO}^1 + \neg\, \mathrm{TR}^1 + \neg\, \mathrm{AC}_*^{1,1}$.

Definition. We define a permutation model $\mathscr{M} = (M, M_1, M_2, \cdots)$ as follows. The domain of objects is the union of two disjoint parts, $M := A \cup B$, namely these:

$$A := 2^\omega \times 2^\omega,$$

$$B := \omega \times \{0,1\}.$$

We write $B = \bigcup_{i<\omega} B_i$, where $B_i = \{(i,0),(i,1)\}$. For the group of permutations, G, we take all permutations $\pi : A \cup B \to A \cup B$ of the following form:

1. If $x \in A$, then $\pi(x) = x$.
2. If $x \in B_i$, then $\pi(x) \in B_i$.

In other words, each $\pi \in G$ is a permutation that preserves the pairs B_i for all $i < \omega$.

In the next lemma, we characterize the concepts of $\mathscr{M}$. Any concept $X \in M_1$ consists of two disjoint parts, namely $A \cap X$ and $B \cap X$. The lemma tells us that the first part can be any arbitrary subset of A, while the second part must not discriminate between members of the pair B_i when $i < \omega$ is sufficiently large.

Lemma 16 (Concepts)**.** *The concepts of $\mathscr{M}$ can be characterized as follows. For any $X \subseteq M$, we have:*

$$X \in M_1 \iff \exists k\, \forall i > k \;\; B_i \cap X \in \{\varnothing, B_i\}.$$

The least such k will be denoted by $k(X)$.

Proof. Suppose that $X \in M_1$, i.e., X is symmetric with some finite support E. Let k be maximal such that $B_k \cap E \neq \varnothing$. (If $B \cap E = \varnothing$, then set $k = -1$.) Then for all $i > k$ we must have $B_i \cap X \in \{\varnothing, B_i\}$, since there is a permutation $\tau_i \in G$ that transposes $(i,0)$ and $(i,1)$ while fixing everything else.

Conversely, suppose that $\exists k\ \forall i > k\ \ B_i \cap X \in \{\varnothing, B_i\}$. Let $k(X)$ be the least such k, and set $E = B_0 \cup B_1 \cup \cdots \cup B_{k(X)}$. Then X is symmetric with support E. $\square$

We now wish to define f (the interpretation of #) so as to satisfy HP; that is, we want

$$f(X) = f(Y) \iff \mathscr{M} \vDash X \approx Y.$$

To understand the construction of f, one has to get a feel for which concepts are equinumerous in $\mathscr{M}$, i.e., which bijections exist in $\mathscr{M}$. The next lemma gives the answer to this question. Roughly speaking, a bijection between concepts $X, Y \in M_1$ is symmetric (and hence occurs in $\mathscr{M}$) iff it preserves the pairs B_i for all sufficiently large $i < \omega$.

Lemma 17 (Bijections). *The bijections in $\mathscr{M}$ can be characterized as follows. Let $X, Y \in M_1$, and let $g : X \to Y$ be any bijection in the metatheory. Then we have:*

$$g \in M_2 \iff \exists k\ \forall i > k\ (B_i \cap X = \varnothing \vee g(B_i) = B_i).$$

Proof. Similar to the previous lemma. Suppose that $g \in M_2$, i.e., g is symmetric with some finite support E. Notice that X (the domain of g) is definable from g without any additional parameters; hence, by inspecting the proof of Lemma 4, we see that X is also symmetric with support E. Let k be maximal such that $B_k \cap E \neq \varnothing$. (If $B \cap E = \varnothing$, then set $k = -1$.) Then for all $i > k$, we have $B_i \cap X \in \{\varnothing, B_i\}$. Take any $i > k$ such that $B_i \cap X = B_i$. We show that $g(B_i) = B_i$. Suppose to the contrary that $g((i, j)) = a \notin B_i$ for some $j \in \{0, 1\}$. Since there is a permutation $\tau_i \in G$ that transposes $(i, 0)$ and $(i, 1)$ while fixing everything else, we must also have $g((i, 1-j)) = a$. But this contradicts the fact that g is a bijection. Thus $g(B_i) \subseteq B_i$, and so $g(B_i) = B_i$.

Conversely, suppose that $\exists k\ \forall i > k\ (B_i \cap X = \varnothing \vee g(B_i) = B_i)$. Take the least such k, and set $E = B_0 \cup B_1 \cup \cdots \cup B_k$. Then g is symmetric with support E. $\square$

This gives us a picture of which concepts $X, Y \in M_1$ are equinumerous in $\mathscr{M}$. Namely, $M \vDash X \approx Y$ just in case the following two conditions are met:

1. $B_i \cap X = B_i \cap Y$ for all sufficiently large $i < \omega$,

2. the union of the "leftover parts" of X (namely, $A \cap X$ and $B_i \cap X$ when i is *not* sufficiently large) has the same cardinality in the metatheory as the union of the "leftover parts" of Y.

Now in order to define f, we need to choose objects to serve as representatives for equivalence classes of equinumerous concepts in $\mathscr{M}$. Roughly speaking, it looks like the equivalence class of a concept X is determined by two pieces of data: the tail of the sequence $(B_i \cap X)_{i<\omega}$, and the cardinality of the leftover part of X. So we would like to define two notions $\mathrm{tail}(X)$ and $\mathrm{head}(X)$ that keep track of this data. However, there is a subtlety here. It seems that we need to choose a cutoff point for "sufficiently large," so as to determine where the tail of X begins, and what the leftover part is. The problem is that the cutoff point is not intrinsic to X; it depends on what other concept we are comparing X with. For example, suppose we are given three concepts $X,Y,Z \in M_1$ such that the sequences $(B_i \cap X)_{i<\omega}$, $(B_i \cap Y)_{i<\omega}$, $(B_i \cap Z)_{i<\omega}$ eventually agree. We may get a different cutoff when comparing $(B_i \cap X)_{i<\omega}$ with $(B_i \cap Y)_{i<\omega}$ than when comparing $(B_i \cap X)_{i<\omega}$ with $(B_i \cap Z)_{i<\omega}$. So, it is not obvious how to define $\mathrm{tail}(X)$ and $\mathrm{head}(X)$ in the way that we originally envisioned.

We can get around this problem with a little bit of ingenuity. Our approach is inspired by some classic puzzles involving equivalence classes of sequences (Bollobás 2022, Problems 112 and 113). The trick is to work with equivalence classes in a non-constructive way, either using the Axiom of Choice in the metatheory, or just working with the equivalence classes directly. This enables us to define $\mathrm{head}(X)$ and $\mathrm{tail}(X)$ non-constructively without specifying any cutoffs.

Firstly, we define $\mathrm{tail}(X) \in 2^\omega$ for each concept $X \in M_1$. Say that two concepts $X,Y \in M_1$ are *tail-equivalent* iff $B_i \cap X = B_i \cap Y$ for all sufficiently large $i < \omega$. Using the Axiom of Choice in the metatheory, let us select a canonical or designated member of each equivalence class of the tail-equivalence relation. We write $\bar{X}$ for the canonical member of the tail-equivalence class of X.

Next, we associate with each $X \in M_1$ a sequence $b(X) \in 2^\omega$:

$$b(X)(i) = \begin{cases} 1 & \text{if } B_i \cap X = B_i, \\ 0 & \text{otherwise.} \end{cases}$$

This sequence mirrors the behavior of $(B_i \cap X)_{i<\omega}$ when i is sufficiently large (namely, when $i > k(X)$). Finally, we define $\mathrm{tail}(X) \in 2^\omega$ as follows:

$$\mathrm{tail}(X) := b(\bar{X}).$$

Thus we have

$$\begin{aligned} \mathrm{tail}(X) = \mathrm{tail}(Y) &\iff \bar{X} = \bar{Y} \\ &\iff X \text{ and } Y \text{ are tail-equivalent} \\ &\iff B_i \cap X = B_i \cap Y \text{ for all sufficiently large } i < \omega. \end{aligned}$$

Secondly, we define $\mathrm{head}(X) \in 2^\omega$ for each concept $X \in M_1$. For any two tail-equivalent concepts $X, Y \in M_1$, let $k(X,Y)$ be the least k such that

$$\forall i > k, \;\; B_i \cap X = B_i \cap Y.$$

(If X, Y are not tail-equivalent, then we assign $k(X,Y)$ some junk value, say $k(X,Y) = 0$.) Let us write $X \sim Y$ iff the following two concepts have the same cardinality in the metatheory:

$$\left(A \cup \bigcup_{i=0}^{k(X,Y)} B_i \right) \cap X, \quad \left(A \cup \bigcup_{i=0}^{k(X,Y)} B_i \right) \cap Y.$$

It is easy to check that $\sim$ is an equivalence relation on concepts. The number of equivalence classes can be no more than the number of possible cardinalities of concepts in M_1. This is at most $2^{\aleph_0}$, because

$$X \in M_1 \implies |X| \leq |A| + |B| = 2^{\aleph_0} + \aleph_0 = 2^{\aleph_0}.$$

So, we may identify equivalence classes of $\sim$ with points in 2^ω. Then we define $\mathrm{head}(X) \in 2^\omega$ to be the equivalence class of X with respect to $\sim$.

Definition. We define $f : M_1 \to A$ as follows:

$$f(X) = (\mathrm{head}(X), \mathrm{tail}(X)).$$

Lemma 18. *f is symmetric.*

Proof. It suffices to show that for any $X \in M_1$ and any $\pi \in G$, we have $\pi(f(X)) = f(\pi(X))$. Since $f(X) \in A$ and π acts like the identity on A, this is equivalent to showing that $f(X) = f(\pi(X))$.

Take any $\pi \in G$ and $X \in M_1$. Since π only moves things around within each pair B_i, we have that X and $\pi(X)$ are tail-equivalent. Hence

$$\mathrm{tail}(X) = \mathrm{tail}(\pi(X)).$$

It remains to show that $\mathrm{head}(X) = \mathrm{head}(\pi(X))$. That is, we want to show that the following two sets have the same cardinality:

$$\left(A \cup \bigcup_{i=0}^{k(X,\pi(X))} B_i\right) \cap X, \quad \left(A \cup \bigcup_{i=0}^{k(X,\pi(X))} B_i\right) \cap \pi(X).$$

But this is certainly true, because π only moves things around within each pair B_i. Thus we have $f(X) = f(\pi(X))$, as desired. $\square$

Corollary 19. *$\mathscr{M}[f]$ is a general $\mathscr{L}^+$-structure.*

Lemma 20 (Equinumerosity)**.** *Let X, Y be concepts in $\mathscr{M}$. Then we have*

$$\mathscr{M} \vDash X \approx Y \iff (\mathrm{head}(X) = \mathrm{head}(Y) \text{ and } \mathrm{tail}(X) = \mathrm{tail}(Y)).$$

Proof. For the ($\Longrightarrow$) direction, suppose that $\mathscr{M} \vDash X \approx Y$, i.e., there is a symmetric bijection $g : X \to Y$. By Lemma 17, we have that for all sufficiently large i, either $B_i \cap X = \varnothing$ or $B_i \cap X = B_i$ and $g(B_i) = B_i$. This implies that $\mathrm{tail}(X) = \mathrm{tail}(Y)$. It also implies that g is a bijection between the sets

$$\left(A \cup \bigcup_{i=0}^{m} B_i\right) \cap X, \quad \left(A \cup \bigcup_{i=0}^{m} B_i\right) \cap Y$$

whenever m is sufficiently large. It follows that the sets

$$\left(A \cup \bigcup_{i=0}^{k(X,Y)} B_i\right) \cap X, \quad \left(A \cup \bigcup_{i=0}^{k(X,Y)} B_i\right) \cap Y$$

have the same cardinality. Hence $\mathrm{head}(X) = \mathrm{head}(Y)$ as well.

For the ($\Longleftarrow$) direction, suppose that $\mathrm{head}(X) = \mathrm{head}(Y)$ and $\mathrm{tail}(X) = \mathrm{tail}(Y)$. Define a bijection $g : X \to Y$ in the metatheory as follows: take some bijection between the sets

$$\left(A \cup \bigcup_{i=0}^{k(X,Y)} B_i\right) \cap X, \quad \left(A \cup \bigcup_{i=0}^{k(X,Y)} B_i\right) \cap Y$$

and extend it to the rest of X by setting $g(x) = x$ for all $x \in \left(\bigcup_{i=k(X,Y)+1}^{\infty} B_i\right) \cap X$. By Lemma 17, this bijection g is symmetric with finite support $E = \bigcup_{i=0}^{k(X,Y)} B_i$. Hence $\mathscr{M} \vDash X \approx Y$. □

Corollary 21. $\mathscr{M}[f] \vDash$ *HP.*

Proof. Immediate from the definition of f and the previous lemma. □

Lemma 22. $\mathscr{M}[f] \vDash \neg AC_*^{1,1}$.

Proof. Fix a sequence $(a_i)_{i<\omega}$ of distinct points in A. Define a dyadic relation R as follows:

$$Rxy \iff (\exists i < \omega)(x = a_i \wedge y \in B_i).$$

This relation R is symmetric, so it occurs in $\mathscr{M}[f]$. Moreover, R is an indexed family of pairwise disjoint sets. However, $\mathscr{M}[f]$ contains no function $h \subseteq R$ defined on $\mathrm{dom}(R) = \{a_i : i < \omega\}$. $\square$

Lemma 23. $\mathscr{M}[f] \vDash \neg\, LO^1$.

Proof. $B \in M_1$, but $\mathscr{M}[f]$ contains no linear order of B. Any linear order of B would have to determine, for each $i < \omega$, which member of B_i is greater and which is lesser. But such a relation cannot be symmetric. $\square$

Lemma 24. $\mathscr{M}[f] \vDash \neg\, TR^1$.

Proof. $A, B \in M_1$, but $\mathscr{M}[f]$ contains no injection from A into B or vice versa. $\square$

This completes the proof of Theorem 15.

Remark. The same construction can be carried out using the first Fraenkel model instead of the second. This gives us a model $\mathscr{N}[g] \vDash \mathrm{HP} + \neg\, \mathrm{LO}^1 + \neg\, \mathrm{TR}^1$. However, we do not seem to obtain $\mathscr{N}[g] \vDash \neg\, \mathrm{AC}_*^{1,1}$ this way.

For the alternative construction, take $N := A \cup B$, where $A = \omega \times \omega$ and $B = \omega$. Let G consist of all permutations π such that $\pi(x) = x$ for all $x \in A$. The symmetric concepts are those $X \subseteq N$ for which either $B \cap X$ is finite or $B \setminus X$ is finite. A bijection is symmetric iff it moves only finitely many points in B.[16] Hence, we obtain that $\mathscr{N} \vDash X \approx Y$ if and only if: (i) $B \cap X$, $B \cap Y$ differ on finitely many points, and (ii) we have

$$|A \cap X| + |B \cap (X \setminus Y)| = |A \cap Y| + |B \cap (Y \setminus X)|,$$

[16] More precisely, if $X, Y \in N_1$ and $f : X \to Y$ is a bijection in the metatheory, then $f \in N_2$ iff $f(x) = x$ for all but finitely many $x \in B$.

where $|\cdot|$ denotes cardinality in the metatheory. It is easy to check that (i) and (ii) are both equivalence relations on concepts X, Y. So we can take equivalence classes as before, and use this to define a function $g : N_1 \to A$. Furthermore, g is symmetric, because $\mathscr{N} \vDash X \approx \pi(X)$ for any $X \in N_1$ and $\pi \in G$. So everything goes through, and we get another permutation model of HP.

The construction does not seem to go through if we start with the Mostowski model and try to add cardinalities in the same way. The problem is that the Mostowski model does not satisfy $X \approx \pi(X)$ for all concepts X and permutations $\pi \in G$. So, the desired interpretation of # is not symmetric, and everything breaks down.

5 Other axioms of infinity

Just as there are many forms of the axiom of choice in second-order logic, so too there are many forms of the axiom of infinity. For reference, see (Asser 1981), (Väänänen 2021, §9.2). There is also a large literature concerning finiteness and infinity in set theory without choice, tracing back to the classic papers (Tarski 1924), (Lévy 1958). See (de la Cruz 2002), (de la Cruz et al. 2006), and the works cited there. Again, we will not consider every possible variation. We shall focus on the simplest and most natural axioms of infinity in the context of second-order logic.

Definition. We define eleven versions of the axiom of infinity in second-order logic. To minimize clutter, let us agree that a "dense" linear or partial order must contain at least two elements.

1. *Pairing:* $|V| > 1$ and $V \times V \approx V$,
2. *Splitting:* $\exists X(X \approx (V \setminus X) \approx V)$,
3. *Dedekind infinity* (DI): there is an injection from V into V that is not a surjection,
4. *Existence of* $\mathbb{N}$: there exist $N, 0, S$ satisfying the axioms of second-order arithmetic,
5. *Linear order with successors* (LOS): there is a nonempty linear order $(A, <)$ in which every

element has a successor element, i.e.,

$$(\forall x \in A)(\exists y \in A)(x < y \wedge \neg \exists z(x < z \wedge z < y)),$$

6. *Partial order with successors* (POS): there is a nonempty partial order in which every element has a successor element (not necessarily unique),
7. *Dense linear order* (DLO): there is a dense linear order,
8. *Dense partial order* (DPO): there is a dense partial order,
9. *Infinite linear order* (ILO): there is a nonempty linear order with no maximal element,
10. *Infinite partial order* (IPO): there is a nonempty partial order with no maximal element,
11. *Stäckel infinity:* there is no double wellordering of V, i.e., no wellordering of V whose converse is also a wellordering.

Intuitively, axioms 1–3 characterize the infinite in terms of special cardinality properties, whereas axioms 5–11 characterize the infinite in terms of special order properties. Among the order-based characterizations, there is a split between discrete and continuous approaches, and a further split depending on whether linear or partial orders are used. The notion of Stäckel infinity is somewhat special; it goes back to (Stäckel 1907) and is discussed in (Mackereth and Avigad 2023).

The deductive relationships between these versions of the axiom of infinity are depicted in Figure 3. Some interesting patterns emerge. In general, cardinality-based axioms of infinity seem to be stronger than order-based ones. Among the order-based axioms, linear orders are stronger than partial orders, and discrete orders are stronger than dense orders. Interestingly, DI seems to be the natural meeting point between cardinality-based and order-based axioms.

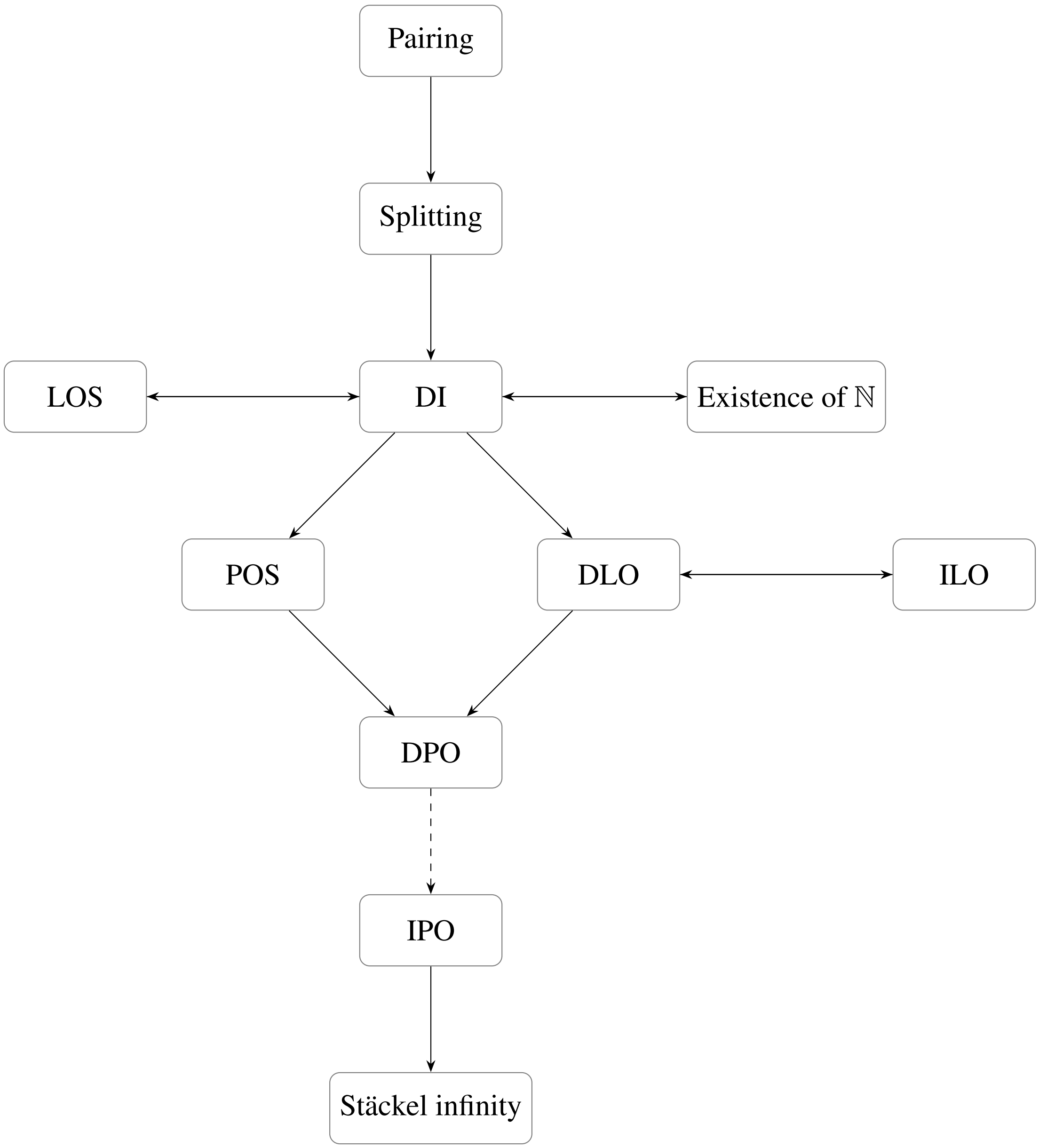

Figure 3: Versions of the axiom of infinity in second-order logic. An arrow $P \longrightarrow Q$ means that $P \vdash_{\mathscr{L}} Q$ and $Q \nvdash_{\mathscr{L}} P$. A double-headed arrow $P \longleftrightarrow Q$ means $P \vdash_{\mathscr{L}} Q$ and $Q \vdash_{\mathscr{L}} P$. A dashed arrow $P \dashrightarrow Q$ means that $P \vdash_{\mathscr{L}} Q$, but I do not know whether $Q \vdash_{\mathscr{L}} P$. If there is no arrow (or path of arrows) between P and Q, this means that $P \nvdash_{\mathscr{L}} Q$ and $Q \nvdash_{\mathscr{L}} P$. For proofs, see (Asser 1981) and the discussion below.

Most of the implications are straightforward to establish, except for two: ILO $\rightarrow$ DLO and POS $\rightarrow$ DPO.[17] We prove (or sketch) these implications in SOL in the next two Propositions.

Proposition 25. *SOL $\vdash$ ILO $\rightarrow$ DLO.*

Proof. Let $(A,<)$ be a linear order with no maximal element. We show that there is a dense linear order. If there exists $X \subseteq A$ such that $(X,<)$ is a linear order with successors, then we are done (because SOL proves LOS $\rightarrow$ DLO). So, we may assume that there is no such X.

Let $\mathrm{Succ}(x,y)$ abbreviate the statement that y is a $<$-successor of x, i.e.,

$$x < y \wedge \neg\exists z(x < z \wedge z < y).$$

Let $B := \{x \in A : \neg\exists y\, \mathrm{Succ}(x,y)\}$. We show that $(B,<)$ is a dense linear order.

First we show that B has at least two elements. Note that B is nonempty, or else A would be a linear order with successors. Fix $b_0 \in B$, and let $C = \{x \in A : x > b_0\}$. Then C is nonempty, because $(A,<)$ has no maximal element. Furthermore $C \cap B \neq \varnothing$, or else $(C,<)$ would be a linear order with successors. Hence B has at least two elements.

Now take any $a,b \in B$ with $a < b$. We show that there exists $c \in B$ such that $a < c < b$. Let $D = \{x \in A : a < x < b\}$. Since $a \in B$, we have that $\neg\mathrm{Succ}(a,b)$ and hence $D \neq \varnothing$. Furthermore $D \cap B \neq \varnothing$, or else $(D,<)$ would be a linear order with successors. The proof is complete. □

Proposition 26. *SOL $\vdash$ POS $\rightarrow$ DPO.*

Proof (sketch). Let (A,R) be a partial order with successors. We show how to define a dense partial order. Fix $a \in A$, and let $A_1 := \{a\}$. For each $i \geq 1$, let A_{i+1} be the set of all R-successors of members of A_i. Think of each set A_i as a block. The idea is to rearrange these blocks into a dense linear order, namely, the order of the dyadic rationals between 0 and 1. This will induce a dense partial order on the set $B := \bigcup_{i\geq 1} A_i$.

[17] The fact that SOL proves DI $\rightarrow$ Existence of $\mathbb{N}$ is essentially due to (Dedekind 1888). To show that SOL proves Pairing $\rightarrow$ Splitting, note that SOL can formalize the Cantor-Schröder-Bernstein Theorem.

Let us define $f : \omega \setminus \{0\} \to \mathbb{Q}$ as follows:

$$f(n) = \frac{2(2^k - n) + 1}{2^k}, \text{ where } k \text{ is minimal such that } 2^k \geq n.$$

This function enumerates the dyadic rationals between 0 and 1. (For convenience, we exclude 0 but include 1.) For each $x \in B$, let $i(x)$ be the index of the block in which x appears, so that $x \in A_{i(x)}$. We would like to define a new partial order (B, R') as follows: for all $x, y \in B$,

$$R'xy \iff f(i(x)) <_{\mathbb{Q}} f(i(y)).$$

This can be done using the fact that SOL + Stäckel infinity interprets first-order arithmetic; see (Mackereth and Avigad 2023, Lemmas 7.43 and 7.46). The basic idea is that arithmetical statements can be coded as statements about Stäckel-finite (i.e., doubly wellordered) concepts, with equality interpreted as equinumerosity. For example, let Sxy denote that y is an R-successor of x. Then we can represent $i(x) = k$ in terms of finite concepts as follows: "for every concept I, if I is doubly wellordered by S with least element a and greatest element x, then $I \approx K$." Intuitively, the natural number k is represented by any concept K that holds of exactly k things. We can define addition and multiplication using finite concepts; for example, $x + y = z$ can be represented by

$$\exists Y'(Y \approx Y' \wedge X \cap Y' = \varnothing \wedge X \cup Y' = Z).$$

To interpret mathematical induction for formulas in the language of first-order arithmetic, we use the principle of induction on Stäckel-finite concepts (Mackereth and Avigad 2023, Lemma 5.32).

Since SOL + POS proves Stäckel infinity, it is able to interpret first-order arithmetic in this way. Now, first-order arithmetic is easily strong enough to formalize the function $f : \mathbb{N} \to \mathbb{Q}$ and the order $<_{\mathbb{Q}}$ and prove that $f(i(x)) <_{\mathbb{Q}} f(i(y))$ defines a dense linear order. So, we can carry out the entire construction in SOL + POS and define the dense partial order that we wanted. □

The non-implications take a bit of work to establish; we will sketch the proofs very briefly.

To show that SOL does not prove Stäckel infinity $\to$ IPO, notice that the first Fraenkel model is a countermodel (i.e., it is a model of SOL + Stäckel infinity + $\neg$ IPO). This follows from the fact that the first Fraenkel model is infinite, but contains no infinite partial order.

To show that SOL does not prove POS $\to$ DLO, notice that the second Fraenkel model is a countermodel. Indeed, the ordering $(m,i) < (n,j) \iff m <_{\mathbb{N}} n$ is a partial order with successors. Furthermore, $<$ is symmetric with empty support, because it never discriminates between members of the same pair. However, the second Fraenkel model contains no infinite linear order. As corollaries, we obtain that SOL proves neither POS $\to$ DI, nor DPO $\to$ DLO.

To show that SOL does not prove DPO $\to$ POS, note that the ordered Mostowski model is a countermodel. As corollaries, we have that SOL proves neither DLO $\to$ DI nor DPO $\to$ POS. For the other two non-implications (viz., SOL $\not\vdash$ DI $\to$ Splitting and SOL $\not\vdash$ Splitting $\to$ Pairing), we sketch the proofs below.

Proposition 27. *SOL does not prove DI $\to$ Splitting.*

Proof. We define a permutation countermodel $\mathscr{M} = (M, M_1, M_2, \cdots)$ as follows. Let $M := A \cup B$ with $A = \omega \times \omega$ and $B = \omega_1$, and let G be the group of all permutations $\pi : A \cup B \to A \cup B$ such that $\pi(x) = x$ for all $x \in A$. Intuitively, $\mathscr{M}$ is full over A and looks similar to the first Fraenkel model over B (but with $B = \omega_1$ instead of $B = \omega$). One can show that for any concept $X \in M_1$, either $B \cap X$ or $B \setminus X$ is finite; that is, X is either finite or cofinite in B. Now, take any $X \in M_1$ and suppose that $\mathscr{M} \vDash X \approx V$. Then X has metatheoretic cardinality $\aleph_1$, and $B \setminus X$ is finite. But then $V \setminus X = (A \setminus X) \cup (B \setminus X)$ has metatheoretic cardinality at most $\aleph_0$, so we cannot have $\mathscr{M} \vDash (V \setminus X) \approx V$. Hence $\mathscr{M} \vDash$ SOL + DI + $\neg$ Splitting, as desired. □

Proposition 28. *SOL does not prove Splitting $\to$ Pairing.*

Proof. We copy the argument of (Lévy 1958, Theorem 7). Define a permutation countermodel $\mathscr{N} = (N, N_1, N_2, \cdots)$ as follows. Let $N := \omega \times \mathbb{Q}$, and let G be the group of all permutations $\pi : N \to N$ of the form $(\mathrm{id}_\omega, \pi')$, where π' is an order-preserving permutation of $\mathbb{Q}$; that is, for all

$x = (i,q) \in N$, we have $\pi(x) = (i, \pi'(q))$. Intuitively, $\mathscr{N}$ consists of countably many copies of the ordered Mostowski model.

First we check that $\mathscr{N} \vDash$ Splitting. Indeed, let $A = \{2n : n \in \omega\}$ and $B = \{2n+1 : n \in \omega\}$. It is not hard to check that $A \times \mathbb{Q}$ and $B \times \mathbb{Q}$ are symmetric, and $\mathscr{N} \vDash V \approx (A \times \mathbb{Q}) \approx (B \times \mathbb{Q})$.

Next we check that $\mathscr{N} \vDash \neg$ Pairing. Suppose to the contrary that $\mathscr{N}$ contains a bijection $f : V \times V \to V$. Let g be the restriction of f to the domain $(\{0\} \times \mathbb{Q}) \times (\{0\} \times \mathbb{Q})$. Note that g is an injective function. Since $\{0\} \times \mathbb{Q}$ is in $\mathscr{N}$, we have that g is in $\mathscr{N}$ (i.e., g is symmetric). Let E be some finite support for g. Take any $a, b \in \mathbb{Q}$ such that $a \neq b$ and $(0,a) \notin E$, $(0,b) \notin E$. Let us write $g((0,a),(0,b)) = (i,c)$ for some $i \in \omega$ and $c \in \mathbb{Q}$. Since $a \neq b$, we have either $c \neq a$ or $c \neq b$. Without loss of generality, suppose $c \neq a$. Now, one can easily devise a permutation $\pi = (\mathrm{id}_\omega, \pi') \in G$ that fixes $E \cup \{(i,c)\}$ pointwise such that $\pi'(a) \neq a$ (see Lemma 10). Then we have:

$$g((0,\pi'(a)),(0,\pi'(b))) = \pi(i,c) = (i,c) = g((0,a),(0,b)).$$

But this contradicts the fact that g is an injective function. QED. □

It is also worth considering the interpretability strength of the various axioms of infinity. We present some facts in the next two propositions.

Proposition 29. *The following theories are mutually interpretable with second-order arithmetic:*

1. *SOL + Pairing,*
2. *SOL + Splitting,*
3. *SOL + DI,*
4. *SOL + Existence of* $\mathbb{N}$*,*
5. *SOL + LOS.*

Proof. We have seen that SOL + DI is mutually interpretable with second-order arithmetic. The proofs of the other facts are straightforward. □

Proposition 30. *SOL + Stäckel infinity is mutually interpretable with first-order arithmetic.*

Proof. See (Mackereth and Avigad 2023, Lemmas 7.43 and 7.46). □

Remark. I do not know the interpretability strength of the other theories. I conjecture that SOL + IPO, SOL + DPO, and SOL + DLO are mutually interpretable with first-order arithmetic (and hence also with each other). We know that SOL + IPO interprets first-order arithmetic, since SOL + IPO proves Stäckel infinity. For the other direction, I conjecture that one can interpret the ordered Mostowski model in first-order arithmetic, similarly to (Mackereth and Avigad 2023, Lemma 7.46). The concepts and relations occurring in the ordered Mostowski model do not seem too complicated; I conjecture that they are definable by Boolean combinations of equations $x = y$ and inequalities $x \leq_{\mathbb{Q}} y$, allowing object parameters (compare Mackereth and Avigad 2023, Lemma 6.40). The other issue is that one cannot formalize the notion of an *arbitrary* order-preserving permutation $\pi : \mathbb{Q} \to \mathbb{Q}$ in first-order arithmetic. So, one hopes to find a subgroup of these permutations that is formalizable in first-order arithmetic and generates the same permutation model. Perhaps it suffices to consider order-preserving permutations of $\mathbb{Q}$ consisting of finitely many linear pieces?

In any case, let us return to Hume's Principle. Our final theorem shows that SOL + HP does not prove Splitting. The idea of the proof is simply that we can add an interpretation of # to the model $\mathscr{M} \vDash \mathrm{DI} + \neg$ Splitting, described above.

Theorem 31. *SOL + HP does not prove Splitting.*

Proof (sketch). Let $\mathscr{M} = (M, M_1, M_2, \cdots)$ be the permutation model described in the proof of Proposition 27. The concepts occurring in $\mathscr{M}$ are precisely those $X \subseteq M$ for which either $B \cap X$ or $B \setminus X$ is finite. Furthermore, the bijections occurring in $\mathscr{M}$ are characterizable as follows: if $X, Y \in M_1$ and $f : X \to Y$ is a bijection in the metatheory, then $f \in M_2$ iff $f(x) = x$ for all but finitely many $x \in B$. Now carry out the construction from the Remark at the end of section 4, but with $B = \omega_1$ instead of $B = \omega$. □

In conclusion, we have seen that SOL + HP does prove DI, but does not prove various natural forms of the axiom of choice, nor any stronger axiom of infinity. Nevertheless SOL + HP proves a bit more than SOL + DI does, because it enables us to select objects as representatives of equivalence classes of equinumerous concepts. This feels like a choice principle, but it seems to be weaker than any form of choice proper. However, there are some versions of the axiom of choice that we did not consider, such as Zorn's Lemma (ZL^1) and the principle that every linearly ordered concept is well ordered (LW^1). Whether HP proves either of these principles remains an open question.[18]

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[18]Thanks to Jeremy Avigad, Patrick Chandler, and Alexander Morgan-Fleming for helpful discussion. Theorems 7 and 12 were conjectured by Jeremy. Special thanks to two anonymous referees for their very helpful comments.

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