

Not Much Higher-Order Vagueness in Williamson's 'logic of clarity'

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Abstract

This paper deals with higher-order vagueness in Williamson's 'logic of clarity'. Its aim is to prove that for 'fixed margin models' $(W, d, \alpha, [\])$ the notion of higher-order vagueness collapses to second-order vagueness. First, it is shown that fixed margin models can be reformulated in terms of similarity structures (W, \sim) . The relation \sim is assumed to be reflexive and symmetric, but not necessarily transitive. Then, it is shown that the structures (W, \sim) come along with naturally defined maps h and s that define a Galois connection on the power set PW of W . These maps can be used to define two distinct boundary operators bd and BD on W . The main theorem of the paper states that higher-order vagueness with respect to bd collapses to second-order vagueness. This does not hold for BD , the iterations of which behave in quite an erratic way. In contrast, the operator bd defines a variety of

tolerance principles that do not fall prey to the sorites paradox and, moreover, do not always satisfy the principles of positive and negative introspection.

Keywords:

Higher-order vagueness, logic of clarity, fixed margin models, similarity structures, graphs, boundary operators, tolerance principles, Galois connection, topology.

1 Introduction

This paper deals with the problem of higher-order vagueness in Williamson's 'logic of clarity'. Its aim is to show that for 'fixed margin models' $(W, d, \alpha, [\])$ (Williamson 1994), the notion of higher-order vagueness completely collapses to second-order vagueness.

We give a new equivalent formulation of these models in terms of similarity structures (W, \sim) . The similarity relation \sim is reflexive and symmetric but not necessarily transitive. It is related to d by $x \sim y \Leftrightarrow d(x, y) \leq \alpha$. More precisely, let W be a set of worlds endowed with a binary similarity relation \sim . If two worlds w and v are similar to each other, this is denoted by $w \sim v$.

According to Williamson, vagueness is due to our lack of knowledge or our ignorance. He appeals to the margin for error principle to explain this ignorance. When the knowledge is inexact there are some margins for error principles. He formulates clearly operator in a fixed margin model as:

$$[\mathbb{C}A] = \{w \in W : \forall x \in W \quad d(w, x) \leq \alpha \rightarrow x \in [A]\}.$$

In this formulation, d is a metric that measures the similarity, α a margin for error and $[A]$ a

set of worlds in which A is true. The ‘fixed margin for error’ model, satisfies *KTB* system. A fixed margin model $(W, d, \alpha, [\])$ is essentially equivalent to a similarity structure (W, \sim) . Define the similarity neighborhood $co(w)$ of $w \in W$ by $co(w) := \{v; w \sim v\}$. Formulas and operators of propositional logic are interpreted in the familiar way as subsets and Boolean operators on the set W . Following Breysse & De Glas(2007) we define an operator h by $hA := \{w; co(w) \subseteq A\}$. Then we will show that hA is just Williamson’s $[CA]$, to be interpreted as ‘Clearly A’ (Williamson 1994, 279). Let PW denote the power set of W . Then the operator h is interpreted as a map $PW \xrightarrow{h} PW$, taking A to hA .

To define the concept of boundary we also need another operator which is in fact the dual of h . We call it s and define it as $sA := ChC(A)$ where CA is a set theoretic complement of A . The operators h and s are interdefinable, namely $hA := CsC(A)$. Now we can define another boundary by $bdA := hsA \cap hs(CA)$.

In section 3, we will prove the following theorem: $bd^3A = bd^2A$.

In Williamson (1994, 1999) he demonstrates that if there is second-order vagueness there will be higher-order vagueness as well. We will show that Williamson’s operator ‘C’ can be redefined as h and with a slightly different approach we will see that higher-order vagueness collapses to the second-order one. Furthermore, we will show that (W, \sim, h, s) satisfies the axioms of *KTB*. The crucial idea is that h and s are Galois-related; i.e., $A \subseteq hB \Leftrightarrow sA \subseteq B$ for all $A, B \in PW$.

The fourth section is devoted to the concept of boundary. We distinguish our concept of boundary, defined in section 3, from two other existing concepts of boundary in the

literature; namely, BDA , defined as the intersection of clearly A and clearly not A and the topological one, $bd_t A$, defined as the intersection of closure of A and the closure of the complement of A . Moreover, for bd and bd_t we make a difference between two concepts: thick and thin boundaries, to explain when the second-order vagueness collapses to the first order vagueness. In the last section we shall discuss that the model, explored in sections 3 and 4, in comparison with other accounts (*Égré* and Bonnay 2010; Bobzien 2012, 2015) can be considered as minimal and rather general in the sense that the axiom of positive introspection $hA \rightarrow hhA$ or of negative introspection $\neg hA \rightarrow h\neg hA$ need not hold for h . Moreover, (W, \sim, h, s) satisfies a weak version of tolerance principle slightly different from Williamson's and Shapiro's definitions.

The outline of the paper is as follows: we first recall the 'logic of clarity', mentioned in the appendix of Williamson(1994). Then in the next step we shall introduce the basic notions and definitions in detail and will prove the central theorem. Section four is devoted to specific cases of boundaries and finally, we discuss the important philosophical role of using Galois connection operators in the higher-order vagueness disputes.

2 Williamson's fixed margin models and Similarity Structures

To set the stage, let us recall Williamson's fixed margin model. Williamson (1994) described vagueness in terms of 'logic of clarity'. In the appendix he introduced a logic for 'clearly' to

define the phenomena of vagueness.

He introduced different models based on fixed and variable margins for error. The latter contains a family of accessibility relations while the former just uses a single accessibility relation. In this paper, we will focus on the fixed one. Moreover, the discussion would be at a formal level and would remain neutral with regards to the sources of vagueness.

Definition 2.1. (*Williamson, 1994: 270-271*)

A fixed margin model is a quadruple $(W, d, \alpha, [\])$. In this model, W is a set of possible worlds, d a metric on W that measures their similarity, α a margin for error and $[A]$ is the set of worlds at which A is true. In fact, $[A]$ maps formulas to subsets of W such that for all formulas A, B :

$$[\neg A] = W - [A] \tag{1}$$

$$[A \wedge B] = [A] \cap [B] \tag{2}$$

$$[\mathbb{C}A] = \{w \in W : \forall x \in W \quad d(w, x) \leq \alpha \rightarrow x \in [A]\} \tag{3}$$

The formulas of a fixed margin model $(W, d, 1, [\])$ with \mathbb{C} operator satisfy the following modal axioms:

(Taut) $\vdash A$ if A is a truth-functional tautology

(K) $\vdash \mathbb{C}(A \supset B) \supset (\mathbb{C}A \supset \mathbb{C}B)$

(T) $\vdash \mathbb{C}A \supset A$

(B) $\vdash \neg A \supset \mathbb{C}\neg A$

(MP) If $\vdash A \supset B$ and A then $\vdash B$

(RN) If $\vdash A$ then $\vdash \mathbb{C}A$

Now let us consider the metric d of a fixed margin model $(W, d, \alpha, [\])$ in more detail.

Williamson observes that d defines a reflexive and symmetric relation R on W :

$$\forall x, y \in W (xRy \text{ iff } d(x, y) \leq \alpha).$$

The modal logic of the fixed margin model is based on the reflexive and symmetric relation R . On the other hand, any reflexive and symmetric accessibility relation R is defined by a metric d and a fixed margin α and there is a metric for any reflexive and symmetric relation. He gives an example of such a metric (Williamson 1994, 270-271):

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ \alpha & \text{if } xRy \text{ but } x \neq y \\ \alpha + 1 & \text{if } \neg(xRy) \end{cases}$$

We will introduce an equivalent formulation of this model. This new formulation is based on *similarity structures*. We will show that Williamson's reference to metrics is not necessary, rather, similarity structures do suffice; what is important for fixed margin models is the relation R .

The upshot of these considerations is that what is logically significant about fixed margin model $(W, d, \alpha, [\])$ is encapsulated in the similarity structure $(W, \sim) = (W, R)$ where $R = R(d, \alpha)$. In the rest of the paper we write (W, \sim) instead of $(W, R(d, \alpha))$.

Finally, let us elucidate the relation between fixed margin models and similarity structures:

Lemma 2.2. *Let (W, d, α) be a fixed margin model.*

We define maps

$$(W, d, \alpha) \begin{array}{c} \xrightarrow{\phi} \\ \xleftarrow{\psi} \end{array} (W, \sim, \alpha)$$

$$(W, d, \alpha) \xrightarrow{\phi} (W, \sim, \alpha)$$

$$\phi(W, d, \alpha) := (W, \sim, \alpha) \text{ with } x \sim y := d(x, y) \leq \alpha$$

$$(W, \sim, \alpha) \xrightarrow{\psi} (W, d, \alpha)$$

$$\psi(W, \sim, \alpha) := (W, d, \alpha) \text{ with } d(x, y) := \begin{cases} 0 & \text{if } x = y \\ \alpha & \text{if } x \sim y \text{ but } x \neq y \\ \alpha + 1 & \text{if otherwise} \end{cases}$$

Often it is useful to represent similarity structures by undirected graphs:

Definition 2.3. *Similarity structures (W, \sim) can be represented as simple undirected graphs $(G(W), E(G))$ where the vertices are members of W and there is an edge between each two different members of W that are similar. $G(W)$ is defined in the following way:*

$$\forall x \forall y ((x, y) \in E(G) \leftrightarrow (x \sim y)).$$

The next definition will be a key definition in the paper.

Definition 2.4. *Let (W, \sim) be a similarity structure and $w \in W$. Define the similarity neighborhood $co(w)$ of w :*

$$co(w) := \{v; w \sim v\}$$

The following example is to familiarize the reader with the definitions introduced thus far.

Example 2.5. Let (W, \sim) be a similarity structure, $W = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $1 \sim 2$, $1 \sim 3$, $2 \sim 3$, $3 \sim 4$, $4 \sim 5$, $5 \sim 6$, $6 \sim 7$, $7 \sim 8$, $8 \sim 4$.

Then (W, \sim) is represented by the graph:

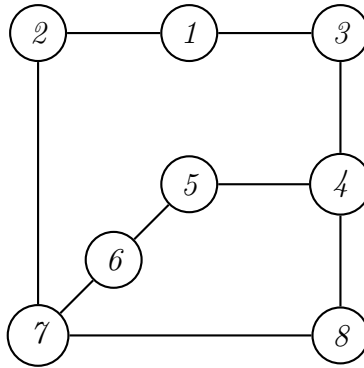


Figure 1.

By definition, the similarity neighborhoods $co(w)$ of $w \in W$ are given by:

$$\begin{aligned} co(1) &= \{1, 2, 3\} & co(2) &= \{1, 2, 7\} \\ co(3) &= \{1, 3, 4\} & co(4) &= \{3, 4, 5, 8\} \\ co(5) &= \{4, 5, 6\} & co(6) &= \{5, 6, 7\} \\ co(7) &= \{2, 6, 7, 8\} & co(8) &= \{4, 7, 8\} \end{aligned}$$

In this paper graphs are used to provide a rich source of examples and counterexamples (See section 4).

3 Similarity Structures and Galois connections

Following Breyse & De Glas(2007) we define hA based on neighborhood $co(w)$ and claim that it is Williamson's clarity operator \mathbb{C} for a fixed margin model.

Definition 3.1. *Let (W, \sim) be a similarity structure. Then for $A \subseteq W$, hA is defined as:*

$$hA := \{w; co(w) \subseteq A\}$$

The operator h is interpreted as a map $PW \xrightarrow{h} PW$, taking A to hA , where PW denotes the power set of W .

It can easily be shown that the map h is distributive with respect to arbitrary intersections, i.e. $h(\bigcap A_i) = \bigcap h(A_i)$, $A_i \in PW$.

The operator h is identical to Williamson's \mathbb{C} operator. The complement of a set $A \subseteq W$ is denoted as $\mathcal{C}A$. Define $s := \mathcal{C}h\mathcal{C}$. s is called the dual of h .

The concatenations of operators h and s , and s and h are denoted as hs and sh respectively.

The main novelty of our approach is based on the observation that h is a component of a Galois connection (h, s) (cf. Giertz et al. 2003).

Definition 3.2. *Let $PW \xrightarrow{h} PW$ be an order-preserving map with respect to set-theoretical inclusion \subseteq . An order-preserving map $PW \xrightarrow{s} PW$ is said to be Galois related to h , denoted by (h, s) iff*

$$\forall A, B \in PW, A \subseteq hB \Leftrightarrow sA \subseteq B.$$

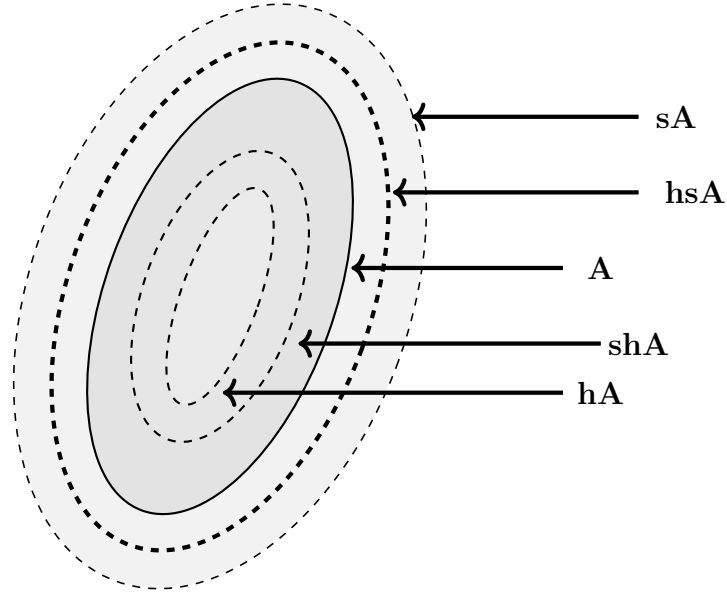


Figure 2. Multi-layer model

Proposition 3.3.

- i) Given a \cap -distributive map h , the map $s := ChC$, is Galois-related to h .
- ii) If h and s are Galois related, then the following series of inclusions will be obtained:

$$...hhA \subseteq shhA \subseteq hA \subseteq shA \subseteq A \subseteq hsA \subseteq sA \subseteq hssA \subseteq ssA...$$

For the details of the proof see Lemmas 3.8 and 3.10. Also (cf. Giertz et al. 2003, 25).

Definition 3.4. Let (W, \sim, h, s) be a similarity structure and (h, s) Galois connection and

$$s^0A = h^0A = A.$$

A layer system, L_k , $k \in \mathbb{Z}$ for $A \subseteq W$ is defined as:

Definition 3.5. Define a binary relation \succ as:

Table 1. Layers L_k

k	$k > 0$	$k \leq 0$
$ k = 2i$	$s^i A - hs^i A$	$h^i A - sh^{i+1} A$
$ k = 2i - 1$	$hs^i A - s^{i-1} A$	$sh^i A - h^i A$

$x \succ y$ iff $\exists i, j \in \mathbb{Z}$; for $j - i \geq 2, x \in L_i$ and $y \in L_j$.

x is said to be distinguishible from y with respect to the predicate A iff $x \succ y$ or $y \succ x$.

Definition 3.6. Let (W, \sim, h, s) be a similarity structure. Define a binary relation \sim_A as:

$$x \sim_A y \text{ iff } \neg (x \succ y) \text{ and } \neg (y \succ x).$$

$x \sim_A y$ is true iff $\exists i, j \in \mathbb{Z}, |i - j| \leq 1, x \in L_i, y \in L_j$ (cf. Rooij 2007, 208).

It can easily be seen that \sim_A is reflexive and symmetric but not transitive.

Proposition 3.7. Let \succ and \sim_A be two binary relations on a set W . Then (\succ, \sim_A) is a semiorder of W (cf. Luce 1956; Rooij 2007).

In the following easy lemma some crucial properties of h , needed for the proof of the pivotal Theorem 3.14, are collected:

Lemma 3.8. Let (W, \sim) be a similarity structure, $A, B \subseteq W$. Then the operator h has the following properties:

(i) $hA \subseteq A$.

(ii) For $A_i \in W$, $h(\bigcap A_i) = \bigcap h(A_i)$.

(iii) $ChChA \subseteq A$.

(iv) if $A \subseteq B$ then $hA \subseteq hB$.

Proposition 3.9. *For the systems (W, \sim, h, s) the Brouwersche scheme B is equivalent to the Galois axiom:*

$$(A \subseteq hB \leftrightarrow sA \subseteq B) \longleftrightarrow shA \subseteq A.$$

Proof. \rightarrow :

Suppose $A \subseteq hB \leftrightarrow sA \subseteq B$. We prove that $shA \subseteq A$.

By assumption, h and s are Galois-related. Thus, by Proposition 3.3, $shA \subseteq A$.

\leftarrow :

Assume $shA \subseteq A$ and $A \subseteq hB$.

$A \subseteq hB$, since s is monotone $sA \subseteq shB$. By the supposition $shB \subseteq B$. So, $sA \subseteq shB \subseteq B$.

Now assume $shA \subseteq A$ and $sA \subseteq B$. By monotonicity of h , $hsA \subseteq hB$. By the assumption, $shA \subseteq A$. On the other hand, by the definition of h and s , $A \subseteq hsA$. So, $A \subseteq hB$. ■

The next lemma gives the most important properties of the operator hs , based on which the boundary is defined.

Lemma 3.10. *Let (W, \sim) be a similarity structure, (h, s) the corresponding Galois connection and $A, B \subseteq W$. Then hs is a closure operator with the following properties:*

- (i) $hs\emptyset = \emptyset, \quad hsW = W.$
- (ii) $hshsA = hsA.$
- (iii) $A \subseteq hsA.$
- (iv) *If $A \subseteq B$ then $hsA \subseteq hsB.$*

For the proof see Breysse & De Glas(2007). It is worth mentioning that in general, hs is neither a topological closure operator, i.e., $hs(A \cup B) \neq hsA \cup hsB$, nor a nucleus, i.e., $hs(A \cap B) \neq hsA \cap hsB$.

Example 3.11. *Consider example 2.4 and let A be a subset of W . We will calculate the operators h, s, hs and sh for it:*

Let $A = \{4, 6, 7, 8\}$. Then $hA = \{8\}$, $sA = \{2, 3, 4, 5, 6, 7, 8\}$, $hsA = \{4, 5, 6, 7, 8\}$ and $sh(A) = \{4, 7, 8\}$.

Now we are in a position to give the exact definition of the boundary bd :

Definition 3.12. *Let (W, \sim) be a similarity structure. The boundary bdA of $A \subseteq W$ is defined as:*

$$bdA := hsA \cap hs(CA).$$

(cf. Breysse & De Glas 2007).

Remark 3.13. Let (W, \sim, h, s) be a similarity structure and $A \subseteq W$. Then bdA is closed; i.e., $hs(bdA) = bdA$, since the intersection of two closed sets is closed.

Now the following theorem can be proved:

Theorem 3.14.

Let (W, \sim, h, s) be a similarity structure and $A \subseteq W$. Then the following holds:

- (i) $bdbdA \subseteq bdA$.
- (ii) $bdbdbdA = bdbdA$.

Proof.

(i):

$$\begin{aligned} bdbdA &= hs(bdA) \cap hs(\mathcal{C}bdA) \quad (\text{By Definition 3.12}) \\ &= bdA \cap hs(\mathcal{C}bdA) \subseteq bdA \quad (\text{By Remark 3.13.}) \end{aligned}$$

(ii):

$$\text{By definition, } bd(A) = hs(A) \cap hs(\mathcal{C}A). \text{ So, } bdbdbdA = hs(bdbdA) \cap hs(\mathcal{C}bdbdA).$$

The boundaries bdA and $bdbdA$ are closed so $hs(bdbdA) = bdbdA$. Therefore, it is enough to show that $hs(\mathcal{C}bdbdA) = 1$.

$$\begin{aligned} hs(\mathcal{C}bdbdA) &= 1 \\ \iff hs(\mathcal{C}(hs(bdA) \cap hs(\mathcal{C}bdA))) &= 1 \\ \iff \mathcal{C}sh(hs(bdA) \cap hs(\mathcal{C}bdA)) &= 1 \quad (shA = \mathcal{C}hs\mathcal{C}A) \\ \iff sh((bdA) \cap hs(\mathcal{C}bdA)) &= 0 \\ \iff h((bdA) \cap hs(\mathcal{C}bdA)) &= 0 \quad (\text{By proposition 3.3. } h \subseteq sh.) \end{aligned}$$

$$\iff h(\text{bd}A \cap \text{hs}(\text{Cbd}A)) = 0 \quad (\text{By Remark 3.13 } \text{hs}(\text{bd}A) = \text{bd}A.)$$

$$\iff \text{hbd}A \cap \text{hhs}(\text{Cbd}A) = 0$$

By (3.3. ii) $\text{hhs}(\text{Cbd}A) \subseteq s(\text{Cbd}A)$. So,

$$\text{hbd}A \cap \text{hhs}(\text{Cbd}A) \subseteq \text{hbd}A \cap s(\text{Cbd}A) = \text{hbd}A \cap \text{Chbd}(A) = 0. \quad \blacksquare$$

Remark 3.15. *The following counterexample shows that in general $\text{bdbd}A \subsetneq \text{bd}A$:*

Take (\mathbb{N}, \sim) with $\mathbb{N} = \{0, 1, 2, \dots\}$, and $n \sim m := |n - m| \leq 1$. Define $A := 2\mathbb{N} = \{0, 2, 4, \dots, 2n, \dots\}$. Then $\text{bd}A = \mathbb{N}$ and $\text{bdbd}A = \text{bd}\mathbb{N} = \emptyset$.

Corollary 3.16. *Let (W, \sim, h, s) be a similarity structure corresponding to a fixed margin model $(W, d, \alpha, [\])$. Denote the n th iteration of the boundary operator by bd^n . Then $\text{bd}^n(A) = \text{bd}^2(A)$, $n \geq 2$.*

Proof. By Theorem 3.14, $\text{bd}^3A = \text{bd}^2A$. By induction on n , one can prove that for any n greater than 3, $\text{bd}^nA = \text{bd}^{n-1}A$. $\text{bd}^3A = \text{bd}^2A$, entails that $\text{bd}^nA = \text{bd}^2A$ for $n \geq 2$. \blacksquare

4 Boundaries

It is expedient to compare the concept of boundary as defined in 3.12., with some other ones that appear in the literature.

Perhaps, the most common one is the following (Bobzien 2012, 2015; *Égré* 2010 ; Williamson 1994):

Definition 4.1. $BDA := \neg\mathbb{C}A \cap \neg\mathbb{C}\neg A$.

Remark 4.2. *Since h is just the clearly operator \mathbb{C} , we can formulate (4.1.) as:*

$$BDA = ChA \cap Ch\mathbb{C}A = s\mathbb{C}A \cap sA.$$

The iterations of the boundary BD generally blow up:

Example 4.3. *Let (\mathbb{N}, \sim) be the similarity structure defined in Remark 3.15. For $A = \{0\}$,*

one obtains:

$$\begin{array}{ll} BDA = \{0, 1\} & BD^7A = \{0, 1, 2, 4, 5, 6, 7\} \\ BD^2A = \{1, 2\} & BD^8A = \{2, 3, 4, 7, 8\} \\ BD^3A = \{1, 2, 3\} & BD^9A = \{1, 2, 4, 5, 6, 7, 8, 9\} \\ BD^4A = \{0, 1, 3, 4\} & BD^{10}A = \{0, 1, 2, 3, 4, 9, 10\} \\ BD^5A = \{1, 2, 3, 4, 5\} & BD^{11}A = \{4, 5, 8, 10, 11\} \\ BD^6A = \{0, 1, 5, 6\} & BD^{12}A = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \end{array}$$

One observes that in some of the iterations of BDA there are some gaps. For example in BD^4A , $\{2\}$ is a gap. Larger gaps occur in BD^6A , $\{2, 3, 4\}$ and in BD^8A , $\{5, 6\}$. The number of gaps increases in $BD^{11}A$ where there are two gaps, $\{6, 7\}$ and $\{9\}$. These gaps in the iterations of BD look quite irregular and that makes the concept of BD implausible.

In contrast to BD which has an erratic behavior, bd is quite well-behaved since by Theorem 3.14., the iterations of the boundary operator bd do not grow so that for n greater than 2, $bd^n = bd^2$.

The relation between BD and bd , defined in 3.14., is succinctly expressed by the following proposition:

Proposition 4.4. $bdA = h(BDA)$.

Proof. Since h is distributive with respect to \cap , $bdA = hsA \cap hsCA = h(sA \cap sCA) = h(BDA)$ (see Lemma 3.8.). ■

In topology, the concept of boundary is also central . Recall that a topological space is given by (X, cl) where $cl : PX \longrightarrow PX$ is a closure operator (Kuratowski 1968).

Definition 4.5. *The topological boundary operator bd_t is defined based on the Kuratowski closure operator.*

$$bd_t A := clA \cap clCA.$$

The dual operator of cl is called the interior kernel operator, denoted by int . It is defined as:
 $int := CclC$.

The boundaries bd and bd_t are somewhat analogous, i.e., h and s are almost analogous to int and cl respectively. Recall that topological closure not only satisfies all the conditions mentioned in Lemma 3.13., but also it is distributive with respect to the union: $cl(A \cup B) = clA \cup clB$. The difference between these operators is discussed in more detail in Breysse & De Glas (2007).

The boundary operators bd and bd_t allow us to distinguish between two kinds of concepts:

Definition 4.6. Let W be a set for which (W, \sim) is a similarity structure and (W, cl) is a topological space and $A \subseteq W$. Then

i. A has a thin boundary bd if $shbdA = \emptyset$. Otherwise, A has a thick boundary.

An analogous distinction can be made for the topological boundary bd_t :

i_t. A has a thin topological boundary bd_t if $int(bd_tA) = \emptyset$. Otherwise, A has a thick boundary.

This distinction helps determine under what conditions the equation $bdbdA = bdA$ holds. The following lemma shows that the second-order vagueness collapses to the first one if the boundary is thin.

Lemma 4.7. Let (W, \sim) be a similarity structure and (W, cl) be a topological space and $A \subseteq W$. Then

i. $bdbdA = bdA$ iff $shbdA = \emptyset$.

ii. $bd_tbd_tA = bd_tA$ iff $intbd_tA = \emptyset$.

Proof. (i): Suppose $bdbdA = bdA$. Then by definition

$$\begin{aligned}
 bdbdA = bdA &\iff hs(bdA) \cap hs(CbdA) = bdA. \\
 &\iff bdA \cap Csh(bdA) = bdA. && \text{(By Remark 3.15.)} \\
 &\iff bdA \subseteq Csh(bdA). \\
 &\iff sh(bdA) \subseteq CbdA. \\
 &sh(bdA) \subseteq bd(A). && \text{(By Proposition 3.3.)}
 \end{aligned}$$

Since $sh(bdA) \subseteq CbdA$ we obtain $sh(bdA) \subseteq bd(A)$. So, $shbdA = \emptyset$.

Now suppose $shbdA = \emptyset$. We prove that $bdbdA = bdA$.

$bdbdA = hs(bdA) \cap hs(CbdA)$. (By Definition 3.12.)

$bdbdA = Csh(CbdA) \cap Csh(bdA)$. (By interdefinability of h and s .)

$bdbdA = C(sh(bdA) \cup sh(CbdA))$.

$bdbdA = C(sh(CbdA))$. (By assumption $shbdA = \emptyset$)

$bdbdA = hs(bdA) = bdA$. (By Remark 3.13.)

(ii) :

—→:

Suppose $intbd_tA = \emptyset$. By Definition 4.5., $bd_tbd_tA = clbd_tA \cap clCbd_t$. And $intA = CclCA$.

Also, bd_tA as an intersection of two closed sets is closed; i.e., $clbd_tA = bd_tA$. So, $bd_tbd_tA =$

$bd_tA \cap Cintbd_tA$. By assumption, $intbd_tA = \emptyset$. Therefore, $bd_tbd_tA = bd_tA$.

←—:

Now suppose $bd_tbd_tA = bd_tA$.

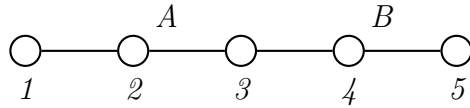
By Definition 4.5., $bd_tbd_tA = clbd_tA \cap clCbd_tA = bd_tA$. Since int and cl are dual and bd_t is closed, $bd_tbd_tA = bd_tA \cap Cintbd_tA = bd_tA$. Therefore, $bd_tA \subseteq Cintbd_t$. This is equivalent to $intbd_tA \subseteq Cbd_tA$. But for the interior kernel operator, $intbd_tA \subseteq bd_tA$. So, $intbd_tA = \emptyset$. ■

For the topological boundary operator bd_tA the class $\{A | intbd_tA = \emptyset\}$ is a non-complete Boolean algebra with respect to \cap, \cup and \mathcal{C} (Stone 1936 ; McKinsey & Tarski 1944). Hence,

the natural question arises whether the analogous result holds for the boundaries bd and BD . The following example shows that for bd , neither the union and intersection of two concepts with thin boundaries nor the negation of a thin boundary concept need have a thin boundary.

Example 4.8. For A and B with thin boundaries, $bd(A \cup B) \neq bdA \cup bdB$.

Let $W = \{1, 2, 3, 4, 5\}$, $A = \{2\}$, $B = \{4\}$.



A and B have thin boundaries. However, $bd(A \cup B)$ is not equal to $bd(A) \cup bd(B)$.

$hs\{2\} = \{2\}$; $hs\{1, 3, 4, 5\} = \{1, 2, 3, 4, 5\}$ so $bdA = hs\{2\} \cap hs\{1, 3, 4, 5\} = \{2\}$ and we can easily see that $bdB = \{4\}$. So $bdA \cup bdB = \{2, 4\}$

$bd(A \cup B) = bd(\{2, 4\}) = hs(\{2, 4\}) \cap hs(\{1, 3, 5\}) = \{1, 2, 3, 4, 5\} \cap \{1, 2, 3, 4, 5\}$

$bd(A \cup B) = W$.

Now it is enough to show that A and B have thin boundaries. According to the definition, A

has a thin boundary if $shbdA = \emptyset$

$hbdA = hbdB = \emptyset$. Therefore, $shbdA = shbdB = \emptyset$.

The result does not hold for BD either. The proof of it is left to the reader.

5 Vagueness and Galois connections

There are many debates in the literature on what the prominent features of vagueness are. However, two main features are identified for vague predicates: they admit borderline cases and they give rise to the sorites paradox (Keefe 2000). Many accounts of vagueness deny the existence of sharp boundaries. In contrast, epistemists like Williamson claim that there is a fact of the matter where the boundaries are, we are just ignorant of it. More precisely, he sticks to the classical logic and contends that vague predicates have well-defined extensions of which we are ignorant. This ignorance is due to the fact that our knowledge is inexact. (Williamson 1994, 216). Inexact knowledge is rooted in the fact that indiscriminability is non-transitive (Williamson, 1990, 1994). When there is inexact knowledge, it seems reasonable to assume some margin for error principle.

(5.1) **Margin for error principle (MEP)** :

A is true in all cases similar to cases in which '*it is known that A*' is true (Williamson 1994, 227).

In contrast, the following principles do not hold:

(5.2) **Positive introspection (PI)**: For any proposition P, If I know that P, I know that I know that P.

(5.3) **Negative introspection (NI)**: if I do not know that P, I know that I do not know P.

When it comes to vagueness, the operator *I know that* is to be replaced by *It is clear that*

and likewise, *I do not know to it is not clear that.*

For systems (W, \sim, h, s) the principles (MEP), (PI) and (NI) are formulated as:

$$(MEP) \quad \forall x, y (hAx \wedge x \sim y \rightarrow Ay)$$

$$(PI) \quad hA \subseteq hhA.$$

$$(NI) \quad \neg hA \subseteq h\neg hA.$$

$(W, d, \alpha, [\])$ and (W, \sim, h, s) satisfy the principle (5.1).

Égré (2010), finds PI and NI desirable. In contrast, it is easy to see from the proposition 3.5. that in (W, \sim, h, s) models PI and NI principles need not hold. Nevertheless, there is higher-order vagueness but by Theorem 3.14., only of first and second order.

One of the most intriguing features of vagueness is that it gives rise to the sorites paradox. (cf. Zardini 2008,2013). Vague predicates are sorites susceptible since they are not sensitive to the small changes. In other words, vague predicates are tolerant to tiny changes so that if a_1 is P and a_2 differs indiscriminably from a_1 then a_2 also is P. Wright(1974) dubbed it the “tolerance principle“. However, this leads to a paradox. Starting from a_1 and applying the tolerance principle, little by little we come to a point a_n that is P but seems clearly not P. The paradox arises since the premises are true and also the rule of inference is valid but the conclusion appears to be false.

Williamson(1994, 2000) claims that the failure of PI resolves the paradox. This failure also explains the higher-order vagueness phenomenon, commonly related to borderline cases. Recently, Pagin(2015) focused on this feature of vagueness with regard to higher-order vagueness

and, in particular, to the tolerance principle which is tightly related to the notion of margin for error (Égré 2015).

Williamson rejects the tolerance principle and gives a weak version of it based on the margin for error principle. For him, vagueness is due to epistemic indiscriminability that is a non-transitive relation and leads to the refutation of PI principle. The failure of PI not only avoids the Sorites paradox but also is associated with higher-order vagueness since it can be the case that *it is clear that P* but not *clear that it is clear that P*.

In the rest of this section we will go through some versions of the tolerance principle based on formulations, proposed in Rooij(2012). Rooij(2012) starts with the following principle of tolerance as P:

$$(P) \quad \forall x, y \in D : (Px \wedge x \sim_p y) \rightarrow Py.$$

Here D is the domain of objects and \sim_p is the semiorder indistinguishability relation with respect to P as defined in Definition 3.6. (Rooij 2012, 207). Rooij mentions different ways of weakening (P) . One of these weak versions is Williamson's. Since his *clearly operator* is just h , we will get:

$$(P_W) \quad \forall x, y \in D : (hPx \wedge x \sim_p y) \rightarrow Py.$$

In words, if x is *clearly P* and y is similar to x , then y is P .

For models (W, \sim, h, s) :

$$(P_{sh}) \quad \forall x, y \in D : (hPx \wedge x \sim_p y) \rightarrow shPy.$$

If the Brouwersche scheme holds, then (P_W) and (P_{sh}) are equivalent.

Shapiro(2006) introduced another weaker version of P. It relies on the concept of antonyms.

\bar{P} is the antonym of P if $\bar{P} \cap P = \emptyset$ and $\bar{P} \cup P \neq D$.

$$(P_S) \quad \forall x, y \in D : (Px \wedge x \sim_p y) \rightarrow \neg \bar{P}y.$$

In words, if x is P and y is similar to x , then one cannot competently judge y in any other manner than x [within the same context](Shapiro 2008 , 6). Shapiro's contextual approach, using the sP as $\diamond P$ would be:

$$(P_S) \quad \forall x, y \in D : (Px \wedge x \sim_p y) \rightarrow sPy.$$

For models (W, \sim, h, s) , (P_{hs}) , we propose a stronger principle than P_s :

$$(P_{hs}) \quad \forall x, y \in D : (Px \wedge x \sim_p y) \rightarrow hsPy.$$

In general, in the multi-layer model, if x is in a layer and y is similar to x then y at most can belong to the next layer. If the B - schema holds, this will be compatible with both Williamson's and Shapiro's views .

Figure 2, shows the relation between the discussed principles of tolerance:

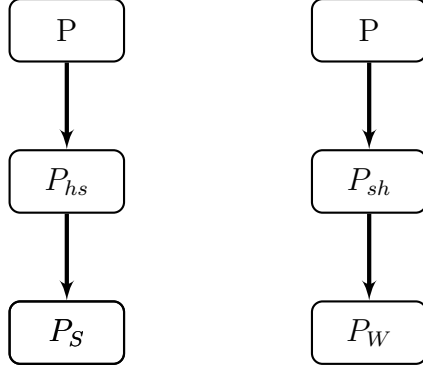


Figure 2. Comparison between principles of tolerance

6 Conclusion

According to Williamson the validity of the margin for error principle MEP and the failure of PI and NI principles explain vagueness, higher-order vagueness, and also avoids the sorites paradox. We focused on the logic he gave for margin for error principle, 'logic of clarity'. One aim of our paper was to show that having a similarity structure is enough. There is no need for a metric. (W, \sim, h, s) is equivalent to Williamson's model $(W, d, \alpha, [\])$, h being just Williamson's clearly operator \mathbb{C} . We have shown that PI and NI principles need not hold. Furthermore, a similarity structure (W, \sim) comes along with a Galois connection (h, s) , and the corresponding modal system satisfies the axioms of KTB. Moreover, this has some interesting consequences: First, it explains why the hierarchy of boundaries stops at the second level and although there are higher-order vagueness, they collapse to the second one and under specific conditions where the boundary is thin they would collapse to the first-order vagueness. Secondly, it suggests a weaker version of tolerance, stronger than both Williamson's and Shapiro's versions.

In summary, it is well worth the effort to have a closer look at the Galois connections and their possible applications to vagueness.

7 References

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