A general note on categorical models of abstract physical theories

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Abstract — Defining an abstract physical theory T as a set of abstract mathematical-logical formulas with a physical interpretation, this note introduces the notion of a categorical model of such a theory T by identifying a model with a small category, whose objects are mathematically concrete set-theoretic models of T that share a common background language, and whose arrows are model isomorphisms. Specifying such a categorical model of an abstract physical theory is then a new application of category theory to theoretical physics, which in a natural way gives rise to new research programs.

1 Introduction

In categorical model theory, the syntactic view on theories—i.e. the view that a theory is a list of axioms expressed in a formal language [1]—is *abandoned*: instead, one identifies the notion of a 'theory' with a category of contexts formed by formulas in the language of the theory [2]. Such an identification is usually motivated by the idea of presenting a theory in a way that is invariant with respect to syntactical choices. By contrast, in this note we *preserve* the syntactic view of a theory: we brush aside the idea that there might be equivalent formulations, but we identify the notion of a 'model' with a category whose objects are set-theoretic structures for a theory and whose arrows are structure isomorphisms.

As the concept of a categorical model of a first-order theory in general may be too broadly construed, we focus at abstract physical theories with a formalization in an existing mathematical-logical framework. The idea is then that a categorical model of such a theory in a natural way incorporates relativity: an object of the category is to be associated with a concrete mathematical model of a physical system in the coordinate system of an observer, and an arrow of the category is to be associated with a transformation that maps a model of a physical system in the coordinate system of one observer to a model of *the same physical system* in the coordinate system of another observer.

The next section formally introduces the notion of an abstract physical theory formalized in a mathematical-logical framework, the sections thereafter introduce categorical models thereof and discuss how these give rise to research programs.

2 Abstract physical theories

The basic idea of an abstract physical theory is that it expresses physical principles in the form of mathematically abstract, well-formed-formulas in a mathematical-logical language. Below we give a formal definition, on which we elaborate in a number of remarks.

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Definition 2.1. Let S be a foundational theory for mathematics, such as ZF, with formal language $\mathcal{L}$; then an **abstract physical theory T formalized within S**, or shortly an **abstract physical theory** T, consists of

(i) the **language** $\mathcal{L}(T)$ for T, which is a sublanguage of $\mathcal{L}$ determined by

- a nonempty set U_T, whose elements are called **individual constants** of T;
- a nonempty set R_T, each element of which is an n-ary **relation** $R \subset (U_T)^n$ of T;

(ii) a collection of **formal axioms** of T:

- for every abstract constant $\phi \in U_T$, an axiom $\exists x(x = \phi)$;
- for every n-ary relation $R \in R_T$, an axiom $\exists x(x = R \wedge R \subset (U_T)^n)$;

(iii) a collection of well-formed formulas in $\mathcal{L}(T)$, which are called the **physical axioms** of T;

(iv) a collection of statements in ordinary language, called the **interpretation rules** of T, which give a physical meaning to the individual constants and relations of T.

Let Σ_T be the total collection of formal and physical axioms of T; a **theorem of** T is then any formula Ψ that can be inferred from Σ_T within the framework of S as in

$$\Sigma_T \vdash_S \Psi \tag{1}$$

and the condition then has to be satisfied that all theorems that can inferred from the physical axioms of T by eliminating all quantifiers, are expressed in terms of abstract constants of U_T. □

Remark 2.2. In the first place an abstract physical theory consists, thus, of a list of well-formed formulas: the use of the term 'theory' is thus justified from the perspective of the aforementioned syntactic view on theories. However, an abstract physical theory is a special kind of theory, namely one whose axioms are expressed in a *mathematical-logical language*: as a consequence, individual constants of the theory are things in the mathematical universe. Such a theory T is, thus, not 'just' a first-order theory: in general, individual constants of a first-order theory *do not have* that ontological status! □

Remark 2.3. In the second place the adjective 'physical' indicates that the collection of axioms of an abstract physical theory is complemented with a collection of interpretation rules: without these, the physical axioms would have no physical meaning whatsoever. That means that in the context of an abstract physical theory, we have to distinguish between the *material object*, i.e. the (postulated) thing in the physical universe that is referred to, and the *formal object*, i.e. the thing in the mathematical universe that refers to the material object. Tegmark's view that mathematics is an external reality [3] is thus rejected: there is no physical reality to the mathematical universe—mathematics provides the language for physics and that's it. □

Remark 2.4. Last but not least, the adjective 'abstract' indicates that the individual constants of an abstract physical theory that refer to components of the physical universe are *abstract objects* in the mathematical universe. Now the ontological status of abstract constants can probably be debated forever, but here the following position is taken. A formal axiom $\exists x(x = \phi)$ of an abstract physical theory T guarantees that there is an object in the mathematical universe whose name is ϕ, but without elaborating on which object that precisely is. The important point here is that we do not have assumed *new objects*: therefore $\mathcal{L}(T)$ is merely a *sublanguage* of $\mathcal{L}$ in which we have assumed *new symbols* for existing objects. Examples of abstract objects in the framework of ZF are *abstract sets*, i.e. sets whose elements are not specified: these stand in contrast to *concrete sets*, i.e. sets whose elements *are* specified (such as the empty set or the set of natural numbers). At this degree of abstractness, the nature of the correspondence specified by an interpretation rule is thus that the formal object *designates* a material object *without* representing its state, that is, *without* containing information of (expectation values of) quantitative properties of the material object: with an abstract physical theory we want to have the largest possible degree of freedom of expression—we want to express physical principles that hold *regardless of* the properties (like position, momentum, etc.) that the involved components of the physical universe have in the reference frame of an observer. □

Example 2.5. The Elementary Process Theory (EPT), published in [4], is an example of a fully specified abstract physical theory. The language $\mathcal{L}$ in which the EPT is formulated is that of set matrix theory, cf. [5]. A mathematically abstract object in this framework is thus, e.g., a 2×1 matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ whose entries x and y are abstract sets. The matrix is then not a set. □

Example 2.6. Newtonian mechanics, classical electrodynamics, special relativity, general relativity, and standard quantum mechanics are examples of theories that *do not* qualify as an abstract physical theory. The crux is that the condition of Def. 2.1 is not satisfied: the individual constants of these theories that refer to components of the physical world are *concrete* mathematical objects, not *abstract* ones as required. As a consequence, the EPT of Ex. 2.5 has a higher degree of abstractness than the theories just mentioned. □

3 Categorical models

An abstract physical theory yields predictions, but due to its degree of abstractness (see Rem. 2.4) these are not experimentally verifiable. E.g. the EPT of Ex. 2.5 predicts that new building blocks will be created from existing ones, but at this level of abstractness it does not predict where the new building blocks can be found: as such, the prediction is not experimentally verifiable. Given that in physics only verifiable predictions are of interest, one might therefore be inclined to think that abstract physical theories are *therefore* not interesting for physics. But that's wrong thinking. A concrete set-theoretic model of an abstract physical theory, namely, *does* yield verifiable predictions: an abstract physical theory can thus be tested by testing its models (refined falsificationism). Below we first give a definition of a concrete set-theoretical model, and then we argue that instead a model of an abstract physical theory is best identified with a category.

Definition 3.1. Let T be an abstract physical theory; then a **concrete set-theoretic model** M of T is an interpretation of the individual constants and relations of T in a concrete set-theoretical domain $\mathcal{D}$ such that the interpretation of the axioms of T in the language of M are true in M. The **interpretation function** is a function $I : \mathcal{L}(T) \to \mathcal{L}(M)$ such that

(i) every abstract object $\phi \in U_T$ that designates a material object is interpreted as a concrete object $I(\phi) \in I[U_T] \subset \mathcal{D}$ representing the state of that object in the reference frame of an observer;

(ii) every n-ary relation $R \subset (U_T)^n$ is interpreted as a relation $I(R) \subset I[U_T]^n$ for which

$$\langle \phi_1, \ldots, \phi_n \rangle \in R \Leftrightarrow \langle I(\phi_1), \ldots, I(\phi_n) \rangle \in I(R) \tag{2}$$

(iii) for any axiom Ψ of T, its interpretation $I(\Psi)$ in the language $\mathcal{L}(M)$ of M is true in M:

$$M \models I(\Psi) \tag{3}$$

□

Remark 3.2. If we are very strict in applying the syntactic view on theories and models—and that *is* our intention here—then specifying a single set-theoretical model M of an abstract physical theory T will only yield verifiable predictions in the coordinate system of **one observer**. For example, in accordance with clause (i) of Def. 3.1 we have that the initial state of a system is a single point-particle at position X_0 in the coordinate system of an observer $\mathcal{O}$ with momentum $\vec{p}_0$, and the model M predicts that after time t the state of the system is that the point-particle is at position X_1 in the coordinate system of $\mathcal{O}$ with momentum $\vec{p}_1$: this is a verifiable prediction. But for another observer $\mathcal{O}'$ the initial state of that same system will be a single point-particle at some position X'_0 in the coordinate system of $\mathcal{O}'$ with momentum $\vec{p}'_0$, and the predicted state of the system will be that the point-particle is at a position X'_1 in the coordinate system of $\mathcal{O}'$ with momentum $\vec{p}'_1$. The model M, however, does not contain the initial state of the system in the coordinate system of $\mathcal{O}'$: it only contains the initial state of the system in the coordinate system of $\mathcal{O}$. And the model M is incapable of predicting what the values of the aforementioned position X'_1 and momentum $\vec{p}'_1$ will be: a single set-theoretic model of an abstract physical theory is thus **insufficient** because it can never predict relativity of spatiotemporal characteristics of motion. That provides the motivation for introducing the notion of a *categorical model* of an abstract physical theory: this does contain a model of a physical system for every observer. □

Definition 3.3. Let T be an abstract physical theory; then a **categorical model** of T is a (small) category $\mathscr{C}$ for which

(i) the collection of objects of $\mathscr{C}$ is a family $\{M_i\}_{i\in F_1}$ of concrete set-theoretic models of T, such that each M_p in $\{M_i\}_{i\in F_1}$ is specified in a common background language $\mathcal{L}(\mathscr{C})$;

(ii) the collection of arrows of $\mathscr{C}$ is a family $\{A_j\}_{j\in F_2}$ of structure isomorphisms, so that for any arrow A_k in $\{A_j\}_{j\in F_2}$ there is a domain $M_p \in \{M_i\}_{i\in F_1}$ with interpretation function I_p and a codomain $M_q \in \{M_i\}_{i\in F_1}$ with interpretation function I_q such that

- A_k bijectively maps the universe $I_p[U_T]$ to the universe $I_q[U_T]$;
- for any n-ary relation $R \subset (U_T)^n$ of T we have

$$(A_k(\alpha_1), \ldots, A_k(\alpha_n)) \in I_q(R) \Leftrightarrow (\alpha_1, \ldots, \alpha_n) \in I_p(R) \tag{4}$$

□

Example 3.4. A categorical model of the EPT (from Ex. 2.5) has been fully specified in [6]. Relativity is then incorporated in the categorical model, as every Lorentz transformations between coordinate systems of observers corresponds with an arrow of the category.

Remark 3.5. The semantic view on theories is that a theory is a collection of models [7]. So from the perspective of this semantic view, the collection of objects of the category $\mathscr{C}$ of Def. 3.3 form a *mathematically concrete* theory T'. It should be noted, however, that T' is constructed by directly specifying the collection of models $\{M_i\}_{i\in F_1}$: that implies that no axiomatization of T' has been developed. In the case of Ex. 3.4 the collection of models corresponds to a 5D unification theory of the EPT and Special Relativity. □

Remark 3.6. Clause (i) of Def. 3.3 resembles 'standard' categorical model theory in the sense that each model M_p in $\{M_i\}_{i\in F_1}$ is a context consisting of formulas in the language $\mathcal{L}(\mathscr{C})$. But not every possible context is an object of $\mathscr{C}$: only those contexts are considered that qualify as a set-theoretic model of T according to Def. 3.1. □

Remark 3.7. Likewise, each arrow A_k in $\{A_j\}_{j\in F_2}$ is an isomorphism, but not every possible isomorphism between the objects of the category $\mathscr{C}$ is an arrow. There are only isomorphisms between models that concern *the same* physical system. To get a grasp of what that means, consider again the physical system of Rem. 3.2. We thus have a model M_1 in which the initial state S_0 of the system is a point-particle with position X_0 and momentum $\vec{p}_0$ in the coordinate system of observer $\mathcal{O}$, and in which the state S_1 of the system is the point-particle with position X_1 and momentum $\vec{p}_1$ in the coordinate system of $\mathcal{O}$. The categorical model then contains a model M'_1 that describes *the same* physical states S_0 and S_1 in the coordinate system of the observer $\mathcal{O}'$: there is, thus, an arrow A_1 that transforms the model M_1 to the model M'_1: if we know M_1 and A_1, then we can predict M'_1—note that this is verifiable because $\mathcal{O}'$ can check whether the particle indeed pops up at position X'_1.

However, under *different* conditions (e.g. different gravitational and electromagnetic fields), the same initial state S_0 will evolve to a *different* state $S_{1'}$: the categorical model thus also contains a model $M_{1'}$ in which the initial state S_0 evolves to a state $S_{1'}$ where the point-particle has position $X_{1'}$ and momentum $\vec{p}_{1'}$ in the coordinate system of $\mathcal{O}$. Although M_1 and $M_{1'}$ are isomorphic *mathematically*, the category contains no arrow $A : M_1 \to M_{1'}$ because M_1 and $M_{1'}$ do not concern the same physical system. But of course, the category $\mathscr{C}$ *does* contain a model $M'_{1'}$ and an arrow $A_2 : M_{1'} \to M'_{1'}$ such that $M'_{1'}$ describes the evolution of the state S_0 to $S_{1'}$ in the coordinate system of the observer $\mathcal{O}'$.

Thus speaking, both M_1 and $M_{1'}$ are models of the abstract physical theory T, but at most one of them corresponds to reality as seen from the perspective of observer $\mathcal{O}$: which one depends on the actual conditions. □

4 Research programs

Once an initial categorical model $\mathscr{C}_0$ has been developed, an abstract physical theory T corresponds in a natural way with a research program in theoretical physics—here the term 'research program' is used in the sense as meant by Lakatos [8]. In [6] a research program based on the EPT has been set forth, but below we specify hard core, positive and negative heuristics of such a research program in general terms.

The **hard core** of the research program consists *at least* of a foundational theory for mathematics that corresponds with the language $\mathcal{L}$ in Def. 2.1 and of the abstract physical theory T: the latter is then considered to be fundamental—that is, the physical axioms of T are the fundamental laws in this research program. This may be supplemented by some examples of how T applies, at an abstract level, to real world problems. This hard core already corresponds to what Kuhn called a paradigm (disciplinary matrix) [9].

The natural **positive heuristic** is to develop successors $\mathscr{C}_1, \mathscr{C}_2, \ldots$ of $\mathscr{C}_0$ that are theoretically and empirically progressive. Lakatos has defined notions of theoretical progression and empirical progression for theories [8], but these notions can be defined similarly for categorical models [6]:

Definition 4.1. Let T be an abstract physical theory; then a categorical model $\mathscr{C}_{n+1}$ of T is **theoretically progressive** compared to a categorical model $\mathscr{C}_n$ of T when not only all observations, which could be expressed as predictions in the language of $\mathscr{C}_n$, can also be expressed in the language of $\mathscr{C}_{n+1}$ but also some observations, which could not be expressed as predictions in the language of $\mathscr{C}_n$, can be expressed in the language of $\mathscr{C}_{n+1}$. Likewise, a categorical model $\mathscr{C}_{n+1}$ of T is **empirically progressive** compared to a categorical model $\mathscr{C}_n$ of T when in the framework of $\mathscr{C}_{n+1}$ predictions can be formulated that are impossible in the framework of $\mathscr{C}_n$ and some of these predictions have been verified. □

The notion of empirical reduction, introduced by Rosaler in [10], is then important for comparing a categorical model $\mathscr{C}$ of an abstract physical theory T to an existing scientific theory T'.

Definition 4.2. Let $\mathscr{C}$ be a categorical model of an abstract physical theory T; then $\mathscr{C}$ **reduces empirically** to an existing scientific theory T' if and only if for every experiment that has confirmed a prediction of T', the experimentally successful predictions of T' can be reproduced by $\mathscr{C}$. □

Note that *only* the empirically successful predictions of T' have to be reproduced by $\mathscr{C}$. Also, note that T' does not have to be an axiomatized theory: Def. 4.2 holds for a scientific theory in the sense of a generally accepted body of explanatory principles that has been tested by the scientific method—in that sense, e.g. quantum electrodynamics (QED) is a scientific theory although it is not axiomatized. With that definition of empirical reduction, we can think of an abstract physical theory T as a **unifying scheme** if it has a categorical model $\mathscr{C}$ that reduces empirically to two existing theories T'_1 and T'_2 that are known to be irreconcilable. For example, suppose the EPT would have a categorical model that reduces empirically to QED and general relativity (GR): although a unification of QED and GR—in the sense of a single theoretical framework in which QED and GR are both universally valid—is impossible, the EPT would then be a unifying scheme with the unifying principles (the physical axioms of the EPT) at a more abstract level.

The natural **negative heuristic** is to refrain from developments that are inconsistent with the physical axioms of the theory T in the hard core. For example, the EPT is inconsistent with standard quantum mechanics (QM): then there is no point in attempting to develop a categorical model of the EPT that *unifies* the EPT and QM. Note that this is something else than developing a categorical model of the EPT that *reduces empirically* to QM!

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