Bunge's Mathematical Structuralism is not a Fiction

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1. Introduction

Bunge's position on the ontological status of mathematical objects has been clear from very early on and has not essentially changed since his first publication on the subject: mathematical objects do not *really* exist[2]. This negative claim is constant throughout his work and follows directly from his materialist outlook. But as Bunge himself has noticed many times, negation comes cheap. And one and the same negative claim can be consistent with many different, incompatible positive theses. Bunge's positive claims, for there is more than one, are somewhat more difficult to put together into one coherent position. Sometimes, Bunge says that mathematical objects *formally* exist. At other times, he emphasizes that they are creations of the human brain and thus, exist only in these brains. Therefore, their mode of existence is a mode of dependency: the existence of mathematical objects depends upon the existence of brains, human brains in communities. At other times, Bunge says that mathematical objects are *fictions*. The goal of this paper is to explore further the type of existence that characterizes mathematics according to Bunge and see whether and how these positive views can be put into a coherent whole.

2. Mathematics as a Science of Structures

Bunge's most explicit and complete presentation of his views on mathematics are in volume 7, part I, of his *Treatise*. (Bunge 1985) I will not systematically go over the basic elements of his position. (See Marquis (2011) for an overview and a critical presentation.) I will focus on the relevant components for my presentation to be self-contained.

Bunge claims that contemporary mathematics is a *formal research field*. This means, basically, that its products are the result of a community of specifically trained individuals who share common methods, techniques, theories and who aims at solving a well-identifiable class of problems with those methods, techniques and theories and, by doing so, produce new definitions, theories, proofs, examples, counter-examples and algorithms according to certain standards of rigor[3]. Thus, mathematics is a science, but a

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[2] It seems that the first implicit expression of this claim came in the first volume of Bunge's *Treatise*. (See Bunge 1974.) The detailed exposition appeared in the first part of the seventh volume of the *Treatise*. (See Bunge 1985.) The fundamentals of the position have not changed since then. (See (Bunge (1997) and Bunge (2016).)

[3] In the present paper, 'mathematics' always refer to *contemporary* mathematics, for this is what Bunge has in mind in his analysis.

science whose objects are *constructs* or *concepts.* Since the latter never come isolated, but in systems, mathematics is the study of *conceptual systems* (Bunge 1985, p.19).

More precisely, mathematics studies the *structure* of these conceptual systems. Thus, and as far as I know this point has never been underlined even by Bunge himself, Bunge is endorsing a form of *structuralism* in mathematics. Here is one passage where Bunge explicitly expresses this view, worth quoting in full:

> Historians of mathematics have noted that, until around mid-nineteenth century, the bulk of mathematical research was concerned with individual constructs, such as particular figures, equations, functions, or algorithms. From then on, and particularly since the mid-20th century, mathematics has been conceived as the study of *conceptual systems*, such as groups of transformations (or even the whole category of groups in general), families of functions (or even entire functional spaces), and topological spaces (such as metric spaces in general). (Caution: Bourbaki, Bernays and others call 'structure' what others, e.g. Hartnett 1963, call 'system'. We stick to our convention … that every structure is the structure *of* some object: that it is the set of all relations among its components – the internal structure – plus those among the latter and the environment or context of the system, which can be empty – the external structure.) (Bunge 1985, p.19)

These systems take a more precise form in mathematics:

> Every mathematical system ("structure") can be characterized in either of two ways: (a) as a set equipped with a structure consisting of one or more operations or functions defined on that set (e.g. Bourbaki 1970); (b) as a collection of objects together with one or more morphisms relating those objects – i.e. a category (e.g. Mac Lane 1971). Actually the second concept subsumes and supersedes the first.) (Bunge 1985, pp.19-20)

At first sight, Bunge seems to be endorsing either a kind of set-theoretical structuralism or a category-theoretical structuralism, though the foregoing quote suggests that he favors a category-theoretic point of view. In fact, the choice between them does not seem to be an issue. Bunge falls back on the current languages of mathematics, his position being that mathematics is about conceptual systems, no matter the particular language used. Indeed, Bunge always comes back to conceptual systems and never develops the set-theoretical or the category-theoretical perspectives. Thus, on the very next pages of the *Treatise*, one reads:

> A moment ago we stated that mathematics studies conceptual systems ("structures"). However, this is only a necessary condition: philosophy and the history of ideas too study conceptual systems, such as cosmologies and mathematical theories. (Bunge 1985, p.21)

This is an odd claim to make, considering that mathematics studies the structure of conceptual systems. Does philosophy and the history of ideas study the structure of conceptual systems *as such*?

> What makes the mathematical study of conceptual systems unique is that (a) it is *purely conceptual* (i.e. does not make essential use of any empirical data or

> procedures) and it involves, at some point or other, (b) *positing or conjecturing the laws* (general patters) satisfied by the members of those conceptual systems, as well as (c) *proving or disproving conclusively* some such conjectures (…).
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> We may then define *contemporary pure mathematics* as the investigation, by conceptual (a priori) means, of problems about conceptual systems, or members of such, with the aim of finding (…) the patterns satisfied by such objects – a finding justified only by rigorous proof. (Bunge 1985, p.22)

There is no mention of sets or categories in this definition. There is no mention of mathematical objects. Notice the very last claim: "finding … the *patterns* satisfied by such objects". We are back at structures. Mathematics is the study of the structures of conceptual systems. It is not entirely clear how mathematics differs from logic according to this definition. Bunge would probably fall back on sets or categories at this point, since these particular concepts allows us to move away from pure logic.

Why has Bunge not developed more explicitly a form of structuralism? In particular, why does Bunge fall back so quickly on mathematical *objects*? On page 23 of the same volume, for instance, we read: "The last problem in our agenda in [sic] whether mathematical objects are discovered or invented." Shouldn't Bunge ask whether *structures* are discovered or invented? Clearly, Bunge does not want to reify structures. As we have seen already, they are always structures of systems and systems are fundamentally made up of objects. That is a basic axiom of his ontology. In the case of conceptual systems, concepts are the objects. Clearly, a conceptual system is itself a concept and, thus, we are not really reifying structures. We are still talking about concepts and constructs. Furthermore, could it be that mathematics *is* formal, precisely because it deals with structures or relations *as such*? Of course, in the end, we will still have to turn to issues of existence, namely the existence of constructs or concepts. We would have nonetheless avoided the whole discussion surrounding the status of mathematical *objects*.

3. Mathematics as an Art?

Indeed, bringing mathematical objects in the picture suddenly brings contemporary mathematics closer to art than to science! To wit:

> ...in our view, mathematics is closer to art than to science as regards its objects and its relation to the real world, as well as regards the role of truth. (Bunge 2006, p.195)

Mathematics is closer to art than to science from the ontological, epistemological and semantical perspectives! Logic seems to be saving the day… Mathematics is decidedly a singular research field. Although the epistemological and the semantical components would deserve a careful treatment, I will simply ignore them in the present paper. From now on, only ontology matters.

Ontologically, mathematical objects and "artistic objects" are on the same plane.

> Mathematical objects are then ontologically on a par with artistic and mytholological [sic] creations: they are all *fictions*. The real number system and the triangle inequality axiom do not exist really any more than Don Quijote or Donald Duck. (Bunge 1985, p.38)

We moved from the structure of conceptual systems to mathematical objects and Donald Duck. Again:

> In short, mathematicians, like abstract painters, writers of fantastic literature, 'abstract' (or rather uniconic) painters, and creators of animated cartoons, deal in fictions. To put it into blasphemous terms: ontologically, Donald Duck is the equal of the most sophisticated nonlinear differential equation, for both exist exclusively in some minds. (Bunge 2006, p.192)

Donald Duck is not the problem. And it is not *a priori* ridiculous to compare Donald Duck to mathematical objects with respect to their ontological status. It is, in fact, rather fashionable these days and has been for some time. It certainly goes in the right direction, but one has to travel carefully to avoid certain pitfalls.

Of course, mathematics is *not* a form of art, despite the foregoing ontological, epistemological and semantical closeness to art. Bunge gave a list of ten differences between the two in numerous publications. (See Bunge 1985, pp.39-40, 1997, pp.63-64 and 2006, pp.204-205. In the last publication, the list contains a few more elements. These were mentioned in the main text of the previous versions. The core of the list has not changed at all between these publications.) We will not go over the differences presented by Bunge. Suffice to say that one element, already mentioned, stands out: Bunge insists on the *necessary* role of reason through logic in mathematics. Thus, it is tempting to say that one of the main differences between mathematical objects and artistic fictions is that whereas both are human creations, products of the imagination, the former is strictly bound by reason in its creation, developments and justification and the latter is not. It is this central role of reason that brings mathematics closer to the scientific territory. Reason is at the core of mathematics. Mathematics *must be* rational. It is the only conceptual domain, together with logic, that *can be fully and autonomously* rational. Even mathematical existence is bound by reason. Rationality is built-in. This is not to say, of course, that mathematics *is* logic. In contrast, artistic fiction does not have to be and, perhaps, cannot be[4] fully and autonomously rational[5].

4. Real Fictions and Mathematical Fictions

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[4] It could certainly be said that art *should not* try to be. An interesting question is where philosophy stands in this framework. Philosophy does not have the same conceptual autonomy as mathematics.

[5] This is not to say, of course, that mathematics and art have nothing in common. Historically, mathematics has been associated to a *technè* and I, for one, have argued that a large part of contemporary mathematics should be thought of as a systematic technology. (Marquis 1997, Marquis 2006) I am here concentrating on the idea that mathematical objects and certain artistic objects, mostly literary 'objects', should be subsumed under the ontological category of *fictions*.

I want to focus the claim that mathematical objects are fictions. As such, this is highly ambiguous and could mean many different things[6]. Do fictions differ from ideal objects? Abstract objects? Imaginary objects? Does Bunge use the term in a different sense than, say, Leibniz when the latter talks about the fact that infinitesimals are fictions? In light of the last paragraph, should Bunge develop a philosophical theory of *logical* fictions and say that mathematical objects are logical fictions? This would make mathematical objects a special *kind* of fiction. I submit that, in fact, the idea of fiction does *not* play a central role in Bunge's philosophy of mathematics and that it could very well be dispensed with[7].

The use of 'fiction' in Bunge's philosophy has two main purposes and they are, in my mind, rhetorical. The first one, already mentioned many times, is to point out that mathematical objects and fictions might very well be *only* constructs, creations of human brains. The second one is to fall back on pretense, on our capacity to treat certain concepts *as if* they had an autonomous existence, thus explaining the prevalence of various forms of Platonism among mathematicians.

> And in a widely publicized interview, the Princeton professor William Thurston stated that "Theorems just kind of exist, you know, just like mountains do". In our view this is an intelligent mistake. It is a mistake because formal existence is radically different from material existence: …. But it is intelligent because, as a matter of fact, the mathematician behaves in many regards *as if* constructs existed by themselves. He can do so because mathematical constructs, though human creations, do not bear the stamp of their creators: they are impersonal or intersubjective (though not objective). (Bunge 1985, p.111)

Thus, Bunge wants to be able to resolve what seems to him to be a 'tension' between two poles: (1) the fact that mathematics is a creation of human brains *and*; (2) the fact that so many mathematicians describe mathematics and its objects as being totally independent of these human brains. Since everybody is familiar with fiction, at least literary fiction like stories, novels and plays, and that fictional character seem to typically resolve that kind of tension – they are undoubtedly created by humans *and* have a certain kind of autonomy –, it appears to be a convincing solution. *Voilà!* End of the story.

Be that as it may, Bunge does not rely on a theory of fiction to clarify the nature of existence of mathematical constructs. He does say quite a few things about existence and existence of mathematical constructs. It *is* surprising that he does not attempt to *derive* the desired properties of mathematical constructs from his conception of formal existence. That would allow him to avoid all reference to fictions, which, in the end, he

[6] Already in 1981, before the publication of Bunge's volume 7 of the *Treatise*, Roberto Torretti had already identified three different kinds of mathematical fictionalism. (Torretti 1981) I must confess that I do not understand his classification and will therefore refrain from using it. His claim that Bunge's position might, in the end, be a form of idealism, is, however, not ridiculous. See also Robert Thomas' excellent papers on fiction and mathematics. (Thomas 2000 and Thomas 2002)

[7] In this particular regard, Bunge's position is not very different from what is now called 'mathematical fictionalism' in the literature. More about this link or, to be more exact, its absence, in the next section.

seems unable to avoid for the reasons just mentioned. There seems to be something wrong with his notion of conceptual existence. Let us see.

5. Modes of Existence

Bunge has always resisted attributing any ontological traction to the existential quantifier and, in particular, Quine's approach according to which 'to be is to be the value of a bound variable' in a well-establish scientific theory[8]. He has steadfastly defended the idea that existence had to be represented by a specific predicate and, in contrast with a large literature in contemporary metaphysics, he has also claimed that there are different modes of existence.

To be real, for Bunge, is to be material or, in other words, it is to be mutable or changeable. This is his definition of *real existence.* Bunge used to oppose the latter to *formal existence.* However, he has recently introduced five different modes of existence: real, phenomenal, conceptual, semiotic, and fantastic. (See (Bunge 2016).) Although it might appear to be a modification of his views, the basic distinction remains between the first mode and the four remaining modes: only the first mode, real existence, is absolute and context-independent; the others are relative and context-dependent. Iset aside the real, the phenomenal and the semiotic and focus on the conceptual and the fantastic.

It is worth recalling Bunge's definition of a construct, as presented in the third volume of the *Treatise*.

Definition: *x* is a *construct* if, and only if,

(1) There exists (really) an animal capable of conceiving *x*;
(2) The animal conceives *x* as a conceptual system or a member of such.

In other words, a thing is a construct if it can be thought by an animal as a conceptual system or in a conceptual system. Bunge treats this as if it clarified the notion of formal existence. As far as I can tell, this is only a definition and says nothing about existence.

A series of remarks is in order. First, the definition is two-dimensional. The notion of construct *depends* upon two ontologically perpendicular realms: 1. The realm of really existing animals and 2. The realm of other constructs.

Second, the definition presupposes distinctions between conceiving, thinking, feeling, imagining, desiring, etc. It is not clear, at least to me, that what I imagine is not a construct. If it is not, what is it? Are internal visual images constructs in Bunge's sense? Internal musical melodies? The last two do not belong to conceptual systems. Or, do they? In what sense?

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[8] The latter criterion is at the source of the vast literature on mathematical fictionalism. (See, for instance, Field (1980), Field (1989), Balaguer (1998), Yablo (2000), Leng (2009).) Indeed, this criterion together with the so-called Quine-Putnam indispensability argument, seemed to provide good reasons for a certain form of Platonism with respect to mathematical objects. In this context, the claim that mathematics is a fiction is taken to follow from the claim that mathematics, like fiction, is not literally true, precisely because in both cases, these discourses literally fail to refer. Bunge has always resisted these Quinian arguments and he also very quickly pushed aside these fictionalist strategies, which he considers to be forms of nominalism and finds inadequate. Lately, Quine's arguments have been criticized and therefore the motivation for this form of mathematical fictionalism has somewhat shifted. See, for instance, Thomasson 2014.

Third, if it is taken as a definition of formal existence, I am not sure I see what we can infer from it. Since it is plausible to imagine that as soon as a human being capable of thinking was alive, that human was *capable* of conceiving mathematics, can we conclude that all mathematical concepts came into existence from that moment on? It all depends what one means by 'capable of conceiving' and 'conceived'. And, of course, no one can verify that mathematical objects came into existence that way.

Fourth, should we add that there was at least one animal that actually or really conceived a particular concept, at least once? Is it enough that only one animal can conceive it? Does it have to be communicated? Or communicable? If so, what does that presuppose, cognitively, culturally and socially?

Fifth, the animal has to conceive the construct x as a system or part of a system. Is it possible *not* to? What would that mean? How is that possible? The notion of an isolated construct seems to be an oxymoron.

Sixth, what about certain concepts that are part of mathematics but that seem to go beyond our capacities of conceiving them at a certain time? The history of mathematics is filled with examples of such constructs: 0, negative numbers, imaginary numbers, non-Euclidean geometries, higher-dimensional geometries, infinities, both great and small. For long periods of time, our best minds thought they had good reasons to doubt the existence/legitimacy of these mathematical objects, even though they were *capable* of conceiving them. For most of these objects and constructs, we now have fully developed theories and very few mathematicians (none?) would nowadays contest the existence of negative or complex numbers, not to mention quaternions, octonions, p-adic numbers, etc.

Seventh, what happens when what seemed to be a perfectly good theory turns out to be inconsistent? Do the constructs of that theory suddenly fail to exist? How does the discovery of an inconsistency affect formal existence? Do we have to suppose consistency?

Eight, to be capable of conceiving x, very often one has to master various cognitive tools, e.g. a written language, certain preliminary concepts y, z, etc. Thus, the capacity to conceive something might in fact depend on a complex network of concrete and conceptual technologies. Whence, conceptual existence might depend on much more than the existence of an animal and its cognitive capacities.

Ninth, one could actually imagine that the (independent) existence of these concrete and conceptual technologies could play a role in the conviction that the constructs that one finally conceives have an independent existence, simply because their existence depends upon a complex network of preexisting concrete and formal entities whose existence is hard to comprehend by a single animal.

Bunge then gives a specific definition of *mathematical existence*: if x is a construct, then *x exists mathematically*$=_{\mathrm{df}}$For some C, C is a set, class or category, such that (i) x is in C, and (ii) C is specified by an exact and consistent theory. (Bunge 1985, p.30)

We now have our answer for objects belonging to an inconsistent theory: they do not exist mathematically[9]. Bunge now fixes an underlying mathematical ontology, to be

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[9] Bunge is of course well aware that we cannot *prove* the consistency of most of our mathematical theories, in particular set theory and for a foundational categorical theory. Should we conclude that we simply cannot *know* that, in the end, our mathematical constructs exist?

(mathematically) is either to be a set, class or a category or be in such an object (in a coherent fashion). We have moved from the existence of constructs *as such* to the existence of *mathematical* constructs *in mathematical* theories. These are very different cases and, from the philosophical point of view, the latter is well understood by mathematicians. Mathematicians know what they mean when they claim that such and such exist in such and such theory. There are *some* debates as to the methods that are legitimate to establish this existence of *some* objects and the nature of these debates is well understood, e.g. infinitely large cardinals for intuitionists and infinitesimals for some classical mathematicians[10]. It is, so to speak, an internal affair. Recently, Bunge has proposed that to exist conceptually means to be a constituent of a conceptual system. (See Bunge 2016, p.228) What about conceptual systems themselves? I suspect that Bunge's answer is the same as the one given in the *Treatise*: conceptual systems are usually parts of larger conceptual systems thus they exist whenever the latter is the case. Surely, this chain must end somewhere: there must be a conceptual system that contains them all. Bunge says that the latter question "makes no more sense than the question "Where is the physical universe?"" (Bunge 1985, p.30) The question of an overall mathematical conceptual system is *the* question of the foundations of mathematics, which we will leave aside.

At the end of the day, Bunge falls back on his fictionist stance. Thus, in his most recent publication, he claims again that "from the fictionist viewpoint, the debate over constructivity is a storm in a teapot. Indeed, whether or not there is a constructive proof of a given mathematical object, this is just as fictitious as Zeus or a talking dog." (Bunge 2016, p.230) Thus, provided that we keep in mind the constraint brought by consistency, mathematical existence boils down to fantastic existence. According to Bunge, something exists fantastically if there is a work of fiction that contains or suggests that thing. (Bunge 2016, p.231) Bunge is clear that this type of existence includes music, plastic art, artistic cinema, as well as mathematics. Indeed, Bunge claims that "mathematicians and theoretical physicists are professional fantasizers. But their fantasies, unlike those of Hyeronimus Bosch or Maurits Escher, are bound by reason." (Bunge 2016, p.232) Is Bunge being merely provocative here or does he *really* believe that mathematicians and theoretical physicists are, to use an image, rational writers?

I claim that Bunge does not need to invoke fictions at all and that, by doing so, he brings in unnecessary difficulties. Bunge *is* right that fictions and mathematics are both *abstract artifacts* and, as such, have an ontological status that differs from, say, electrons and trees. Thus, we preserve the basic ontological claim that the existence of mathematical concepts is a *dependent* existence. This, of course, is true of all abstract artifacts. I claim, however, that the *type* of dependence of mathematical concepts is not the same as the type of dependence of fictions. Moreover, we do not need the pretense, the *as if*, to make sense of their existence and their properties. One only needs to understand mathematics, how it is learned and how it is done, without any pretense.

6. Doubts about Fictions

[10] It is well known that Cantor and Russell resisted the introduction of infinitesimals for purely ideological reasons, even when they were perfectly acceptable objects in algebra at the time. See Ehrlich 2006.

There is a huge literature on fictions and their properties which I cannot do justice to in such a short paper. There is, however, one objection brought forward recently by Amie Thomasson that hits a soft spot.

When we deal with fictions, we do indeed *pretend* that certain objects or events are such and such. As I have indicated, this is one of the reasons Bunge appeals to fictions and draws a parallel between mathematical objects and fictions. For instance, when we go to the opera to see a performance of Puccini's *La Bohème*, we pretend that what we see is taking place in Paris around 1830. Of course, we are not committed to the claim that we actually *see* Paris on the stage. We pretend that the singer who plays *Mimi* dies, etc. We are willing to say that Mimi does die *in the opera*, although we know that the singer impersonating her does not. We are not committed to the *actual* death of the singer. What this pretense amounts to and how it varies has been analyzed in the literature and we do not need to go into details for our purposes. If we can pretend that someone dies on a stage, then we can certainly pretend that the Monster Group exists on some stage, so to speak. So far, so good.

The objection then goes as follows: when dealing with a fictional discourse, we *know* when we pretend and we know when we do not. In other words, we know which parts of the discourse have to be taken *as if* they were true and which ones are to be taken literally, e.g. that Puccini composed *La Bohème* is literally true, that Mimi dies is not[11]. Not knowing how and when to make the difference is a sign of psychopathology. If mathematical objects are fictions, we should be able to make similar distinctions in this case too. The fact is, we do not. Here is how Thomasson herself puts the objection:

> In the case of works of fiction or children's games of make-believe, there is a clear contrast to be drawn between committing oneself to the real content (the truth about the props) and committing oneself to the literal content: a difference between being committed to stumps versus bears, words on pages versus deaths on train tracks. That difference, however, is not obvious for the fictionalist about disputed ontological entities such a social entities, numbers, events, and properties. Committing oneself to the vows and paperwork being undertaken does seem to commit oneself to being married. Similarly, to the extent that it sounds redundant in English to say "there are five stumps *and* the number of stumps is five", being committed to the first claim does seem to commit one to the second, and so to there being a number. … But then we cannot (…) take the latter claims, explicitly about numbers or propositions, to be *merely* pretending while the former are committing. (Thomasson 2014, p.190)

In other words, in the case of natural numbers, the Monster Group, the homotopy type of the 3-sphere, is there a difference between the pretense and some underlying, literally true discourse? When I read about the Monster Group, I don't say to myself "I know that there is no such thing as the Monster Group, there is only these marks on paper. But the story (theory) says that such and such is true and I am willing to pretend that it is true, although I know that it is not." In fact, Bunge is suggesting that most mathematicians, at least those like Thurston, are pretending, but are not aware of the pretense. In the

[11] Of course, that is one of the reasons why art is so powerful: even though we *know* that it is all a pretense, we*feel* emotions just as strongly as when it *is* real. Some people simply *cannot* watch horror movies, although they *know* they are watching movies, i.e. fictions.

foregoing quote, Bunge says that mathematicians *behave as if* mathematical objects existed by themselves. Thus, Bunge explains mathematicians' behavior by attributing them an attitude towards the objects of their thought. Is this some sort of anthropological explanation? We study how mathematicians behave and the best explanation we come up with is that they pretend that what they talk about really exist. When we question them, they are not aware of this pretense. Notice that Thurston did not say "I know that mathematical objects are not like mountains, but I pretend they are. It allows me to do beautiful mathematics." However, he *did* say "*kind* of exist *like* mountains", indicating that he is well aware that there *is* a difference between the two, an obvious difference.

There is an additional puzzle. There *are* mathematicians who claim that mathematical objects are constructed and mental entities. They might even accept the claim that mathematical objects are fictions. However, I suspect that they would resist the statement that we thus pretend that they exist autonomously and can therefore be treated like any ordinary or real objects, that is ordinary classical logic can be used without restrictions. Imagine asking an intuitionist: why can't you pretend that mathematical objects exist autonomously, like all other mathematicians do, and use classical logic? Imagine telling an intuitionist: you might have not noticed this, but your classical colleagues do not *really* believe that mathematical objects exist, they only *pretend* that they do. As far as I am aware, neither the intuitionist nor the classical mathematician *decide* to pretend or know that they *have to* pretend.

I don't know what it would mean, in the case of mathematics, to stop pretending and fall prey to the literal interpretation of mathematical discourse (which, for ordinary fiction, leads us towards psychopathology). Would someone start attributing *real* properties to mathematical objects? Would the Monster Group be *really* frightening? We never have to tell our children "well, you know, we are sorry, we never told you this, but in fact, numbers do not *really* exist", whereas those of us who have decided to do as if Santa Claus existed had to have a conversation or at least make a verification at some point that our children have picked up on reality. Some children are *really* disappointed to learn that Santa Claus does not exist. It is an interesting thought experiment to imagine a child crying after learning that numbers do not *really* exist. In fact, no one has ever feigned that numbers *really* exist! What would *that* mean[12]?

Thus, at the very least, Bunge has to tell us how pretense works in the case of mathematical objects and how it differs from how pretense works in the case of ordinary fictions. If we are to explain the behavior of mathematicians by saying that they pretend that mathematical constructs exist by themselves, we have to be able to say how this pretense comes about, how it works and why it is the best explanation for that behavior.As far as I know, Bungenever went further than to suggest that one *could* explain this behavior that way. Perhaps one can. Perhaps there is a simpler explanation,

[12] That seems to be an easy exercise in Bunge's framework. If we were to feign that mathematical objects *really*existed, then it means that mathematical objects could be in various states. What exactly these states would be, that would have to be determined. Would they be more like physical objects or living organisms? It is up to your imagination to decide.I suspect that in some cases, the 'reality' would be expressed more in terms of an independence from the mind, the will of the subject, in contrast with the objects that we create. But this shows, once again, that if we pretend that mathematical objects really exist, we do it in a very selective fashion without having learned anything about it.

and still within a materialist framework. We have to go back to the notion of dependent existence.

7. Abstract Artifacts and Varieties of Dependent Existence

To understand the various types of dependent existence, let us stick with fictions and literary works a little further. Amie Thomasson has developed an interesting theory of dependent existence that she has applied to clarify the ontological status of fictional characters and fictional works, among other things[13]. (See Thomasson 1999.)

According to Thomasson, the existence of a fictional character depends on (1) the creative act of a (really) existing author or authors and (2) on the existence of a literary work. In turn, the existence of a literary work depends on the acts of its creator or creators, but it also depends on some copy or memory of it and a (really) existing competent reader.Thus, the structure of dependence of a fictional character is a complex network of real existents, acts and intentions. In Thomasson's words, fictional characters "should be entities that depend on the creative acts of authors to bring them into existence and on some concrete individuals such as copies of texts and a capable audience in order to remain in existence." (Thomasson 1999, p.12)

There are subtle points that we need not go into for our purposes[14]. The following remarks will suffice. According to this analysis, fictional characters have a history: they are born in certain historical, cultural and social contexts and by the act of a real human or a group of humans. In the case of literary fictions, there are writings or oral traditions, more generally literary works, in which the character first appear. It is important to notice, however, that for the character to be, that particular object, the original literary work, does not have to survive. Any copy or any faithful memory of it will do and will allow for the fictional character to exist. There is thus a certain independence from *particular* and *specific* real objects for them to exist and continue to exist. We all recognize that they can survive their creator and even the original book or work by which they were introduced. It is obvious that, according to Thomasson, fictional existence does not amount to the capacity of a human to conceive a fictional character. There has to be a creator (or creators) who not only creates the character, but does so by doing, building something, namely what we call a literary work. For the latter to be possible, one has to have a language and, in most cases, a written language (although one can argue that the latter is not necessary, as the various oral traditions clearly indicate), together with specific cognitive capacities, for instance a powerful enough memory (where, in most cases, one will find mnemonic tricks to help remember the stories) or the capacity to read a certain language and the latter has to be possessed by other humans afterwards. There is, clearly, an intrinsic *social* component at work in this picture, since the existence

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[13] Thomasson seems to have move away from the specifics of her earlier theory. I stick to it simply because it provides an ontological analysis of fictions as dependent entities, thus an analysis that is close to Bunge's claims. I am *not* claiming that it is the most adequate analysis. In fact, I would be inclined to address these issues more in the spirit of Thomasson's recent work. That is another matter.

[14] Thomasson offers an interesting and rich classification of artefacts based on certain properties of the dependence relation. We refer the reader to her book for more.

depends on more than one human and even some cultural elements, which goes hand in hand with (neuro)*biological* capacities[15].

This takes care of most of our intuitions about the existence of fictional characters. Thomasson introduces a more general framework to treat the different kinds of ontological dependence. It is worth looking at the dependence relations that she uses.

Thomasson's ontological categories

In her theory, Thomasson offers an explication oftypes of ontological dependence by combining four notions of dependence: rigid dependence, generic dependence, constant dependence and historical dependence. Thomasson introduces the distinction between constant dependence and historical dependence as follows:

> We can begin by distinguishing constant dependence, a relation such that one entity requires that the other entity exist at every time at which it exists, from historical dependence or dependence for coming into existence, a relation such that one entity requires that the other entity exist at some time prior to or coincident with every time at which it exists. There are not all of the different possible cases of dependence but merely describe some of the most interesting and general cases of dependence. (Thomasson 1999, p.29)

Clearly, historical dependence is weaker than constant dependence. In other words, if *x* is constantly dependent on *y*, then *x* is also historically dependent on *y.* As we have already indicated, fictional characters historically depend on a creator or a group of creators to be, but they do not constantly depend on that creator. Examples of constant dependence are numerous. If consciousness is an emergent property of brains, then my consciousness is constantly dependent on my brain.

When the relation of historical dependence rests on a particular individual or a particular group of individuals, Thomasson qualifies this relation as being *rigid.* This qualification can be applied both for the constant case and the historical case. For example, I am rigidly historically dependent on my parents and *La Bohème* is rigidly historically dependent on Puccini's existence. My consciousness is rigidly constantly dependent on my brain. There are relations of dependence that are not rigid, but rather *generic.* In this case, the relation does not depend on a particular, singular individual, the latter being understood in a broad sense. To use Thomasson's example, "a given sample of alcohol is rigidly historically dependent on the sugar from which it is formed, it is merely *generically* historically dependent on some yeast (or other appropriate catalyst)." (Thomasson 1999, p.33) An example of a generic constant dependence is provided by the existence of a University. At any moment that the Université de Montréal exists, there must be persons who work at this particular institution, people who teaches, do research

[15]Otavio Bueno has sketched a form of mathematical fictionalism based on Thomasson's views that is strikingly close to Bunge's. Bueno defends the idea that mathematical entities are *like* fictional characters since, according to him, they are created in a particular context and in a particular time and their existence depends upon the existence of written papers and competent readers. He even adopts an existence predicate and distinguishes it from the existential quantifier. However, in the end, his position differs both from Bunge's position and from Thomasson's. We cannot do it justice in such a short paper. See Bueno 2009.

and other people that are registered as students, who attend classes, go to the library, etc. Of course, there is no particular person whom the University's continued existence requires.

Notice that Thomasson is well aware that there may be other cases of dependence and she does not claim to cover all possible cases[16]. The strongest relation of dependence is the category of rigid constant dependence (*RCD*). It entails all the others. Thus, it entails the categories of rigid historical dependence (*RHD*) and rigid dependence (*RD*). This can be pictured thus: $R \to R \to R$. In turn, rigid constant dependence entails generic constant dependence (*GCD*). There is an obvious line of entailments between the generic dependences: the generic constant dependence (*GCD*) entails the generic historic dependence (*GHD*) which, in turn, entails the generic dependence (*GD*). Hence, the complete picture looks like this:

$$\begin{array}{ccccc} R & \to & R & \to & R \\ \downarrow & & \downarrow & & \downarrow \\ G & \to & G & \to & G \end{array}$$

Table 1 Dependence Relations

We are not done. We have introduced the relations of dependence in general. We now specify two types of dependence that are ontologically fundamental for our purposes: the dependence on material or spatiotemporal entities and the dependence on mental states[17]. Each type of dependence yields a two-dimensional grid of 10 categories each therefore the whole space of ontological categories is four-dimensional with 100 possible categories. There is no need to present the whole system here. We refer the reader to Thomasson's book. (Thomasson 1999, chap. 8.)

In a materialist framework, material entities are *not* dependent on mental states, neither rigidly nor generically. Mathematical realists or Platonists would probably claim that mathematical objects, although not material, neither depend on material entities nor do they depend on mental states. Thus, both material and mathematical objects would belong to the category of objects that are neither rigidly dependent nor generically

[16] Thomasson argues that the relations of dependence, constant dependence and historical dependence are all reflexive and transitive. This suggests that the resulting ontology could be formalized using the mathematical theory of categories, by representing the relation of dependence by a morphism between objects. The links between the kinds of dependence can be represented by functors. In fact, the distinction between rigid dependence and generic dependence can also be captured via a specific type of adjunction. This is not surprising given the fact that mathematical functions capture a form of dependence. We even talk about dependent and independent variables. We leave this project for another paper.

[17] We will stick to the terminology of mental states instead of brain states, despite the fact that we are in a materialist ontology. The reason for this choice is that the term 'mental states' already suggests a certain independence from particular brains but still indicates a clear and well-understood realm of discourse. Of course, the question as to how mental states depend on brain states is fundamental and, at some point, we might be ready to talk about brain states. See, for instance, Piazza & Izard 2009. By saying that constructs are *equivalence classes* of brain states, Bunge himself introduces a different identity criterion for mental states than for brain states, thus introducing the possibility of treating them as a genuine category.

dependent on mental states[18]. We are interested in the case where mathematical objects depend on mental states *and* on material entities, for this seems to be consistent with Bunge's claims. Where should they be in this framework?

One of the fascinating aspects of Thomasson's categories is that it opens the door to a multiplicity of ontological categories between the concrete and the abstract, a distinction that is usually considered to be a dichotomy. Of course, the latter depends on how these two terms are interpreted. Thomasson puts the distinction squarely within the space of material dependence. It seems reasonable to claim that concrete objects rigidly constantly depend on themselves. What about abstract objects or entities more generally? An obvious possibility in the present framework is to interpret the property of being abstract as being *historically independent* from material entities, which leaves open the possibility of being *generically* and/or *rigidly* dependent on material entities.

Thomasson herself proposes to take the weakest definition allowed by her system, namely that an entity is *abstract* if it is *not* rigidly constantly dependent on material (spatiotemporal) entities. This still leaves three possibilities: abstract entities can be rigidly historically dependent on material entities, rigidly dependent on material entities or, finally, not rigidly dependent on material entities. In the latter case, being abstract is not one single category. There are now various kinds of abstract entities. Nonetheless, if we accept the foregoing characterizations of concrete and abstract objects, we get back the familiar 'abstract/concrete' dichotomy.

On the one hand, fictional characters depend rigidly historically on mental states, since they rigidly depend on historical actors for their creation. They *also* belong to the category of generic constantly dependent entities, since after their creation, their existence depends on the mental states of agents that are capable of understanding the works in which they appear. This takes care of the dependence on mental entities. On the other hand, fictional characters are part of literary works. As such, they constantly depend on some material entities, i.e. copies of the work, and also rigidly depend historically on certain material entities to bring them into existence, i.e. the particular author who wrote the work at a particular time in a particular sociocultural context. Thus, the literary work of art itself, and the fictional characters it talks about have no spatiotemporal properties, but the work was created at a specific moment by a specific individual (or individuals). It is therefore an abstract object that nonetheless has a dependent existence and the dependence is both mental and material.

Thomasson mentions another case of abstract entity worth comparing with fictions and that will be useful for our own purposes: technological artifacts, like the telephone, the computer, etc. In these cases, it is not one specific object that we refer to, but a *type* of object with a specific *function*. In many cases, the type has many different material instantiations. Their existence is generically historically dependent on mental states, for they did not exist before a certain time, they had to be *invented*, but their coming into being does not depend on one unique and specific individual. The telephone

[18] Thomasson herself distinguishes two opposites: the mental-material and the real-ideal. The first one would be reflected in the space of mental dependence, the material being independent of anything mental, and the second one would be placed in the space of spatiotemporal dependence, the ideal – which, from a Platonic point of view, would include numbers and similar entities – being independent of anything real. Notice that the material and the ideal are thus characterized purely negatively. (Thomasson 1999, p.125)

was invented independently by many different people and so was the computer. There has to be *someone* to do it, but it does not have to be that particular person. Similarly, one could argue, as Thomasson does, that they generically constantly depend on material entities. If we were to lose all known exemplars of eight-tracks cassettes and machines, together with their plans and designs, then this technological artifact would cease to exist.

8. Mathematical Constructs as Abstract Artifacts

What about mathematical objects, or to borrow Bunge's terminology, mathematical constructs? They are undoubtedly abstract in any sense of that word and, thus, in particular in the sense proposed by Thomasson: they lack a spatiotemporal location or, in her terminology, they are not rigidly constantly dependent on any material entity. We can be more precise.

Let me immediately state in Thomasson's terminology what seems to be one of the claims repeatedly made by Bunge: mathematics is at the very least generically constantly dependent on mental entities. Although it does not depend on one particular mathematician for its existence, Bunge claims that mathematics requires the existence of some person capable of understanding it to continue to exist. Note that it is, in principle, absolutely impossible to verify this claim.

Is it generically constantly dependent on material entities? I claim that it is. Mathematics needs to be told, written, drawn, etc. It has always been accompanied by physical embodiments, tools (stones, wooden marks, compass, ruler, abacus, calculators and, nowadays, computers), symbolic system and notational devices of all kinds. Of course, it does *not* depend on one *particular* such material entity, but it *does* depend socially and culturally on the presence of symbolic representation in one form or another. Thus, if humanity were to disappear and all writings, marks, concrete models and mathematical tools were to disappear with it, mathematics would cease to be. Note once again, that it is, in principle, absolutely impossible to verify this claim.

I claim that there is an important difference with the case of technological artifacts described in the preceding section and it has to do with the role or the kind of dependence at play between the material entities involved in mathematics. The material objects embodying technological artifacts *are* real, genuine exemplars of these artifacts. A real, concrete and functioning turntable *is* just a turntable: it is a *real* token of the type. What I hold in my hand *is* a screwdriver. It was invented, build just to do what it does. It is a real token of the type. A drawn triangle is *not* a triangle, a constructed wooden dodecahedron is *not* a dodecahedron. The symbol 'π' is not the number π. No symbol, no sequence of digits, even thought of as a type, can be the number π, since the latter is irrational and transcendental. A written proof of a theorem is *not* the proof. The latter is an abstract object. There are no *real* token of mathematical types.

It is of course because of these facts that mathematical objects are traditionally considered to be *ideal* objects. Their dependence on material entities is of a different nature. We move away from technological artifacts and we move back to fictions. Most of mathematics is *written* and presents itself as a text. It can also be told, usually next to a blackboard or a piece of paper, thus with some written marks. Some mathematical texts are more akin to musical partitions in the sense that the notation tells you how to *do* things: it has to be performed, in the case of music, on an instrument, in the case of mathematics, in one's head or on a piece of paper. But one does not read a mathematical text like one reads a short story or a musical partition. When reading a mathematical paper, one usually needs a pen and a piece of paper and writes as she reads. While someone can take notes while reading a short story for various reasons, it is rarely in order to *understand* the story that someone would do it.

This brings us to the question of the rigid historical dependence of mathematics on humans and here we touch upon an important difference with fictions. Many mathematical theorems, proofs, constructions, theories, algorithms, etc.are identified by the name of the mathematician or mathematicians that introduced them: Gauss fundamental theorem of algebra, Gödel's incompleteness theorems, Wiles proof of the Shimura-Weil-Taniyama conjecture and, as a consequence, of Fermat's last 'theorem', Hamilton's quaternions, the Hopffibration, Grothen diecktoposes, Buchberger's algorithm, etc. It would be very easy to extend the list almost indefinitely. Still, although for sociological reasons the correct attribution of these items to their creators is important, they are nonetheless often thought as being ultimately independent of their creators, in contrast with, say, Offred in Margaret Atwood's novel *The Handmaid's Tale*.

I submit that this behavior, to go back to Bunge's terminology, is attributable to fundamental aspects of mathematical knowledge. First, a concept is sometimes introduced independently and differently by different mathematicians. These various presentations of the concept are then shown to be 'the same', in an appropriate sense of the latter term. One standard example of that phenomenon is the notion of computable functions. This shows that, in some cases at least, the notion of dependence is generic instead of being rigid.

Second, as already observed by Robert Thomas, a mathematical paper does not describe a series of events that happen in some possible space. It is not a narrative in the usual sense of that word. (See Thomas 2000 and 2002.) Writing a mathematical paper is not like writing a fiction. The language of mathematics is such that it is possible for a mathematician to write a paper and erase any trace of a narrator or any reference to mental or material entities. Reading a mathematical paper is not like reading a fiction. In a sense, reading a mathematical paper is comparable to reading a blueprint: it guides you through concepts, examples, constructions, statements, proofs, computations, etc. via certain conventional signs and notational systems. Any competent mathematician who masters the concepts and the language used in a paper or a conference *reconstructs* n her mind the conceptual system that is presented. The particular way of describing that conceptual system does not matter (although the particular language might). Once it is understood and mastered, the mathematical content is in some sense *entirely* assimilated by the reader. It is from then on his or hers. Completely. There is nothing that seems idiosyncratic to the author, there is nothing that escapes the reader. What is more, the content can be *completely* represented in a different manner, even rewritten in a different language or framework. It can be extended by following its necessary logical conclusions. It can also be enriched, transformed, generalized, abstracted, applied, etc. Although a different writer can very well extend a given work, as we have convincingly seen in the latest extensions of Larsson's *Millenium series* by Lagercrantz, the new writer still has to adopt a style, characters, a history, conventions, etc. to go on and, as the example shows clearly, Lagercrantz's remarkable achievement cannot be presented as the *logical* and *necessary* extension of Larsson's work, nor can it be said that he has generalized it, or abstracted from it, or applied it, or expressed it by using different concepts.

Third, the natural numbers and elementary geometry have a special epistemological status which contribute to the conviction that mathematical objects have a distinctive ontological status. According to recent research in cognitive science, innate, non-linguistic, and universal cognitive capacities underlie the development of the natural number concept and of elementary geometry. (See, for instance, Dehaene et. al. 2011, Dillon et. al. 2013.) The fact that these capacities are innate, universal and non-linguistic and that they serve as the cognitive bedrock for numbers and geometry certainly fuels the belief that what we are referring to in these cases is *in*dependent of mental entities or capacities. They are just given. Of course, this reinforces the idea that mathematics is not rigidly and historically dependent on mental entities. However, these findings certainly do not allow us to conclude that number *theory* and geometry as a *theory* are innate, non-linguistic and universal. That is where the dependences kick in and that we start attributing the creation of concepts, conjectures, proofs, calculations, algorithms to particular mathematicians.

These reasons explain why, at the end of the day, mathematicians refrain from claiming that mathematics is rigidly historically dependent on mental entities and even generically constantly dependent on mental entities. The case against a constantly historic dependence on material entities seems too easy to mention.

If I am correct, it is wrong to say that mathematical objects are fictions. In fact, *even ontologically*, there are substantial differences between the two. Thus, *pace* Bunge, mathematical objects are *not* on an ontological par with fictional characters. That could be received as the bad news. The good news is that we do not need to talk about fictions nor do we need to talk about mathematical objects in any deep ontological sense.

9. Structuralism and Mathematical Objects

As it is clear from the definition given in section 5 above, that Bunge tries to combine two different relations of dependence when he deals with mathematics. On the one hand, he clearly believes that constructs in general depend on brain states and therefore mathematical constructs in particular depend on brain states. This relation of dependence leads him to the claim that mathematical objects are *ficta*. On the other hand, mathematical constructs depend on other concepts in a singular way. There is a *conceptual* dependence between mathematical concepts whose nature is unique to mathematics[19]. When Bunge moves to this type of dependence, he switch to mathematical concepts as being conceptual systems or part of conceptual systems. This dualism is in fact inevitable if one wants to develop a form of structuralism *within a materialist framework*. The challenges consist in identifying the correct relations of dependence in both cases and how they should be articulated together into a coherent whole. I claim that the correct *foundational* or *metamathematical stance* is indeed a form of structuralism. This takes care of the conceptual dependence.

As to the dependence on mental/brain states, if one looks carefully at the remarks I have made in the previous section, then it should be clear that 1) there are some

[19]The reader might want to include logic here. That is another issue. For a long time, this conviction was captured by the claim that mathematics is analytic.

existence claims made concerning the ontological status of mathematical objects that go beyond what we can, even in principle, verify and, therefore, I suggest that we simply discard the underlying questions as being pseudo questions and 2) what *can* be said, over and above the internal existential questions settled by mathematicians themselves, can be taken care of by investigating mathematical practice and the human and social sciences, in particular the cognitive sciences, but not only them. Needless to say, I will only make sketchy remarks in this section.

Mathematical structuralism is the claim that mathematics is about structures. This thesis can be spelled out completely, through a metamathematical analysis of the notion of structure. Such an analysis was provided by Bourbaki already in the 1950s in a set-theoretical framework, although it was dismissed by most logicians, philosophers and historians of mathematics for various reasons[20]. A more recent and improved version was presented by the logician Michael Makkai (Makkai 1998.) It is not necessary to present the technical details to understand the basic ideas underlying this form of structuralism.

The fundamental idea is straightforward. For a mathematical theory to be a structuralist theory, it should be possible to prove that the following claim is a meta-theorem: given any property P in the given language $\mathcal{L}$ of the theory T, for all objects X, Y of the theory, if $P(X)$ and $X \cong Y$, then $P(Y)$. In words, a theory is a *structuralist theory* if the provable properties of the theory are only those that are invariant under the proper notion of isomorphism. This says precisely that mathematics is *about* the properties and relations expressed in the proper language and that the underlying objects merely fill in the places to be filled in the relations of the theory. The specific nature of the objects is totally irrelevant. It is in this sense that mathematical objects are not the central concern and that they are always part of a system[21].

10. Conclusion

It is ironic, in the present context, that the appropriate language for structuralism is a logic with *dependent* sorts (or types). This dependence reproduces the fact that a structure depends, for its existence, on previously given objects or systems. We are here dealing with a form of *conceptual* dependence. The appropriate language reflects a clear ontological hierarchy.

Structures themselves are given by a formula φ in a language $\mathcal{L}$. They are then interpreted in a system in which the notion of isomorphism plays *the key* role. Makkai's system covers set-theoretical structures, categorical structures, bicategorical structures, …, *n*-categorical structures, up to ω-categorical structures. Whether one has to go on is an open problem. How the foundational framework has to be develop and what will exist within it has to be settled by mathematicians and logicians. A philosopher must take note and see what follows. But it is first and foremost an internal affair to logic and mathematics.

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[20] One interesting exception is Erhard Scheibe who, in his tribute to Bunge in 1981, presents an analysis of invariance and covariance of physical theories based on Bourbaki's analysis. See Scheibe 1981.

[21] Makkai has developed a structuralist set theory using his framework. We refer the reader to his paper for details. See Makkai 2013 for the technical presentation, and Marquis 2012, Marquis 2018, for more on the philosophical ideas involved in the project.

How about the dependences of mathematical structures on mental/brain states and material entities? As I have already indicated, claims that mathematical structures originate with humans and that they would cease to exist if the latter were to disappear with the material production that comes with it are unverifiable. These claims are intuitively plausible for fictions and technological artifacts. They might be true, they might be false for mathematical structures. There is no way to know and it seems that there are just as many people who believe that they are false than they are true. We submit that it does not matter and that we do not have and cannot have the philosophical tools to settle the issue. They are, of course, consequences of a materialist ontology and, as such, have to be acknowledged. Furthermore, ontological issues pertaining to abstract objects have concrete consequences. What *can* be settled are the types of dependence of mathematics on mental/brain states and material objects in its history, as it is practiced, as it is taught. Questions pertaining to these dependences *can* and *should* be investigated. The answers make a difference to real issues. This should not be a surprise to Bunge's readers: he has been calling attention to the real impact of ontological decisions all along.

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