

# Essays in Philosophy

General Editor: Michael F. Goodman

Published by Humboldt State University, Arcata, California

<http://www.humboldt.edu/~mfg1/opening.html>

*Logic, Logic, and Logic.* George Boolos. Harvard University Press, 1998. ix + 443 pages. Hardcover \$45, paperback \$22.95. ISBN 0-674-53767-X.\*

Cancer robbed our community of an outstanding philosopher. One is tempted to say “philosopher and logician,” but as Richard Cartwright remarked in his eulogy for George Boolos, “he would have not been altogether happy with the description: accurate, no doubt, but faintly redundant—a little like describing someone as ‘mathematician and algebraist.’”

While the title of this book is redundant, its content is not. Completed by colleagues, students and friends, this posthumous collection is not only a testament to Boolos’ legacy but also to his logical virtuosity. George Boolos made significant contributions in every area of logic in which he worked. The volume is a treasure trove of insightful observations and elegant formal contributions to central questions in the philosophy of logic. The book is divided into three sections: Studies on Set Theory and the Nature of Logic, Frege Studies, and Various Logical Studies and Lighter Papers.

The first section contains Boolos’ seminal essay “The Iterative Conception of Set,” (article 1) published in the early 1970s, which introduced philosophers to the underlying intuitive conception of set articulated by Zermelo’s axioms. Most philosophers had regarded these axioms as *ad hoc* attempts to avoid Russell’s, Burali-Forti’s, and other set-theoretical paradoxes, but Boolos demonstrates how a core of Zermelo’s axioms (excluding the Axioms of Choice and Replacement) can be logically deduced from a set of axioms characterizing the idea of the formation of sets in stages over time.

The iterative conception was, however, known to mathematicians. It is now standard to articulate this conception in intuitive accounts of the subject (see, for example, Enderton’s text *The Elements of Set Theory* (1977), Schoenfeld’s survey article in Barwise (ed.) *The Handbook of Mathematical Logic* (1977), and Devlin’s *The Joy of Sets* (1993)). Hao Wang in *From Mathematics to Philosophy* (1974) shows how this conception was already implicit in Cantor’s and Zermelo’s writings and in Gödel’s proof of the consistency of the continuum hypothesis. Indeed, Wang reports that Gödel, in contrast to Boolos,

argued that the axioms of choice and replacement do follow from the iterative conception. In article 7, an introduction to a posthumously published lecture by Gödel, Boolos takes issue with Gödel's Platonistic claim that the axioms of ZFC (Zermelo Frankel set theory with Choice) "force themselves upon us as true." Even if the axioms articulate a natural and compelling conception of set, they need not correspond to anything objectively real.

Article 2 contains Boolos' defense of Fraenkel's, in contrast to Zermelo's, position that first-order but not second-order logic is applicable to set theory. Boolos criticizes the view of Charles Parsons (and D. A. Martin) that it makes sense to use second-order quantifiers when first-order quantifiers range over entities that do not form a set. Boolos' answer to the title of article 8, "Must We Believe in Set Theory?" is 'no': the phenomenological argument (due to Gödel) does not imply that the axioms of set theory correspond to something real, and the indispensability argument (due to Carnap) that mathematics is required by our best physical theory, is dismissed as "rubbish." Boolos laments not having time to discuss the *predicative* set theory of Solomon Feferman and *cabalistic* views of Penelope Maddy (and D. A. Martin, John Steel, Robert Solovay, and W. Hugh Woodin).

The first section also contains Boolos' papers on the logic of plurals and second-order logic. Second-order, unlike first-order, logic can express such mathematically useful notions as coextensiveness, equinumerosity, and the ancestral. Nevertheless, many logicians argued that second-order logic should not be regarded as logic because the failure of the completeness theorem cast doubts on the claim that it could serve as a model of deductive reasoning. The question of what is and is not logic aside, Boolos examines the issues of whether second-order logic is of use in formalizing many natural language sentences that cannot be captured in first-order logic such as the following:

- (1) Some relative of each villager and some relative of each townsman hate each other (Hintikka).
- (2) The richer the country, the more powerful is one of its officials (Barwise).
- (3) Some critics admire only one another (Kaplan).

Boolos' plural quantifiers: '*some things, the U's are such that...*' adds to the expressive vocabulary of logic. Plural quantifiers provide an alternative interpretation of second-order logic in which second-order entities are subsets of the universe. In his excellent introductory remarks to the various sections, John Burgess points out that plural quantification is limited by the fact that plurals provide a monadic, but not dyadic, second-order quantification.

Section II contains Boolos's work in forging a new direction in Frege studies. Boolos claims that when Frege was confronted with the derivation of Russell's paradox from his Basic Law V, he "grievously undervalued his actual achievement" and mistakenly regarded the paradox as invalidating the whole of his formal work. Instead, Boolos' work vindicates Frege. Boolos claims that Frege deserves a place in the philosophical pantheon beside Descartes, Leibniz, and Kant: "What is sad is not so much that Frege's system turned out to be vulnerable to Russell's paradox as that both he and we failed to realize how valuable his actual accomplishment was. Frege proved the first great theorem of logic: arithmetic can be derived from the number principle."

Boolos (with student Richard Heck) shows that arithmetic can be logically derived from Hume's Principle. Frege's Basic Law V, which leads to the inconsistency of Russell's paradox,

$$(Vb) \quad \forall F \forall G (\#F = \#G \leftrightarrow \forall x (Fx \leftrightarrow Gx)) ,$$

is formalistically similar to the consistent number principle, dubbed "Hume's Principle" because Frege cited a passage from Hume's *Treatise* (Book I, Part III, Section I) in support of the principle:

$$(HP) \quad \forall F \forall G (\#F = \#G \leftrightarrow F \approx G) .$$

Crispin Wright (1983) "shows the beautiful, deep, and surprising result that arithmetic is interpretable in Frege arithmetic, a theory whose sole non-logical axiom is HP." Boolos takes issue with Wright, however, on whether Hume's Principle is analytic (article 19). Boolos argues that the claim Hume's principle is analytic is defective in the same way as the analogous claim that "the present king of France is royal" is analytic. An analytic truth shouldn't imply the existence of the present king of France. Similarly, Boolos is skeptical of HP's existential commitments and considers a hypothetical revision. As a compromise, Boolos offers Gödel's notion of analyticity, not as "true in virtue of meaning," but as true "owing to the nature of the concepts occurs therein" which "form an objective reality of their own, which we cannot create or change, but only perceive and describe." Boolos criticizes this idea. One of the most interesting objections is that Wright's number  $\#[x: x = x]$ , *anti-zero*, would be the number of all the sets there are, which would make Frege Arithmetic inconsistent with ZF set theory, in which there is no (cardinal) number that is the number of all the sets there are.

The third section contains "lighter works" on an intriguing array of problems. Boolos discusses the principle of mathematical induction in relation to the sorites paradoxes (Article 22, "Zooming Down the Slippery Slope"), and article 25 "A Curious Inference"

returns to the question of the relationship between first- and second-order logic. Boolos uses the exponentially expanding Ackerman-type function to show that even if there is a derivation in first-order logic of a second-order theorem, the proof can be intractably long. Article 26 "A New Proof of Gödel's Incompleteness Theorem" uses the Berry paradox and the notion of the length of proof and of algorithmic randomness (in addition to Boolos's reference, see Gregory Chaitin's "Randomness and Mathematical Proof," *Scientific American* (May, 1975) and his recent *The Unknowable* (1999)). Article 27 "On 'Seeing' the Truth of the Gödel Sentence," an invited commentary on Roger Penrose's *The Emperor's New Mind* (1989), diagnoses the fallacy in Penrose's argument that no "mechanical" procedure could match our abilities to "see" the truth of mathematical assertions, like the Gödel sentences, which are unprovable within the system yet must be true assuming the system is consistent.

Article 29 "The Hardest Logical Puzzle Ever" involves a discussion of a logical puzzle devised by Raymond Smullyan. There are three gods A, B, and C. One always speaks truly, one always speaks falsely, and one speaks truly or falsely in a random manner. You are to ask three yes-no questions and then determine definitively who is who. A complication is that they answer "da" or "ja" but you don't know which word means "yes" and which means "no." Boolos discusses a series of three simpler, but enticing, puzzles. The first is as follows: "Noting their locations, I place two aces and a jack face down on a table, in a row; you do not see which card is placed where. Your problem is to point to one of the three cards and then ask me a single yes-no question, from the answer to which you can, with certainty, identify one of the three cards as an ace. If you have pointed to one of the aces, I will answer your question truthfully. If you have pointed to the jack, I will answer your question yes or no completely at random." Boolos claims that our ability to reason about alternative possibilities would be almost completely paralyzed if we were denied the use of the law of excluded middle. (However, we leave it as an exercise to the reader to show that allowing a sentence that is neither true nor false renders the above puzzle trivial.)

Boolos is best known for his work on modal provability logic. Boolos' *The Logic of Provability* (1993) is regarded as the single best volume on the subject, which surpasses his *The Unprovability of Consistency* (1979). A simple modal propositional proof of Gödel's Second Incompleteness Theorem occurs in the last article, a peculiar piece whimsically entitled "Gödel's Second Incompleteness Theorem Explained in Words of One Syllable." The afterward by John Burgess surveys Boolos' work in the modal provability logic.

George Boolos was responsible for the growth of a distinctive intellectual culture in philosophy at MIT, according to Judith Jarvis Thomson, who characterized this culture

as one in which “clarity is crucial; grandeur, scope, reach, significance, importance—all that is lucky fallout if it so falls out, but not essential” (<http://web.mit.edu/philos/www/boolos.html>).

Boolos’ volume is characterized by both clarity and significance. His writing is also witty. I conclude with an anecdote by Judy Thomson in her eulogy: “George had worked on this thesis with Hilary Putnam, who was here then, and with James [Thomson]. At the end of his defense, Hilary said ‘That’s all well and good internally, Mr. Boolos, but in what relation does your thesis stand to the universe?’ What do you reply when your thesis supervisor asks you that question at your defense? If you’re George, then quick as a flash you reply, flatly, ‘It’s part of it.’”

Gary Mar  
Department of Philosophy  
SUNY at Stony Brook

---

\* I wish to thank my students Erik Closson, Trina Kokalis, and David Meieren, and my colleague Peter Ludlow for helpful suggestions.