Mathematical Abstraction, Conceptual Variation and Identity

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ABSTRACT. One of the key features of modern mathematics is the adoption of the abstract method. Our goal in this paper is to propose an explication of that method that is rooted in the history of the subject.

1 Introduction

The main purpose of this paper is to sketch a theory of mathematical abstraction as it appeared in 20th century mathematics. Indeed, mathematics in the 20th century is marked by what is called the abstract approach. To wit:

> One of the amazing features of twentieth century mathematics has been its recognition of the power of the abstract approach. This has given rise to a large body of new results and problems and has, in fact, led us to open up whole new areas of mathematics whose existence had not even been suspected. (Herstein, 1975, 1)

This quote is taken from the opening page of a standard textbook in abstract algebra. I would not say, however, that one of the amazing features of modern mathematics is the "recognition" of its power. For that suggests that the abstract approach existed before the twentieth century and that mathematicians then came to realize how powerful the approach is. In fact, the abstract approach was *created* in the late 19th and early 20th centuries and it was *developed* in the 20th century. The recognition of its power came along with its development.

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Together with its power, Herstein emphasizes the capacity of the abstract approach to bring to existence whole new areas of mathematics. As is well known, the face of mathematics changed radically during the 20th century, largely because of the abstract approach. The existence of these new areas of mathematics goes hand in hand with the power of the approach. Were these new areas not powerful—no matter how one understands that latter notion—, they would not be taken to refer to genuinely new mathematical objects.

If we were to ask mathematicians what is the abstract approach, they would probably point to obvious examples. Herstein, for one, would simply invite us to study his textbook: after a brief introduction to set theory, mappings and the integers, follow chapters on group theory, ring theory, vector spaces and modules, fields and linear transformations. These are all familiar topics in contemporary mathematics. But why are they part of the abstract approach? My goal in this paper is to elucidate what it is for a mathematical theory to be considered abstract and how this feature contributes to the approach that turned out to be and still is so powerful. Along the way, we should be able to clarify what this power amounts to in conceptual terms.

Given that there is in contemporary philosophy of mathematics a large literature surrounding the nature of abstract entities, let me state explicitly what I am *not* doing in this paper.

First, I am not entering the *ontological* debate. What I have to say bears no relationship with contemporary discussions surrounding abstract entities or the abstract/concrete distinction, e.g., that abstract entities are causally inert or lack any spatio-temporal coordinates, etc. (For a general discussion of the issues involved in this debate, in particular issues related to the nominalism/platonism debate, see for instance (Burgess & Rosen, 1997).) The whole discussion I am about to launch unfolds entirely *within* the realm of abstract entities, no matter how these are defined. It is my profound belief that abstraction in mathematics is solely an epistemological issue and that the abstract character of mathematics is *not* an ontological property but rather derives from epistemological features of mathematical knowledge itself. I am not so much concerned with abstract objects than with the *process* of abstraction and the abstract *method*. Some mathematical objects, or rather mathematical concepts, are *abstracted*. They do not inherit a dubious ontological status for that reason. Mathematicians also talk about concrete mathematical entities and, by the latter, they don't mean an abacus or a compass.

> This text attempts a different approach, letting the abstract concepts emerge gradually from less abstract problems about

> geometry, polynomials, numbers, etc. This is how the subject evolved historically. This is how all good mathematics evolves—abstraction and generalization is forced upon us as we attempt to understand the "concrete" and the particular. (Solomon, 2003, 3)

Thus, I am after a specific process that we find (mostly but not exclusively) in mathematics and that plays a key role in the development of contemporary mathematics.

Second, even though I have pushed aside the ontological issues involved, it is undeniable that abstraction is a multifaceted and polysemous concept. Abstraction has a long history, it is basically as old as Western philosophy itself. On the contemporary scene, it comes in various flavours and textures. It is sometimes analysed as being psychological in nature, at other times, as being purely logical and at other times, as being epistemological. It is also central in the teaching of higher mathematics.[1] My hope is to unearth what seems to me to be key features of *mathematical* abstraction as it actually developed in the 20th century.[2]

Third, I am focussing here on abstraction and the abstract method in the *practice* of mathematics and not in the foundations of mathematics or in its logical analysis. I am *not* merely claiming that the abstract character of modern mathematics emerged as a by-product from the usage of the axiomatic method within *formal* systems and that it is the formal aspect of language which is responsible of the abstract character of mathematics. That approach would equate being abstract with being formal, the latter term referring to formal languages. In other words, I will not identify being abstract with what can be studied apart from any particular interpretation. This is certainly one possible and plausible interpretation of the abstract nature of modern mathematics and, in fact, there is a grain of truth in that picture, as I will try to show. Although it contains a part of the analysis, it fails to include some important parts, in particular the inherent dynamics and recursive aspect of the abstraction process in modern mathematics.

[1] For the historical aspects, see (Cleary, 1995; Walmsley, 2000; Jesseph, 1993), for the psychological aspects, see, for instance (Piaget, 1977; Barsalou, 2003, 2005; Houdé, 2009), for the logical components, see (Lorenzen, 1965; Fine, 2002; Tennant, 2004; Antonelli, 2010), for some epistemological and logical components, see (Weyl, 1949; Pollard, 1987; Arbib, 1990; Simon, 1990; Ferrari, 2003) and, finally, for pedagogical reflexions on the subject, see (Piaget, 1977; Dubinsky, 1991; Frorer *et al.*, 1997; Hazzan, 1999; Mitchelmore & White, 2004).

[2] I recommend Sinaceur's interesting discussion of the various facets of mathematical abstraction (see Sinaceur, 2014). Unfortunately, time constraints did not allow me to incorporate elements of her analysis in my present work.

In a nutshell, in the practice of mathematics, the abstract method progressively became a method with many different functions: it was used to *solve* problems, to introduce new concepts and organizing principles, and even to install norms of construction. Furthermore, the method is such that it works in a recursive fashion. I believe that the abstract nature of modern mathematics is better captured by a faithful description of the process of abstraction inherent to the actual historical development of modern mathematics and not merely by the description of axiomatic systems (together with their underlying logic). The latter are a result of the former.

2 Very brief historical remarks

I have to limit myself to sketchy and impressionistic remarks in this paper. For, the history is complex and convoluted. The roots of the abstract method certainly go back to the end of the 18th century with Euler, Lagrange and others, and the beginning of the 19th century with Gauss, Galois, Abel, Dirichlet, Riemann, Dedekind, etc. My aim is not to unearth these roots, but merely bring to the fore certain elements that were inherent to the genesis of the abstract approach. We fortunately have some serious studies that allow us to have a good grasp of the main historical components involved.[3]

It seems reasonably safe to claim that the abstract approach made its *official* and *general* appearance in 1930, in a famous and extremely influential book, namely Van der Waerden *Moderne Algera*. Here is the opening sentence:

> The "abstract," "formal," or "axiomatic" direction, to which the fresh impetus in algebra is due, has led to a number of new formulations of ideas, insight into new interrelations, and far-reaching results, especially in *group theory, field theory, valuation theory, ideal theory*, and the *theory of hypercomplex numbers.* (van der Waerden, 1991, ix)

The first three words of the book are, to my mind at least, striking: "abstract", "formal", or "axiomatic". Notice the "or"; it is not an "and". It is as if van der Waerden considered them to be almost synonyms and perhaps, since they are in quotes, not quite clear. I do believe, however, that he was quite clear about the fact that the axiomatic direction was taking a new orientation, breaking away from its traditional philosophical

[3]See, for instance, (Bernkopf, 1966; Browder, 1975; Dieudonné, 1981; Birkhoff & Kreyszig, 1984; Wussing, 1984; Kleiner, 1996, 1999*a*,*b*; Corry, 1996; Smithies, 1997; Corry, 2000; Epple, 2003; Corry, 2007; Dorier, 1995, 2000; Gray & Parshall, 2007; Moore, 1995, 2007).

status. The identification between the abstract, formal and axiomatic was common in the 1930s and it remains so to this day. I will give only one quote, but it would be easy to multiply them:

> It is abstraction—more than anything else—that characterizes the mathematics of the twentieth century. There is both power and elegance in the axiomatic method, attributes that can and should be appreciated by students early in their mathematical careers and even if they happen to be confronting contemporary abstract mathematics in a serious way for the very first time. (Watkins, 2007, ix)

When van der Waerden wrote his book, he did not make a clear distinction between the formal, the abstract and the axiomatic. They all came together, fused and converged into one general method which was considered new and powerful. It is easy to show that the formal, and the axiomatic are not necessarily tied to the abstract method. In fact, the abstract method does not *necessarily* rely on the formal and the axiomatic methods. But historically, the formal and the axiomatic methods were combined in a certain manner by mathematicians and it became modern mathematics. Once this combination was found and shown to be elegant and powerful, it transformed the way mathematics was done and progressively led to a second wave of abstraction that came in the 1950s and 1960s and that is still going on today, once again transforming mathematics profoundly.

It is well-known that the first discipline to appear as an abstract theory was group theory. It will be enough for me to underline but one element in this historical process.

> The mathematical literature of the nineteenth century, and especially the work of decisive importance for the evolution of the abstract group concept written at the century's end, make it abundantly clear that that development had *three* equally important historical roots, namely, the theory of algebraic equations, number theory, and geometry. Abstract group theory was the result of a gradual process of abstraction from implicit and explicit group-theoretic methods and concepts involving the interaction of its three historical roots. I stress that my inclusion of number theory and geometry among the sources of causal tendencies for the development of abstract group theory is grounded in the historical record and is not the result of a backward projection of modern group theoretic thought. (Wussing, 1984, 16)

This is an interesting and surprising empirical fact: *three* equally important domains had to be available for the abstraction to take place. It seems to be a minimum, at least at this early stage. When only two are available, mathematicians will rather consider an analogy or a generalization, not an abstraction. One telling example of this is the work of Dedekind and Weber on algebraic numbers and algebraic functions (see Corfield, 2003, chap. 4). This appears to be an important cognitive component of the story. In fact, even for the whole field of algebra, it seems that three different and equally important theories had to be available for the community to consider that algebra as a whole could go in the direction of the abstract method. However, I should stress that it might not be *necessary*, it might simply reflect a cognitive trait in some individuals or communities.

> Her [i.e., Noether] work had a greater overall impact on algebra than Steinitz's, if only due to the fact that, having appeared about ten years later, it showed that Steinitz's program applied not only for the particular case worked out by him, but for many other significant cases as well. Group theory was thus the first algebraic discipline to be abstractly investigated, and field theory the first discipline that arose from the research of numerical domain into an abstract, structural subject. The study of ideal theory in an abstract ring consolidated the idea that a more general conception lay behind all this: the conception that algebra should be concerned, as a discipline, with the study of algebraic structures. (Corry, 1996, 251)

Corry here suggests that after three examples, mathematicians tend to generalize. I would say that, in this particular case, they were ready to abstract. Be that as it may, from the algebraic side, we have the development of group theory, field theory and ring theory, the latter accompanied by ideal theory and module theory. There were developments on the geometric side that were also important. Thus, metric spaces appeared early on the scene in the work of Fréchet, who was soon followed by Hausdorff on topological spaces and Banach on Banach spaces. The history of vector spaces is more convoluted but certainly belongs here. Finally, two theories that have a somewhat different path but that certainly belong to the picture, if only because they bring in different components to it, the theory of Boolean algebras and lattice theory. It is also worth mentioning at this point that Bourbaki considered that there were *three* mother structures: order structures, topological structures and algebraic structures. Underlying this abstract method, one finds, of course, set theory and, to a certain extent, logic. In the second wave of abstraction, the most important and salient example of the use of the abstract method is certainly the categorical foundations of algebraic geometry provided by Grothendieck and his school in the 1960s. The other striking example along these lines is Quillen's work in homotopical algebra which can be seen as the bedrock of abstract homotopy theory.

Let us come back to the formal and the axiomatic methods and their role in the rise of the abstract method. Historically, both a formal standpoint and the axiomatic method were available. I claim that both were diverted from their original purposes and became key components of the abstract method, to the extent that the latter was more or less identified with them, as we have seen. Let us consider them briefly in turn.

2.1 Symbolic formalism and algebra

Algebra is customarily associated with the manipulation of signs, letters, that are used to represent quantities and are manipulated according to given, explicit rules. Nowadays, we take for granted various symbolic conventions and rules of manipulation associated with various calculus. Needless to say, the introduction of these symbolisms has itself an intricate and philosophically important history.[4] One driving analogy emerged towards the end of the 18th century between rules of manipulations of arithmetic and rules of manipulations of differentials and operations in general. It goes back at least to Lagrange and was developed by Lacroix, Arbogast, Brisson, Franais, Servois on the French side, and using the work by the French as a springboard, by Woodhouse, Babbage, Peacock, Gregory, Boole, DeMorgan on the English side. It became known as symbolic algebra or the calculus of operations. But the key element is that it became a *formal* method.

> Symbolic algebra represented a movement away from algebra as universal arithmetic to a purely formal algebra. It emphasized the importance of structure over meaning, and acknowledged what has been called the *principle of mathematical freedom*. This principle implies that algebra deals with arbitrary, meaningless symbols, mathematicians create the rules regarding the manipulation of those symbols, and the interpretation follows rather than precedes the algebraic manipulation. (Allaire & Bradley, 2002, 403)

It has been argued that this view goes back in the philosophical literature at least to Berkeley.[5] One striking expression of this view is found in the British algebraist Peacock:[6]

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[4] See, for instance, (Serfati, 2002, 2005).

[5] See, for instance, (Detlefsen, 2005).

[6] The view clearly goes back to Woodhouse as early as 1803. But it seems that his book had almost no impact, apart from the fact that Babbage apparently learned a lot

> Algebra may be considered, in its most general form, as *the science which treats of the combinations of arbitrary signs and symbols by means of defined through arbitrary laws*: for we may *assume* any laws for the combination and incorporation of such symbols, so long as our assumptions are independent, and therefore not inconsistent with each other... (Peacock, 1830, 71, § 78)

It is easy to multiply the quotes of the so-called Cambridge algebraists. I will restrain myself to two:[7]

> ... symbolical algebra is... the science which treats of the combination of operations defined not by their nature,... but by the laws of combination to which they are subject... [W]e suppose the existence of classes of unknown operations subject to the same laws. (Gregory, 1840, 210) quoted by (Allaire & Bradley, 2002, 404)

And in Boole:

> They who are acquainted with the present state of the theory of Symbolic Algebra, are aware, that the validity of the processes of analysis does not depend upon the interpretation of the symbols which are employed, but solely upon the laws of their combination. (Boole, 1847, 3) quoted by (Allaire & Bradley, 2002, 400)

These sound extraordinarily modern to our ears. However, we have to be very careful not to read our conception of algebra, in particular abstract algebra in them, for it is definitely not. To mention but one clear case, Peacock would not include in algebra a non-commutative system, since it differs as such from arithmetic.[8]

For our purposes, it is sufficient to underline one aspect of the theory: it is seen as a *general* method, not as an *abstract* method. As Allaire & Bradley puts it "What can be proved for a class generally, holds for all specific operations in that class" (Allaire & Bradley, 2002, 407). But, I hasten to add, a (limited but genuine) form of abstraction appeared in the

from it. It should also be noted that Peacock was one of his students (see Koppelman, 1971).

[7] For more, see, for instance, (Koppelman, 1971; Allaire & Bradley, 2002).

[8] I should add that, in this respect, I disagree with the view proposed by Koppelman, who claims that the work done by the English algebraists fostered "an abstract view and clearly influencing many of the men who were to give, in the 1840's the beginnings of an abstract definition of algebra" (Koppelman, 1971, 188). I would say that they developed a *formal* view of algebra and not an *abstract* view. I hope the next sections will allow the reader to see why I would make this nuanced claim.

writings of some mathematicians of that period. Here is one of the most striking passage under the pen of the French mathematician Servois:

> Along the way, other links between the differential, the difference, variation and numbers emerged; it was necessary to find its cause, and everything is fortunately explained, when after having stripped, by a severe abstraction, these functions of their specific qualities, one only has to consider the two properties they have in common, being *distributive and commutative between them.* Servois, quoted by (Koppelman, 1971, 175) (my translation)

However, this seems to be the exception rather than the norm. The emphasis, at that time, was rather on the analogy underlying operations and numbers.

Many developments in mathematics in the 19th century contributed to the emergence of various shades of formalism: projective geometry, non-euclidean geometry, complex numbers, quaternions, octonions and hypercomplex numbers and, of course, the theory of invariants. One of the legacies of the 18th century was the status of negative numbers! In each case, there were problems attributing sense and reference to the symbols manipulated or, in the case of invariants, particularly in the so-called algorithmic school of Paul Gordan, there were series of manipulations that were used in order to obtain results which could not be justified except as pure rules of computations.[9]

It is therefore not hard to find both in the view or image and in the body of algebra the formal component explicitly mentioned.[10] Thus, in two important works by Weber we find a clear and undeniable endorsement of symbolic formalism. First, in his book on Galois theory, one reads:

> In the following an attempt is made to present Galois theory of algebraic equations in a way which include equally well all cases in which this theory might by used. Thus we present it here as a direct consequence of the group concept illuminated by the field concept, as a *formal structure completely without reference* to any numerical interpretation of the elements used. (Weber, 1893, 521) quoted by (Corry, 1996, 36) (my emphasis)

We find a similar claim in the abridged edition of his famous textbook on algebra:

[9] See, for instance, (McLarty, 2011).

[10] The distinction between the image of a discipline and its body comes from (Corry, 1996), who attributes it to (Elkana, 1981).

> In analysis one is accustomed to understand a "variable" as a sign which takes successively different values. Algebra uses the word variable as well but in a different sense. Here it is a mere calculating symbol [*Rechnungssymbole*] with which one operates by the rules of calculating with letters [*Buchstabenrechnung*]. (Weber, 1912, 47), quoted by (McLarty, 2011, 105)

What is the point? As the foregoing quotes show, towards the end of the 19th and at the beginning of the 20th century, it was becoming possible to divorce symbols and their rules from a specific, fixed content, a definite meaning. Algebra was, in some of its areas, considered to be formal. Notice that I did not say abstract, for I claim that this is different. By 1910, field theory and group theory were already considered to be abstract and for good reasons. But the abstract method was not quite in place yet.

2.2 The axiomatic method and the abstract method

Mathematicians talk about the axiomatic method and the abstract method as if they were interchangeable. Of course, this is simply false. The axiomatic method as nothing to do *per se* with the abstract method. Suffice it to mention Euclidean geometry, the paradigmatic example of an axiomatic theory. Euclidean geometry is certainly not considered to be an example of the abstract method.

The main point to make here is that, historically, the axiomatic method was the only known mode of presentation that could perform the function required by the abstract method: to provide a clear statement of a set of properties chosen in the process of abstraction.[11] It is well-known that this was not the main nor the only function of the axiomatic method in the late 19th and early 20th centuries. Nowadays, other modes of presentation can be and are used, e.g., the graphical language of sketches in categorical logic.

3 Mathematical Abstraction and the abstract method: putting the pieces together

What I am going to describe constitutes, it seems to me, the main route to the abstract method. It should be kept in mind that all four components have to be present for the process to be a full process of abstraction. The order is not crucial.

First, there is a domain of distinct types of entities, at least three distinct types of entities, which becomes a domain of variation *within which there are nonetheless invariant features.* It is crucial that the three domains be considered to be sufficiently different, that the domain of variation be a

[11] For more on the axiomatic method in 20th century mathematics, see Schlimm (2013).

domain of *significant* variation. This interplay of variation and invariance opens the door to the possibility of abstracting.

Second, for the invariant features to be abstracted, one has to take a formal stance with respect to the individual objects of the various domains. In other words, one has to forget what one is talking about, the meaning of the signs involved and treat them purely formally. Thus, the abstract and the formal are sometimes confused.

Third, the invariant features are abstracted and, at this stage, one needs a method to present these features and be able to investigate them in an autonomous fashion. It is at this moment that the axiomatic method appears to be exactly what one needs. The axioms capture the invariant features and one then uses logic to investigate what can be known from them. Notice that in some cases, the axioms might stipulate properties that are not obviously in the distinct domains as such. It is once the property is enunciated in a language that one can, in some cases, convince oneself that, indeed, it is a property of the entities given.

Fourth, a new criterion of identity for the abstracted entities has to be discovered and fixed. In turn, fixing a criterion of identity is possible if and only if linguistic resources are available for these properties and criteria of identity to be expressed. The new criterion of identity is almost always discovered after the abstract entities have been introduced, for a developed theory has to be available in order to make sure that the criterion of identity has the right properties. The invariant component is from then on circumscribed clearly and independently of the original entities. These are seen to be instances of these new types and are studied as such, that is, there is a shift of attention from the old criterion of identity and its associated properties to the new criterion and its associated properties. Almost always, it is then possible to discover and construct new, unforeseen instances of these new abstract entities. Thus, the domain of variation can expand and is never fixed once and for all. In more philosophical terms, once the new types have been fixed, known examples become tokens of the type and new, unforeseen tokens can be constructed or discovered. However, very quickly the shift of attention draws mathematicians towards intrinsic problems, or one might say pure problems, of the new field, for instance, problems of classification or decompositions into well-organized patterns will become central.

Let me underline immediately that arbitrary sets and functions between them played a key part in the development of the abstract method, particularly in the fourth step. Once the focus shifts towards the abstract entities themselves, one needs to talk of unspecified elements that are determined by the properties stated by the axioms. Sets and functions introduced earlier were perfect candidates for that role.

3.1 Domain of significant variation

I now need to clarify what I mean by a domain of variation and a domain of *significant* variation. The best way to introduce these ideas is by giving an example.

One of the very first cases of an extraordinarily successful abstraction in the history of modern mathematics is certainly that of metric spaces, introduced by Fréchet around 1906 in the context of functional analysis.[12] What I want to emphasize in this case is the range of the domain of variation and the fact that it is a domain of significant variation. I think it is fair to say that at that point, mathematicians did not think in terms of abstract sets in the sense of a collection of faceless points. In the context of geometry, mathematicians were thinking of manifolds, either as subspaces of spaces of real or complex points. In the case of Fréchet, he was dealing with these usual manifolds, namely $\mathbb{R}, \mathbb{R}^2, \ldots, \mathbb{R}^n, \ldots, \mathbb{C}, \mathbb{C}^2, \ldots, \mathbb{C}^n$ together with functions between them on the one hand, and infinite-dimensional functional spaces together with operators between them on the other hand. In his thesis, Fréchet gives four examples of functional spaces that satisfy his axioms. See (Fréchet, 1906) or Taylor (1982). Here they are.

1. Let J be a closed interval of the real line $\mathbb{R}$ and consider the space $\mathbb{R}^J$ of continuous functions $f : J \to \mathbb{R}$. A metric on $\mathbb{R}^J$ is defined by
$$d(f, g) = max(|f(x) - g(x)|) \qquad \forall x \in J.$$
2. Consider the space $E_\infty = \mathbb{R}^{\mathbb{N}}$ of infinite sequences $x = (x_1, x_2, \ldots)$ of real numbers. A metric on E_∞ is given by
$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$
3. A space of parametrized curves in $\mathbb{R}^3$ with the standard Euclidean metric between points. Using the latter, Fréchet defines a metric between the curves.
4. Finally, let A be a complex plane region whose boundary consists of one or more contours. Let $\{A_n\}$ be a sequence of bounded regions such that $A_n \subset int(A_{n+1})$ and $A_n \subset int(A)$ and such that any given bounded region in the interior of A is in the interior of

[12] It is known that Fréchet knew about the case of groups and that it provided at least guidelines and a model of what could be achieved by moving up the ladder of abstraction.

some A_n for n sufficiently large. Consider the space $\{f : int(A) \to \mathbb{C} | f \text{ is holomorphic}\}$ and let

$$M_n(f, g) = max(|f(z) - g(z)|) \qquad \text{when } z \text{ is in the closure of } A_n.$$

The metric between two such functions is then defined by

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{M_n(f, g)}{1 + M_n(f, g)}.$$

Although I haven't described the third example in detail, what is striking is how different they are from one another and, perhaps even more so, from the spaces of points $\mathbb{R}^n$ and $\mathbb{C}^n$. The main point is this: if we did not know about the metric involved in each case, we might not think that these entities have something in common. Indeed, we are accustomed to attribute certain properties to real functions: continuity, differentiability, roots, maximum, minimum, etc., we represent the graph of a function as a one-dimensional path in the codomain, thus as something that necessarily has a length, we think of a real function as a systematic relation of dependence between two or more properties, as a quantity that varies according to a certain pattern or whose variation depends on a another variation. A function is essentially thought of as being dynamic. The four examples given by Fréchet are of this kind. A (real) point is, well, a point. It has none of the properties of a function. Thus, the properties of the elements of $\mathbb{R}$ and even $\mathbb{R}^n$ are incommensurable with the properties of the elements of a functional space. I want to insist on the fact that given the properties of functions and given that we think of functions with their properties, it is hard to conceive of a *space* of functions, that is treating the latter as being points. It is as if we were trying to think of the properties of functions and forget about them at the same time. Of course, as soon as we have succeeded in thinking of them as spaces, we stumble upon what is certainly seen as being the main difference between these spaces and the usual spaces of points: the examples given above are *infinite* dimensional. Thus, we also have two different types: finite dimensional spaces on one side and infinite dimensional spaces on the other.

We immediately see how the introduction of functional spaces increases substantially the domain of variation. There is also a considerable amount of variation between the four examples themselves. It is hard to see what infinite sequences of real numbers might have in common with parametrized curves in three dimensional Euclidean space, for instance. They seem to belong to different categories of thought. It is only when they are thought as being genuine spaces that we allow ourselves to attribute them similar properties. Here is how Fréchet himself came to characterize the general situation:

> At first sight such an undertaking might be considered as absurd. How can we speak of a geometry in a space whose "points" are of an undefined nature, when we do not know if the elements are numbers, curves, surfaces, functions, series, sets, etc.? (Fréchet, 1951, 152)

By introducing an abstract level of analysis, one can specify the domain of application of geometrical ideas. It is important to note that this range might turn out to be much larger than anticipated. By considering all these cases as being genuine spaces, one has at the same time a language and a universe of interpretation for these terms in which it makes sense to consider these seemingly different geometric entities as being nonetheless entities of the same type. Notice that it is impossible at this stage to think of the abstraction process in terms of an equivalence relation. One has to have the properties that will be abstracted in order to define the criterion of identity between the abstract entities. In other words, the criterion of identity can not be given *a priori* but is derived from the theory. In fact, many of the relevant properties of the spaces will only emerge while the theory is constructed and developed.

The very same analysis can be given for group theory, field theory and ring theory. I cannot present the details in such a short paper. I will merely give pointers towards the relevant features in each case. Describing the domain of variation and seeing how significant that variation is turns out to be rather easy. It is important to keep in mind that, at this stage, what I want to underline is *not* what these domains have in common, that is that they are groups or fields or rings, but, on the contrary, how much they differ at a very basic mathematical level.

As we have already seen in the foregoing quote about the genesis of group theory, there had to be three different theories, the theory of algebraic equations, number theory and geometry, for the abstract point of view to emerge as such. These three domains, from the point of view of the practice, when one consider the nature of the entities and their properties in each case, are, in some sense, orthogonal. Algebra, number theory and geometry: these were, in the 19th century, about different entities having different properties and studied with different methods altogether. One would not think of transferring properties of algebraic equations to numbers – how many roots does it have? –, or properties of numbers to geometric figures – is this triangle prime? Thus, once again, one has to systematically *ignore* most of what one has learned about these entities, how one ought to think about these entities and their properties.

The case of fields is just as clear. Weber gave an axiomatic presentation of the concept of field in 1893. However, in that paper, his goal was not to develop field theory, rather he found it convenient to use the concept in his presentation of Galois theory. But for the record, it is worth mentioning that Weber includes in his examples algebraic numbers (number theory), algebraic functions (algebraic geometry), Galois's finite fields (algebra) and Kronecker's "congruence fields" $K[x]/(p(x))$, where K is a field and $p(x)$ is irreducible over K (algebra). Notice the variation already, but that is not quite enough, for in these cases, one can rather think in terms of analogies between the various domains. It is nowadays acknowledged that the *abstract* theory of fields appeared on the scene with the publication of Steinitz's groundbreaking paper on the algebraic theory of fields in 1910. As Steinitz himself explicitly acknowledges, it was Hensel's p-adic numbers that sparked his investigation.

> I was led into this general research especially by Hensel's *Theory of Algebraic Numbers*, whose starting point is the field of p-adic numbers, a field *which counts neither as a field of functions nor as a field of numbers in the usual sense of the word.* Steinitz, quoted by (Kleiner, 1999*b*, 861) (my emphasis)

One should show why the field of p-adic numbers introduces a *significant* variation. I will unfortunately have to rely on Steinitz's words in the context of the present paper.

The history of abstract ring theory is convoluted and would deserve a whole section in itself. We can set aside Fraenkel's work on rings, since although it constitute an important step towards the theory, it fails to do so for interesting reasons that we simply cannot cover here. (But see Corry, 2000, for a nice analysis.) In a sense, one of the problems of ring theory was precisely that the domain of variation was too wide and varied for the construction of the *theory.* Two separate historical strands leading to abstract ring theory have to be distinguished: commutative rings and non-commutative rings. Commutative ring theory originates from algebraic number theory, invariant theory and algebraic geometry and it is this strand that led to Noether's ground breaking work. Non-commutative ring theory comes from the theory of hypercomplex number systems, nowadays called finite dimensional algebras, and there are numerous different cases of these. It would be necessary to focus our attention on Noether's work, but we have to leave this to another study. (See (Kleiner, 1996), (Swetz *et al.*, 1995), (Corry, 2000) and (McLarty, 2011) for instance.)

These examples illustrate clearly what it is to start with a domain of significant variation. In all cases, we have mathematical systems that have different, even incompatible properties, e.g., being finite/infinite dimensional, being discrete/continuous, etc. I should point out that in these particular examples, the systems considered are build from below so to speak, that is from specific elements, their properties and operations on these elements or relations between them. Once they are looked at from the abstract point of view, these elements and their specific individual properties become totally superfluous. Thus, what varies fundamentally, at this stage, is that each element has, so to speak, a myriad of properties, a whole individuality. In the process of abstraction, these specific individual properties are almost all ignored in favor of properties that relate these individuals together, properties of parts and how they are related to one another and to the whole. Finding the latter property is not a trivial matter and very often new properties, relational properties, have to be found and emerge during the abstraction process itself. Furthermore, these systems certainly cannot, at first, be considered as being even *possibly* identical, not even as being instances of a unique type with *its* criterion of identity, different from all the specific criteria used for the individual systems. It is impossible to tell that some of these systems might turn out to be *identical* when considered as instances of a new type.

One last remark about a domain of variation is necessary. Axiom systems automatically yield a range of variation, at the syntactical level. But it is seldom fruitful. It took a very long time and a considerable amount of ingenuity before mathematicians considered it possible to obtain a significant domain of variation from the axioms of Euclidean geometry. The strategy here is simple: simply ignore some of the axioms and see whether you get something interesting. But this strategy seldom yields genuinely interesting results. One might simply get a more general framework that does not perform any real work. However, as in the case of Euclidean geometry, what might be taken to be a sterile enterprise can reveal vast and unforeseen possibilities. Hilbert's axiomatization of Euclidean geometry is a remarkable example of a successful, systematic, organization of a domain that, at the same time, characterizes adequately specific domains *and* deals with a domain of variation properly. There are other cases in algebra, e.g., monoids and groups or rings and commutative rings, but also in other fields, e.g., generalized cohomology theories like K-theory, where deleting an axiom still captures a rich domain of variation. The fact is, this method, if it is a method at all, rarely yields interesting fruits: subtracting an axiom at random does not necessarily provide a new, interesting theory. In all the cases we have just mentioned, the domain of variation was already known when the axioms were set up and therefore one knew, in some sense, which axioms could be removed fruitfully. As far as I know, removing an axiom in the definition of a topological space does not yield any interesting *geometric* system.[13] The same is true for the notion of category (in contrast with the notion of group) and probably many others as well.

4 The point of the abstract method

When Stephan Banach introduced the spaces that now bear his name, he justified the use of the abstract method thusly:

> The aim of the present work is to establish certain theorems valid in different functional domains, which I specify in what follows. Nevertheless, in order not to have to prove them for each particular domain, I have chosen to take a different route...; I consider sets of elements about which I postulate certain properties; I deduce from them certain theorems, and I then prove for each particular functional domain that the postulates adopted are true for it. Banach, quoted by (Moore, 1995, 280)

This is the strategy adopted by most mathematicians afterwards. The abstract method leads to two different methodological levels: first, one proves certain results for the abstract entities themselves, for the types so to speak, and then one shows that domains of interest are tokens of these types and therefore automatically satisfy the properties stated in the theorems proved.

However, this characterization fails to reveal the real import of the method and why it is mathematically and philosophically so important. For, as such, Banach's claim merely says that the abstract approach is a form of generalization and a more economical method. This is indeed the case, but it does not go at the heart of the method, its real strength or power. It should be pointed out that by taking the abstract method, it is sometimes possible to treat a domain of exotic or unusual entities as if they were known. For instance, once p-adic numbers are seen as being a field and that it is possible to prove results about fields from pure field-theoretic properties, one can dispense with trying to manipulate p-adic numbers, with some unusual operations or properties. This is clearly one benefit of the method. But, again, it is not the main force.

The abstract method is taken to yield a *conceptual* analysis of mathematics: one talks one the group-concept, the ring-concept, the vector-space

[13] Of course, it might yield an interesting algebraic structure, e.g., an inf-lattice. It is true that Hausdorff included the separability condition that now bears his name in his first axiomatization and that removing it still yield a coherent and interesting geometric notion, although some might still want to debate this last point.

concept. Mathematics is then organized around these concepts which *unify* in a deep way various domains of mathematics that were, and for good reasons, believed to be unrelated.

> In the wake of these developments has come not only a new mathematics but a fresh outlook, and along with this, simple new proofs of difficult classical results. The isolation of a problem into its basic essentials has often revealed for us the proper setting, in the whole scheme of things, of results considered to have been special and apart and has shown us interrelations between areas previously thought to have been unconnected. (Herstein, 1975, 1–2)

This is one of the main epistemological claims I want to make here: the introduction of a level of abstraction is seen as a way of clarifying and distilling what, in some cases, has become a complex domain or, in other cases, exhibits similarities, parallels indicating the possibility of an underlying common framework. The previous disjunction is clearly not exclusive. The new abstract level not only simplifies the situation but it also yields a better control and understanding of the concepts involved. As Herstein puts it: it reveals the proper setting for the solution of various problems. The axiomatic method is a part of that process. Axiomatization should be seen, in this light, as a form of design. Axioms capture either a common structure or common properties leading to a better control and understanding of the features at work. The axiomatic method is thus used as a sieve, a filter in these processes. It brings to the fore the Archimedean points upon which solutions to given problems work. What was previously immersed in a mountain of irrelevant details is unearthed and shown to constitute the mechanisms making concepts work together. This is precisely why we feel justified in speaking of abstraction. As I have said, the process leads to new mathematics, conceptually systematic and organized according to clear principles. I claim that this way of using the axiomatic method has evolved in contemporary mathematics to become a standard method.

As any contemporary mathematician knows too well, to work abstractly is to work with mathematical entities in a certain manner. This was already clear to Weber:

> We can ... combine all isomorphic groups into a single class of groups that is itself a group whose elements are the generic concepts obtained by making one general concept out of the corresponding elements of the individual isomorphic groups. The individual isomorphic groups are then to be regarded as different

> representatives of the generic concept, and it makes no difference which representative is used to study the properties of the group. Weber in 1893, quoted in (Wussing, 1984, 248)

It should be said, however, that in some cases, one wants to keep track of specific *isomorphisms* between groups and they are just as significant as the groups themselves. Weber is nonetheless expressing an extraordinarily modern point of view in 1893.

It is tempting to reduce abstraction to a particular case of generalization. Generalization is usually assumed to be a clear and simple process: it is purely logical and consists in inferring a universally quantified proposition $\forall x P(x)$ from a list of particulars having a property $P(a), P(b), \ldots, P(n)$. Let me immediately emphasize the fact that this simply does *not* cover all cases of generalizations that occur in mathematics. A simple example is provided by the concept of integral and its various generalizations in the last half of the 19th century (see Villeneuve, 2008, for details). For one thing, an integral is an operation and it is not propositions about the integral that were generalized but the operation itself. This is but one example. At the conceptual level, it is the relationships between abstraction and generalization that have to be clarified. It certainly seems possible to generalize without abstracting. Think of various theorems that are generalized although without leading to more abstract results. For instance, the passage from the definition of continuity of a function $f : \mathbb{R} \to \mathbb{R}$ at a point to the notion of continuity over an interval $[a, b] \subset \mathbb{R}$ is a simple generalization that is certainly not an abstraction. The same could be said for the generalization of theorems of real analysis to theorems of complex analysis. Abstraction seems to always involve a form of generalization.

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