



On the representational role of Euclidean diagrams: representing *qua* samples

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Abstract

We advance a theory of the representational role of Euclidean diagrams according to which they are samples of co-exact features. We contrast our theory with two other conceptions, the instantial conception and Macbeth's iconic view, with respect to how well they accommodate three fundamental constraints on theories of the Euclidean diagrammatic practice—(i) that Euclidean diagrams are used in proofs whose results are wholly general, (ii) that Euclidean diagrams indicate the co-exact features that the geometer is allowed to infer from them and (iii) that Euclidean diagrams play the same role in both direct proofs and indirect proofs by *reductio*—and argue that our view is the one best suited to account for them. We conclude by illustrating the virtues of our conception of Euclidean diagrams as samples by means of an analysis of Saccheri's quadrilateral.

Keywords Euclidean diagrams · *Reductio ad absurdum* · Co-exact information · Iconicity · Samples

1 Introduction

Regardless of having been the undisputed paradigm of mathematical precision for more than a millennium, Euclid's *Elements* and its pervasive use of diagrams in geometric proofs has given rise to much philosophical perplexity. In response to the increasing suspicion with which visual representations in mathematics were regarded, philosophers and mathematicians alike have wondered which role diagrams are supposed to be playing in the geometric proofs of the *Elements*. This paper offers an account of this role and argues that it does better than two alternative views. In order to do that,

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it is first necessary to introduce and motivate three guiding constraints. Any adequate account of diagram use in Euclidean geometry must at least be compatible with and, ideally, explain:

The generality constraint: particular diagrams can be used in proofs whose results are completely general.

The co-exact constraint: Euclidean diagrams underwrite inferences only about *co-exact* relationships, and these relationships are *indicated* in them.

The uniformity constraint: any adequate theory of Euclidean diagrams' role must provide a uniform account of their use in both direct as well as indirect (*reductio*) proofs.¹

The *generality constraint* should be familiar to anyone acquainted with the literature on mathematical diagrams. The particularity of geometrical diagrams seems intuitively to be at odds with the generality of the geometrical propositions reached at the end of a Euclidean proof. An adequate account of Euclidean diagram use must thus be minimally compatible with that fact, and ideally, to explain it.

The *co-exact constraint* is based on two features of Manders' (2008a, b) analysis of the Euclidean diagrammatic practice. The first one, that the diagrams underwrite inferences only about *co-exact* relationships, is more familiar and has proven greatly influential in subsequent research. The second one, that these relationships are *indicated* in the diagrams, is less discussed and remains at most implicit in recent discussion. In the following, we explain these two features. Co-exact relationships, at a first pass, concern those aspects of a diagram that are fairly stable across a range of variations of the diagram and not eliminable by improving the drawing; some prominent examples are mereological and topological relationships that spatial regions and shapes can stand in (e.g. containment, interiority, exteriority, intersection relations); this type of information should be contrasted with what Manders dubs 'exact' information, which paradigmatically include metrical relationships such as equalities/proportionalities, parallelisms between line segments, the magnitudes of angles etc. As this author showed, diagrams are ever only used as a source of justification in Euclidean proofs in inferences concerning co-exact relationships.

Another important finding from Manders' analysis (that is often glossed over) is how the relationship between Euclidean diagrams and co-exact features is of a distinct type than that to exact features. Manders says that a diagram may be "subject to" the exact conditions prescribed by the text, but that the diagram "indicates" co-exact conditions (e.g. 2008b, p. 119). Although he consistently uses that terminology throughout his text, he never goes as far as providing an explicit definition of what he means, but it is undoubtedly connected to the fact that the relationship between a diagram and co-exact features is *sui generis*, and this allows that these features (unlike the exact ones) may be: *directly read off* from the diagram (ibid., pp. 88–89, 91, 94), inferred

¹ The proofs by *reductio* are pervasive in the *Elements*. Among the 85 proofs in books I and III, 21 are by *reductio*: I.6, I.7, I.14, I.26, I.27, I.39, I.40, III.2, III.4, III.5, III.6, III.7, III.8, III.10, III.11, III.12, III.13, III.16, III.18, III.19, III.23, III.24, III.27.

on the basis of the diagram's *appearance*² (ibid. p. 89), inferred on the basis of what the diagram *looks like* (2008a, p. 74).

The co-exact constraint thus follows from the fact that Euclidean diagrams only underwrite inferences about a special type of information, and also that they relate to that information in a specially direct manner. We conjecture that Manders is alluding to the fact that, when a Euclidean diagram underwrites an inference about co-exact features such as an intersection between two lines or the contiguity of two angles, then the diagram *directly presents* this intersection or contiguity. For one example, in the well-known proof I.1 (Fig. 1), the existence of an intersection between shapes (BCD and ACE) is underwritten by, among other things, the fact that the diagram presents an intersection between drawn shapes, i.e. in the construction of the diagram, the drawn shapes cross each other—furthermore, this intersection is resilient to a significant range of deformations of the drawing, entailing that it is a co-exact aspect. The existence of that intersection is directly read off from the diagram's appearance, in Manders' sense, without having been introduced textually.

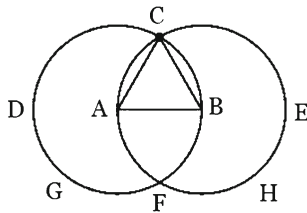


Fig. 1 Proposition I.1 of *Elements*

PROP. I. – PROBLEM

On a given finite right line (AB) to construct an equilateral triangle.

Sol. – With A as centre, and AB as radius, describe the circle BCD (Post. III).

With B as centre, and BA as radius, describe the circle ACE, cutting the former circle in C. Join CA, CB (Post. I.). Then ABC is the equilateral triangle required.

Dem. – Because A is the centre of the circle BCD, AC is equal to AB (Def. XXXII.). Again, because B is the centre of the circle ACE, BC is equal to BA. Hence we have proved. And $AC = AB$, $BC = AB$. But things which are equal to the same are equal to one another (Axiom I.); therefore AC is equal to BC; therefore the three lines AB, BC, CA are equal to one another. Hence the triangle ABC is equilateral (Def. XXI.); and it is described on the given line AB, which was required to be done. (Heath 1956, p. 8)

Thus, a geometer can infer the existence of an intersection between shapes by inspecting the diagram and *seeing* that it presents a drawn intersection between drawn

² Indeed, Manders understands Euclidean diagrams' "appearance" in a technical sense that roughly comprises its co-exact features ("we define the appearance or topology of a diagram to comprise the inclusions and contiguities of regions, segments and points in the diagram", Manders 2008b, p. 89). This is further evidence that Manders takes the relationship of the diagram and its co-exact features to be direct in a sense which its relationship to exact features is not.

curves. We thus propose to understand Manders' talk of a diagram *indicating* a co-exact feature as meaning that the diagram *directly presents* that feature, i.e. that this feature can be inferred on the basis of the diagram's appearance.

Contrast that with other well-known types of diagrams: Euler and Venn diagrams. In these logical diagrams, abstract relations between sets are represented by means of drawn spatial relations. Thus, a Euler diagram constituted by a circular figure B inside of a circular figure A (see Fig. 2), means that all members of the set to which B refers are members of the set to which A refers. In Venn diagrams, we additionally have X-marks, indicating non-emptiness of the corresponding set, and shaded regions, indicating emptiness.³ According to the Venn diagram below (Fig. 3), *all C are A* and *some B is C*—allowing one to infer that *some B is A*, a syllogism of the type *Darii*.

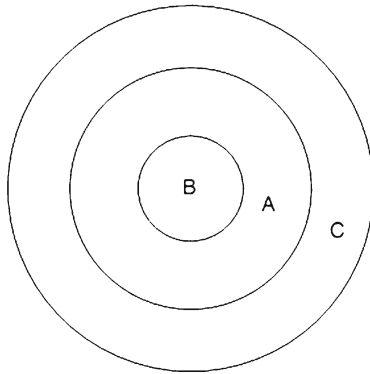


Fig. 2 Euler Diagram

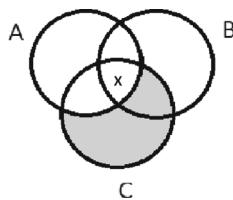


Fig. 3 Venn Diagram

In both Euler and Venn diagrams, concrete spatial relationships between drawn shapes stand for abstract relationships between sets. In Euclidean diagrams, on the other hand, some of the relationships indicated in the diagram are of the same type as

³ One difference between Euler and Venn diagrams is that, in the latter, the representation of a base for categorical propositions (closed overlapping figures) is independent from the representation of the categorical propositions (shading and x-marks). In practice, this means that the content of a Venn diagram can be changed without having to draw a completely new representation (e.g. one can just change which regions are shaded). For more on this, see Sautter (2019).

those that are underwritten by them and employed in the geometrical proof. Thus, the theory we seek must be sensible to the fact that Euclidean diagrams presents some of the properties whose inference they allow. We'll have more to say about this constraint in Sects. 3 and 4.⁴

Finally, *the uniformity constraint* is based on the idea that, if one does not clear up the division of labor between textual and diagrammatic parts of a Euclidean proof, it can become difficult to understand how diagrams can be employed in proofs that involve a contradiction. If one thinks, for example, that the drawn figures are merely a visual translation of the textual part of the proof, it is hard to avoid the conclusion that the figure exhibits a contradictory image—a difficult position to sustain. In summary, an account of the role of diagrams in the *Elements* must allow for a uniform account of their use in direct proofs and in proofs by *reductio*.⁵

Based on these three constraints, we suggest, in Sect. 2, that a view taking diagrams to be a semantic counterpart of the textual part of the proof—i.e. a visual translation of what is asserted by the text—faces problems with respect to the *generality* and the *uniformity constraints*. In Sect. 3, we analyze Macbeth's (2010) conception of Euclidean diagrams as icons with non-natural meaning of ideal geometrical objects. We show that, regardless of overcoming some of the problems faced by the previous view, it fails in properly accommodating the *co-exact constraint*.

Motivated by the limitations of these two conceptions, in Sect. 4 we advance a new understanding of the representational role of diagrams within the Euclidean mathematical practice. Influenced by Goodman (1968) and Lassalle Casanave (2013), we develop the view that Euclidean diagrams are *samples* (a type of representation rarely discussed in the philosophy of mathematics' literature) and show how it explains the three guiding constraints of our investigation. Finally, in Sect. 5 we illustrate the virtues of our proposal by means of an analysis of the very peculiar case of Saccheri's quadrilateral (Saccheri 2014 [1733]). Our aim there is showing that, if one understands Euclidean diagrams as samples, then one can readily explain the multi-tasking role that a diagram can have in a proof such as Saccheri's.

2 The instancial conception

When one observes the structure of the proofs of the *Elements* and their division in textual and diagrammatic parts, it is natural to imagine that the things being said by the text are supposed to be true of the drawing that accompanies it. In fact, the language used by the textual part of Euclidean proofs often seems to refer directly to the drawing at its side, as when it is written 'let AB be the given finite straight line' or 'let ABC be a right-angled triangle' etc. It is also natural to understand the text as providing

⁴ To our knowledge, Manders' insight about indication has never been properly evaluated by other authors. Carter (2019) does not cite Manders' specific point about indication, but she remarks that some representations, including Euclidean diagrams, not only show the existence of connections between objects, but also show the particular type of connection that relates them. A two-dimensional depiction of a graph, for example, shows which objects or sets of objects are related to each other (by means of drawn lines connecting them), but not the particular relation in question. A Euclidean diagram, on the other hand, not only shows that some figures are related, but also how they are related.

⁵ This constraint is also in line with Manders' (already classic) strategy (Manders 2008a, b).

instructions that seem to be performed in the drawing itself, such as constructing an equilateral triangle (e.g. proposition I.1) or cutting off a straight line (e.g. proposition I.3). Thus, there could be reasons to argue in favor of the idea that Euclidean diagrams are visual counterparts of the textual parts of the proofs. This is the core thesis of what we'll call 'the instantial conception'.

If tenable, this conception would allow us to explain the co-exact constraint: if the diagrams are themselves what the textual part of the Euclidean proofs refer to, it is not surprising that some of the properties invoked in the course of geometrical reasoning are indicated in the diagram. Additionally, this conception promises an account of both the role of diagrams and the nature of the subject matter of Euclidean geometry without presupposing any commitment to ideal objects (e.g. perfect circumferences). This would allow one to sidestep traditional metaphysical and epistemological problems often associated with Platonic views.

Regardless of that, the instantial conception has well-known shortcomings. One obvious problem is the *prima facie* incompatibility between the non-empirical (non-perceptible) character of the ideal objects presented in the first Euclidean definitions—"a point is that which has no part", "a line is a breadthless length"—and the concrete and imperfect nature of physical diagrams.⁶ To be sure, there are ways by means of which one could try to dissolve that incompatibility,⁷ but even if they work, the instantial view is still plagued by other serious difficulties.

More specifically, the instantial conception faces significant complications in accounting for the *generality constraint* and faces even more serious difficulties regarding the *uniformity constraint*. As far as generality is concerned, it is easy to see that a conception taking the subject matter of Euclidean proofs as consisting in concrete diagrams faces special difficulties in explaining how the propositions of the *Elements* are supposed to be wholly general. How is it, for example, that a geometric proof can be taken as applying with full generality to any possible triangle if any concrete construction of a triangular figure will have either two or three acute angles?⁸

Even more serious, as was noticed by Manders (2008b), is the fact that an instantial conception would not seem to be able to account for the existence, and indeed, pervasiveness of diagrammatic proofs by *reductio ad absurdum*:

Long-standing philosophical difficulties, on the nature of geometric objects and our knowledge of them, arise from the assumption that the geometrical text is in an ordinary sense true of the diagram or a 'perfect counterpart'. These difficulties

⁶ This criticism of the instantial view was pointed out already in antiquity. Its roots can be traced back to Plato and Aristotle's commentary on pre-Euclidean geometry as well as in Proclus' famous commentary of Euclidean geometry (Proclus 1970, Book 2, Chapter 1).

⁷ There have been proposals for different interpretations of the role and nature of the Euclidean definitions according to which they are not taken as introducing points without parts or lines without breadth, but as e.g. interpretation rules for 'reading' the diagrams (Ferreirós 2016; Netz 1999).

⁸ Netz seems to favor a view related to the instantial conception and acknowledges the pressing issue of explaining the generality of the *Elements'* proofs. In response to that problem, Netz tries to defend a conception of generality as repeatability. For more on that, see Netz (1999, chapter 6). Another philosopher who might have been sympathetic to something close to the instantial conception was Berkeley. Sherry (1993, p. 214) ascribes to him the "thesis that geometrical diagrams are the very ideas with which geometrical theorems are concerned" (however, see Brook 2012 for a dissenting interpretation of Berkeley's views on geometry).

aside, a genuinely semantic relationship between the geometrical diagram and text is incompatible with the successful use of diagrams in proof by contradiction: *reductio* contexts serve precisely to assemble a body of assertions which patently could not together be true (Manders 2008b, p. 84).

The reason for this problem is clear: in some proofs by *reductio*, the textual part of the proof describes impossible arrangements of figures—e.g. proof III.10 (*ekthesis*) starts with the requirement to draw a circle that cuts another in four points (Fig. 4). If the diagrams that accompany Euclidean proofs are conceived as visual counterparts of the textual part of the proof, the inevitable conclusion would be that some of these diagrams are drawings of impossible geometrical arrangements. Using proof III.10 as an example, one who thinks that the diagram is the subject matter of the textual part of the proof would have to concede that the diagram accompanying it is literally composed by circles that cut each other in more than two points. But this suggestion is obviously unacceptable—as Manders says, this diagram “is simply impossible” (2008b, p. 109). This means that the instantial view is not able to account for *the uniformity constraint*.

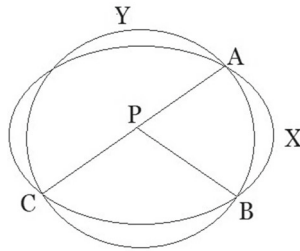


Fig. 4 Proposition III.10 of *Elements*

PROP. X. – THEOREM

If two circles have more than two points common, they must coincide.

Dem. – Let X be one of the circles; and if possible let another circle Y have three points, A, B, C, in common with X, without coinciding with it. Find P, the centre of X. Join PA, PB, PC. Then since P is the centre of X, the three lines PA, PB, PC are equal to one another. Again, since Y is a circle and P a point, from which three equal lines PA, PB, PC can be drawn to its circumference, P must be the centre of Y. Hence X and Y are concentric, which [v.] is impossible. (Heath 1956, p. 76)

In summary, not only the instantial conception faces important difficulties in accounting for the generality of the Euclidean proofs, but also seems incompatible with the existence of diagrammatic proofs by *reductio*. Thus, if we accept the generality and uniformity constraints, we ought to abandon the idea that diagrams are semantic counterparts of the textual part of Euclidean proofs, as the view under examination would have it. If, however, the proofs are not about the concrete drawings, what can they be about? One way of trying to overcome the obstacles faced by the instantial

conception is to take both the textual and the diagrammatic part of Euclidean proofs as standing for something else. In taking diagrams as representations, new ways are opened for explaining how they can be used in the process of proving wholly general propositions as well as their use in proofs by *reductio*. A recent theory that develops these insights is Macbeth's (2010), which we discuss in the following section.

3 Macbeth and icons with non-natural meaning

Macbeth (2010) presents a rich and insightful examination of the Euclidean geometrical practice that includes a particular proposal about the representational role of Euclidean diagrams. Macbeth argues that they are icons with non-natural meaning. Non-natural meaning is the concept introduced by Grice (1957) to mark the fact that some signs represent what they do only in virtue of their producers' intentions, a feature which Macbeth takes to be important in explaining how Euclidean diagrams work. Even more important to our discussion is Macbeth's claim that they are icons. Following Peirce, she conceives of icons as representations that in "some way resemble what they signify" (Macbeth, p. 245). Euclidean diagrams, Macbeth argues, function as a specific type of icon that resembles its content not in appearance, but "in the relations of [its] parts, that is, in virtue of a homomorphism" (*ibid.*). The physical diagrams thus resemble the geometrical entities they represent not because of how they look like, but because of how they are structured (e.g. the parts of one are organized in a way that resembles how the parts of the other are).

Macbeth takes very seriously the idea that, if the subject matter of Euclidean geometry are things of a non-perceptible nature, such as those described by the initial Euclidean definitions, then it is inconceivable that a concrete diagram could ever instantiate them or even visually resemble them. These ideal objects are, by their very nature, not instantiable by concrete constructions. Any diagram that could be concretely constructed will have all sorts of perceptible properties which are incompatible with them: concrete lines drawn on paper or carved on wax tablets will inevitably have breadth (not to even mention depth); concrete points, produced by a slight touch of the tip of the pen on paper or of the stylus on wax, will inevitably have an area that could be divided into smaller parts. A diagram can, however, be taken to *iconically* signify something that it neither instantiates nor looks like.

Macbeth's view of diagrams as icons finds inspiration in Peirce, although it seems to have already been interestingly discussed in earlier work by Leibniz.⁹ It is based on the observation that some logical and mathematical signs have their parts organized in such a way that resemble how the parts of their logical and/or mathematical contents are. That kind of resemblance is independent from visual resemblance. Macbeth's suggestion is that a diagram might represent a geometrical object if the parts of the former are organized in a way that corresponds to the organization of the parts of the latter:

⁹ For a historically informed study of Leibniz's views on symbolic knowledge, see Esquisabel (2012, pp. 1–50).

For example, on such an account a drawn circle serves as an icon of a geometrical circle not in virtue of any similarity in appearance between the two but because there is likeness in the relationship of that parts of the drawing, specifically in the relation of the [drawn] points on the drawn circumference to the drawn center, on the one hand, and the relation of the corresponding parts of the geometrical figure, on the other. (Macbeth 2010, pp. 245–246)

Macbeth’s hypothesis has many virtues. Treating Euclidean diagrams as icons with non-natural meaning would, in principle, allow for an account of *the generality constraint*. Since diagrams are representations whose non-natural meaning crucially depends on their producer’s intentions, their contents can be, she argues, wholly general. Macbeth also believes that her iconic conception is capable of accommodating the use of diagrams in proofs by *reductio*. In her paper, she analyzes proof III.10 and argues that conceiving diagrams as icons of the impossible situation to which the proof refers to presents no substantial problem. The key point of her argument is that, in order to function as an icon, the diagram does not need to resemble its content in appearance. In her own words (Macbeth 2010), “it is specially obvious that we do not *picture* the hypothesized situation [e.g. that a circle cuts another in four points], which is of course impossible, but instead formulate in the diagram the content of that hypothesis” (p. 262). Macbeth’s view would then be able to account for *the uniformity constraint*.

We do think, however, that if all one has to say about Euclidean diagrams’ representational role is that they are icons, then one fails to illuminate (or even worse, might obscure) the *co-exact constraint*. In order to explain what we mean, it is interesting to pause and reflect one more time about paradigmatic instances of icons that signify what they do by means of a likeness in the relation of their parts. By means of doing so, we aim to show that identifying Euclidean diagrams with this type of representations is tantamount to committing an oversimplification.

A paradigmatic example of icons with non-natural meaning in the sense invoked by Macbeth are Euler diagrams (Fig. 2).¹⁰ As pointed out before, in Euler diagrams the containment relation between drawn shapes represents the abstract containment relation between sets. Here, it is important to notice that the relation of containment between the shapes in the diagram—a spatial relation—is not the same relation of containment that is represented—an abstract set-theoretic one. Indeed, while the choice of the spatial containment relation is a suitable one for the purpose of representing (among other things) the subset relation, it carries with it a certain degree of conventionality.¹¹

¹⁰ We do not have space to go over many interesting features of Macbeth’s analysis of Euclidean geometry, most of them going way beyond her conception of them as icons with non-natural meaning. In Macbeth (2014, p. 67), for example, we find an important distinction between Euclidean diagrams, on the one hand, and Euler and Venn diagrams, on the other: “[in Euler and Venn] one pictures the given information in a way that serves implicitly to picture also the desired result. [...] Diagrammatic reasoning in Euclid, we will see, is not like this. [...] Instead, much as a calculation in Arabic numeration does, a Euclidean diagram formulates content in a mathematically tractable way, in a way enabling one to reason in the system of signs in a step-wise fashion from the given starting point to the desired endpoint.” This a valuable distinction but does not correspond to the co-exact constraint (as can be seen by the fact that it groups Euclidean diagrams together with Arabic numeration). For more on this, we refer the reader to Macbeth (2010, 2014).

¹¹ Mancosu (2012, pp. 8–9, our translation) echoes our point: “However, in the case under discussion [of an Euler diagram that represents sets], there is no spatial similarity between the diagram and the represented

Compare this case with that of Euclidean diagrams. In the Euclidean practice, when two drawn figures in a Euclidean diagram intersect (as in proposition I.1), the geometer is authorized to extract the information that there exists an intersection point. The sense in which the lines in the drawing intersect is the same sense invoked when the proof mentions an intersection: intersections stands for intersections. Let us go over another example: from a Euclidean diagram where a figure is included in another (or a line segment that is a proper part of another), the geometer is authorized to extract mereological information about part-whole relationships. This allows the geometer to infer that one of the figures is greater than the other. See e.g. proof I.6 (Fig. 5), where a piece of information employed in the proof—that triangle ACB is greater than triangle DBC—is directly readable off a relationship indicated in the diagram, i.e. that triangle DBC is a proper part of triangle ACB.¹²

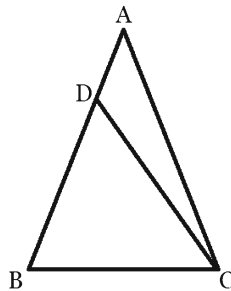


Fig. 5 Proposition I.6 of *Elements*. The information that the triangle DBC is a proper part of the triangle ACB, and thus, that the former is smaller than the latter—‘the less to the greater, which is absurd’—comes from observing that the drawn triangle DBC is a proper part of the drawn triangle ABC

PROP. VI. – THEOREM

If two angles (B, C) of a triangle be equal, the sides (AC, AB) opposite to them are also equal.

Footnote 11 continued

situations, e.g. between a circle and a set of objects. Besides, a relation between a set of objects is an abstract relation of inclusion. The diagram represents that abstract relation by means of the spatial relation of inclusion between two circles. There is no “natural” relation of [visual] similarity between the two relations. In effect, elements of conventionality are necessary to establish the relation”.

¹² Interestingly, this is one of the few cases where metrical information can be extracted from the diagram. When a diagram presents part-whole relations between figures, angles or segments, Euclid’s fifth common notion allows the geometer to infer that some of these figures are greater or lesser than the others. As Manders affirms: “geometrical reasoning frequently obtains inequalities directly from the diagram, but (corresponding to the restriction for equality) only when an object in the diagram is a proper part of another, rather than from any kind of indirect comparison” (2008b, p. 91). For a more detailed discussion of this point see Lassalle Casanave and Seoane (2016). As Mumma comments, it is not difficult to conjecture why Euclid would have restricted diagrammatic inferences in that way: “generating the symbols which comprise it ought to be straightforward and unproblematic. Yet there seems to be room for doubt whether one has succeeded in constructing a diagram according to its exact specifications perfectly. The compass may have slipped slightly, or the ruler may have taken a tiny nudge. In constraining himself to the co-exact properties of diagrams, Euclid is constraining himself to those properties stable under such perturbation” (2010, p. 11).

Dem. – If AB , AC are not equal one must be greater than the other. Suppose AB is the greater, and that the part BD is equal to AC . Join CD (Post. I.). Then the two triangles DBC , ACB have BD equal to AC , and BC common to both. Therefore the two sides DB , BC in one are equal to the two sides AC , CB in the other; and the angle DBC in one is equal to the angle ACB in the other (hyp). Therefore [IV.] the triangle DBC is equal to the triangle ACB —the less to the greater, which is absurd; *hence* AC , AB are not unequal, that is, they are equal (Heath 1956, p. 13)

The difference between the representational role of Euclidean diagrams and the other logical diagrams should now be evident: when a geometer makes an inference on the basis of a co-exact aspect of a Euclidean diagram, it is always the case that some features presented by the diagram are of the same type as the information that is inferred and then employed in the proof. We'll see other examples in the next sections, as when information about exteriority (e.g. that some angle is external to a figure) or non-adjacency (e.g. that some angle is not adjacent to another) is inferred on the basis of these very same features being presented by the drawing. As we have seen, this representational property is absent in other logical diagrams. The syntax of Euler or Venn diagrams is constituted by spatial relationships while their semantics deals with abstract set-theoretical relations. Euclidean diagrams, on the other hand, are such that their syntax and semantics seem to, at least with respect to co-exact information, involve the same types of relationships. This transparent connection between syntax and semantics is central to the co-exact constraint, including Manders' insight about indication, and is a key feature of their representational role that Macbeth's view fails to be sensitive to.

To be fair, Macbeth seems to sometimes point out that Euclidean diagrams cannot be reduced to mere icons with non-natural meaning. In a few places, when discussing the impossibility of drawing points and lines in accordance with the Euclidean definitions, Macbeth notices that things seem to be distinct as far as circles are concerned:

A drawn circle is, again, a slightly different case because drawn circles do look roughly circular; that is, there is a look that geometrical circles can be said to have. (Macbeth 2010, pp. 248–249)

We conjecture that, here and elsewhere in her paper,¹³ Macbeth is striving to hold on to the idea that the relation between a Euclidean diagram and the content it represents cannot be reduced to a mere resemblance in the organization of their parts. The implicit idea in these passages is that a circular figure not only resembles a circle in that sense, but also instantiates it in some way or another, or somehow looks like it. However, the manner that Macbeth chooses to express this intuition makes her position unstable. As pointed out by Mancosu (2012, p. 15, our translation), if geometrical circles are constructed by means of lines and points, and if these lines and points have

¹³ In another point, Macbeth (2010) admits that a circular drawing might be taken to resemble a circle either because it is an icon of a circle, as her conception would have it, but also because a circular drawing is an instance of a circle itself: "A drawn circle is roughly circular; it looks like a circle just as a dog looks like a dog. But a dog looks like a dog because it is a dog, that is, a particular instance of doggy nature (as we can think of it). A drawn circle, I have suggested, can look like a circle for either of two reasons. It can look like a circle for the same reason that a dog looks like a dog, namely, because it is a circle, a particular instance of circle nature. Or it can look like a circle because it is an icon with non-natural meaning that is intended to resemble a circle first and foremost in the relation of its parts" (p. 246).

no appearance and are not instantiable by a concrete drawing, then how could the opposite be true of circles?

Macbeth's position seems to me highly unstable. If points and lines are imperceptible, invisible [...] so are circles, triangles etc. If, on the other hand, circles and triangles are capable of being exemplified by drawn diagrams, then so are points and segments [...] In conclusion, if a circle is a platonic geometric object [i.e. as the Euclidean definitions make it seem], no diagram can look like it in the sense in which a photography looks like the photographed object, i.e. there is no similarity of appearance.

In summary, Macbeth's conception faces the following dilemma: if she limits herself to the thesis that Euclidean diagrams are icons with non-natural meaning of geometrical objects, then she seems to be glossing over Manders' insights about the contrast between what diagrams are subject to, and what they indicate; consequently, it obscures the *co-exact constraint*. If, on the other hand, Macbeth tries to argue that, besides that, some Euclidean diagrams are also instances of geometrical objects—or at least that some of the former visually resemble some of the latter—her conception becomes unstable and in need of a robust metaphysical explanation of how it could be the case that ideal objects constituted by non-perceptible things could themselves have a perceptible appearance.

Let us take stock. There are at least two general lessons that we could extract from the limitations faced by both the instantial conception and Macbeth's view. The limitations of the former suggest that concrete Euclidean diagrams should be conceived as representations—by going representational we clear the road for an explanation of the *generality constraint* as well as their use in proofs by *reductio*. The limitations of the latter suggest that the type of representation that Euclidean diagrams are cannot be fully accounted by a merely conventional association between features of the sign and features of the content, or, at least, not as conventional as the relation between e.g. a Euler diagram and the set-theoretical relations it represents. Thus, our conclusions so far seem to force us to look elsewhere for a different type of representational relation in order to properly characterize Euclidean diagrams.

4 Euclidean diagrams as samples

Which kind of representation could help us understand the representational role of diagrams in Euclidean plane geometry? Goodman's (1968) influential discussion of notational systems seems to afford us an interesting possibility: samples. A sample is an object that is used to represent a property (or a set of properties) that it itself possesses. 'Sample' should be understood with its ordinary meaning, an intuitive example being that of a colored swatch of cloth used by a tailor to show her clients which tonality and texture she intends to make them a dress with. Imagine that this tailor has a booklet of small swatches of cloth—a booklet whose pages are swatches of distinct colors and textures. A red swatch, for example, can, in this context, be used to represent the specific kind of red that it itself possesses. How this works is utterly non-mysterious. The red swatch is a paradigmatic example of that color—it is literally red—and, for

that very reason, can be used to represent it. The same goes for texture, weave, pattern etc.¹⁴

The idea that Euclidean diagrams could be conceived as samples was first advanced by Lassalle Casanave (2013) as a way of understanding how they can be used in proofs whose results are general as well as in proofs by *reductio*. Discussing the example of the booklet of swatches, Lassalle Casanave observes that “the condition of a sample involves some kind of generality. In effect, a sample can, for example, be a sample of a texture of a fabric, but not of a piece of fabric in particular” (2013, p. 25, our translation). This author is here emphasizing the fact that samples have a content which is wholly general. Samples represent properties or kinds, but never particular objects. It does not make sense to think, for example, that a miniature of the Eiffel tower could be used as a sample of the real tower. One could use the miniature as a sample of a wrought-iron lattice tower, of towers in general, or of some other property exemplified by the tower, but not of the *particular* tower sitting in Paris. Samples are particular objects, but their content is, by necessity, general. If this idea can be fleshed out for Euclidean diagrams, then we seem to have a promising way of accounting for the generality principle.

An additional reason to conceive of Euclidean diagrams as samples is how well it fits the demands of the co-exact constraint. As we have seen, a sample not only represents a certain content, but the sample itself, *qua* concrete object, realizes that content in a certain way. It is precisely for that reason that e.g. a red swatch of cloth can be used as a sample of red. In a slogan, a sample *is* what it represents. This means that samples appear to be exactly the type of representation that we were looking for: if Euclidean diagrams are samples of their contents, then they instantiate their content in a manner which would readily explain why this content is not only represented but also indicated.

Before moving forward with our positive proposal, it is important to make two preliminary distinctions. First, between diagrams and diagrammatic entries. In the course of some Euclidean proofs, during the *Ekthesis* (setting out) and *Kataskeuè* (construction), several constructions are performed before one reaches the final “static” diagram presented in the pages of the *Elements*. Let us reserve the term ‘diagram’ for this final “static” result, and call ‘diagrammatic entries’ the successive dynamic steps that precede it. Most Euclidean proofs involve several diagrammatic entries distinct from the final diagram; but not all—e.g. in proof I.28, the diagrammatic entry constructed in the *Ekthesis* coincides with the “static” diagram. Both the final diagram and the diagrammatic entries can be said to represent or, as in our view, to be samples.

A second important preliminary distinction is between a diagram (or diagrammatic entry) and the figures which constitute some of them. Which one represents as samples: the diagram taken as a whole or the individual figures? In order to answer that question, notice, for starters, that, while some diagrams are composed by several figures (e.g. I.1, III.10), and while others are composed by a single figure (e.g. I.19), some are void of any figures, such as the line-based diagrams of proofs I.13, I.14, I.15, I.28, I.29, I.30, and I.31. Since our proposal is about the role of Euclidean diagrams in general,

¹⁴ This is an adaptation of Goodman’s original example (1968, p. 53).

it is the diagram or diagrammatic entries, be they composed of figures or not, which are being claimed to *primarily* represent as samples.¹⁵

Naturally, the important question now is: what would Euclidean diagrams and diagrammatic entries be a sample of? Two natural possibilities come to mind. Either they are samples of points, lines and geometrical circles in their diverse configurations or they are samples only of a restricted set of co-exact relationships. According to the first possibility, they would be samples of the geometrical shapes Euclid is concerned with; in other words, they would simultaneously indicate and represent all of the relevant properties these shapes are supposed to have, and these would not only include their co-exact aspects but also their exact ones. At first glance, this might strike one as a promising view. Indeed, if one is exclusively focused on the direct proofs of the *Elements*, one gets the impression that Euclidean diagrams not only indicate the co-exact information employed in the proof, but also correspond to the exact information introduced textually.

However, as we have seen in the case of some proofs by *reductio*, it is not uncommon for a Euclidean diagram to be very much unlike that which is the topic of the proof. But the issue is even more serious than we made it to be: not only for the case of proofs by *reductio*, but also for the case of direct proofs, Euclidean diagrams do not need to correspond to the exact information introduced textually. Take proof I.1 one more time for the sake of illustration. Even though the text assures the geometer that the two enclosed shapes in the diagram are circular, the diagram could be drawn with shapes that visibly depart from circularity (Fig. 6). In order to play their representational role, Euclidean diagrams must unequivocally exemplify their co-exact aspects and nothing more—in the case of proof I.1, they must only exemplify regions enclosed by curves dividing the space into an outside and an inside region as well as their intersections:

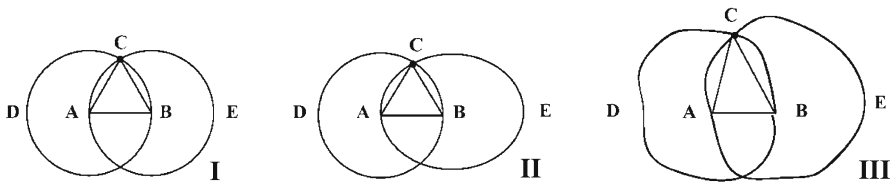


Fig. 6 The variations in the manner of drawing the enclosed figures—in case II, one of the curves does not look like a circle; in case III, the curves do not even seem to have a regular shape—is not detrimental to the validity of the proof since the information that these figures are circles is given by the text. More importantly, notice that the co-exact aspects remain stable under the three distinct configurations, i.e. they are invariant to the deformations and improvements on the diagram

In proofs by *reductio*, the issue is, of course, even more dramatic. Indeed, in these proofs it is often the case that some exact property *suggested* by the diagram needs to be ignored in favor of some textual stipulation—it is not at all uncommon for a mismatch to appear between how the diagram looks and what the text prescribes. Take, for example, the interesting case of proof I.27 (Fig. 7), where one must ignore the diagram's suggestion that there exist two angles corresponding to the letters EBG

¹⁵ We thank an anonymous referee for pressing us to clarify these and several other important points.

and FDG, and requires the geometer to take the diagram as being just like the textual part independently stipulates, i.e. taking EG and FG as straight lines and EGF as a triangle.

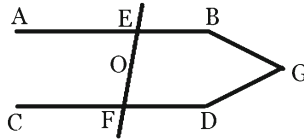


Fig. 7 Proposition I.27 of *Elements*

PROP. XXVII – THEOREM

If a right line (EF) intersecting two right lines (AB, CD) makes the alternate angles (AEF, EFD) equal to each other, these lines are parallel.

Dem. – If AB and CD are not parallel they must meet, if produced, at some finite distance: if possible let them meet in G; then the figure EGF is a triangle, and the angle AEF is an exterior angle, and EFD a non-adjacent interior angle. Hence [XVI.] AEF is greater than EFD; but it is also equal to it (hyp.), that is, both equal and greater, which is absurd. *Hence AB and CD are parallel.* (Heath 1956, p. 29).

If the assumption that the diagram is a sample of geometrical objects and all of their features is difficult to sustain for proofs like I.27, it is even more implausible for others, such as the aforementioned III.10.¹⁶ If one insists that Euclidean diagrams are samples of both exact and co-exact features, then one has to conclude that some diagrams are samples of impossible arrangements. This unfortunate conclusion is problematic for reasons quite similar to those that led to the rejection of the instantial conception.

Thus, it is more appropriate to restrict the representational role of Euclidean diagrams as samples of co-exact properties.¹⁷ Claiming that Euclidean diagrams are samples of a limited set of co-exact relationships that can be directly inferred from them should not be taken as if we were reducing the richness of that type of representation. Instead, we are making sure that the representational role of Euclidean diagrams is completely in line with how these representations are actually used in the Euclidean mathematical practice. Given that, as Manders has shown, Euclidean diagrams are only used as a source of co-exact information, it should not be surprising that this is the type of information that they first and foremost represent.

An analogy might help. Even if a sample of the color red in a page of the tailor's booklet might have some shape (e.g. it could be in the shape of a diamond), if the tailor is using it merely as a sample of the color that the dress will have, then the client commits a mistake if she infers that the tailor intends to make her a diamond-shaped dress. That mistake betrays a failure to understand what the red diamond is being used

¹⁶ Another proof in which the predicament would be the same is III.13.

¹⁷ Lassalle Casanave (2013, p. 27) draws his main conclusion when considering cases of proofs by *reductio*, namely, that if Euclidean diagrams are supposed to be a sample of anything, then they are samples of topological and mereological properties.

as a sample of. Things are similar in the case of Euclidean diagrams. Even if some diagrams may seem to suggest some exact properties (e.g. two drawn lines may look roughly parallel, a shape may look roughly circular), it is an error to understand their use in Euclidean proofs as samples of them.

Taking a particular diagram as a paradigmatic example of a set of co-exact information allows one who uses it as a sample to infer that it possesses characteristics that will be shared among all suitable diagrams of the relevant arrangement. With this idea, it seems clear to us that physical and particular diagrams allow the geometer to attain wholly general conclusions, i.e. to account for *the generality constraint*.¹⁸ As should be already clear, a conception of Euclidean diagrams as samples of co-exact properties would also allow us to easily accommodate *the co-exact constraint*. Samples possess the properties that they represent, and this is more than enough to explain why the way that they represent these properties is sharply distinct from e.g. the way a Euler diagram represents the subset relation.

What about the proofs by *reductio*? In effect, if we take Euclidean diagrams as samples of co-exact aspects, it is easy to understand how they are used in the context of these proofs. In the *Ekthesis* of proof III.10, for example, the diagrammatic entry is to be seen as a sample of two curves (each distinguishing between an interior and an exterior region) that intersect each other in four places. That these curves are circles (not ellipses, for example) is an exact information and, thus, comes from the textual part of the proof, not from the diagram. The diagram is not a sample of a contradictory arrangement. The contradiction only emerges from the interplay of exact and co-exact information.

In conclusion, we have tried to show that a theory according to which Euclidean diagrams represent *qua* samples of co-exact properties has the resources to accommodate the three constraints that have guided our discussion. The next and last section provides an illustration of our conception by means of a case study of Saccheri's quadrilateral (2014 [1733]). As we will see, this case is particularly interesting because Saccheri—besides following the standard Euclidean methodology—uses the same diagram to prove results about three distinct and incompatible figures. This proof was historically seen as a puzzle to instantialist conceptions of Euclidean diagrams or conceptions that

¹⁸ The present conception could also help explain why the diagrams in the *Elements* have never led Euclid to commit a fallacy. A fallacy is obtained when a general proposition is based on accidental aspects of a particular diagrammatic configuration. Understanding the representational nature of diagrams as samples of a restricted set of properties—that, as we have explained, are not specific to particular configurations, but representative of all possible configurations of the same proof—give us a new way of explaining why there are no errors in Euclidean proofs. The error, we suggest, can be committed: (1) when an exact property is extracted from a diagram or (2) when a property that is particular to one diagrammatic configuration (but not present in others) of a proof is extracted. In our terms, these errors occur when one fails to adequately understand the type of properties that a diagram exhibits as sample, e.g. when the agent is tempted to extract metrical aspects that are suggested by a diagram, but which are only accidental. One interesting example come from the famous fallacy that all triangles are isosceles, which was used by critics as a symptom of the misleading nature of Euclidean proofs. As some analyses of this fallacious proof make clear, one of its steps involve extracting an aspect of the diagram which is particular to a misleading diagrammatic configuration of the proof, but not present in more accurate ones (the details and refutation of the proof can be checked in Manders 2008b, pp. 94–96). In other words, this fallacious proof employs an aspect of the diagram that is not co-exact and which the diagram does not represent as sample. In the *Elements*, this misunderstanding never happens.

tried to take them as approximate representations of geometrical figures—how could the same diagram be an instance or an approximate representation of three incompatible figures?¹⁹ We will see that our conception of Euclidean diagrams as samples easily accounts for the peculiar diagram use in Saccheri’s proof.

5 Saccheri’s quadrilateral

Saccheri was a late seventeenth century, early eighteenth century Jesuit priest, philosopher and mathematician who is nowadays mostly well-known for his book *Euclid Vindicated from Every Blemish* (2014 [1733], henceforth *Euclid Vindicated*), which contains his notorious attempt to prove that Euclid’s fifth postulate logically follows from the previous four. Saccheri’s (ultimately failed) strategy involved an investigation, by means of the usual Euclidean methodology, of the properties of a bi-rectangular isosceles quadrilateral, i.e. a quadrilateral constructed by taking a straight line, AB, and then constructing two equal straight lines AC and BD, each perpendicular to AB (see Fig. 8). This figure has then become known as ‘Saccheri’s quadrilateral’. Notice that we are given the magnitude of Saccheri’s quadrilateral’s two base angles (both are right), but not of its summit angles. Furthermore, we are given the equality of two of its opposing sides, AC and BD, but not of the two others, AB and CD.



Fig. 8 *Ekthesis* of the proofs of propositions I and III

The first two propositions proved by Saccheri are straightforward and involve almost nothing more than the application of Euclid’s congruence propositions I.4 and I.8. Proposition I establishes that, if ABDC is a Saccheri quadrilateral, that the angles $\angle C$ and $\angle D$ are equal. As for Proposition II, the geometer is asked to bisect sides AB and CD and then to join their midpoints in a segment; the proposition finally establishes that this segment is perpendicular to both AB and CD. The proof of these two propositions do not involve any move that would be alien to a geometer familiar with Book I of the *Elements*; they are run-of-the-mill direct proofs in the same style as Euclid’s simplest ones. It is when Saccheri introduces Proposition III and its subsequent proof that things become really interesting—specially as far as the role of diagrams go.

Proposition III of *Euclid Vindicated* is actually three-fold; it establishes that: (i) if the angles $\angle C$ and $\angle D$ are right (‘the right hypothesis’), then $AB = CD$, (ii) if $\angle C$ and $\angle D$

¹⁹ For more on how this proof presents a problem to views that take diagrams as being approximate representations of geometric figures, see Sherry (2009).

are obtuse ('the obtuse hypothesis'), then $AB > CD$; and if $\angle C$ and $\angle D$ are acute ('the acute hypothesis'), then $AB < CD$.²⁰ Of central importance to the proof at this point is the realization that the right hypothesis is itself equivalent to Euclid's fifth postulate. Saccheri then hoped that Proposition III would provide the resources necessary for the following crucial steps towards his main—although ultimately unfulfilled—ambition: establishing that the right hypothesis, and thus the fifth postulate, could be proved even if one assumed the incompatible obtuse and acute hypotheses.²¹

The three-fold proof of this proposition shares, in its three stages, a common *Ekthesis* where the following diagrammatic entry is constructed: a region bounded by four lines distinguishing between an interior and an exterior region. In the course of proving, by *reductio*, that if the right hypothesis is true, then $AB = CD$, Saccheri asks us to suppose that $AB < CD$, and then to take a piece DK from DC such that $AB = DK$, and then, to join AK (this is the first auxiliary construction, see Fig. 9). By proposition I, the angles $\angle BAK$ and $\angle DKA$ can be proved equal. However, this result swiftly entails a contradiction. $\angle DKA$ is external and opposite to the right angle $\angle DCA$, and thus, greater than it according to Euclid's Proposition I.16.²² But $\angle DKA$ was proven equal to $\angle BAK$, which, as Saccheri himself says, is 'by construction less than the assumed right angle [$\angle BAC$]' (p. 73). This shows that it is not the case that $AB < CD$. An analogous proof (left to the reader) would show that, given the right hypothesis, $AB > CD$ also cannot be the case. This is the first step of Proposition III and shows that if the right hypothesis holds, then $AB = CD$:

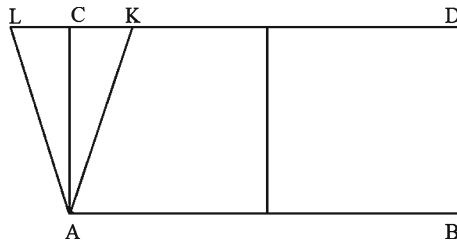


Fig. 9 *Ekthesis* + auxiliary constructions (Proposition III)

Saccheri's procedure for proving that the right hypothesis entails that $AB = CD$ should not be surprising to a geometer familiar with Euclid's proofs by *reductio*.

²⁰ Saccheri's propositions V, VI and VII establish that, if any of these three hypotheses is true even in a single case, then always in every case it alone is true.

²¹ A common misconception about Saccheri's project is that he intended to prove the fifth postulate by *reductio*, i.e. by showing that the obtuse and acute hypotheses are self-contradictory. Instead, as De Risi's (2014, pp. 36–41) historically informed analysis shows, Saccheri intended to show that the fifth postulate can be derived from its own negation, a method of proof known at the time as *consequentia mirabilis* and which Saccheri regarded, contrary to the *reductio* method, as capable of positively establishing the evident status of a proposition. It is debatable whether the two methods are really independent, but Saccheri seems to have believed so. In any case, Saccheri never succeeded in deriving the right hypothesis from the acute one, although some of his results were later re-evaluated as precursors to non-Euclidean geometries.

²² Euclid's I.16: "if any side of a triangle be produced, the exterior angle is greater than either of the interior non-adjacent angles" (Heath 1956, p. 19).

Indeed, it is strikingly similar to Euclid's proof I.6 (see Fig. 5). The interplay of exact and co-exact aspects should leave no doubt that Saccheri's methodology is wholly Euclidean. The metric properties of Saccheri's quadrilateral are given by the textual part of the proof. On the other hand, the diagrammatic entries indicate many important co-exact aspects, such as the fact that angle $\angle DKA$ is external to the triangle CKA and non-adjacent to the triangle's internal angles $\angle KAC$ and $\angle DCA$ (a fact that allows for the employment of Euclid's proposition I.16), or the fact that $\angle BAC$ is a proper part of $\angle BAK$ (this crucial bit of information is justified by considering the diagrammatic entry and *seeing* that the latter angle is a proper part of the former).

Now, the aspect of Saccheri's proof of Proposition III that is most relevant to our purposes: after establishing this first result, Saccheri then asks the geometer to discard the right hypothesis and then to subject the initial diagrammatic entry—the same one constructed in the *Ekthesis*—to the obtuse hypothesis. That is, without performing any change in that initial diagrammatic entry, Saccheri asks us to subject it to distinct exact stipulations. The angles $\angle C$ and $\angle D$ which, in the first part of the proof, were taken as being right, are now supposed to be taken as being obtuse. The details of the proof should not concern us here; as said above, Saccheri ends up showing that, under this assumption, $AB > CD$. Finally, in the third part of the proof, Saccheri again asks us to discard the current hypothesis about the magnitude of the summit angles and, without changing anything in the initial diagrammatic entry, to assume and subject it to the distinct and incompatible acute hypothesis (then showing that this hypothesis entails that $AB < CD$). In summary, Saccheri employs one and the same diagrammatic entry in proofs that demonstrate theorems about distinct (and incompatible) figures: a Saccheri quadrilateral whose summit angles are right (and whose lower and upper sides are equal), obtuse (with lower side greater than the upper side), or acute (with upper side greater than lower side).

We suggest that a conception of Euclidean diagrams and diagrammatic entries as samples of co-exact information shows how one can use one and the same diagram (or entry) to prove theorems about multiple distinct and incompatible figures.²³ This is our explanation: in Saccheri's three-part proof of Proposition III, we start from the same diagrammatic entry that represents *qua* sample the same co-exact relationships (e.g. that the region $ABDC$ is enclosed by four curves). This entry is then supplemented with auxiliary constructions and these constructions then lead to the emergence of more co-exact aspects—such as the aforementioned externality of some angles in relation to triangles that have popped up, the division of some angles into lesser parts etc. In any case, the most important fact to our discussion is that the same initial diagrammatic entry is independently combined with three distinct sets of textual stipulations introducing incompatible exact information (e.g. that the summit angles are right, acute or obtuse). Thus, there is one sense in which Saccheri's initial diagrammatic entry is representing the same thing in all of the three stages of the proof in which it is employed: it is representing as sample the same co-exact aspects. When one discards one hypothesis and assumes another, there is no modification to the co-exact features

²³ For a discussion of a similar case where identical diagrams subject to distinct textual stipulations are used in distinct proofs in Euclid's *Elements*, see Lassalle Casanave (manuscript) on proofs I.28 and I.29. Differently from Saccheri's proof, where the relevant diagrammatic entry contains a figure, the line-based diagrams of these proofs contain no figures.

of the diagrammatic entry—e.g. no exteriority relations are modified when one moves from the right hypothesis to another. Conversely, there is also a sense in which the three employments of Saccheri's initial diagrammatic entry are subject to distinct information: since each employment is accompanied by distinct textual stipulations, we are asked to take the diagram as being just like the exact information introduced by each textual part stipulates (even if its appearance suggests otherwise).

In our own terms, Saccheri's initial diagrammatic entry is, in all three stages of the proof, a sample of the same co-exact properties. This accounts for the sense in which Saccheri's initial diagrammatic entry seems to be representing the same thing in all three contexts. However, it is not a sample of the metrical properties introduced by the text, such as the particular magnitudes of the summit angles. This explains how the same diagrammatic entry can be combined with distinct sets of textual stipulations and then be used to prove results about distinct and incompatible figures.

Clearing up the representational status of Euclidean diagrams as samples illuminates the possibility of giving them a multi-tasking role, such as in Saccheri's case. The distinction between what a Euclidean diagram indicates and what it is subject to by means of textual stipulations is suitably taken care of by our conception of diagrams and diagrammatic entries as samples of co-exact properties.

6 Conclusion

In conclusion, taking Euclidean diagrams to be samples of their co-exact aspects allows us to account for the three constraints which, as we have assumed, are essential to an account of their representational role. It allows us to account for the *generality constraint* because a sample of some property p is not only a particular instance of p , but a representation of p whose content is wholly general. This view also allows us to account for the *co-exact constraint* of Euclidean diagrams. Samples directly present the properties that they represent, and this is more than enough to explain how the way that they represent these properties is sharply distinct from e.g. the way a Euler diagram represents the subset relation. Finally, this account allows us to attain the *uniformity constraint*. As we have argued, Euclidean diagrams can be appropriately understood as functioning as a sample of co-exact relationships in *both* direct and indirect proofs.

We have concluded by showcasing the virtues of our approach by means of an analysis of the multi-tasking role of the diagram of Saccheri's quadrilateral. By arguing that a diagram indicates its co-exact aspects (in virtue of being a sample of them) but is merely subject to its exact aspects, one shows how the same diagram can be employed in demonstrations that involve distinct (and sometimes incompatible) textual stipulations.

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