THE HISTORY OF CATEGORICAL LOGIC 1963–1977

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Categorical logic, as its name indicates, is logic in the setting of category theory. But this description does not say much. Most readers would probably find more instructive to learn that categorical logic *is* algebraic logic, pure and simple. It is logic in an algebraic dressing. Just as algebraic logic encodes propositional logic in its different guises (classical, intuitionistic, etc.) by their Lindenbaum-Tarski algebras (Boolean algebras, Heyting algebras and so on), categorical logic encodes first-order and higher-order logics (classical, intuitionistic, etc.) by categories with additional properties and structure (Boolean categories, Heyting categories and so on). Thus, from the purely technical point of view, categorical logic constitutes a generalization of the algebraic encoding of propositional logic to first-order, higher-order and other logics. Furthermore, we shall present and discuss arguments (given by the main actors) to show that this encoding constitutes the correct generalization of the well-known algebraic encoding of propositional logics by the Lindenbaum-Tarski algebras.

The proper algebraic structures are not only categories, but also morphisms between categories, mainly functors and more specially adjoint functors. A key example is provided by the striking fact that quantifiers, which were the stumbling block to the proper algebraic generalization of propositional logic, can be seen to be adjoint functors and thus entirely within the categorical framework. As is usually the case when algebraic techniques are imported and developed within a field, e.g. geometry and topology, vast generalizations and unification become possible. Furthermore, unexpected concepts and results show up along the way, often allowing a better understanding of known concepts and results.

Categorical logic is not merely a convenient tool or a powerful framework. Again, as is usually the case when algebraic techniques are imported and used in a field, the very nature of the field has to be thought over. Furthermore, various results shed a new light on what was assumed to be obvious or, what turns out to be often the same on careful analysis, totally obscure. Thus, categorical logic is philosophically relevant in more than one way. The way it encodes logical concepts and operations reveals important, even essential, aspects and properties of these concepts and operations. Again, as soon as quantifiers are seen as adjoint functors, the traditional question of the nature of variables in logic receives a satisfactory analysis.

Furthermore, many results obtained via categorical techniques have clear and essential philosophical implications. The systematic development of higher-order logic, type theory under a different name, and various completeness theorems are the most obvious candidates. But there is much more. Many important questions concerning the foundations of mathematics and the very nature of mathematical knowledge are inescapable. In particular, issues related to abstraction and the nature of mathematical objects emerges naturally from categorical logic.

This paper covers the period that can be qualified as the birth and the constitution of categorical logic, that is the time span between 1963 and 1977. No one will deny that categorical logic started with Bill Lawvere's Ph.D. thesis written in 1963 under S. Eilenberg's supervision and widely circulated afterwards. (It is now available on-line on the TAC web-site.) In his thesis, Lawvere offered a categorical version of *algebraic* theories. He also suggested that the category of categories could be taken as a foundation for mathematics and that sets could be analyzed in a categorical manner. In the years that followed, Lawvere tried to extend his analysis and sketched a categorical version of first-order theories under the name of *elementary theories*. Then, in 1969, in collaboration with Myles Tierney, Lawvere introduced the notion of an elementary topos, making an explicit connection with higher-order logic and type theories. Both Lawvere and Tierney were aiming at an elementary, that is first-order, axiomatic presentation of what are now called *Grothendieck toposes*, a special type of categories introduced by Alexandre Grothendieck in the context of algebraic geometry and sheaf theory. Soon after, connections with intuitionistic analysis, recursive functions, completeness theorems for various logical systems, differential geometry, constructive mathematics were made. We have decided to end our coverage in 1977 for the following reasons. First, we had to stop somewhere, otherwise we would have to write a book. Second, and this is a more serious reason, three independent events in 1977 mark more or less a turning point in the history of categorical logic. First, the book *First-Order Categorical Logic*, by Makkai and Reyes appears, a book that more or less codifies the work done by the Montreal school in the period 1970-1974 and now constitutes the core of categorical first-order logic. Second, the same year witnesses the publication of Johnstone's *Topos Theory*, the first systematic and comprehensive presentation of topos theory as it was known in 1974-75. (The reader should compare this edition with Johnstone's recent *Sketches of an Elephant*, a comprehensive reference on topos theory in three volumes.) Third, 1977 was also the year of the Durham meeting on applications of sheaf theory to logic, algebra and analysis, whose proceedings were published in 1979. We submit that around the end of the nineteen seventies, categorical logic was on firm ground and could be developed in various directions, which is precisely what happened, from theoretical computer science, modal logic and other areas.

The usual warnings, caveat and apologies are now necessary. It is impossible to cover, even in a long article such as this one and for such a short time period, all events involved in the history of categorical logic. This paper is but a first attempt at a more precise and detailed history of a complicated and fascinating period in the history of ideas. We hope that it will stimulate more work on the topic. We hasten to add that it also reflects our interests and (hopefully not too limited) knowledge of the field. It is our hope that it will nonetheless be useful to logicians and philosophers alike. We sincerely apologize to mathematicians, logicians and philosophers whose names ought to have appeared in this history but have not because of our ignorance.

1 THE BIRTH OF CATEGORY THEORY AND ITS EARLY DEVELOPMENTS

Category theory as a discipline in itself and was born in the context of algebraic topology in the nineteen forties. We will briefly sketch the history of category theory before the advent of categorical logic and rehearse the fundamental notions of the theory required for the exposition of the following sections.

1.1 Category theory: its origins

We will here only rehearse the ingredients required for the history of categorical logic. The reader is referred to [Landry and Marquis, 2005], [Marquis, 2006] and [Krömer, 2007] for more details.

Category theory made its official public appearance in 1945 in the paper entitled "General Theory of Natural Equivalences" written by Samuel Eilenberg and Saunders Mac Lane. This "off beat" and "far out" paper, as Mac Lane came to qualify it later [Mac Lane, 2002, 130], was meant to provide an autonomous framework for the concept of *natural transformation*, a concept whose generality, pervasiveness and usefulness had become clear to both of them during their collaboration on the clarification of an unsuspected link between group extensions and homology groups. Such a general, pervasive and conceptually useful notion seemed to deserve a precise, rigorous, systematic and abstract treatment.

Eilenberg and Mac Lane decided to devise an *axiomatic* framework in which the notion of natural transformation would receive an entirely general and autonomous definition. This is where *categories* came in. Informally, a natural transformation is a family of maps that provides a systematic "translation" or a "deformation" between two systems of interrelated entities within a given framework. But in order to give a precise definition of natural transformations, one needs to clarify the *systematic* nature of these deformations, that is, one has to specify *what* these deformations depend upon and *how* they depend upon it. Eilenberg and Mac Lane introduced what they called *functors* — the term was borrowed from Carnap — so that one could say between *what* the natural transformations were acting: a natural transformation is a family of maps between functors. Clearly, one has to define the notion of a functor: the concept of *category* was tailored for that purpose. The *systematic* nature of natural transformations was also made clear by categories themselves. Thus categories were introduced in 1945 and, as Mac Lane reported (see [Mac Lane, 2002]), Eilenberg believed that their paper would be the only paper written on "pure" category theory.

As we have already mentioned, Eilenberg and Mac Lane gave a purely *axiomatic* definition of category in their original paper. It is worth mentioning that they explicitly avoided using a set-theoretical terminology and notation in the axioms themselves. Here is how their definition unfolds (only with a slightly different notation):

A *category* $\mathcal{C}$ is an aggregate of abstract elements X, called the *objects* of the category, and abstract elements f, called *mappings* of the category. Certain pairs of mappings f, g of $\mathcal{C}$ determine uniquely a product mapping $g \circ f$, satisfying the axioms C_1, C_2, C_3 below. Corresponding to each object X of $\mathcal{C}$, there is a unique mapping, denoted by 1_X satisfying the axioms C_4 and C_5. The axioms are:

C_1 The triple product $h \circ (g \circ f)$ is defined if and only if $(h \circ g) \circ f$ is defined. When either is defined, the associative law

$$h \circ (g \circ f) = (h \circ g) \circ f$$

holds. This triple product will be written as $h \circ g \circ f$.

C_2 The triple product $h \circ g \circ f$ is defined whenever both products $h \circ g$ and $g \circ f$ are defined. A mapping 1 of $\mathcal{C}$ will be called an *identity* of $\mathcal{C}$ if and only if the existence of any product $1 \circ f$ and $g \circ 1$ implies that $1 \circ f = f$ and $g \circ 1 = g$.

C_3 For each mapping f of $\mathcal{C}$ there is at least one identity 1_r such that $f \circ 1_r$ is defined, and at least one identity 1_l such that $1_l \circ g$ is defined.

C_4 The mapping 1_X corresponding to each object X is an identity.

C_5 For each identity 1 of $\mathcal{C}$ there is a unique object X of $\mathcal{C}$ such that $1_X = 1$.

The last two axioms "assert that the rule $X \longrightarrow 1_X$ provides a one-to-one correspondence between the set of all objects of the category and the set of all its identities. It is thus clear that the objects play a secondary role, and could be entirely omitted from the definition of a category. However, the manipulation of the applications would be slightly less convenient were this done." [Eilenberg and Mac Lane, 1945, 238] Thus, from a theoretical point of view, a category is determined by its mappings, but from a practical point of view, it is convenient to distinguish the objects from the mappings. Eilenberg and Mac Lane then state as a lemma that each mapping f has a unique domain (source) X and a unique codomain (target or range) Y and write $f : X \longrightarrow Y$.

Eilenberg and Mac Lane proceed to define *equivalences in* a category, nowadays called *isomorphisms*, thus: a mapping f is an *isomorphism* if it has an inverse, i.e. if there is a mapping g such that $g \circ f$ and $f \circ g$ are defined and are identities. Two objects X_1 and X_2 are said to be *isomorphic* if there is an isomorphism between them.

Eilenberg and Mac Lane gave four basic examples of categories: the category *Set* of sets with functions between them, the category *Top* of topological spaces with continuous functions, the category $TopGrp$ of topological groups with continuous homomorphisms and the category Ban of Banach spaces with linear transformations with norm at most 1. This is a surprisingly short list of examples. They give more examples by defining the notion of a subcategory in the obvious fashion. Thus, they point out that given a category $\mathcal{C}$, the subcategory composed of the same objects as $\mathcal{C}$ but with mappings only the isomorphisms is a category, nowadays called a *groupoid*. The category of finite sets is also mentioned as well as other subcategories of the category of sets, e.g. for a fixed cardinal k, there is a category of all sets of power less than k together with all the mappings. By restricting the mappings between sets to be onto or injective, one obtains different subcategories of sets. Similarly, if one restricts the continuous maps to open maps between topological spaces, then one obtains a different subcategory of topological spaces. In §11 of their paper, Eilenberg and Mac Lane observe that any group G can be thought of as a category: it has only one object and its mappings are the elements of the group. They also point out in §20 that any preorder P can be viewed as a category.

It should be emphasized how truly secondary categories were for Eilenberg and Mac Lane at that point. In that respect, categories had an ambiguous status. It is clear that categories are *conceptually* required for the systematic and rigorous definition of natural transformations, but at the same time, they cannot be legitimate mathematical entities unless certain precautions are taken with respect to their size. Eilenberg and Mac Lane explicitly recognized this fact in §6 where they discuss foundational issues related to categories, e.g. the category of all sets is not a set, thus not a legitimate entity from the standard set-theoretical point of view.

> It should be observed first that the whole concept of a category is essentially an auxiliary one; our basic concepts are essentially those of a *functor* and of a natural transformation (. . .). The idea of a category is required only by the precept that every function should have a definite class as domain and a definite class as range, for the categories are provided as the domains and ranges of functors. Thus one could drop the category concept altogether and adopt an even more intuitive standpoint, in which a functor such as "Hom" is not defined over the category of "all" groups, but for each particular pair of groups which may be given. The standpoint would suffice for the applications, inasmuch as none of our developments will involve elaborate constructions on the categories themselves. [Eilenberg and Mac Lane, 1945, 247]

Although the definition of category was conceptually necessary, categories themselves were not doing any mathematical work. They were simply a way of systematizing the required data. However, it did not take long, approximately ten to fifteen years, before elaborate constructions on categories themselves became essential and thus the question concerning their nature became more pressing and we indeed see, for instance, Mac Lane, going back to the problem of the foundations of categories in the late fifties and early sixties. (See [Mac Lane, 1961].)

Eilenberg and Mac Lane's definition of a functor is given for n arguments. We will give the definition of a functor with one argument.

A *functor* F between categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functions, an *object-function* which associates to each object X of $\mathcal{C}$ an object $Z = F(X)$ in $\mathcal{D}$ and a *mapping function* which associates to each mapping f of $\mathcal{C}$ a mapping $h = F(f)$, such that

$$1. \quad F(1_X) = 1_{F(X)}$$

$$2. \quad F(g \circ f) = F(g) \circ F(f)$$

Such a functor is said to be *covariant*. Whenever a functor satisfies the equality

$$2^o. \quad F(g \circ f) = F(f) \circ F(g)$$

instead of 2, it is said to be *contravariant*.

Functors with the same domain category and the same codomain category can be connected to one another systematically or "naturally". This is precisely what the notion of natural transformation captures. Here is Eilenberg and Mac Lane's definition, restricted to functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ in one argument.

A *natural transformation* $\tau : F \longrightarrow G$ between functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ is a function that associates to each object X of $\mathcal{C}$ a mapping $\tau_X : F(X) \longrightarrow G(X)$ of $\mathcal{D}$ such that for any mapping $f : X \longrightarrow Y$, the following diagram commutes

$$\begin{tikzcd}
F(X) \arrow[r, "\tau_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\tau_Y"'] & G(Y)
\end{tikzcd}$$

that is, $G(f) \circ \tau_X = \tau_Y \circ F(f)$.

If each τ_X is an isomorphism, then τ is said to be a *natural isomorphism* (Eilenberg and Mac Lane said *natural equivalence*).

Given functors and natural transformations, it is possible to define categories of functors: its objects are functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and its mappings are natural transformations $\tau : F \longrightarrow G$. Eilenberg and Mac Lane find categories of functors "useful chiefly in simplifying the statements and proofs of various facts about functors" [Eilenberg and Mac Lane, 1942, 250] and not in themselves. This is one of the key elements that was about to change drastically in the following years.

Eilenberg and Mac Lane defined two other important notions in their original paper: the dual $\mathcal{C}^o$ of a category $\mathcal{C}$ in §13 and limits and colimits for directed sets in §21 and §22. Dual categories play an important conceptual role in category theory and categorical logic. Given a category $\mathcal{C}$, the dual category $\mathcal{C}^o$ has as its objects those of $\mathcal{C}$; the mappings f^o of $\mathcal{C}^o$ are in one-to-one correspondence $f \rightleftarrows f^o$ with the mappings of $\mathcal{C}$. If $f : X \longrightarrow Y$ is in $\mathcal{C}$, then $f^o : Y \longrightarrow X$ is in $\mathcal{C}^o$. The composition law is defined by the equation $f^o \circ g^o = (g \circ f)^o$, whenever $g \circ f$ is defined in $\mathcal{C}$.

Before we move on, let us now quickly underline what one does *not* find in Eilenberg and Mac Lane's paper. First, although the notion of a subcategory is clearly defined in the paper, properties of the inclusion functor, or for that matter, basic properties of functors in general, for instance being faithful, full and essentially surjective, are not identified.

Although Eilenberg and Mac Lane did define the notion of isomorphism of categories, they did *not* define the notion of *equivalence* of categories. The distinction between the two concepts might seem to be formally subtle, but it is crucial in the applications of category theory. The notion of isomorphism between categories is just the same as the notion of isomorphism between objects in a category: two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be *isomorphic* if there is an isomorphism between them, that is if there are functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ such that $G \circ F = 1_{\mathcal{C}}$ and $F \circ G = 1_{\mathcal{D}}$, where $1_{\mathcal{C}}$ and $1_{\mathcal{D}}$ denote the obvious identity functors. Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be *equivalent* if there is an equivalence between them, that is if there are functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ and natural isomorphisms $\tau : G \circ F \xrightarrow{\sim} 1_{\mathcal{C}}$ and $\rho : F \circ G \xrightarrow{\sim} 1_{\mathcal{D}}$. Thus, in the case of an equivalence, composing the functors F and G does not yield the identity functors, but there are *systematic translations*, namely natural isomorphisms, of the compositions to the identity functors. From the point of view of category theory, the notion of equivalence of categories is fundamental.

Although Eilenberg and Mac Lane introduced functor categories, they do not mention the possibility of a category of categories nor do they notice that natural transformations compose in two different ways. Of course, they did not need these concepts and therefore did not have to consider them at all. Interestingly enough, *all* these notions — functors with specific properties, equivalence of categories and, *in a certain sense*, the category of categories — will play a crucial role in the development of categorical logic in the nineteen sixties. But it can certainly be said without hesitation that the construction that will occupy the center stage of the development of categorical logic is the construction of functor categories.

1.2 Category theory from 1945 until 1963

We will now sketch the development of category theory from 1945 until 1963, underlying the points that will prove to be indispensable for the development of categorical logic.

Although, Eilenberg and Mac Lane introduced and defined the basic concepts of category theory, we believe that it is reasonable to claim that they did not introduce category *theory* as such. Category *theory* started in the late fifties and early sixties. For the *theory* to get off the ground, properties of categories and functors had to be introduced and used systematically. Those arose naturally in specific applications.

Eilenberg and Steenrod quickly applied category theory to algebraic topology. Their book *Foundations of Algebraic Topology* was published in 1952, but it circulated in the form of notes well before that date. It was extremely influential in many different ways. First, because it provided a systematic presentation of algebraic topology and clarified immensely how algebraic topology had to be organized and developed. Second, many students learned algebraic topology from their book and thus assimilated basic categorical concepts along the way. Eilenberg and Steenrod did not develop category theory itself, all the definitions are taken directly from Eilenberg and Mac Lane's paper, but they did use it in such a way that *diagrams* became a fundamental tool in the proofs of various results. Third, homology and cohomology theories were now *functors* and comparing theories amounted to looking at natural transformations between them. Thus, some mathematical objects were now best thought of as being functors between categories.

The same remarks apply to the equally influential book by Cartan and Eilenberg *Homological Algebra*, published in 1956. This book more or less created the subject of homological algebra. It relied heavily on the language of categories and in the use of diagrams in proofs.

Both books brought a shift of focus with respect to the original paper by Eilenberg and Mac Lane. First, it can be said that functors were moving to the front stage or were at the very least just as important as natural transformations. Second, both presented an obvious problem that was about to become a fundamental and general heuristic principle: to find the appropriate category to define and develop a certain aspect of mathematics. In the first case, it was the appropriate setting to express clearly and precisely the duality between homology and cohomology theories. In the second case, it was the search for the appropriate setting for the notion of derived functor. This last heuristic principle led to a second fundamental shift: categories could now be considered in themselves, not only as domains and codomains of functors, but as formal contexts with specific categorical properties in which one could identify, define and develop a specific portion of mathematics.

Thus, these last two problems led to the use of categories within an axiomatic framework. More precisely, it became imperative to define certain concepts, develop specific theories by stipulating that a category satisfied appropriate categorical properties. [Buchsbaum, 1955], [Grothendieck, 1957] and [Heller, 1958] defined the notion of an Abelian category in which a large portion of homological algebra can be carried out from a purely abstract or formal point of view. Furthermore, functor categories became a prominent tool in various fields: algebraic topology, especially homotopy theory, homological algebra, algebraic geometry and more and more constructions on categories were required and seen to be useful. Finally, in 1958, Daniel Kan introduced the notion of adjoint functor, arguably the core concept of the whole theory and which plays a key role in categorical logic as well.

Informally and as a rough heuristic guide, adjoint functors can be thought of as conceptual inverses. The formal definition goes as follows. Two functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ together with natural transformations $\eta : 1_{\mathcal{C}} \longrightarrow G \circ F$ and $\xi : F \circ G \longrightarrow 1_{\mathcal{D}}$ determine *an adjunction* if $G\xi \circ \eta G = 1_G$ and $\xi F \circ F\eta = 1_F$, that is the following triangles commute:

$$\begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\xi"] \\ & G \end{tikzcd}$$

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\xi F"] \\ & F \end{tikzcd}$$

In the foregoing adjunction the functor F is said to be a *left adjoint* to G, denoted by $F \dashv G$, and G is said to be a *right adjoint* to F. As is usual in category theory, adjoints are determined up to a unique isomorphism, that is if $F \dashv G$ and $F \dashv G'$, then there is a unique natural isomorphism $G \longrightarrow G'$.

When Kan introduced the concept of adjoint functor in 1958, he immediately saw the generality of the concept, its usefulness and power in unifying various and apparently different results. Many important theorems and almost all fundamental mathematical constructions can be cast in the frame of adjoint functors. It is also a surprise to see that many fundamental constructions simply appear as adjoints to basic functors. (See [Mac Lane, 1998]; [Adamek *et al.*, 1990]; [Taylor, 1999] for many examples.)

By the time Grothendieck and Kan wrote their seminal papers, in 1955-56, constructions on categories, and in particular the construction and use of functor categories, had become pivotal. Their use of functor categories may not be totally unrelated to the fact that both mathematicians were coming from mathematical fields, namely functional analysis and homotopy theory respectively, in which functional spaces and their properties played a key role.

Adjoint functors and functor categories occupied right from the start a central role in categorical logic. The idea that logical operations, all logical operations, should appear as adjoints to basic functors was one of Lawvere's convictions and motivation.

2 LAUNCHING THE PROGRAM: 1963–1969

> When they introduced the theory of categories in 1945, Eilenberg and Mac Lane suggested the possibility of "functorizing" the study of general algebraic systems. The author has carried out the first steps of this program, making extensive use of the theory of adjoint functors, as introduced by Kan and refined by Freyd.[Lawvere, 1963, 869]

2.1 Basic principles

We now turn to logic proper. It is unquestionable that one person sparked the whole program of thinking about logic and the foundations of mathematics *in general* in a categorical spirit: F. William Lawvere. He did so in his very first work, namely his PhD thesis defended at Columbia under Eilenberg's supervision in 1963. It can certainly be said that the thesis already contains all the basic ideas that have guided Lawvere throughout his career and that have influenced the categorical community greatly. These ideas are in a nutshell:

1. To use the category of categories as a framework for mathematics, i.e. the category of categories should be the foundations of mathematics;

2. Every aspect of mathematics should be representable in one way or another in that framework; in other words, categories constitute the background to mathematical thinking in the sense that, in this framework, essential features of that thinking are revealed;

3. Mathematical objects and mathematical constructions should be thought of as *functors* in that framework;

4. In particular, sets always appear in a category, there are no such thing as sets by themselves, in fact there is no such thing as a mathematical concept by itself;

5. But sets form categories and the latter categories play a key role in the category of categories, i.e. in mathematics;

6. Adjoint functors occupy a key position in mathematics and in the development of mathematics; one of the guiding principles of the development of mathematics should be "look for adjoints to given functors"; in that way foundational studies are directly linked to mathematical practice and the distinction between foundational studies and mathematical studies is a matter of degree and direction, it is not a qualitative distinction;

7. As the foregoing quote clearly indicates, Lawvere is going back to the claim made by Eilenberg and Mac Lane that the "invariant character of a mathematical discipline can be formulated in these terms" [i.e. in terms of functoriality], [Eilenberg and Mac Lane, 1945, 145]. But now, in order to reveal this invariant character, extensive use of adjoint functors is made.

8. The invariant content of a mathematical theory is the "objective" content of that theory; this is expressed at various moments throughout his publications. To wit:

    > As posets often need to be deepened to categories to *accurately reflect the content of thought*, so should inverses, in the sense of group theory, often be replaced by adjoints. Adjoints retain the

> virtue of being uniquely determined reversal attempts, and very often exist when inverses do not. [Lawvere, 1994, 47] (our emphasis)

9. Not only sets should be treated in a categorical framework, but also logical aspects of the foundations of mathematics should be treated categorically, in as much as they have an objective content. In particular, the logical and the foundational are directly revealed by adjoint functors.

As we have said, these ideas, as well as others, are more or less implicit in Lawvere's thesis. Lawvere's goal is general in the sense that it aims at incorporating the whole of mathematics. Furthermore, Lawvere's *usage* of categories reflects a change in their *status* among category theorists in the sixties and seventies. Lawvere recognizes explicitly that categories defined axiomatically constitute autonomous *kinds* or *types* and are, as such, independent of any underlying set-theoretical structures and structure preserving functions. Furthermore, categories become polymorphic: in addition to their usual role, they become the algebraic descriptions of formal systems and, as such, can be thought of as formal systems; but they also provide the underlying framework for semantics and, as such, can be thought of as universe of interpretations.

A summary of the main results of the thesis was communicated to the *Proceedings of the National Academy of Sciences* by Mac Lane and published in 1963. Essentially the same summary was presented at Berkeley in 1963 at a symposium on model theory and later published in 1965 in the proceedings of the meeting. Mac Lane also communicated Lawvere's axiomatization of the elementary theory of the category of sets in the same *Proceedings* in 1964. That axiomatization was not in the thesis as such. The following year, Lawvere presented an explicit axiomatization of the category of categories, published again in the *Proceeding*s in 1966. Another paper, published in 1968, gives an account of the main elements of the thesis together with some new extensions.

In the early and mid-sixties, a certain methodological shift in attention can be detected in the work done by many category theorists. Following Kan and Grothendieck, certain mathematical theories are developed within categories, e.g. homotopy theory, algebra, and the development of these theories is done using the properties of categories more directly. (To mention but a few cases, all published in 1963: Bénabou with his categories with multiplication, C. Ehresmann with his structured categories and Eckmann and Hilton with their group-like structures in categories.) At the same time, adjoints receive more attention and are used more systematically, in particular to define and characterize various structures. Examples are once again provided by Bénabou's work on categories with multiplication, but others, slightly different examples can be found in Eilenberg and Moore and Kleisli on triples, as well as Eilenberg and Kelly with the notion of *closed category.* (See [Bénabou, 1963]; [Eilenberg and Kelly, 1966]; [Kleisli, 1965]). Needless to say, Grothendieck's work in algebraic geometry, based on the use of sheaf theory, is extraordinarily influential, but it is not in pure category theory.

Going back to his original program of clarifying the conceptual content of semantics, Lawvere realized that certain types of categories can be *defined* purely by stipulating that certain adjoint functors to *given elementary* functors exist. The definitions can be given by this data and nothing else. In a loose sense, defining a category via the existence of adjoints amounts to the claim that certain basic conceptual operations can be represented in that category. This in itself would probably not be of foundational relevance, were it not for the fact that the categories so defined correspond in a precise technical sense to *logical* concepts and theories. Thus the existence of certain adjoints to specific elementary functors amounts to a specification of logical structures and resources. With these ideas and results in his pocket, Lawvere could see that a program of "functorizing" the study of mathematical concepts in general could be formulated.

The presentation of these fundamental facts and the program that ensued were made at various conferences in the mid-sixties, published in the form of abstracts in 1966 and a series of papers published in 1969 and 1970. Among the latter, the paper entitled *Adjoints in Foundations* deserves special attention for its general philosophical orientation. It contains the seeds of a categorical program in logic and the foundations of mathematics. [Lawvere, 1969a] Two other papers of that period also contain important parts of that program: first the paper on diagonal arguments presented in 1968 and published in 1969 and the paper on quantifiers and the comprehension schema as adjoints also presented in 1968 and published in 1970. [Lawvere, 1969b; Lawvere, 1970a] The two abstracts published in 1966 are also revealing and influential, for they concentrate on first-order logic. [Lawvere, 1966; Lawvere, 1967] Finally, the discovery of the notion of elementary topos in collaboration with Tierney in 1969/70 provided the general framework in which the whole program could be cast and opened vast and rich possibilities that were unforeseen.[Lawvere, 1970b]

We will now look more carefully at the details of this program. We will start with Lawvere's study of algebraic categories, look briefly at the elementary theory of the category of sets and then move to the so-called elementary theories. We will ignore Lawvere's axiomatization of the category of categories, since it did not have a direct impact on the development of categorical logic. Lawvere's work on the category of categories and the category of sets did not have the same fate as his work on universal algebra. Despite the fact that Lawvere's work on the category of categories suffered from a slight technical flaw, both it and his work on the category of sets were essentially metamathematical and category theory was not yet seen as a potentially useful framework for the latter. Studies on the category of categories that followed Lawvere's pioneering work were mathematically motivated and we speculate that no one saw what to do with the category of sets. It simply did not have a clear function. His work on algebraic theories, however, inspired much of what was to follow in logic, including Lawvere's own work, and it still constitutes the starting point of what are now called "doctrines", a term suggested to Lawvere by John Beck, in categorical logic and the categorical approach to universal algebra.

2.2 Lawvere's thesis: 1963

> Lawvere's imaginative thesis at Columbia University, 1963 contained his categorical description of algebraic theories, his proposal to treat sets without elements and a number of other ideas. I was stunned when I first saw it; in the spring of 1963, Sammy and I happened to get on the same airplane from Washington to New York. He handed me the just completed thesis, told me that I was the *reader*, and went to sleep. I didn't. ([Mac Lane, 1988, 346].)

> Essentially, algebraic theories are an invariant notion of which the usual formalism with operations and equations may be regarded as "presentation". [Lawvere, 1963, ii]

The main concept of Lawvere's thesis is the notion of *algebraic category.* The main result of the thesis is a *categorical* characterization of algebraic categories. Together with algebraic categories, Lawvere also introduced *algebraic theories* and *algebraic functors.* The three notions are intimately connected to one another. As Lawvere pointed out himself, there is a strong analogy between the way his work is developed and the theory of sheaves that had been just introduced at that time: in the same way that Grothendieck had provided an abstract characterization of categories of sheaves on topological spaces, Lawvere's goal was to characterize algebraic categories in a similar manner. The main tool of the thesis and what provides the key to the connections between these notions are adjoint functors. Thus, they constitute the methodological core of the thesis and of the whole approach. The framework is presented as a new foundation for universal algebra. In the very first chapter of the thesis, thus the underlying context of the work, there is a sketch of a first-order theory of the category of categories. Within that context sets are defined in categorical terms, the notion of equivalence of categories is given, as well as the category of small categories, the category of large categories, the category of finite sets, the category of small sets, and the category of large sets. A categorical version of the Peano postulates is also given.[1] But the bulk of that chapter is occupied by the presentation and development of the notions of adjoint functors and limits. In the second chapter, algebraic theories are introduced and the category of algebraic theories is defined and various properties of the category are proved, e.g. the existence of an adjoint that corresponds to the existence of free algebraic theories. Chapter three deals with algebraic categories (and here Lawvere explicitly exploits the analogy with sheaves). The notions of algebraic semantics and algebraic structure are defined and a categorical characterization of algebraic categories is given. Chapter IV deals with algebraic functors and their adjoints. Finally, in chapter V, particular cases and extensions are considered.

Let us look at the central concepts and results of the thesis and see how the invariant content of universal algebra is analyzed.

[1] These axioms are sometimes called the Peano-Lawvere axioms. See for instance [Birkhoff and Mac Lane, 1967, 67].

First comes the notion of an algebraic theory. A group is usually thought of as a set together with some specified operations, e.g. multiplication, inverse and unit, satisfying certain identities, e.g. associativity, unit law and inverses. Formally, this is encoded by a signature, e.g. $(\times, (-)^{-1}, e)$ or some other, with the standard axioms. But it is clear that the signature can in general vary and so does the choice of axioms. These choices determine, though, the *same* theory, in the sense that all the definable operations and all the theorems are the same.[2] Lawvere's idea is to define a category which will encode all the information at once, thus independently of the choice of signature and axioms and call *that* category the theory. Thus, the latter category would in some sense be the objective encoding of the theory, independent of any presentation of it. This is in stark contrast with the traditional logical approach in which a theory T depends directly on a choice of primitive symbols, its signature, and a choice of axioms. Strictly speaking, changing the axioms changes the theory, although, for instance we still talk about *the* theory of groups.

Here is the general definition:[3]

Definition: an *algebraic theory* is a (small) category $\mathcal{A}$ such that:

(i) the objects of $\mathcal{A}$ is a denumerable set $\{A^0, A^1, A^2, \ldots, A^n, \ldots\}$ of distinct objects;

(ii) each object A^n is the product of A^1 with itself n times; thus, for each n, the projection maps $\pi_i^{(n)} : A^n \longrightarrow A$, for $i = 0, 1, \ldots, n-1$, exist;

(iii) for any n morphism $A^m \xrightarrow{\vartheta_i} A$, for $i = 0, 1, \ldots, n-1$, in $\mathcal{A}$, there exists exactly one morphism $A^m \xrightarrow{\langle \vartheta_0, \vartheta_1, \ldots, \vartheta_{n-1} \rangle} A^n$ such that $\pi_i^{(n)} \circ \langle \vartheta_0, \vartheta_1, \ldots, \vartheta_{n-1} \rangle = \vartheta_i$, for $i = 0, 1, \ldots, n-1$.

The arbitrary morphisms $A^n \xrightarrow{\eta} A$ are the *n-ary operations* of $\mathcal{A}$.

Thus, the theory, thought of as a category, contains all the possible operations systematically.

The underlying motivation is very simple and makes perfect sense once the notion of an *algebra of type* $\mathcal{A}$, also called an $\mathcal{A}$*-algebra*, has been given: it is simply a product preserving functor from an algebraic theory to the category of sets, $F : \mathcal{A} \longrightarrow Set$. Thus, $F(A)$ picks a set, and $F(A^n)$ is simply an n-fold product of $F(A)$, i.e. $F(A) \times \cdots \times F(A)$ n-times. An operation $A^n \xrightarrow{\eta} A$ becomes a standard set-theoretical operation $F(A) \times \cdots \times F(A) \xrightarrow{F(\eta)} F(A)$. Notice that

[2]The second author recalls that in one of his courses in the late 60's at Berkeley, Tarski told the class that there wasn't such a thing as *the* theory of groups, but there were infinitely many theories of groups.

[3]We are here presenting the direct definition given by Lawvere in his published papers. In his thesis, Lawvere defines the *category* of algebraic theories as a subcategory of the category of finite sets, which is itself a category in the category of categories. Pareigis' presentation is, in this respect, more faithful to Lawvere's original work. See [Pareigis, 1970].

an algebra of type $\mathcal{A}$ *is* a *functor*. It is also called a *model* of the theory $\mathcal{A}$. Thus, in particular, if $\mathcal{A}$ is the theory of groups, then each and every group *is* a functor.

Two categories can now be defined: 1. The category $\Im$ of algebraic theories whose objects are algebraic theories, morphisms are functors preserving products and taking 1 to 1; 2. The functor category $Set^{(\mathcal{A})}$ of all *product preserving* functors $\mathcal{A} \longrightarrow Set$, which can be thought of as the category of models of the theory $\mathcal{A}$. The latter category is called an *algebraic category.*

A morphism $f : \mathcal{A} \longrightarrow \mathcal{B}$ of algebraic theories induces a functor $Set^{(f)}$: $Set^{(\mathcal{B})} \longrightarrow Set^{(\mathcal{A})}$ by composition. The latter functor is called an *algebraic functor.* Furthermore, for any algebraic category $Set^{(\mathcal{A})}$, there is an obvious forgetful functor $U_{\mathcal{A}} : Set^{(\mathcal{A})} \longrightarrow Set$ which sends to each object $F(A^n)$ the underlying set and to each morphism the underlying set map. Notice that the forgetful functor is an algebraic functor.

Algebraic functors lead to another construction, named *algebraic semantics*:it assigns to each algebraic theory $\mathcal{A}$, the forgetful functor $U_{\mathcal{A}}$ and to each morphism $f : \mathcal{A} \longrightarrow \mathcal{B}$ of algebraic theories the algebraic functor $Set^{(f)} : Set^{(\mathcal{B})} \longrightarrow Set^{(\mathcal{A})}$. This is in fact itself a functor, sometimes called the *semantic functor*,

$$\mathrm{S} : \Im^{op} \longrightarrow \mathrm{K}$$

where K is the category of algebraic categories.

With these definitions, Lawvere's main results are:

1. Every algebraic functor has a (left) adjoint. This is the conceptual and unified formulation of various constructions in universal algebra, e.g. free algebras, tensor algebras, monoid rings, etc.

2. Algebraic semantics has a (left) adjoint, which can be called *algebraic structure.* This means that it is possible to recover an algebraic theory from the semantics, i.e. from the category of models.

3. The categorical characterization of algebraic categories: if $\mathcal{C}$ is a category with finite limits, has an abstractly finite regular projective generator G and every precongruence in $\mathcal{C}$ is a congruence, then there is an algebraic theory $\mathcal{A}$ and an equivalence $\Phi : \mathcal{C} \longrightarrow Set^{(\mathcal{A})}$. (There is no need to specify what the second and third conditions mean here. They are technical conditions that we do not have to look into.)

These results were not only remarkable for what they accomplished, but also for the research avenues they opened. These three points in themselves will become guidelines in categorical logic. There were some obvious generalizations that were taken up rapidly by others and not so obvious generalizations that had to wait for other concepts to be fully worked out.

1. The obvious generalization was to consider infinitary operations and related work in universal algebra. This was done quickly by Linton. (See [Linton, 1966b], [Linton, 1966a].)

2. One of the great advantages of the categorical language is that it is possible to replace the category *Set* of sets by an arbitrary category $\mathcal{C}$ with appropriate properties. [Eilenberg and Moore, 1965] and [Barr and Beck, 1966] used triples (monads) to extend Lawvere's work over an arbitraty base category. [Bénabou, 1968] considered the case of many-sorted theories. The identification of the properties of $\mathcal{C}$ required to do the work, expressed in categorical terms, leads to a *classification* of logical categories in *categorical* terms. The category *Set* of sets becomes a special, but very important, case of a type of category defined abstractly. Lawvere has given a characterization of algebraic categories. Further work lead to characterizations of similar categories, i.e. categorical characterization of semantic frameworks.

3. An algebraic theory as defined by Lawvere can be thought of as a data type.[4] Lawvere's work shows how syntactical information of a specific kind can be encoded by categories. The search for a proper generalization to cover all types of logical theories, not only the algebraic or equational case, is irresistible. More specifically, the task is to find a general procedure to move from a theory written in a given formal system to a category that would be the invariant formulation of the latter. The notion of algebraic theory was specifically tailored for algebraic structures and it is not clear how one can go from there to other cases, e.g. cases with quantifiers and relations. In particular, Lawvere considered single-sorted theories and a generalization to many-sorted seems natural, although in traditional logical presentations, we are used to the single-sorted case.

4. Once an element of one or all the previous points have been settled, the next task consists in looking at the various adjoint situations and see what one can obtain from them. For instance, it appears clearly from Lawvere's work that the adjoint situation is a special case of an algebraic duality and its importance is due to the fact that it is the very first case of such a duality where the category of sets appears as the dualizing object.

Mathematicians and logicians took up these tasks soon after Lawvere's thesis. Important results were obtained in the late sixties by Lawvere himself, but also by Lambek, Freyd, Linton, Isbell and by Gabriel and Ulmer. (See [Gabriel and Ulmer, 1971].) Parallel and independent developments were obtained by Ehresman and his students, most notably Bénabou, also during that period. Ehresman's motivation was different and was mainly oriented towards the foundations of differential geometry, but it lead him to the notion of sketch which was developed by his students C. Lair and R. Guitart in the seventies and independently rediscovered by M. Makkai and R. Paré in the eighties. (See [Makkai and Paré, 1989].)

[4]This is more than a metaphor. There is indeed a formal connection with data bases. See for instance [Plotkin, 2000].

2.3 The Elementary theory of the category of sets

Let us now briefly turn to Lawvere's axiomatization of the category of sets.

We should emphasize right from the start the fact that Lawvere never tried to *eliminate* sets altogether from the foundational landscape, but rather he tried to provide a *categorical* analysis of sets, that is a characterization of the category of sets. Mac Lane, for one, at first thought that the idea was absurd.

> ... and then [Lawvere] conceived the idea of giving a direct axiomatic description of the category of all categories. In particular, he proposed to do set theory without using the elements of a set. His attempt to explain this idea to Eilenberg did not succeed; I happened to be spending a semester in New York (at the Rockefeller University), so Sammy asked me to listen to Lawvere's idea. I did listen, and at the end I told him "Bill, you can't do that. Elements are absolutely essential to set theory." After that year, Lawvere went to California. [Mac Lane, 1988, 342]

Lawvere's basic insight is that, even for sets, one can know them by looking at the invariant content of the category of sets.

> Thus we seem to have partially demonstrated that even in foundations, not Substance but invariant Form is the carrier of the relevant mathematical information. [Lawvere, 1964, 1506]

> Some years ago I began an introductory course on Set Theory by attempting to explain the *invariant content* of the category of sets, for which I had formulated an axiomatic description. [Lawvere, 1994, 5] (our emphasis)

Let us briefly look at this invariant content, at least as Lawvere saw it in the early sixties.

Lawvere assumes the standard axioms for a category and then postulates that the category of sets has a terminal object 1, an initial object 0, binary products and coproducts, equalizers and coequalizers, thus all finite limits and colimits (axiom 1). He also assumes that it is Cartesian closed, i.e. that the object B^A, together with the known morphism and universal property, exist for any A and B (axiom 2). These two axioms are "structural" and are satisfied by many categories. Thus, as such they do not characterize a category of sets.

The next axiom is known as the Peano-Lawvere characterization of the natural numbers. It is therefore an axiom of infinity, but it contains more when it is assumed with the previous axioms.

Axiom 3: There is an object N together with morphisms $1 \xrightarrow{z} \mathrm{N} \xrightarrow{s} \mathrm{N}$ such that given any object X together with mappings $1 \xrightarrow{x_0} \mathrm{X} \xrightarrow{s} \mathrm{X}$, there is a unique morphism $\mathrm{N} \xrightarrow{x} \mathrm{X}$ such that $x_0 = z \circ x$ and $x \circ t = s \circ x$.

This is now the standard characterization of the natural number system by a universal property. A theorem on primitive recursion follows from it and the preceding axioms. However, Lawvere points out that it is satisfied, as well as the first two axioms, by the category **1**, i.e. the category with one object and one morphism. Thus, more is needed, both structurally and in terms of existence.

The next axiom is false in the category of categories, for it says that the terminal object 1 is a *generator*, i.e. if the morphisms $A \underset{g}{\overset{f}{\rightrightarrows}} B$ are different, then there is a morphism $x : 1 \longrightarrow A$ such that $f \circ x \neq g \circ x$. In more colloquial language, this axiom states that if f and g are different, then there is an element x, or a point, in A such that f and g are different on that element.

The fifth axiom is the axiom of choice. The last three axioms are *not* expressed in a categorical fashion. Surprisingly perhaps, Lawvere reintroduced the $\in$ notation in his axiomatization. Since there is a one-to-one correspondence between the morphisms $x : 1 \longrightarrow A$ and the elements $x \in A$ in the universe of sets, it seems reasonable to say that x is an *element* of A if and only if $x : 1 \longrightarrow A$. The remaining axioms are then expressed as follows:

Axiom 6: If A is not an initial object, then A has elements.

Axiom 7: An element of a sum is a member of one of the injections.

Axiom 8: There exists an object with more than one element.

Lawvere underlined the fact that the first seven axioms are satisfied by the category **1** with one morphism, thus the need for the eighth axiom. One could certainly object to the introduction of the $\in$ relation and the notion of element in a categorical framework and claim that this part of the axiomatization does not capture the invariant form of the category of sets.

Be that as it may, the claim that the invariant form of the concept of set is captured by the axioms is substantiated by a metatheorem and its corollary. Indeed, the metatheorem asserts that any two categories satisfying the eight axioms are equivalent, thus in particular any (complete) category satisfying the eight axioms is equivalent to the category of sets. Lawvere is in a position to state precisely what it means to be "the" category of sets: the properties are (mainly) expressed in categorical terms and the invariance amounts to the claim that any other category satisfying these properties is equivalent to it.

Lawvere's work did not open the door to further investigation along similar lines. The category of sets was not taken as a foundational framework. It was not studied and explored. Although there is no clear explanation of this fact, Johnstone suggested that the category of sets is simply too "rigid":

> In retrospect, the answer is that Lawvere's axioms were too specialized: the category of sets is an extremely useful object to have as a foundation for mathematics, but as a subject of axiomatic study it is not (*pace* the activity of Martin, Solovay *et al.*!) tremendously interesting

> — it is too "rigid" to have any internal structure. [Johnstone, 1977, xiii]

It is not exactly clear why it is such "an extremely useful object to have as a foundation for mathematics" if nothing is developed in it and no one proceeds to do more research within it. It seems that one of the most common reactions at the time was that ETCS was merely a translation in categorical terms of the standard axioms of set theory and that, as such, did not provide any genuinely new insight into the nature of sets (in contrast with the analysis provided with the axioms of elementary toposes, which was about to appear). Sociological and historical factors must probably be invoked also. Lawvere's paper appeared in 1964, very shortly after Cohen's proof of the independence of the continuum hypothesis was published. (See [Cohen, 1963].) Cohen's result, and probably most notably his method of forcing, deservedly attracted much attention. There is very little doubt that one of the key advantages of topos theory over ETCS is precisely that the former bridges the gap between a categorical description of sets and the method of forcing, whereas the latter is a category of sets satisfying the axiom of choice. It should be recorded that Miles Tierney's proof of the independence of the continuum hypothesis in a topos theoretical setting is done with respect to Lawvere's ETCS. The latter is taken to be the categorical expression of ZF set theory, reinforcing the impression that ETCS is nothing more than a translation.

Although the ETCS did not attract much attention, the general program was launched and it did not take long for Lawvere and others to see how logic could be and perhaps should be developed in a categorical setting.

2.4 Categorical logic: the program

> Categorical logic, in a very broad sense, can be seen to derive from the completeness and exactness properties of the category of sets, in a manner paralleling the earlier development of Abelian categories. [Makkai and Paré, 1989, 5]

Lawvere pursued his work in the categorification of logic and presented two papers at the Meeting of the Association of Symbolic Logic, one in 1965 and the other in 1966. Abstracts of these talks were published in *The Journal of Symbolic Logic* in the following year. The first one was received the 17 August 1965 and the second almost a year later, the 12 August 1966. They were published a year later. (See [Lawvere, 1966; Lawvere, 1967].) The second note offers a simplification of the first attempt. It is entirely clear that Lawvere is trying to extend his results obtained for algebraic theories to first-order theories in general. In these two notes, three elements stand out. First, it is the first time in print that the existential quantifier is presented as an adjoint. Second, every first-order theory with equality corresponds to a category with certain properties that Lawvere calls an *elementary theory*. In particular, certain elements of how semantics of first order languages ought to be done in a category are suggested. Third, in the second abstract, Lawvere sketches a categorical proof of the completeness theorem that amounts to the existence of specific adjoint functors. These abstracts were in fact influential and did serve as guides in the late sixties and early seventies for a categorical analysis of first-order logic. Here are the essential elements.

We will here present the notions as they are defined in 1966, for it is the latter that were referred to by various logicians who followed Lawvere's footsteps.

An *elementary theory* $\mathbb{T}$ is a small category with finite products, including the empty product 1, satisfying three conditions:

1. There is a distinguished object B together with two morphisms $1 \underset{\perp}{\overset{\top}{\rightrightarrows}} B$ such that for all objects X, the morphisms $\langle 1_X, \top_X \rangle : X \longrightarrow X \times B$ and $\langle 1_X, \perp_X \rangle : X \longrightarrow X \times B$, where $\top_X : X \longrightarrow 1 \xrightarrow{\top} B$ and $\perp_X : X \longrightarrow 1 \xrightarrow{\perp} B$, constitute a categorical coproduct of X with itself, that is $X \times B = X + X$; it follows that for each object X of $\mathbb{T}$, the functor $Hom(X, B)$[5] is in fact a Boolean algebra;

2. For any $f : X \longrightarrow Y, \varphi : X \longrightarrow B$, there is a morphism $\exists_f[\varphi] : Y \longrightarrow B$ such that for all $\psi : Y \longrightarrow B, \exists_f[\varphi] \leq_Y \psi$ if and only if $\varphi \leq_X \psi \circ f$;

3. There is a second distinguished object A of $\mathbb{T}$ such that every object X of $\mathbb{T}$ has a unique representation of the form $A^n \times B^k$, where n, k are finite non negative integers.

Morphisms $A^n \longrightarrow A$ are thought of as n-ary terms and morphisms $A^n \longrightarrow B$ are thought of as n-ary formulas. In particular, $Hom_{\mathbb{T}}(1, B)$ is the set of sentences of $\mathbb{T}$. A *model* of $\mathbb{T}$ is any product preserving functor $M : \mathbb{T} \longrightarrow Set$ which takes B to a two-element sets and which takes each $\exists_f[\varphi]$ to the image of φ under f, that is if, as above $f : X \longrightarrow Y$ and $\varphi : X \longrightarrow B$, then identifying $M(B)$ with the two-elements set $\mathbf{2}$, we have that an element $y \in M(Y)$ is in $M(\exists_f[\varphi])^{-1}(1)$ if and only if *there exists* $x \in M(\varphi)^{-1}(1)$ such that $M(f)(x) = y$. The set $M(A)$ is called the *universe* of the model. Morphisms of models are natural transformations, thus yielding a category $Mod_{Set}(\mathbb{T})$ of set-models. Notice that the category Set satisfies all the axioms, with the only exception that it is not small. A morphism of theories $I : \mathbb{T} \longrightarrow \mathbb{T}'$ is a product preserving functor that preserves quantification. Lawvere notices that there is an induced functor $I^* : Mod_{Set}(\mathbb{T}') \longrightarrow Mod_{Set}(\mathbb{T})$ which preserves universes.

In his first abstract, Lawvere extends the claim we find in his thesis to elementary theories as follows. Let $\mathfrak{T}$ be the category of elementary theories with morphisms of theories $I : \mathbb{T} \longrightarrow \mathbb{T}'$ and $\mathfrak{M}$ the category of set-models. Then, as indicated in the foregoing paragraph, there is a functor $\Phi : \mathfrak{T}^{op} \longrightarrow \mathfrak{M}$, called *elementary semantics* and Lawvere claims that it has an adjoint, called *elementary structure*. No details are given.

[5] $Hom(X, Y)$ denotes the collection of morphisms from X to Y in the category.

The main claim of the second abstract is a completeness theorem. The precise statement of the theorem is not given in the abstract. Lawvere enumerates three conditions that constitute the core of the *proof* of the theorem: it sketches how, given an elementary theory $\mathbb{T}$, a model M of $\mathbb{T}$ can be constructed.

As we will see, the notion of an elementary theory served as a guide in the search for the correct categorical characterization of first-order logic. Also, the description of the quantifiers as adjoints to substitution, more generally the description of the logical operations as adjoints and the basic element of the semantics left a definitive impact. Interestingly enough, Lawvere himself suggests, in his first abstract, that a given theory $\mathbb{T}$, as a category, could be thought of as the *"Sinn"* of a theory presented in a first order language, and the category $Mod_{Set}(\mathbb{T})$ could be thought as the *"Bedeutung"* of the theory. Whether this can be defended and how it compares to other analyses of sense and reference, for instance Frege's, remains to be clarified and as far as we know, no one took it up afterwards. (But look at Bell's *Epilogue* in his textbook on topos theory, i.e. [Bell, 1988, chapter 8]. Awodey's paper on structuralism is also relevant here, [Awodey, 1996].)

Here is how quantifiers can be thought of as adjoints to substitution.Recall that if $f : X \longrightarrow Y$ is a function between sets and $B \subseteq Y$, then the pullback of B along f is the subset of X defined by $f^*(B) = \{x \in X : f(x) \in B\}$. By considering B as a predicate of Y, say $B(y)$, the pullback may be considered as the predicate of X obtained from $B(y)$ by substituting $f(x)$ for y. Lawvere's remark was that the pull-back formation

$$f^* : \wp(Y) \longrightarrow \wp(X)$$

which is a functor between posets has a left adjoint and a right adjoint

$$\exists_f \dashv f^* \dashv \forall_f$$

Indeed, one can easily check that

$$\begin{aligned}\exists_f(A) &= \{y \in B : \exists_x f(x) = y \text{ and } x \in A\} \text{ and}\\ \forall_f(A) &= \{y \in B : \forall_x f(x) = y \text{ implies } x \in A\}.\end{aligned}$$

This was a key observation that convinced many mathematicians that this was the right analysis of quantifiers. They arise naturally as adjoints to an elementary operation, namely substitution, which appears as *the* basic operation of first-order logic, contrary to the classical view which defines this operation by recursion, as a derived one. By the way, this recursive definition is not universally valid. In particular, it gives wrong results in some non-classical logics (e.g. co-Heyting). Again, apart from the intrinsic interest of the conceptual analysis it provides, this point of view generalizes to categories other than the category of sets. Once the *categorical* properties of sets used in the characterization have been identified and defined, it is possible to transpose the analysis in new contexts, i.e. in other categories.

After 1966, Lawvere's own published work extends the connections between categories and logic. Three papers published in 1969 and 1970 are extremely important since they contain the seeds and the statement of a vast foundational program which was taken up and is still alive. One of the surprising features of these papers is that none of them contain a bibliography. There is no reference to other papers, books or earlier related work, which we believe can be taken as an sign of the originality of the work presented. Here are the basic methodological elements underlying this program.

1. The use of adjoint functors is emphasized both in practice and also from a more general point of view. In his paper *Diagonal arguments and Cartesian closed categories*, Lawvere defines a CC category as a category $\mathcal{C}$ equipped with three kinds of right adjoints:

1. a right adjoint 1 to the unique $\mathcal{C} \longrightarrow 1$;
2. a right adjoint $(- \times -)$ to the diagonal functor $\Delta : \mathcal{C} \longrightarrow \mathcal{C} \times \mathcal{C}$;
3. and for each object A in $\mathcal{C}$, a right adjoint $(-)^A$ to the functor $A \times - : \mathcal{C} \longrightarrow \mathcal{C}$, [Lawvere, 1969b]].

Before that paper, adjoints were used by Lawvere to show that any category satisfying certain specifications was equivalent to a fixed category of interest or to establish certain properties of given functors in a context. In this Lawvere was following the examples of Abelian categories and sheaves. As we have seen, the axioms for the category of sets did not mention adjoints explicitly and the notion of elementary theories stipulated the existence of two distinguished *objects* of a category. Right from the beginning, Lawvere emphasized the fact that Cartesian closed categories are algebraic versions of type theories. Notice two important facts in the foregoing characterization of a CC category: first, the first two adjoints are adjoints to basic functors or structural functors and as such arise "naturally" and express fundamental connections; second, the third adjoint depends on the second and, in this way, there is a hierarchy or an order in the way the fundamental operations are defined or related to one another. This is a key element of the analysis of logical operations in a categorical setting.

2. In connection with the role of adjoint functors, there is an *explicit* recognition of the levels of abstraction introduced by category theory, more specifically the fact that category theory now allows for purely abstract characterizations of mathematical domains.

> More recently, the search for universals has also taken a conceptual turn in the form of Category Theory, which began with viewing as a new mathematical object the totality of all morphisms of the mathematical objects of a given species A, and then recognizing that these new mathematical objects all belong to a common non-trivial species C *which is independent* of *A*. [Lawvere, 1969a, 281]. (our emphasis)

What is described by Lawvere should be clear : in the beginning, mathematicians started with already defined mathematical objects and structure-preserving functions and moved to a new object, namely the *category* of these objects of species *A*.

But it was soon realized that such a category participated in a different species, namely a category of type C, which can be described *independently* of A. The first historical example of this phenomenon was provided by Abelian categories: one started with Abelian groups (or modules over a commutative ring), the latter constituting the objects of species A, moved to the category of Abelian groups (or modules over a commutative ring) and then to an Abelian category, which is a category of type C. From a conceptual and as well as a foundational point of view, the crucial step is that the latter can be described independently of the former. Thus, although a category of type C arises from given structures, and in the last century, that meant more often than not, structured sets with structure-preserving functions between them, once the abstract description has been given, it is possible to develop mathematics directly from categories of type C. Lawvere makes a bold generalization: he sees this case as a *general* phenomenon, even as a framework that should guide the development and analysis of mathematics. Of course, Abelian categories did not constitute his only example, e.g. algebraic theories and categories, Cartesian closed categories and hyperdoctrines, etc.

3. Lawvere uses Cartesian closed categories to present an analysis of well-known diagonal arguments, i.e. those of Cantor, Russell, Gödel and Tarski.The motivation is similar to the one indicated in the foregoing paragraph: these diagonal arguments are similar and thus seem to form a *species* of argument. Lawvere hopes to be able to disclose the common abstract structure underlying them. This abstract structure takes the form of a fixed point theorem based on the properties of Cartesian closed categories. (For an excellent presentation of these results together with work inspired by Lawvere along these lines, see [Yanofsky, 2003].) In the process of his analysis, Lawvere introduces an object of truth-values **2**, for this object appears in one way or another in all the arguments. This object, which will be called *the object of truth-values* and had already been introduced in the context of elementary theories, as well as the Cartesian closed structure, will become pivotal in his characterization of the notion of elementary topos in 1969. However, the object of truth-values will constitute an *obstacle* for the characterization of first-order logic.

4. In all three papers, Lawvere suggests extensions of his earlier work on algebraic theories to theories written in higher-order type theories and, as a special case, first-order theories. We should point out immediately that he was no longer alone in looking for connections between categories and logic. Lambek's work on categorical analysis of deductive systems has to be mentioned at this stage. (See [Lambek, 1968a; Lambek, 1969; Lambek, 1972].) We will get back to Lambek's work later.) Lawvere's extensions are based on the following fundamental facts:

1.1. As we have already seen, the logical quantifiers can be presented as adjoint functors to the simple and fundamental operation of substitution (the analysis is extended in [Lawvere, 1969c; Lawvere, 1970a]);

1.2. The comprehension principle can be presented as an adjoint functor in a proper context [Lawvere, 1969a; Lawvere, 1970a];

1.3. Lawvere sketches how one can construct a category from a given theory formalized in higher-order logic [Lawvere, 1969b; Lawvere, 1970a].

We have to underline the fact once more that a categorical analysis of logical systems not only provided a novel and unifying understanding of logical operations and systems, but by the same token, it initiated a shift in the status of categories *themselves*. It is now possible to *identify* a type of category with a type of deductive system. The claim that category theory can be seen as a language can now be made more precise: category theory can be seen as a *formal* language for mathematics. However, it should also be emphasized that in 1969, these were all programmatic claims.

5. All these constructions are incorporated in a general framework that constitutes nothing less than the scaffolding of categorical logic as it developed afterwards.[6] This framework is presented in very broad strokes at the beginning and the end of the papers entitled "Adjointness in Foundations" and is itself of considerable philosophical interest.

Lawvere identified two aspects, which he qualifies as being "dual", since they appear to obey some sort of general duality inherent to mathematics, namely the Formal and the Conceptual. The Formal is more or less identified with the manipulation of symbols, either in deduction or calculations, whereas the Conceptual is identified with the content of these symbols, the subject matter of the Formal or what they refer to. Thus, at first sight, Lawvere's terminology coincides with the classical distinction between the syntax and the semantics of formal languages. However, Lawvere has the actual practice of mathematics in mind and therefore does not equate his distinction with the fundamental metamathematical distinction. In fact, he sees foundational research as being part of mathematics: "Being itself part of Mathematics, Foundations also partakes of the Formal-Conceptual duality." [Lawvere, 1969a, 281]. Thus, the syntax of a logical system is part of the Formal, whereas the semantics is part of the Conceptual. However, his presentation of the semantics is somewhat odd: "Naturally the formal tendency in Foundations can also deal with the conceptual aspect of mathematics, as when the semantics of a formalized theory $\mathbb{T}$ is viewed itself as another formalized theory $\mathbb{T}'$, or in a somewhat different way, as in attempts to formalize the study of the category of categories." [Lawvere, 1969a, 281]. Category theory is clearly put on the conceptual side of mathematics and, in fact, one can see that Lawvere sees his work on the foundations of universal algebra and the subsequent work on Cartesian closed categories and extensions thereof as being part of the *conceptualization* of the formal aspect of mathematics.[7] Indeed, he claims explicitly that "Foundations may conceptualize the formal aspect of mathematics, leading to Boolean algebras, cylindric and polyadic algebras, and certain of the structures discussed

[6] At least before categorical logic became a standard tool in theoretical computer science. But even in this context, one can see the influence of Lawvere's suggestions.

[7] Indeed, in the nineties Lawevere published, in collaboration with Steve Schanuel, a book entitled "Conceptual Mathematics" (see [Lawvere and Schanuel, 1997]).

below." [Lawvere, 1969a, 282]. At the center of this conceptualization appear adjoint functors.

Adjoints are present in Foundations in two senses.

1. Lawvere introduces Cartesian closed categories and what he calls hyperdoctrines in that paper. The main property of these two concepts is that they are *entirely* given by adjoint functors. However, as we have already mentioned, Cartesian closed categories and hyperdoctrines correspond in a precise technical sense to logical frameworks. Thus, both Cartesian closed categories and hyperdoctrines are categorical codifications of logical structures, the algebraic counterpart of these structures. Adjoint functors are used to *define* the conceptual content of foundations;

2. However, adjoint functors also play a more general role. The Formal and the Conceptual mentioned above should be, but we are clearly at a programmatic stage here, related by adjoint functors. Lawvere is in fact more precise in the way these adjoints should show up and here we see Lawvere generalizing the work contained in his thesis. First, Lawvere suggests that one should consider *categories* of models of a theory, thus framing model theory in the context of category theory.[8] More specifically, a model, in the standard model theoretic sense, can de described as a *functor* from a category $\mathbb{T}$ to the category Set of sets. The category of such models is then a subcategory of the functor category $Set^{\mathbb{T}}$, or also written $\mathrm{Mod}(\mathbb{T}, Set)$ or, $Mod_{Set}(\mathbb{T})$ as we have already done. Second, the category $\mathbb{T}$ is the categorical encoding of a given formal theory. In Lawvere's own term "The *invariant* notion of theory here appropriate has, in all cases considered by the author, been expressed most naturally by identifying a theory T itself with a category of a certain sort" [Lawvere, 1969a]. We emphasize the fact that Lawvere is looking, once again, for an *invariant* notion of a theory and that this invariant notion is provided by a category. Third, and adjoint functors enter the picture explicitly at this stage, in principle there ought to be an adjoint pair of functors encapsulating the general duality expressed above

$$\text{Theories}^{op} \underset{\text{structure}}{\overset{\text{semantics}}{\rightleftarrows}} \text{Conceptual}.$$

The conceptual is identified with the category of models of $\mathbb{T}$.[9] However, the formal is *not* identified with the invariant formulation of the theory, since clearly there are aspects of the formal, e.g. specific rules of computation or derivation, that are inherent to a formal framework. Therefore, Lawvere suggests that there

[8] We have to point out that Freyd had made explicit connections between category theory and model theory before. See [Freyd, 1965].

[9] We are not entirely faithful to Lawvere here. He takes the conceptual to be the functor category $\mathrm{Mod}(Cat, [Set^{\mathbb{T}}])$, which is of course slightly more general than what we have been describing.

is a further adjoint situation, left unspecified this time

$$\text{Formal} \rightleftarrows \text{Theories}$$

which describes "the presentation of the invariant theories by means of the formalized languages appropriate to the species" [Lawvere, 1969a, 295]. Since adjoint functors compose, one should get a family of adjoint functors

$$\text{Formal}^{op} \rightleftarrows \text{Conceptual}$$

which we started with.

As such, this description is completely programmatic. It is clearly a bold generalization of Lawvere's thesis. It is given as a completely general framework for foundational research. It is taken as being faithful to the essential elements of mathematical knowledge. Logicians who were about to enter the scene picked up that program and started to develop it systematically. It lead to what were called afterwards "categorical doctrines" and is presented as such by Kock and Reyes in their survey paper (see [Kock and Reyes, 1977]). Thus, we strongly disagree with the claim made by Corry:

> Lawvere himself proposed in an article of 1969 to connect the concept of duality, and other categorical concepts, with the epistemological issues related to the philosophy of mathematics. In order to do that, he identified two "dual aspects" of mathematical knowledge — the conceptual and the formal aspects — which appear in many domains of mathematics. .... Now Lawvere proposed to dedicate efforts to develop the second aspect, the conceptual one, embodied in category theory. This proposal, however, remained at the programmatic level and no one seems to have developed it further. [Corry, 1996, 388].

It *is* true that different mathematicians may have interpreted the formal and the conceptual according to their own convictions, but the mathematical content of Lawvere's proposal, including the sketch of the *mathematical* duality involved, lead very quickly to a host of important results. The fact is, no one had to quote Lawvere explicitly or say that their work was part of that program — and in fact, no one did quote Lawvere's paper —, for in a sense the program was already implicit in the *manner* Lawvere had set up his own work on universal algebra and that work, as we have seen, called for various generalizations and expansions that were taken up by various mathematicians.

6. One has to contrast categorical logic with other attempts at developing an algebraic framework for logic around the same time, e.g. Halmos's polyadic algebras and Tarski et. al.'s cylindrical algebras. The advantages of the categorical approach over the latter have been explicitly argued by Joyal and Reyes in the mid-seventies:

> 1. the concept of category is used in all branches of mathematics, whereas the structure of polyadic algebra is exotic. Thus, we may hope to extend the application of logic in mathematics

2. certain categories, used in different fields of mathematics are in fact theories and is useful [sic!] to consider them as such;

3. as we shall see in this paper, constructions which are actually used in model theory are specializations of general categorical constructions (hence logic is no exception to the generalized use of categories in mathematics). [Joyal and Reyes, 1976?, 5]

These considerations boil down to one fundamental fact: whereas cylindric algebras and polyadic algebras are isolated in the conceptual realm of mathematics, categories are omnipresent. The heuristic gain of using categories is therefore clear and powerful.

There are indications that the gain of categorical logic is more than heuristic. It is well known that the algebraic expression of propositional logic is given by lattice theory and in the latter adjoint functors are usually called Galois connections. Disagreements appear when higher-order operations like quantifiers are considered, in other words, disagreements appear as to how to generalize the algebraic framework to higher-order logical operations. The fact that categories are a generalization of posets suggests that they might yield the correct generalization. Indeed:

i. Concepts and results about propositional logic are special cases of concepts and results in the categorical setting;

ii. It is possible to extend naturally *proofs* in the propositional setting to the categorical setting;

iii. It is possible to obtain *new* results in the categorical setting;

iv. It is possible to make contacts with other areas of mathematics either by using results of different fields in the new context or by applying the new results in different fields.

It is a remarkable fact that the categorical machinery introduced for algebraic topology, homological algebra, homotopy theory and algebraic geometry constitutes at the same time the proper setting for an algebraic analysis of logic.

Thus, towards the end of the sixties and early seventies, the following 'dictionary' was being elaborated:

Logic	*Category Theory*
(many-sorted) theory	category
Sort	Object (of a category)
(sorted) formula	Sub-object
(sorted) term	Morphism
Interpretation	Functor
Set-theoretical model	Functor into Sets
Homomorphism	Natural transformation

In this table, category refers to categories with finite limits (pull-backs and terminal object) with possibly further structure satisfying some exactness conditions. This extra structure is determined by the choice of the logical connectives considered, as we will see.

3 1969–1970 ELEMENTARY TOPOS THEORY

From the logical point of view, Lawvere's thesis and subsequent work had introduced a starting point in two senses. First, it is the first systematic attempt to give a categorical version of logical concepts and second, the theories at first captured by Lawvere were purely equational theories. When Lawvere and Tierney introduced the concept of elementary topos, they introduced, in some sense, the other end of the spectrum, namely higher order theories or type theories. Of course, the relationships between elementary toposes and type theories still had to be clarified fully, but it was clear to everyone that there were intermediate cases to consider and characterize, or as they became called at that time, there were 'doctrines' to be defined categorically, in particular, the doctrine of first-order logic.

3.1 Elementary topos theory: 1969–1970

> The unity of opposites in the title is essentially that between logic and geometry, and there are compelling reasons for maintaining that geometry is the leading aspect. At the same time, in the present joint work with Myles Tierney there are important influences in the other direction... [Lawvere, 1970b, 329]

> The program of investigating the connections between algebraic geometry and "intuitionistic" logic under the guidance of the form of objective dialiectics known as category theory was discussed and moved forward at a conference in January 1971 at Halifax, ... (...) Briefly we may say that the notion of topos summarizes in objective categorical form the essence of "higher-order logic" (we will explain below how the logical operators become morphisms in a topos) with no axiom of extensionality. This amounts to a natural and useful generalization

> of set theory to the consideration of "sets which internally develop". In a basic example of algebraic geometry, the development may be viewed as taking place along a parameter which varies over "rings of definition"; in a basic example from intuitionistic logic, the parameter is interpreted as varying over "stages of knowledge". [Lawvere, 1972, 1–3].

As the first sentence of the first foregoing quote indicates, in the early seventies, Lawvere insisted again and again on the dual nature of toposes: a geometric aspect and a logical aspect. As the quote suggests clearly, the geometric aspect was thought to be more important than the logical aspect at the beginning and that, to a certain extent, it subsumed the logical aspect. Even to this day, Lawvere presents the situation in similar terms:

> In spite of its geometric origin, topos theory has in recent years sometimes been perceived as a branch of logic, partly because of the contributions to the clarification of logic and set theory which it has made possible. However, the orientation of many topos theorists could perhaps be more accurately summarized by the observation that what is usually called mathematical logic can be viewed as a branch of algebraic geometry, and it is useful to make this branch explicit in itself. [Lawvere, 2000, 717].

Going back to the first quote above, an explicit ideology associated with category theory can be discerned. Indeed, Lawvere himself talks about "unity of opposites" and of the "form of objective dialectics". This paper, published in the *Actes du Congrès International des Mathématiciens 1970*, was the first public presentation of elementary topos theory. In the bibliography, for this paper contains a short and cryptic bibliography, the first reference is to Mao Tsetung and his work entitled *On contradiction — Where do correct ideas come from?* Notice also that the title of the paper — *Quantifiers and Sheaves* — does not refer directly to toposes. This was probably not the original or the intended title of the paper, for as we have seen the first sentence of the paper refers to "the unity of opposites in the title". It is worth quoting a short passage of the first paragraph, for it weaves all these elements together in an interesting manner.

> We first sum up the *principal contradictions* of the Grothendieck-Giraud-Verdier theory of topos in terms of four or five adjoint functors, significantly generalizing the theory to free it of reliance on an *external* notion of infinite limit (*in particular enabling one to claim that in a sense logic is a special case of geometry*). (our emphasis)

Notice that the axiomatization of the notion of elementary topos is not only seen as a significant generalization of the notion introduced by Grothendieck, but also as a way of giving the concept an autonomy, as a way of liberating the notion of topos from its reliance on an external notion.

For Lawvere, then, category theory was the objective form of dialectical materialism and adjoint functors were the exact formulation of principal contradictions. Lawvere felt that the logical and the geometric aspects of toposes were its main contradictions. Geometry was considered to be progressive and Logic was, at least for some time and in some circles, considered to be reactionary. Thus, many category theorists came to the conclusion that it was preferable to avoid the logical aspect altogether, by subsuming it under the geometrical aspect. This was certainly not Lawvere's own proposal if only because it would not be faithful to a dialectical outlook, it became natural to think that way in the early seventies. In particular, using logical techniques to solve a problem was sometimes considered to be reactionary or fascist. It is even said that set theory was considered to be essentially bourgeois since it is founded on the relationship of belonging. This background ideology might partly explain the difficult relationships that category theorists developed with the other parts of the mathematical and logical communities. Political ideas are rarely thought of as being inherent to mathematical theories.

Nowadays, people recognize that two faces are simply not enough to capture toposes, that toposes are multifaceted objects with many complementary aspects. (See, for instance, the introduction in [Johnstone, 2002].) The geometric aspect of toposes comes directly from their origins: Grothendieck defined toposes in the early sixties in the context of algebraic geometry via the use of sheaves. Grothendieck presented them as a generalization of the concept of topological spaces and, in fact, as potentially being the real object of study of topology. Grothendieck had already noted that a topos inherited many of the properties of the category of sets and, as such, could be considered to be a generalization of the latter. Thus, a topos was at the same time a generalization of the idea of a topological space and of the idea of a category of sets. However, Grothendieck and his students did not see or perhaps pay attention to its logical aspect.

Lawvere suggested thinking of a topos as a universe of variable sets, among which the standard sets are conceived as being without variation, or constant. He later suggested thinking of the standard sets as being abstract. (See [Lawvere, 1975a; Lawvere, 1976; Lawvere, 1994; Lawvere, 2003].) Once this viewpoint is adopted, one starts looking at the elementary properties of a topos as a universe of variable sets. As we have seen, Lawvere had already tried to axiomatize the universe of sets, or of constant sets. He had a larger class of universes to capture.

> When the main contradictions of a thing have been found, the scientific procedure is to summarize them in slogans which one then constantly uses as an ideological weapon for the further development and transformation of the thing. Doing this for "set theory" requires taking account of the experience that the main pairs of opposing tendencies in mathematics take the form of adjoint functors, and frees us of the mathematically irrelevant traces ($\in$) left behind by the process of accumulating ($\cup$) the power set (P) at each stage of a metaphysical "construction". Further, experience with sheaves, permutation representations, alge-

> braic spaces, etc., shows that a "set theory" for geometry should apply not only to *abstract* sets divorced from time, space, ring of definition, etc., but also to more general sets which do in fact develop along such parameters. [Lawvere, 1970b, 329].

The hope was that these "geometric sets", or sets for geometry, would provide an adequate foundation for analysis, in particular functional analysis, as is illustrated by the very last sentence of the same paper: "In any topos satisfying (ω) [an additional condition satisfied by the topos of the so-called evolutive sets] each definition of the real numbers yields a definite object, but it is not yet known what theorems of analysis can be proved about it." [Lawvere, 1970b, 334].

Already in 1967, Lawvere was moving towards the notion of an elementary topos (as can be witnessed in his 1969 paper on diagonal arguments). As he himself made clear, his motivation was coming from continuum mechanics, whereas Myles Tierney's motivation was coming from sheaf theory.

> What was the impetus which demanded the simplification and generalization of Grothendieck's concept of topos, if indeed the applications to logic and set theory were not decisive? Tierney had wanted sheaf theory to be axiomatized for efficient use in algebraic topology. My own motivation came from my earlier study of physics. The foundation of the continuum physics of general materials, in the spirit of Truesdell, Noll, and others, involves powerful and clear physical ideas which unfortunately have been submerged under a mathematical apparatus including not only Cauchy sequences and countably additive measures, but also ad hoc choices of charts for manifolds and of inverse limits of Sobolev Hilbert spaces, to get at the simple nuclear spaces of intensively and extensively variable quantities. But, as Fichera lamented, all this apparatus may well be helpful in the solution of certain problems, but can the problems themselves and the needed axioms be stated in a direct and clear manner? And might this not lead to a simpler, equally rigorous account? [Lawvere, 2000, 726].

Thus in 1969, in collaboration with Myles Tierney, the correct axiomatization of the notion came about. Although the axioms given by Lawvere in 1970 at the Nice Congress are redundant, for the assumption that a topos has all finite colimits will be shown quickly afterwards to be a consequence of the other axioms, it contains the key ingredients of the characterization. In a nutshell, an elementary topos is a Cartesian closed category $\mathcal{E}$ with a subobject classifier. Formally, the definition is as follows.

DEFINITION
An *elementary topos* is a category $\mathcal{E}$ such that

1. it has pullbacks;
2. it has a terminal object 1;

3. the functor $X \times - : \mathcal{E} \longrightarrow \mathcal{E}$ has a right adjoint, denoted by $(-)^X : \mathcal{E} \longrightarrow \mathcal{E}$;

4. it has an object Ω, called the *subobject classifier*, together with a monic arrow $\top : 1 \rightarrowtail \Omega$ such that for any monic $m : A \rightarrowtail X$ there is a unique morphism $\phi : X \longrightarrow \Omega$ in $\mathcal{E}$ for which the following square is a pullback:

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, tail, "m"'] & 1 \arrow[d, tail, "\top"] \\ X \arrow[r, "\phi"'] & \Omega \end{tikzcd}$$

The morphism $\phi : X \longrightarrow \Omega$ is called the *characteristic morphism of* A and we will sometimes denote it by ϕ_A.

The third axiom could be replaced by the following:

(3′) it has, for each object X, an object PX and an arrow $\in_X : X \times PX \longrightarrow \Omega$ such that for every arrow $f : X \times Y \longrightarrow \Omega$ there is a unique arrow $g : Y \longrightarrow PX$ for which the following diagram commutes:

$$\begin{tikzcd} Y \arrow[d, "g"'] & X \times Y \arrow[r, "f"] \arrow[d, "1 \times g"'] & \Omega \arrow[d, equal] \\ P(X) & X \times P(X) \arrow[r, "\epsilon_X"'] & \Omega \end{tikzcd}$$

Notice that the definition can be given entirely in terms of the existence of adjoint functors to given functors. The notion of being a Cartesian closed category, as we have already mentioned, was introduced earlier by Lawvere in this spirit. The existence of the subobject classifer Ω can also be presented via the existence of an adjoint functor to a given functor. Since the adjoints whose existence is required by the axioms are adjoints to "structural" functors whose nature is fundamental or elementary, depending on the point of view, the notion of elementary topos ought to be seen as arising naturally from elementary constraints. We have to emphasize this fact once again: in a categorical set up, certain notions arise naturally from elementary functors in a hierarchical manner. They are not ad hoc nor do they depend on complicated contraptions. This is taken by many as being an indication of the fundamental character of the analysis provided by categorical logic and of the notions themselves.

Examples of toposes abound. The category of sets, as defined by the axioms of ZFC for instance, is a topos. Given a set X, the functor $X \times -$ is defined in the obvious way and its right adjoint is given by the set $X^Y = \{f : Y \longrightarrow X\}$. The subobject classifier Ω can be taken to be the set $2 = \{0, 1\}$ together with the map $\top : 1 \longrightarrow 2$, defined by $\top(*) = 1$. It should be observed that these sets are *canonical* choices and that from a purely categorical point of view, any isomorphic set with the appropriate function would do just as well. Thus, there is nothing special about the set $2 = \{0, 1\}$, any two elements set would do just as well (with an appropriate function picking out an element into it).

The other examples of toposes are more involved, but there is a class of examples that have the same general form: given a topos $\mathcal{E}$ and a small category $\mathcal{C}$ ("small" in a sense that can be made precise with respect to the topos $\mathcal{E}$) the functor category $\mathcal{E}^{\mathcal{C}^{op}}$ is a topos. Notice that it is a functor category. Specific important cases of this construction are the following. Let $\mathcal{E}$ be the category *Set* of sets and $\mathcal{C}$ be the lattice of open sets $\mathcal{O}(X)$ of a topological space X. The functor category $Set^{\mathcal{O}(X)^{op}}$ is a topos, called the topos of *presheaves* over X. There is an important subcategory of $Set^{\mathcal{O}(X)^{op}}$, denoted by $Sh(X)$, the category of *sheaves* over X and it can also be shown to be a topos (we will come back to sheaves in a very short while). A related class of toposes, especially important from the point of view of the history of logic, are those of the form $Set^{P^{op}}$, where P is a poset. Lawvere and Tierney saw that the usual models of intuitionistic logic and mathematics known at that time as well as the forcing method of Cohen were specific cases of these toposes. Finally, let $\mathcal{C}$ be a group $\mathcal{G}$ or, more generally a groupoid, then the functor category $Set^{\mathcal{G}^{op}}$, also denoted by $B\mathcal{G}$ and called the *category of representations of* $\mathcal{G}$, can also be shown to be a topos. These examples were all mentioned by Lawvere in his conference at the International Congress of Mathematicians held in Nice in 1970 (see [Lawvere, 1970b]).

Lawvere emphasized right from the start the logical structure of toposes. First, for any category $\mathcal{C}$ with finite limits, it is possible to define an *internal lattice object* L of $\mathcal{C}$ as an object of $\mathcal{C}$ together with two morphisms making certain diagrams commute. (See, for details, [Mac Lane and Moerdijk, 1994, IV.8].) If, furthermore, there is an additional binary morphism $\Rightarrow: L \times L \longrightarrow L$ satisfying the usual identities, then L is an *internal Heyting algebra*. It can be shown that for any object X of a topos $\mathcal{E}$, the power object PX is an internal Heyting algebra. In particular, it follows from the definition that $\Omega \approx P1$ and thus that the subobject classifier Ω is also an internal Heyting algebra. (See [Mac Lane and Moerdijk, 1994, 201] for details.) Thus all the propositional operations are definable in a topos $\mathcal{E}$ as morphisms of the subobject classifier: $1 \xrightarrow{\top} \Omega$, $\Omega \xrightarrow{\neg} \Omega$, $\bot = \neg \circ \top : 1 \rightarrow \Omega$, $\Omega \times \Omega \xrightarrow{\wedge} \Omega$, $\Omega \times \Omega \xrightarrow{\Rightarrow} \Omega$, $\Omega \times \Omega \xrightarrow{\vee} \Omega$. Furthermore, for any morphism $f : X \longrightarrow Y$ of $\mathcal{E}$, there is a morphism $Pf : PY \longrightarrow PX$ and the latter has a left adjoint $\exists_f : PX \longrightarrow PY$ and a right adjoint $\forall_f : PX \longrightarrow PY$ or, equivalently, for any morphism $f : X \longrightarrow Y$ of $\mathcal{E}$, there are morphisms $\Omega^X \xrightarrow{\forall_f} \Omega^Y$ and $\Omega^X \xrightarrow{\exists_f} \Omega^Y$. Hence, quantifiers are also definable in any elementary topos. Needless to say, the logic of an arbitrary topos $\mathcal{E}$ is intuitionistic.

In certain specific toposes $\mathcal{E}$, the power object $P\Omega$ can be shown to be a Boolean algebra. In this case, we say that $\mathcal{E}$ is *Boolean*. Lawvere and Tierney noted that this condition is equivalent to the following:

1. the negation operator $\Omega \xrightarrow{\neg} \Omega$ satisfies $\neg \circ \neg = 1_{\Omega}$;

2. for every object X of $\mathcal{E}$, the Heyting algebra PX is a Boolean algebra;

3. every subobject is complemented, i.e. for every AX in $\mathcal{E}$, $A \vee \neg A = X$;

4. the morphisms $1 \xrightarrow{\top} \Omega$ and $1 \xrightarrow{\perp} \Omega$ induce an isomorphism $1 + 1 \simeq \Omega$, which means that the subobject classifier is two-valued.

In 1971, Radu Diaconescu gave a sufficient condition for a topos $\mathcal{E}$ to be Boolean. We say that a topos satisfies the axiom of choice (AC) if every epimorphism $p : X \longrightarrow Y$ has a section $s : Y \longrightarrow X$, that is $p \circ s = 1_Y$. It can easily be seen that this is only a different way of formulating the usual axiom of choice expressed in term of the existence of a choice function defined on a family of disjoint sets. Diaconescu showed that any topos $\mathcal{E}$ satisfying AC is necessarily Boolean. It was quickly observed afterwards that this statement has an "internal" version, in the sense that this property can be expressed solely in terms of the internal language of the topos. (We will come back to the internal language later.) A topos $\mathcal{E}$ satisfies the *internal axiom of choice* (IAC) if for any epimorphism $p : X \longrightarrow Y$ and for any object Z of $\mathcal{E}$, the morphism $X^Z \longrightarrow Y^Z$ is also an epimorphism. Any topos $\mathcal{E}$ satisfying AC also satisfies IAC, but the converse is false. It was shown that any topos $\mathcal{E}$ satisfying IAC is necessarily Boolean.

Given the foregoing definition of an elementary topos, it is natural to define a morphism of toposes as a functor preserving all the specified structure. Lawvere and Tierney called these morphisms *logical morphisms*. More specifically, a functor $T : \mathcal{E} \longrightarrow \mathcal{E}'$ which preserves, up to isomorphism, all finite limits, the subobject classifier and the exponential is a *logical morphism*. Such morphisms preserve the logical structure of a topos "on the nose".

However, Lawvere and Tierney considered and used primarily a different class of morphisms between toposes, those coming from the geometric side of toposes and already defined and heavily used by the Grothendieck school. The definition, directly lifted from sheaf theory, goes thusly: a *geometric morphism* $f : \mathcal{F} \longrightarrow \mathcal{E}$ between toposes is a pair of functors $f^* : \mathcal{E} \longrightarrow \mathcal{F}$ and $f_* : \mathcal{F} \longrightarrow \mathcal{E}$ such that $f^* \dashv f_*$ and f^* is left exact, that is it preserves all finite limits. The functor $f^* : \mathcal{E} \longrightarrow \mathcal{F}$ is called the *inverse image part* of the geometric morphism and the functor $f_* : \mathcal{F} \longrightarrow \mathcal{E}$ is called the *direct image part* of the geometric morphism.

Lawvere and Tierney then claimed that any geometric morphism $f : \mathcal{F} \longrightarrow \mathcal{E}$ between toposes can be factored into

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "f"] \arrow[dr, "f'"'] & & \mathcal{E} \\ & \mathcal{E}' \arrow[ur, "f''"'] & \end{tikzcd}$$

where f' and f'' are geometric morphisms such that f''_* is full and faithful while f'^* reflects isomorphisms. Lawvere and Tierney observed that the geometric morphism f'' is entirely determined by a single morphism $j : \Omega \longrightarrow \Omega$ in $\mathcal{E}$. This morphism, in turn, determines what Lawvere and Tierney called a *Grothendieck topology*, now also known as a *Lawvere-Tierney topology.* The formal definition is as follows.

DEFINITION
Let $\mathcal{E}$ be a topos and Ω be its subobject classifier. A *Lawvere-Tierney topology on* $\mathcal{E}$ is a morphism $j : \Omega \longrightarrow \Omega$ in $\mathcal{E}$ satisfying the following three properties:

(i) $j \circ \top = \top$;

(ii) $j \circ j = j$;

(iii) $j \circ \wedge = \wedge \circ (j \times j)$.

This definition was central to Lawvere and Tierney since it allowed them to make the connection to sheaf theory and Grothendieck toposes and, thus, putting the geometry at the forefront of the theory. For, one would not be able to see from the axioms of a topos *alone* that they *have* this essentially geometric character. (In the same way, perhaps, that one would not be able to see the *geometric* content of Lie groups from their axioms alone.) After all, the properties stated in the axioms are very much *set*-like or *logical*: taking products and finite limits in general, having the possibility to classify subobjects and forming function spaces (or power sets, depending on the choice of axioms). The definition of a Lawvere–Tierney topology does not look geometrical either. Since the subobject classifier Ω can be thought of as an object of truth-values, the morphism j can be viewed in a loose sense as a modal operator. However, the main examples of such topologies are given by sheaves of sets over a site, the latter being a category equipped with a Grothendieck topology which, in turn, is a categorical generalization of the notion of topology. It is precisely this connection between logic and algebraic geometry that Lawvere and Tierney were trying to capture: to give an elementary characterization of toposes in terms of adjoint functors that would allow them to recover Grothendieck's toposes. But not all examples of such topologies are geometrical in that sense. A more logical example is the double negation topology: as we have mentioned, any topos $\mathcal{E}$ has a negation operator $\Omega \xrightarrow{\neg} \Omega$ and it is easy to verify that the double negation $\Omega \xrightarrow{\neg\circ\neg} \Omega$ is a Lawvere-Tierney topology.

Before we give the definition of a j-sheaf, recall that a sheaf is, informally, a tool that allows one to look at a concept of local character from a global point of view. In other words, an object is a sheaf if it is possible to extend a local notion to the whole space in which it is defined.

The formal connection with sheaves comes in two steps. First, the morphism j determines a closure operator on the subobjects of each object X of $\mathcal{E}$. Given a subobject $A \longrightarrow X$, its *j-closure* $\bar{A}$ is given by the following pullback:

$$\begin{tikzcd} \bar{A} \arrow[r] \arrow[d] & 1 \arrow[d, "\top"] \\ X \arrow[r, "j \circ \phi_A"'] & \Omega \end{tikzcd}$$

It can be shown that j is a Lawvere–Tierney topology if and only if the closure operator $A \mapsto \bar{A}$ has the following properties for all subobjects A, B of X : $A \wedge \bar{A} = A, \bar{\bar{A}} = \bar{A}, \overline{A \wedge B} = \overline{A} \wedge \overline{B}$.

Second, the closure operator allows us to characterize 'concepts of a local character' as follows. Given a monomorphism $i : A \longrightarrow X$, we say that i is a *dense monomorphism* if $\bar{A} = X$. Whenever such a dense monomorphism exists, the subobject A is said to be *dense in* X. A dense subobject represents a concept of local character. A j-sheaf is an object F of $\mathcal{E}$ such that any representation of a concept of local character in it can be extended to the whole space in which the concept lies. Formally, we say that an object F of $\mathcal{E}$ is a *j-sheaf* (for the Lawvere–Tierney topology) if for every dense monomorphism $i : A \longrightarrow X$ in $\mathcal{E}$ and every morphism $f : A \longrightarrow F$, there is a unique morphism $g : X \longrightarrow F$ making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "f"'] & X \arrow[d, "g"] \\ & F \end{tikzcd}$$

The full subcategory $\mathrm{Sh}_j\mathcal{E}$ of j-sheaves of $\mathcal{E}$ can be shown to be a topos. As Lawvere and Tierney have indicated, every Grothendieck topos arises in this way. Moreover, there is a *geometric morphism* $\mathrm{Sh}_j\mathcal{E} \longrightarrow \mathcal{E}$.

At that point, Lawvere indicates in his paper how Cohen's proof of the independence of the continuum hypothesis from the axioms of ZF could be translated into the language of topos theory. It was left to Tierney to present the full proof of this translation in 1972. (See [Tierney, 1972].)

This was the first public and printed presentation of elementary topos theory. The study and presentation of elementary toposes were taken up quickly afterwards by various mathematicians: Peter Freyd, Jean Bénabou, Gavin Wraith, Anders Kock, André Joyal, Chris Mikkelsen.

It is interesting to see what the first volume published in 1972 and dedicated to toposes, algebraic geometry and logic contains. (See [Lawvere, 1972].) Apart from Lawvere's introduction, which goes over the basic definition and concepts we have mentioned and Tierney's paper on the independence of the continuum hypothesis in the topos theoretical framework, there is nothing about the interactions between toposes, algebraic geometry and logic! The remaining papers are in order of appearance: Giraud on the classifying topos (*not* in the logical sense of that expression), Lambek on deductive systems and categories, Goodman and Myhill on a formalization of Bishop's constructive mathematics, Scott's paper on continuous lattices , Bucur on the applications of formalism of duality in algebraic geometry and Illusie on cotangent complex and deformations of torsors and group schemes . All these papers touch on issues that are clearly relevant and will even become important in their own right, e.g. classifying toposes or domain theory, but they do not constitute direct contributions to the subject at hand. This was about to change rapidly.

4 FOCUSING ON FIRST-ORDER LOGIC

> Part I of this volume consists of three of the first papers on functorial model theory, developing concretely the approach to algebraic logic according to which a "theory" (understood in a sense invariant with respect to various "presentations" by means of particular atomic formulas and particular axioms) is actually a category T having certain properties P and a model of T is any set-valued P-preserving functor. As a rough general principle, one could choose for P any collection of categorical properties which the category of sets satisfies, the choice determining the "doctrine" of theories of kind P, which is thus a (non-full) subcategory of the category of small categories. For example, the doctrine of universal algebra thus springs from the fact that the category of sets has the property P of having finite Cartesian products, while the doctrine of higher-order logic springs from the property of being a topos. [Lawvere, 1975a, 3].

Lawvere had emphasized right from the start the idea of replacing the category of sets by an arbitrary topos to develop mathematics, in particular analysis. The first publication on the application of topos theoretical methods to analysis appeared in 1974, but the ideas had been presented already in 1972. (See [Kock and Mikkelsen, 1974].) At the same time, there was a lot of work to do. Very quickly, presentations of topos theory became available. (See [Freyd, 1972; Kock and Wraith, 1971; Wraith, 1975] (presented in 1973)].) Since the axioms of topos theory are very much set-like, early works were concerned with clarifying the relationships between toposes and set theory. Of course, in this case, it is necessary to add axioms to the theory, e.g. an axiom of infinity and an axiom to guarantee that the toposes considered were Boolean. Thus, J. C. Cole and W. Mitchell both explored categories of sets and models of set theory independently and about the same time (see [Cole, 1973], in fact his PhD thesis defended at the University of Sussex in 1971; [Mitchell, 1972] (also presented in 1971)) and Osius followed closely. (Osius presented his results in 1973; they were published in 1974 and 1975.) As we will see, Mitchell introduced a method that had a direct impact on the development of categorical logic. Around the same time, Marta Bunge gave a topos theoretical proof of the independence of Souslin hypothesis in set theory and Kock, Lecouturier and Mikkelsen investigated the concepts of finiteness in toposes. (See [Bunge, 1974] and [Kock *et al.*, 1975].)

As far as *logic*, and in particular first-order logic, is concerned, the development of its history is extremely difficult to document accurately. When Gray wrote his history of sheaf theory, he already lamented on the situation:

> Perhaps the main aspect which is difficult to document in published works is the connection with logic. The best sources are Lawvere [1975] and [1976], together with Reyes [1974; 1975], and [1976], Lambek [unpublished when Gray wrote] and Makkai and Reyes [1976] and [1977], and also the articles in this volume. Unfortunately one of the most influential figures in this development, A. Joyal, has thus far not given us a written record of his work; however see [Labelle, 1971]. Besides topics discussed at the present meeting, future developments seem to be going in the directions of Bénabou [1975] . . . , and recent unpublished work of Cole and Tierney on pseudolimits in the category of topoi. [Gray, 1979, 63].

This was presented in 1977 and published in 1979! When one consults the bibliographical sources mentioned in this paragrah, one discovers how poor and inaccessible these sources are in general. Lawvere's papers are important in themselves and do refer to Joyal, Bénabou, Reyes and others, but they do not allow one to reconstruct precisely who did what when and while [Reyes, 1974] and [Reyes, 1977] were easily available, [Reyes, 1975/1976] was not. Labelle [1971] was just as difficult to obtain, although Labelle's long exposition was influential among students at Montreal in the early seventies. Bénabou's paper was about fibered categories and, although relevant, could certainly not constitute an introduction to, or a survey of, his work in the area. The Makkai and Reyes papers published in 1976 already build up on previous material, only sketch proofs and they certainly do not constitute and entry point to the field. The fact is, two of the main actors, namely Jean Bénabou in Paris and André Joyal in Montréal, never even wanted to publish their results. It seems that they simply did not believe in publishing at that time. Recall that we are in the late sixties and early seventies and events like *Mai 68* in Paris and the *Crise d'octobre* in 1970 in Québec were still very much in the air. Bénabou and Joyal did present their results during colloquia, seminars and at informal gatherings and thus people learned and knew what they had done, but the details of these contributions and the manner in which they were done are not available. What we do have is a list of talks given by Bénabou and Joyal during this period, collected from various bibliographical sources, mostly theses written during that period. First, here is the list of talks given by Bénabou and his students.

indexCeleyrette, J.

Here is the list of Joyal's talks:

Bénabou and Celeyrette	*Généralités sur les topos de Lawvere et Tierney*	Paris	1971
Bénabou	*Structures syntaxiques*	Montreal	April 1973
	Catégories et logiques faibles	Oberwolfach	April 1973
Michel Coste	*Logique du premier ordre dans les topos élémentaires*	Paris	1973
	Logique d'ordre supérieur dans les topos élémentaires	Paris	November 1973

The lists are not complete. Notice that these talks, except for the first one in Joyal's list, were given after the introduction of elementary topos theory. There was work done before, as we will see. Some of this work, for instance Joyal's recursive universes (also called 'arithmetic universes'), have simply vanished for a long period of time (they are now coming back in circulation). They are mentioned *en passant* in some of Lawvere's papers, but there is no available systematic exposition of what they were and how they were developed.

There was a particularly important period in April 1973. Lawvere was in Montreal for that month, away from Perugia. Dana Schlomiuk had invited many category theorists for a month during which there were many seminars and discussions. It was an intense informal gathering and not an organized, planned and disciplined meeting. Another decisive event for the dissemination of categorical logic was the meeting of the summer of 1974, called *Séminaire de mathématiques supérieures*, organized by Shuichi Takahashi. During this meeting, André Joyal gave a series of talks on all the topics mentioned above, Gonzalo Reyes gave a series of talks on what was to become his collaborative book with Mihaly Makkai, Jean Bénabou gave a series of talks on fibered categories. Freyd presented tau-categories, what were to become *allegories* much later. (See [Freyd and Scedrov, 1990].) Eilenberg started a "bilingual' talk: one sentence in french, one sentence in english and alternating between the languages until the audience pleaded him to stick to one language. Also in the audience, a contingent of Italians coming with Lawvere: Meloni, Carboni, Riccioli (Feit). As far as we know, there is no official record of that meeting either.

The first *published* results appeared in 1973, although the work was done in 1970: one of Lawvere's student, Hugo Volger, presented a categorical version of a first-order logic and of the completeness theorem by following the leads given earlier by Lawvere himself, e.g. [Lawvere, 1966; Lawvere, 1967]; also in 1971, Orville Keane, in his PhD thesis (under Freyd's supervision at the University of Pennsylvania), gave a categorical characterization of universal Horn theories. (See [Keane, 1975].)

Joyal	*Dogmes Booléens*	Montreal	4 talks, December 1969, January 1970
	Topos	Montreal	24 April 1971
	Topos et Logique	Montreal	5 October and 27 October 1971
	Théorèmes de descente finitaire; théorème de complétude	Montreal	Summer 1972
	Indépendance de l'hypothèse du continu	Montreal	7 November 1972
	Une méthode uniforme pour démontrer les théorèmes de Gödel, Kripke, Freyd, Mitchell, Barr	Montreal	14 November 1972
	Homotopie dans les topos (le théorème de Giraud)	Montreal	28 November 1972
	Qu'est-ce qu'une théorie équationnelle générale?	Montreal	January 1973
	Modèle universel d'une théorie équationnelle générale	Montreal	January 1973
	Théorie catégorique des fonctions récursives	Montreal	February 1973
	Forcing=Interprétation dans un Topos (I et II)	Montreal	23 February and 2 March 1973
	Univers récursifs	Montreal	March 1973
	Sans titre (distributive lattices and topology without points)	Montreal	5 March 1973
	Une nouvelle preuve du théorème de Barr	Montreal	24 April 1973

Joyal	*Théorème d'incomplétude de Gödel et Univers arithmétique*	Amiens	9 July 1973
	Le théorème d'incomplétude de Gödel	Montreal	9 and 16 October 1973
	Nouveaux fondements de l'analyse	Montreal	12 December 1973
	Sur un sujet non-déterminé	Montreal	9 March 1974
	Théorie formelle des topos et langage d'ordre supérieur	Montreal	2 April 1974
	Algèbre constructive	Montreal	15 November 1974
	Algèbre constructive et théorie des faisceaux	Montreal	2 December and 9 December 1974
	Théorème de Chevalley-Tarski	Montreal	10 July 1975

4.1 Volger's work

It is instructive to look more closely at Volger's work, for although he obtained significant results, the community did not adopt them. Many category theorists felt that the notions defined were not the "right" notions, mostly because his results were not easily generalizable to non-Boolean cases. Volger modified the definitions given in the first paper "because my friends in Aarhus insisted that the construction should work in this [e.g. non-Boolean] more general case" [Volger, 1975b, 87]. Alas for Volger, the modified definitions suffered the same fate as the original ones.

In his first paper, Volger introduced three types of categories: logical categories, elementary theories and semantical categories. Elementary theories were meant to be the categorical encoding of first-order theories with equality, as Lawvere had already presented them. Logical categories were introduced to allow a generalization to higher-order logic and, finally, semantical categories were supposed to play the role of categories of sets, but the notion is weaker than the concept of topos. We will restrict ourselves to the notion of logical categories, for it contains the seeds of the demise of these notions and even of Volger's subsequent attempt to rescue them.

Volger's definition is based on Lawvere's notion of an elementary theory. However, Volger added two further conditions, for his "proof of the completeness theorem requires the addition of two new conditions to the original definition of elementary theories." [Volger, 1975a, 52]. A category $\mathcal{C}$ is said to be *logical* if it satisfies the following conditions:

i. $\mathcal{C}$ has finite products (in particular a terminal object 1);

ii. $\mathcal{C}$ has an object Ω which is a Boolean algebra with mappings $1 \underset{\bot}{\overset{\top}{\rightrightarrows}} \Omega$;

iii. For every $f : X \longrightarrow Y$ in $\mathcal{C}$, there is a functor $\exists_f : Hom_{\mathcal{C}}(X, \Omega) \longrightarrow Hom_C(Y, \Omega)$, left adjoint to the functor $Hom(f, \Omega) : Hom(Y, \Omega) \longrightarrow Hom(X, \Omega)$;

iv. Given the following pullback

$$\begin{array}{ccc} X & \xrightarrow{(X,f)} & X \times Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times Y} \\ Y & \xrightarrow[\Delta_Y]{} & X \times Y \end{array}$$

then we have that $\exists_{(X,f)} [1_Y f] = \exists_{\Delta_Y} [1_Y] (f \times Y)$, where $1_Y = \top \circ !_Y$ and given the other pullback

$$\begin{array}{ccc} X \times Z & \xrightarrow{\pi_Z} & Z \\ {\scriptstyle 1_X \times g}\downarrow & & \downarrow{\scriptstyle g} \\ X \times Y & \xrightarrow[\pi_Y]{} & Y \end{array}$$

then we have that $\exists_{\pi_Y} [\varphi] g = \exists_{\pi_Z} [\varphi (X \times g)]$.

v. a) Let $e_Y = \exists_{\Delta_Y} [1_Y]$ denote the *equality on* Y. If $e_Y(f_1, f_2) = 1_X$, then $f_1 = f_2$;

b) $e_\Omega = \Leftrightarrow$, where $\Leftrightarrow$ denotes the bi-implication.

This is Volger's working definition, for all the results in his paper are for logical categories. Notice that the distinguished object A, what was meant to become the universe of interpretation, has now disappeared. The only distinguished object left is the Boolean algebra Ω. Notice that the category $\mathcal{S}$ of sets is a logical category but that an arbitrary topos $\mathcal{E}$ is not (which is somewhat disappointing given the inherent logical structure of an arbitrary topos). A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between logical categories is said to be *logical* if it preserves finite products, the Boolean algebra object Ω together with 0, 1, $\neg$, $\wedge$ and quantification. A *Set-model* of a logical category $\mathcal{C}$ is a logical functor $M : \mathcal{C} \longrightarrow Set$ and a natural transformation $\alpha : M \longrightarrow N$ between two *Set*-models is called a $\mathcal{C}$-*embedding*.

However, Volger also gives a slightly modified definition of elementary theories. An *elementary theory* $\mathbb{T}$ is a category such that

i) $\mathbb{T}$ has two basic objects A and Ω such that every object X different from Ω has a specified representation as a finite power A^n of A, and $Hom_{\mathbb{T}}(\Omega, X)$ is empty.

ii) $Hom_{\mathbb{T}}(A^n, \Omega)$ is a Boolean algebra for every object A^n and $Hom_{\mathbb{T}}(f, \Omega)$ is a Boolean homomorphism for every $f : A^n \longrightarrow A^m$.

iii) For every $f : A^n \longrightarrow A^m$ in $\mathbb{T}$, there exists an existential quantifier $\exists_f$ which satisfies the conditions iii), iv) and v) of the definition of a logical category.

Although all the results are given for logical categories, Volger remarks that all the proofs given still hold, provided that appropriate but slight modifications are made. The most important result, naturally, is the completeness theorem for logical categories, formulated as follows:

THEOREM.
Let $\mathcal{C}$ be a small logical category such that $\perp_X \neq \top_X$ and $\exists_{!_X}[\top_X] = \top$ for every object X of $\mathcal{C}$.

1. For every pair of morphisms $f, g : X \longrightarrow Y$ there exists a *Set*-model M such that $M(f) \neq M(g)$
2. There exists a *Set*-model M such that $\mathrm{card}(M(X)) \leq \mathrm{card}(\mathcal{C})$ for every object X of $\mathcal{C}$.

The proof is a categorical adaptation of Henkin's completeness proof to a categorical framework along the lines indicated by Lawvere in his second abstract. The basic idea is to construct an extension $I : \mathcal{C} \longrightarrow \mathcal{C}'$ for the given logical category $\mathcal{C}$ for which $Hom_{\mathcal{C}'}(1, -) : \mathcal{C}' \longrightarrow Set$ has a model, in fact a canonical model. One obtains the required model by composition. It is shown that a logical category $\mathcal{C}$ has a canonical model if and only if 1) it is *maximally consistent*, that is $Hom_{\mathcal{C}}(1, \Omega) = \{0, 1\}$ and 2) it is *rich*, that is for every $\varphi : X \longrightarrow \Omega$ such that $\exists_{!_X}[\varphi] = \top$, there exists $k : 1 \longrightarrow X$ with $\varphi \circ k = \top$.

As we have mentioned, Volger's proposal, although technically irreproachable, was not received favorably. The main problem was that it was too contrived, perhaps too *ad hoc*. While Volger was following Lawvere's path and suggestions, André Joyal together with Gonzalo Reyes were looking at categorical logic from a slightly different perspective.

4.2 The Montreal school

Category theory had a distinctive status in Montreal in the late sixties and early seventies. Joachim Lambek, who graduated from McGill under Zassenhaus in 1950, was already a distinguished algebraist at McGill in the sixties. He had spent his 1965–66 sabbatical year in Zurich at the ETH, where Beno Eckmann had asked him to give a graduate seminar in category theory, a subject that he was learning at the time. In his audience, he found Fritz Ulmer, John Beck, John Gray and Bill Lawvere. Also showing up were Eckmann himself, Peter Hilton, Pierre André and Paul Bernays. His seminar led to his Springer Lecture Notes *Completion of Categories* published in 1966. It is also during period that Lambek wrote his first paper on category theory, a generalization of Tarski's fixpoint theorem to categories. (See [Lambek, 1968b].) Before coming back to McGill, he invited one of the post-doctoral students at the ETH, Marta Bunge, who came in the fall of 1966 and was soon to join faculty. She had officially worked under Freyd's supervision but she had in fact been supervised both by Freyd and Lawvere and done category theory right from the start. Michael Barr, who started as an homological algebraist under the supervision of David Harrison at the University of Pennsylvania, had spent two years at Columbia as a post-doctoral fellow where he met Eilenberg before moving to Urbana where he found John Beck, John Gray, Alex Heller and Max Kelly and then spent the fall of 1967 in Zurich with Eckmann. He joined the McGill faculty in the fall of 1968. The logician Mihàly Makkai, who contributed rapidly to the development of categorical logic, arrived in Montreal in the fall of 1973 at the *Centre de recherches mathématiques* of the Université de Montréal and joined the McGill mathematics faculty in the fall of 1974.

At the Université de Montréal, one finds an unusual, interesting and perhaps unique combination of algebraic logic and category theory that might explain in part why Lawvere's ideas quickly found a good reception. Maurice Labbé, a student of Church from Princeton in the early fifties, acted as chair of the department during most of the sixties. Another student from Princeton, also supervised by Church and who graduated in 1959, Aubert Daigneault, took interest in Lawvere's approach right from the beginning. Léon LeBlanc, who graduated in 1960 from Chicago under Halmos' supervision in algebraic logic, was there in the early sixties until his untimely death in 1968. Thus, in the sixties, Daigneault and Leblanc were active in algebraic logic. Gonzalo Reyes, after completing his thesis in model theory at Berkeley under Craig's supervision, was hired by the Université de Montréal in 1967, in replacement of Léon Leblanc. In the late sixties, there was also an important number of category theorists. Jean Maranda, who graduated from McGill in 1953 under Zassenhaus, was closely following developments in category theory and had published some papers on the subject in the sixties, but died in 1971. (See [Maranda, 1962; Maranda, 1965; Maranda, 1966].) Two students who had written Ph.D. theses on category theory under Lambek's supervision were also hired in the sixties: first, Pierre Berthiaume in 1964 and then Dana Schlomiuk in 1967. When Reyes arrived, André Joyal was a graduate student at the same institution. Joyal arrived in 1963 as an undergraduate student and did his masters between 1967 and 1969, under Q.I. Rahman's supervision. Joyal joined the mathematics faculty of the *Université du Québec à Montréal* (UQAM) in 1970. Also important for his influence on colleagues was Shuichi Takahashi who was interested in category theory, topos theory and logic, among other things, already in the sixties. (See, for instance [Takahashi, 1965; Takahashi, 1969; Takahashi, 1974].) There were also a number of post-doctoral students, for instance Michel Jean who had also graduated from Berkeley under Henkin and Craig's supervision and Diana Dubrovsky, a student of Yiannis Moschovakis coming from UCLA, participating in the seminars.

There was an intense activity in Montreal in the seventies. McGill University, the Université de Montréal and the Université du Québec à Montréal, had their own seminars in category theory and there was also a Montreal Category Theory research seminar held at that time during weekends. Freyd, Lawvere, Beck, Tierney and, sometimes, even Eilenberg would show up at the latter seminar. Often, when no one was ready to speak, Joyal would go to the blackboard and present new ideas, new results and new theories. Although a student of Adrian Mathias at Cambridge, Robert Seely spent the year 1974–1975 in Montréal and interacted with Joyal, Reyes, Makkai, Lambek and their students. He came back to Montréal in 1977. Philip Scott, who had written his thesis under the supervision of Denis Higgs at the University of Waterloo, came to McGill in the fall of 1976 as a post-doctoral student. Reflecting all this activity, eight theses, Master's and PhDs, were written on categorical logic between 1973 and 1977 at the Université de Montréal. Thus, categorical logic was a very active field in Montreal from the early seventies until about 1977, the year of the publication of Makkai and Reyes book on categorical logic and of the Durham meeting on applications of sheaves. Afterwards, for reasons that are not entirely clear, there are very few theses submitted in categorical logic. André Joyal concentrated his efforts on the conceptual foundations of combinatorics, leading to his beautiful and influential work on species of structures. (See [Joyal, 1981].) Gonzalo Reyes, in collaboration with Anders Kock, started to work actively in synthetic differential geometry. (See, for instance, [Kock and Reyes, 1979].) Logic moved from the French speaking part of Montreal to the English speaking universities, mainly McGill University where Makkai, Lambek, Scott and Seely continued to do original and important work. (Scott left for Ottawa in 1982.) Reyes came back to logic in the late eighties and nineties in collaboration with the psychologist John Macnamara and Joyal also returned to foundational issues in the nineties, mainly in collaboration with Ieke Moerdijk [Macnamara and Reyes, 1994]; [Joyal and Moerdijk, 1997]. However, in contrast with the situation that prevailed in the seventies, few graduate students followed.

The influence of Grothendieck's work on logical research done during the seventies has to be underlined, and this, for two completely different reasons. The first reason is purely conceptual: many of the logical concepts developed were aimed at finding a purely logical translation of concepts of algebraic geometry. Thus, for instance, we read in the introduction of Makkai and Reyes:

> Finally, our treatment of categorical logic is geared towards establishing a link with Grothendieck's theory of (Grothendieck) topoi as it is exposed in SGA4. One of our main points is that some of the fundamental properties of notions in this theory (notably the notions of

> *topos*, *coherence* of, and in, topoi and *pretopos*) are purely logical.... It is a very interesting fact that notions originally developed for the purposes of (abstract) algebraic geometry turn out to be intimately related to logic and model theory. Compared to other existing versions of algebraic logic, categorical logic has the distinction of being concerned with objects that appear in mathematical practice. [Makkai and Reyes, 1977, 3].

Thus, Grothendieck's work was seen as central to development of algebraic geometry (and coincidentally to homological algebra, sheaf theory and category theory itself) and was part of the development of mathematics. Key notions of categorical logic were directly related to these developments, in fact they were seen as a conceptual clarification of some of it, e.g. coherent topos or the notion of a site, and thus justified almost as such.

The second reason is sociological but just as important: while category theorists were developing topos theory and categorical logic with toposes in mind, in France, Grothendieck had quit the mathematical scene altogether in 1970 and his successors were in fact playing down the importance of some of his mathematical ideas, in particular the concept of (Grothendieck) topos. Whereas the notion of schemes, Grothendieck's analysis of the foundations of algebraic geometry, slowly but surely came to constitute the bedrock of algebraic geometry — as witnessed by [Shafarevich, 1974; Hartshorne, 1977] and more recently [Eisenbud and Harris, 2000] —, toposes were more or less banished from the mathematical establishment, even from algebraic geometry. Indeed, they are not even defined in Shafarevich and Hartshorne. Thus, although categorical logic was directly linked to important ideas of Grothendieck's program, it was not linked to the ideas that the mathematical community was judging as being central. Hence, in a sense, the attempt to build bridges between logic and algebraic geometry did not bear its fruits, although mostly it seems, for sociological reasons.

4.3 The background

Let us come back to the late sixites and early seventies. Among Joyal's multiple interests in late sixties was Algebraic Geometry that he was studying by himself in the formidable treatise EGA, *Éléments de géométrie algébrique*, of Grothendieck and Dieudonné and SGA, *Séminaire de Géométrique Algébrique*, in particular SGA4 on topos theory, which was held in 1963/64 and circulated in limited form before its publication by Springer in the *Lecture Notes*. Sheaf theory and more generally category theory were omnipresent in these treatises. Reyes learned category theory including early work of Lawvere (along with some algebraic geometry) from Joyal, who in turn learned model theory *à la* Berkeley, in particular some of the results and ideas included in his PhD thesis, from Reyes. There are some indirect indications of the work done in 1969 and 1970, before the introduction of the notion of elementary topos hit the community.

We have already indicated that in 1968, Aubert Daigneault had started to look for relationships between Lawvere's elementary theories and polyadic and cylindric algebras. He considered both approaches in a categorical framework and proved some equivalences the category of elementary theories and the category of polyadic algebras. As far as we know, Daigneault's paper is the only published paper whose purpose is to investigate properties of Lawvere's elementary theories. Daigneault's motivation, in this paper, was explicitly "to call attention to Lawvere's important contribution to Algebraic Logic". [Daigneault, 1969/1970, 307]. Thus, Lawvere's contribution is ackowledged as being a contribution to *algebraic logic*. It is worth mentioning that Volger refers explicitly to Daigneault's paper in the introduction of his article on logical categories. In fact, Daigneault's proof of the equivalence between elemetary theories and polyadic algebras is taken by Volger as evidence in favor of the algebraic character of Lawvere's work.

> This concept of an elementary theory may also be viewed as an algebraization of first-order logic by categorical means in the following sense. The elementary theory and the structure preserving functors between them correspond to polyadic algebras and homomorphisms between them. ... The connections between elementary theories and polyadic algebras have been studied by Daigneault.... [Volger, 1975a, 51–52].

Daigneault uses Lawevere's second characterization of elementary theories to construct the relevant categories and the equivalence between them. Recall that in Lawvere's analysis, Boolean algebras play a key role and the existential quantifier is part of the data. Daigneault reformulates Lawvere's completeness theorem (and uses it) as follows:

THEOREM.
For any elementary theory $\mathbb{T}$ and $\varphi : 1 \longrightarrow B$, where B is the distinguished Boolean algebra (Volger's Ω) in $\mathbb{T}$ such that $\varphi \neq \bot$, there exists a model M of $\mathbb{T}$ such that $M(\varphi) = \top$.

Daigneault also gives the representation theorem for polyadic algebras. Let X be a non-empty set and I a denumerably infinite set (taken to represent individual variables). Then it is noted that the Boolean algebras of all functions $X^I \longrightarrow 2$ is a polyadic algebra C_X. The representation theorem is then:

Theorem: For any polyadic algebra P and any $q \in P$ such that $q \neq 0$ (the smallest element of P), there exists a non empty set X and a 2-valued representation $f : P \longrightarrow C_X$ such that $f(q) = 1$.

Although Daigneault uses Lawvere's theorem in his paper, he points out that Lawvere's completeness theorem could be reduced to the representation theorem for polyadic algebras and ultimately to the ordinary completeness theorem for polyadic algebras. Thus, already in 1968, one can see the connections between the various theorems, a theme that will become central in the early 1970's.

A second paper of Daigneault deserves to be looked at carefully, for it contains a key component of what is about to come. It is his paper entitled *Injective*

Envelopes, published in the *American Mathematical Monthly* in 1969. [Daigneault, 1969]. In this paper, Daigneault gives conditions that are sufficient to ensure the existence of injective envelopes in a category. The existence of enough injectives played a key role in the development of abelian categories ten years earlier. In his paper, Daigneault gives a list of six equivalent conditions on a category $\mathcal{C}$ that are sufficient in order that, for any object of $\mathcal{C}$, there exists an injective essential extension. The first of these conditions is that for each morphism $f : X \longrightarrow Y$, there exists an epi-mono factorization, i.e. there is an epimorphism $g : X \longrightarrow Z$ and a monomorphism $h : Z \longrightarrow Y$ such that $f = h \circ g$. This condition is satisfied by any abelian category. But it is satisfied in some non-additive categories as well. In particular, Daigneault considers the category of Boolean algebras with unit preserving homomorphism and shows that it satisfies the given conditions. (It should be mentioned at this point that the Stone duality theorem, and thus the category *Stone* of Stone spaces, is used to prove that surjective homomorphisms are indeed epimorphisms.) It is also shown that the injectives are precisely the complete Boolean algebras. Although Daigneault's paper makes no connection with Lawvere's elementary theories nor to polyadic algebras, it is known that the concept of an extension of a map played a role in the latter. The proof that complete Boolean algebras are injectives is attributed to Vincent Papillon, Reyes' first graduate student.

Vincent Papillon Master's thesis is entitled *Quelques aspects de l'enveloppe injective d'une algèbre de Boole*. [Papillon, 1969] The first chapter of the thesis starts off by presenting the results of the foregoing paper. In the remaining two chapters of the thesis, Papillon gives two different constructions of the injective envelope of a Boolean algebra: one follows Reyes's alternative definition of a complete ring of quotients of a ring A and the other follows Joyal's suggestion of using the Stone space associated to the Boolean algebra A. The thesis was submitted in April 1969.

We mention Daigneault's work and Papillon's thesis for two reasons. First, it shows that certain categories, which were not necessarily abelian, satisfying certain simple exactness conditions like the existence of images, had interesting consequences, e.g. the existence of injective envelopes. Furthermore, the conditions revealed here are directly linked to the category of Boolean algebras, certainly relevant for classical propositional logic. Indeed, if one were to start with classical propositional theories, construct their Lindenbaum–Tarski algebras and consider the resulting category of such algebras, one would obtain a category satisfying these exactness conditions. Second, as Papillon's thesis shows clearly, Joyal and Reyes were probably aware of Lawvere's elementary theories and they were immersed in category theory, Boolean algebras and Stone spaces.

Indeed, in August of 1969, the *Notices of the American Mathematical Society* received an abstract from Joyal entitled *Boolean algebras as functors*. A γ-complete Boolan algebra, for γ an infinite regular cardinal, is a Booleal algebra closed under sups of power less than γ. A γ-field is a Boolean algebra isomorphic to a γ-complete field of sets. A γ-representable Boolean algebra, is a Boolean algebra isomorphic to a quotient of a γ-field by a γ-ideal. Let Set_γ denote the full subcategory of the category Set of sets, generated by sets of power less than γ. The functor category Set^{Set_γ} where the functors $Set_\gamma \longrightarrow Set$ commute with limits is equivalent to the category of γ-fields with γ-isomorphisms. When limits are restricted to diagrams of power less than γ, then there is an equivalence between the functor category Set^{Set_γ} and the category of γ-representable Boolean algebras (and γ-homomorphisms). Joyal obtains a similar equivalence between the category of *complete* Boolean algebras with a functor category Set^{Set_γ}, thus showing that Boolean algebras can be described as functors. A particular case of this result is attributed to Lawvere. What is interesting here is the *kind* of result Joyal is after: to be able to represent certain fundamental mathematical concepts as *functors*. As we have seen in the foregoing sections, this was very much in the spirit of Lawvere's work.

Approximately one year later, on November 4 1970, the same *Notices* receive five abstracts from Joyal. Three of them are put in the section of algebra and two of them are placed in the section on logic and foundations. The first three are entitled *Spectral Spaces and Distributive Lattices*, *Spectral Spaces II* and *Cohomology of Spectral Spaces* respectively. They are directly related to Grothendieck's work in algebraic geometry, in particular to results about schemes and sheaves. The two abstracts on logic and foundations are entitled *Polyadic Spaces and Elementary Theories* and *Functors which preserve elementary operations*. Thus, Joyal indeed knew about Lawvere's elementary theories from early on and was pursuing their study. Despite the fact that they were inserted in different sections by the editors of the *Notices* and that their titles suggest that they are about different topics altogether, they are in fact related to one another, illustrating how, already before the advent of elementary toposes, Joyal was connecting algebraic geometry and logic together. Furthermore, it is clear that Joyal's earlier work on Boolean algebras and Stone spaces played a role in his subsequent work. Let us start with the two abstracts on logic and foundations. The first abstract is about polyadic *spaces*, not algebras, and elementary theories. Let Set_0 be the category of *finite* sets. A *polyadic space* is a functor $E : Set_0^{op} \longrightarrow Stone$, where $Stone$ is the category of Stone spaces and continuous maps such that

i) For each $f : n \longrightarrow m$ in Set_0, $E(f) : E(m) \longrightarrow E(n)$ is open and

ii) E transforms push-out squares in Set_0 into quasi pull-back squares in $Stone$.

A *morphism* $\alpha : E \longrightarrow E'$ of polyadic spaces is a natural transformation such that for any $f : n \longrightarrow m$ in Set_0, the square

$$\begin{array}{ccc} E(m) & \xrightarrow{E(f)} & E(n) \\ {\scriptstyle \alpha_m}\downarrow & & \downarrow{\scriptstyle \alpha_n} \\ E'(m) & \xrightarrow[E'(f)]{} & E'(n) \end{array}$$

is a quasi pull-back. (Recall, that given a category $\mathcal{C}$ with pullbacks, a square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

if the morphism $P \longrightarrow X \times_Z Y$ is an epimorphism.)

Given a finite set X, Joyal defines a polyadic space $\tilde{X}$ by $\tilde{X}(n) = \beta(X^n)$, the Stone-Čech compactification of X^n, and a *model* of a polyadic space E(based on X) to be a morphism $\widetilde{X} \longrightarrow E$. He then claims that it is possible to associate canonically a polyadic space to an elementary theory and vice-versa. No details are given. It is asserted that under this correspondence, the two concepts of model coincide and that classical theorems of logic are interpreted and proven. We are not told which classical theorems of logic are interpreted and proven. Finally, and this is an important remark, it is claimed, without any detail, that polyadic spaces arise naturally in algebraic geometry.

The second abstract does not mention polyadic spaces. But it contains a fundamental result that will turn out to be central later on. Consider the category $\mathcal{E} = Set^{Set}$ with functors $Set \longrightarrow Set$ preserving finite limits and finite (disjoint) sums. It is claimed that these functors are precisely those preserving elementary operations of first-order logic. We are not told precisely what the latter operations are, except for projection and complement. But this is not the result reported, it is simply mentioned in the opening sentence of the abstract. The statement of the main result requires additional data. Let $\mathcal{U}$ denote the category whose objects are pairs $(I, \mathfrak{F})$, with $\mathfrak{F}$ an ultrafilter over the set I and whose morphisms are mappings preserving ultrafilters. Given two objects $(I, \mathfrak{F})$ and $(J, \mathfrak{G})$, define an equivalence relation on morphisms $(I, \mathfrak{F}) \longrightarrow (J, \mathfrak{G})$ by $f \equiv g$ if and only if $\exists F \in \mathfrak{F}$ such that $f|F = g|F$. Let $\mathcal{U}P$, called the category of *ultrapoints*, be the quotient category $\mathcal{U}/\equiv$. The main theorem is that there is an equivalence between $\mathcal{E}^{op}$ and the category of pro-objects of $\mathcal{U}P$ of ultrapoints. This is in itself an interesting duality theorem, but Joyal points out immediately afterwards that "the straightforward proof of this theorem can be adapted to prove a theorem of Keisler on elementary extensions of complete structures" [Joyal, 1971b, 967]. The theorem refered to here appeared in 1960 and states that an elementary class K is closed with respect to colimits if and only if K is characterized by a set of sentences constructed from finite conjunctions, finite disjunctions and existential quantifiers. We will give a more precise statement of the theorem later. Keisler's proof is model theoretical whereas Joyal's result stated here is purely categorical. The reference to the category of pro-objects is revealing, for the latter construction was introduced by Grothendieck and Deligne in SGA4, thus showing that Joyal was already well versed in Grothendieck toposes and related constructions.

This work was seen by Joyal and Reyes as being part of the search for an algebraic approach to logic. Indeed, in his paper published in 1972, Reyes used

Joyal's work on Boolean algebras and talks about "Joyal's functorial approach to algebraic logic", which refers to the first abstract quoted above. In the bibliography of the same paper, Reyes announced a paper in preparation entitled "Éléments de logique algébrique", which never saw the day.

Let us now turn to the three abstracts in algebraic geometry. A *spectral space* X is a topological space satisfying two conditions: i) the quasi-compact open subsets of X form a base closed under finite (possibly empty) intersections; ii) every irreducible closed subset F possesses one and only one generic point x, i.e. $\overline{\{x\}} = F$. A morphism of spectral spaces is a quasi-compact mapping. A theorem asserting that a topological space X is a spectral space if and only if X is a limit of finite T_0-spaces is stated. The main theorem is, once more, an equivalence of categories between the opposite of the category $DLat$ of distributive lattices and the category S_p of spectral spaces. Nowadays, the so-called category of coherent locales is used instead of the category of spectral spaces to prove the duality theorem. (See [Johnstone, 1982, 65].) The link to Stone's duality theorem is immediate. Furthermore, as Joyal mentions himself, the underlying space of a quasi-compact quasi-separated scheme is a spectral space, making explicit once more the connection with algebraic geometry *à la* Grothendieck, this time with EGA.

The second abstract is exactly in the same vein: the main theorem asserts the existence of a right adjoint to the inclusion functor $Stone \longrightarrow S_p$. Furthermore, there is a covariant full embedding from the category of spectral spaces and the category of *ordered* Stone spaces. Finally, Joyal claims that a Stone space is ordered if and only if it is a limit of finite partially ordered sets. (See [Johnstone, 1982, 75] for details.)

Only the last abstract is directly related to algebraic geometry: in it, Joyal generalizes results of Grothendieck on the cohomology of quasi-compact schemes to spectral spaces.

What these abstracts show clearly is that Joyal was juggling with three conceptual realms during this period: algebraic geometry, including Grothendieck toposes, Stone-type dualities and Lawvere's elementary theories in the spirit of algebraic logic. Thus, when Lawvere and Tierney introduced the concept of elementary topos, Joyal already had results that could easily be seen to be directly related to that development. The notion of elementary topos simply provided the general framework to develop these ideas systematically.

Joyal, but not Reyes, was present at the meeting held at Dalhousie between January 16 and January 19 1971. The goal of this meeting, according to Lawvere, was to investigate

> ... the connecctions between algebraic geometry and " intuitionistic " logic under the guidance of the form of objective dialectics known as category theory....
>
> Our own hopes in the success of the above general program were strengthened by initial progress in carrying out a more special program

> which will be outlined in the introduction. This is the development on the basis of elementary (first-order) axioms of a theory of " toposes " just good enough to be applicable not only to sheaf theory, algebraic spaces, global spectrum, etc. as originally envisaged by Grothendieck, Giraud, Verdier, and Hakim but also to Kripke semantics, abstract proof theory, and the Cohen-Scott-Solovay method for obtaining independence results in set theory. [Lawvere, 1972, 1].

According to Lawvere, seventy mathematicians participated in this meeting, among others Jean Bénabou, Marta Bunge, Jim Lambek, William Mitchell, Dana Schlomiuk, Dana Scott, Hugo Volger, to mention but those who are directly involved in our story. In the concept of elementary topos, geometry and logic are woven together in one structure, thus providing a natural setting for logic and algebraic geometry.

Let us now look carefully at the characterization of first-order logic developed by Joyal and Reyes.

4.4 From logical theories to categories

Joyal and Reyes realized that Volger's mistake had been to include the object Ω in the data of a logical category. They came to the conclusion that this object does not belong to first-order logic, neither in its syntax nor in its semantics. The key methodological insight was to start from a logical theory and build a category from it. The beauty of this approach is that in this way, one introduces, simultaneously, a hierarchy of logical theories together with a hierarchy of categories. Furthermore, the very first type of category thus obtained turned out to be an important type of abstract categories that had just been introduced and developed by Michael Barr for entirely different purposes, namely *regular categories*. (See [Barr *et al.*, 1971].) In addition, Joyal and Reyes made connections with elementary toposes and Joyal gave a uniform categorical treatment of the various completeness theorems. In fact, Lawvere reported that much himself in the introduction of the volume where Volger's papers were published. Indeed, in the very first page of the introduction, we read that the "much-researched intermediate doctrine of (classical) first-order logic corresponds to the fact P that the category of sets has finite limits, complements of subsets, and images of mappings (related by the condition of being a "regular" category. . .)." [Lawvere, 1975a, 3]. Volger himself acknowledges Joyal's work and remarks that Joyal has shown that his semantical categories are regular categories. (See [Volger, 1975a, 82].)

Joyal and Reyes started a joint paper that was finally published under Reyes's name in 1974, but a preliminary version of it was already circulating in 1972. (See [Reyes, 1974].) This is one of the very few published articles that present Joyal's work during that period. In the introduction, Reyes explicitly acknowledges Joyal's contribution. However, there is more literature that gives a portrait of the situation: as we have already mentioned, eight theses, Master's and PhDs, were written at the Université de Montréal on categorical logic between 1973 and 1977. Here is the complete list with the authors, subject, supervisors, degree and year:

Name	Title	Supervisor	Degree and year
Jean Dionne	Des théories élémentaires aux categories conceptuelles	Gonzalo Reyes	MSc July 1973
André Boileau	Les multiples splendeurs du forcing	Gonzalo Reyes	MSc December 1973
Christiane Rousseau	Triples, localization et topos	Pierre Berthiaume	MSc July 1974
Roch Ouellet	Axiomatization de la logique interne du premier ordre des topos, version inclusive et multisorte	Aubert Daigneault	PhD August 1974
Monique Robitaille-Giguère	Modèles d'une catégorie logique dans des topos de préfaisceaux et d'ensembles de Heyting	Gonzalo Reyes	MSc January 1975
Wedad Antonius	Théories cohérentes et prétopos	Gonzalo Reyes	MSc June 1975
André Boileau	Types versus Topos	Gonzalo Reyes	PhD July 1976
Christiane Rousseau	Théorie des topos et analyse complexe	Dana Schlomiuk	PhD April 1977

We assume that the work presented in Dionne, Ouellet, Robitaille-Giguère, Antonius and Boileau's Ph.D. thesis *roughly* reflects the chronological development and assimilation of the work done by Joyal and Reyes, although with a time shift. Also relevant are Reyes' later publications in 1977 and 1978 as well as Makkai and Reyes's papers published in 1976 which builds on the work done by Reyes and Joyal. Boileau's thesis is particularly interesting since it gives a general portrait of the work done in categorical logic at the time. Indeed, he refers to almost all the results presented in the theses by his colleagues.

Joyal and Reyes's strategy was to *start with* a first-order theory $\mathbb{T}$, *construct* a *small* category $\mathcal{C}_{\mathbb{T}}$ from it and then characterize in an abstract manner the type of category thus obtained. One of the goals was then to work with categorical methods to obtain significant results about the theory $\mathbb{T}$ or the underlying logic. There was an additional philosophical motivation. The very first formal step in the construction is to identify formulas of a theory $\mathbb{T}$ in a given language relative to their logical strength. More specifically, given a language $\mathcal{L}$ and a theory $\mathbb{T}$ in $\mathcal{L}$, one defines an equivalence relation between formulas of $\mathcal{L}$ by

$$\varphi \sim \psi \text{ if and only if } \mathbb{T} \vdash (\forall x_1) \cdots (\forall x_n)(\varphi \leftrightarrow \psi)$$

where $x_1, \ldots, x_n$ are the free variables of φ and ψ. The objects of the constructured category are the equivalence classes determined by the equivalence relation. One quickly recognizes that this construction is the categorical extension of the usual Lindenbaum–Tarski construction for propositional logic. One of the key insights here, which builds on Lawvere's work on quantifiers, is the fact that the Lindenbaum-Tarski construction can be lifted in a natural way so that when applied to first-order logic, it yields a category. Since categories are, among other things, generalizations of partial orders, this constitutes evidence that categorical logic provides a natural setting for the algebraic treatment of logic. From the point of view of Joyal, Reyes and other categorical logicians of that period, one of the underlying motivations was that logically equivalent formulas "mean" the same thing, that is we are dealing with *concepts* instead of their specific linguistic expressions. For that reason, these categories were labeled *conceptual categories* or *categories of concepts* (or sometimes *syntactical categories*). Morphisms between concepts are also equivalence classes of functional relations definable in the language.

As Jean Dionne underlies in the introduction of his thesis, it was believed that by replacing the theory $\mathbb{T}$ by the category $\mathcal{C}_{\mathbb{T}}$, the object of study would be at the same time simpler and more complete. The latter fact amounts to the claim that categories allow for more relations between concepts than the usual set-theoretical inclusion. Furthermore, models of theories can now be taken in various categories, in particular in toposes. The very last sentence of Dionne's thesis summarizes the belief that a categorical presentation $\mathcal{C}_{\mathbb{T}}$ of a theory $\mathbb{T}$ in a given formal system allows logicians to concentrate on the conceptual aspects of a theory and less on the syntactical details of a chosen presentation:

> Our method is therefore general enough to apply to a large variety of languages and we can conclude that it constitutes the true link between logic in the traditional sense and categorical logic : with the advantage that we can now work without using the axioms, rules of inference, formal demonstrations and the rest. We can even forget about variables, which were always inconvenient. Moving to categories, we can do logic in an " abstract " fashion in terms of objects, arrows, functors and diagrams, which is, among all the advantages mentioned in this work, not the least, since the logician is naturally more interested by concepts than by the indices to give to variables. [Dionne, 1973, 110] (our translation)

Thus, in the early seventies, the belief was that categorical methods were more objective or invariant than logical methods and that they would fruitfully replace logical methods. As far as we can tell, it was in April 1973, while he was in Montreal, that Bénabou showed that in some cases logical methods, via the so-called internal language, were more simple and direct than categorical methods to prove categorical properties of categories, in particular of certain toposes.

As we have already said, Joyal and Reyes' approach lead to a hierarchy of conceptual categories, the first of which resulting from the weakest first-order logical language yielding a category in the foregoing construction: regular categories. It was also quickly conjectured that the converse would be true, i.e. that one could start from a regular category $\mathcal{C}$ and obtain a theory $\mathbb{T}_{\mathcal{C}}$ in the appropriate language. Furthermore, it was common at that point to consider logic in a many-sorted language. (Indeed, Lawvere, Lambek, Bénabou and as we will see Mitchell had already indicated that many-sorted languages were natural a categorical context.) For the presentation of many-sorted languages, Dionne refers to Feferman's lecture notes in proof theory published a few years earlier. (See [Feferman, 1968].) The same reference is found in [Makkai and Reyes, 1976b; Reyes, 1977; Makkai and Reyes, 1977]. We will here essentially follow [Boileau, 1976], for he gives a general and unified presentation that, as we have indicated, constitutes in our opinion a comprehensive synthesis of the work done by Joyal and Reyes between 1971 and 1973 approximately. (For a slightly different presentation and terminology, see [Johnstone, 2002].)

A (first-order) *primitive language* $\mathcal{L}_0$ is given by:

1. A set $\mathcal{S}$ of sorts. The set $\mathcal{T}$ of *types* is build from the set of sorts as follows:
   (a) if $S \in \mathcal{S}$, then $S \in \mathcal{T}$;
   (b) if $S_1, ..., S_n \in \mathcal{T}$, then $S_1 \times \cdots \times S_n \in \mathcal{T}$, where n can be 0; whenever it is 0, we denote it by the symbol 1; (notice that $S_1 \times \cdots \times S_n$ is merely a formal expression, not a product);
   (c) nothing else is a type.
2. A set $\mathcal{F}$ of *function symbols*, to each function symbol f, we associate its *type* $S_1 \times \cdots \times S_n \times S$ (with the last sort having a distinguished status); we write $f : S_1 \times \cdots \times S_n \longrightarrow S$ to indicate the type of f. Whenever n is 0, f is a *constant* of type S and we write $1 \xrightarrow{f} X$.
3. A set $\mathcal{R}$ of *relation symbols*, each such relation symbol R has a *type* $S_1 \times \cdots \times S_n$; we write $R \rightarrowtail S_1 \times \cdots \times S_n$ to indicate that R has type $S_1 \times \cdots \times S_n$.

Terms and formulas are defined by recursion as follows:

Definition of terms:

1. Every variable of type S is a term of type S;
2. If $f : S_1 \times \cdots \times S_n \longrightarrow S$ is a function symbol and $t_1, \ldots, t_n$ are terms of type $S_1, \ldots, S_n$ respectively, then $f(t_1, \ldots, t_n)$ is a term of type S; (if f is a constant, we simply write f);

Definition of formulas:

1. If t_1 and t_2 are terms of type S, then $t_1 = t_2$ is a formula;

2. If $R \rightarrowtail S_1 \times \cdots \times S_n$ is a relation symbol and $t_1, \ldots, t_n$ are terms of type $S_1, \ldots, S_n$, then $R(t_1, \ldots, t_n)$ is a formula;

3. $\top$ is a formula ('true' is a formula);

The notions of free variable, bound variable, closed formula and closed terms are defined in the usual manner. This completes the definition of $\mathcal{L}_0$. There are no logical connectives so far. But there are logical rules associated to it, as we will see.

However, interpreting logical systems in a categorical framework forced categorical logicians to reconsider basic and simple assumptions in a different light. Indeed, in a set-theoretical setting, sorts are assumed to be non-empty. In a categorical setting, the situation is different. For one thing, in certain toposes, in particular toposes of sheaves, there are objects without elements that are nonetheless inhabited. To illustrate the situation, consider the simple case of the topos Set^2 whose objects are pairs $\langle X, Y \rangle$ of sets and a morphism $\langle f, g \rangle : \langle X, Y \rangle \longrightarrow \langle V, W \rangle$ is given by a pair of functions $f : X \longrightarrow V$ and $g : Y \longrightarrow W$. Consider now any object of the form $\langle \emptyset, X \rangle$ where X is any non-empty set. Such an object is not empty (nor is it an initial object in the language of categories), but it is not like an arbitrary object $\langle X, Y \rangle$ either. From the categorical point of view, an element x of a set X corresponds to a morphism $1 \xrightarrow{x} X$ from the terminal object 1, that is any singleton set, to the set X in the category Set of sets. It is therefore reasonable to generalize this fact to categories with a terminal object (or finite limits in general): a (global) element of an object X in a category $\mathcal{C}$ is a morphism $1 \xrightarrow{x} X$. In the topos Set^2, the terminal object is the pair $\langle 1, 1 \rangle$ and there is no morphism $\langle 1, 1 \rangle \longrightarrow \langle \emptyset, X \rangle$. In this sense, we can say that the latter object does not have any (global) element, although it is not empty. In his introduction to the volume on model theory and toposes, Lawvere underlines this fact and comments rather harshly on the classical solution:

> Since a variable set may be partly empty and partly non-empty, the traditional model-theoretic banishment of empty models cannot be maintained, bringing to light a certain difficulty which the banishment obscured. Some claim that this difficulty is the "fact" that "entailment is not transitive", contrary to mathematical experience. However, the actual "difficulty" is that the traditional logical way of dealing with variables is inappropriate and hence should be abandoned. This traditional method (which by the way is probably one of the reasons why most mathematicians feel that a logical presentation of a theory is an absurd machine strangely unrelated to the theory or its subject matter) consists of declaring that there is one set I of variables on which all finitary relations depend, albeit vacuously on most of them; ... [Lawvere, 1975b, 5]

Various people, among others Mostowski, Hailperin in the fifties, and Bénabou and Joyal in 1973, had observed the failure of the transitivity of entailment with empty sorts. (See [Mostowski, 1951; Hailperin, 1953].) A simple case is as follows: the following two sequents are always valid, no matter what:

$$\forall x P(x) \vdash P(x)$$

$$P(x) \vdash \exists x P(x)$$

Thus, by transitivity of entailment, we obtain:

$$\forall x P(x) \vdash \exists x P(x)$$

which *is* an invalid entailment when interpreted in an empty sort. Mostowski suggested to simply dropping the transitivity of entailment. Hailpering gave an axiomatization in which entailments are restricted to sentences. Ideally, one would keep the transitivity of entailments and give a presentation of a logical system for arbitraty formulas.

As far as we can tell, the current solution to the problem was first presented in print in Roch Ouellet's PhD thesis defended at the Université de Montréal in 1974. [Ouellet, 1974] (See also [Ouellet, 1981].) He explicitly attributes the key observation leading to the solution to Joyal. The solution consists in introducing the notion of the *support of a sequent* or what is now called a *context*: a *context* is a finite list $\vec{x} = x_1, \ldots, x_n$ of distinct variables. Notice that a context can be empty, i.e. n can be 0. The *type* of a context $\vec{x}$ is the string of (not necessarily distinct) sorts of the variables appearing in it. A *term-in-context* is an expression of the form $\vec{x} \cdot t$ where t is a term and $\vec{x}$ is a context containing all the free variables of t. Similarly, a *formula-in-context* is an expression of the form $\vec{x} \cdot \phi$, where ϕ is a formula and $\vec{x}$ is a context containing all the free variables occurring in ϕ.

The notion of logical consequence is then defined with respect to a context $\vec{x}$. A *sequent* is a sequence of symbols of the form $(\phi \vdash_{\vec{x}} \psi)$, where ϕ and ψ are formulas and $\vec{x}$ is a context containing all the free variables of ϕ and ψ. It is now easy to see that the foregoing failure of transitivity is eliminated, since the sequent of the conclusion, namely $\forall x P(x) \vdash \exists x P(x)$, has an empty context whereas the premises do not. Hence, if we require, as we will do, that entailments have to be done over uniform contexts or that a change of context has to obey certain restrictions, the conclusion will not follow. It should be emphasized that, as is done by Ouellet himself in his thesis, the process of finding a solution was guided all along by categorical semantics. Thus, in this case, the correct syntactical constraint was derived from categorical constraints.

Recall that in a sequent calculus, a deductive rule has the form $\frac{\Gamma}{\alpha}(R)$, where Γ is a (possibly empty) list of sequents and α is a sequent. The horizontal bar means that the rule R allows the move from Γ to α. The intended meaning is that whenever the sequents of Γ are valid, then α is valid. When Γ is empty, then in this case α is a logical axiom. Here are the axioms and rules of $\mathcal{L}_0$, which are in fact the rules of *equational logic*:

1. Structural rules:

1.1. $\phi \vdash_{\vec{x}} \phi$ (identity axiom)

1.2. $\dfrac{\phi \vdash_{\vec{x}} \psi \quad \psi \vdash_{\vec{x}} \theta}{\phi \vdash_{\vec{x}} \theta}$ (cut)

1.3. $\frac{\phi \vdash_{\vec{x}} \psi}{\phi[\vec{s}/\vec{x}] \vdash_{\vec{y}} \psi[\vec{s}/\vec{x}]}$ (substitution) where $\vec{s} = s_1, \ldots, s_n$ is a list of (not necessarily distinct) terms of the same length and type as the context $\vec{x}$, $\phi\,[\vec{s}/\vec{x}]$ denotes the usual operation of simultaneous substitution with the usual proviso and $\vec{y}$ is any string of variables including all the variables occurring in the terms $\vec{s}$.

2. Logical rule:

   2.1. $\phi \vdash_{\vec{x}} \top$;

3. Rule for equality:

   3.1. $\top \vdash_{\vec{x}} x = x$.

The next step consists in introducing logical connectives in a certain order. The first extension of $\mathcal{L}_0$ is denoted by $\mathcal{L}_1$ by Boileau, but we will denote it by $\mathcal{L}_{reg}$ for *regular logic*.

Definition of formulas of $\mathcal{L}_{reg}$: formulas of $\mathcal{L}_{reg}$ consist of formulas of $\mathcal{L}_0$ together with the following additional clauses:

4. If ϕ and ψ are formulas, then $(\phi \wedge \psi)$ is a formula;

5. If ϕ is a formula, then $\exists x \phi$ is formula (where x is variable of some type; we sometimes write $(\exists x : S)\phi$ to indicate the type of x).

Axioms and rules of $\mathcal{L}_{reg}$: the axioms and rules of $\mathcal{L}_{reg}$ are those of $\mathcal{L}_0$ together with:

2. Logical rules:

3. $\phi \wedge \psi \vdash_{\vec{x}} \phi \quad \phi \wedge \psi \vdash_{\vec{x}} \psi \quad \dfrac{\phi \vdash_{\vec{x}} \psi \quad \phi \vdash_{\vec{x}} \theta}{\phi \vdash_{\vec{x}} \psi \wedge \theta}$;

4. $$\frac{\phi \vdash_{\vec{x},y} \psi}{\exists y \phi \vdash_{\vec{x}} \psi}$$ where y is not free in ψ and where a double line indicates that we can go either way;

5. $\phi \wedge \exists y \psi \vdash_{\vec{x}} \exists y\,(\varphi \wedge \psi)$ where y is not free in ϕ.

6. Rule for equality:

   3.2. $((x_1 = y_1) \wedge \ldots \wedge (x_n = y_n)) \wedge \phi \vdash_{\vec{z}} \phi\,[\vec{y}/\vec{x}]$ where $\vec{z}$ is a context containing $\vec{x}$, $\vec{y}$ and the free variables of ϕ.

A *regular theory* $\mathbb{T}$ is a set of regular sequents, i.e. sequents in which the formulas are all regular formulas. The elements of $\mathbb{T}$ are called the *axioms* of $\mathbb{T}$. There are no examples of regular theories in [Dionne, 1973], nor in [Reyes, 1974]. Of course, there are numerous examples of regular *categories*.

The next extension quickly became the center of attention because of its metalogical properties. It is the language $\mathcal{L}_{coh}$, labeled $\mathcal{L}_2$ by Boileau, of *coherent logic*.

Definition of formulas of $\mathcal{L}_{coh}$: formulas of $\mathcal{L}_{coh}$ consist of formulas of $\mathcal{L}_{reg}$ together with the additional clauses:

6. $\perp$ is a formula ('false' is a formula);

7. If ϕ and ψ are formulas, then $(\phi \vee \psi)$ is a formula.

Axioms and rules of $\mathcal{L}_{coh}$ are those of $\mathcal{L}_{reg}$ together with:

2. 2.1. $\perp \vdash_{\vec{x}} \phi$;

   2.2. $\phi \vdash_{\vec{x}} \phi \vee \psi \quad \psi \vdash_{\vec{x}} \phi \vee \psi \quad \frac{\phi \vdash_{\vec{x}} \theta \quad \psi \vdash_{\vec{x}} \theta}{\phi \vee \psi \vdash_{\vec{x}} \theta}$;

   2.3. $\phi \wedge (\psi \vee \theta) \vdash_{\vec{x}} (\phi \wedge \psi) \vee (\phi \wedge \theta)$.

A *coherent theory* $\mathbb{T}$ is a set of coherent sequents. Although the language of coherent logic might seem to be rather weak at first, it was quickly seen that many mathematical theories are in fact axiomatized in that language. Thus, every equational theory is coherent, e.g. the theory of groups, the theory of rings, Boolean algebras, etc., provided one writes the axioms appropriately. The theory of fields of characteristic p, $p \geq 0$, is coherent, as well as the theory of algebraically closed fields. It was proved that any classical theory can be translated in the language of coherent logic provided that the latter is enriched with sufficiently many relational symbols. The latter claim is a corollary of the proof of the completeness theorem for coherent theories attributed to Makkai by Antonius. (See [Antonius, 1975, 35–36].) Coherent logic was seen has having an interesting position with respect to intuitionistic logic and classical logic. In the words of Reyes:

> If the reader looks at the formal system for coherent logic ..., he will notice that all axioms as well as rules of inference are intuitionistically valid (...) as well as classically valid. The distinction between the intuitionistic and classical interpretations of logical operations become irrelevant for this "absolute" logic.
>
> Furthermore, this logic besides being a part [sic], may be considered as a generalization of classical logic. Indeed, any classical theory may be rendered coherent by extending the language [Antonius]. The idea is trivial and may be seen from the following example : take
>
> $$\exists x \exists y (x \neq y)$$

> as our theory. Adding a new binary relation symbol D (to be thought of as $\neq$), the desired coherent theory is
>
> $$\begin{array}{l} \top \vdash \exists x \exists y D(x,y) \\ \top \vdash x = y \vee D(x,y) \\ x = y \wedge D(x,y) \vdash \bot \end{array}$$
>
> The point is : the models which respect the coherent logic of the new theory are the usual classical models (respecting the full logic) of the old one. [Reyes, 1977, 23].

(Reyes did not specify the support of the sequents.) As we can see from the last sentence, the charms of coherent logic were particularly obvious from the model-theoretic point of view. But we should point out that the insistence on the formal system itself is something that came during the academic year 1973–74, for absolutely nothing is said about it before that date. As we have already pointed out, they were merely used as a springboard in order to rise to the categorical level.

The next two levels are the intuitionistic and the classical levels.

Definition of formulas of $\mathcal{L}_{int}$ (Boileau's $\mathcal{L}_3$): the formulas of $\mathcal{L}_{int}$ are those of $\mathcal{L}_{coh}$ together with:

8. If ϕ and ψ are formulas, then $(\phi \Rightarrow \psi)$ is a formula;
9. If ϕ is a formula, then $\forall x \phi$ is formula (or $(\forall x : S)\phi$).

Axioms and rules of $\mathcal{L}_{int}$ are those of $\mathcal{L}_{coh}$ together with:

2.8 $\dfrac{\phi \wedge \psi \vdash_{\vec{x}} \theta}{\psi \vdash_{\vec{x}} \phi \Rightarrow \theta}$

2.9 $\dfrac{\phi \vdash_{\vec{x},y} \psi}{\phi \vdash_{\vec{x}} \forall y \psi}$ where y is not free in ϕ.

Intuitionistic theories are defined in the obvious manner.

Finally, we come to classical first-order logic $\mathcal{L}_{Bool}$ (Boileau's $\mathcal{L}_4$). We now allow formulas of the form $\neg\phi$, which can be defined as $\phi \Rightarrow \bot$.

Axioms and rules of $\mathcal{L}_{Bool}$ are those of $\mathcal{L}_{int}$ together with:

2.10 $\dfrac{\phi \wedge \psi \vdash_{\vec{x}} \bot}{\psi \vdash_{\vec{x}} \neg\phi}$; a special case of the rule 2.8;

2.11 $\top \vdash_{\vec{x}} \phi \vee \neg\phi$.

A *proof* in these systems is defined in the usual fashion. We will write $\mathrm{T} \vdash: \alpha$, where α is a sequent $\phi \vdash_{\vec{x}} \psi$, to indicate that there is a proof of α from the axioms of $\mathbb{T}$. Whenever we will have to specify the underlying logic, we will write for instance $\mathbb{T} \underset{coh}{\vdash:} \alpha$, meaning that the proof is in coherent logic.

Dionne 1973 describes these languages for one-sorted systems and sketches the extension to many-sorted systems in the conclusion of his thesis. Furthermore, coherent logic does not receive a name: it is simply called regular logic with stable sup (for reasons that will appear clear once we look at categories corresponding to this language). This is also how they are called in Reyes [1974]. (Remember that the latter was already circulating in 1972.) This does suggest that in 1972 and 1973, coherent logic had not quite acquired its status or at least no one thought that it deserved to receive a name.

There is an obvious partial order between the foregoing logical systems: define $\mathcal{L}_i \leq \mathcal{L}_k$ if and only if $\mathcal{L}_i \subseteq \mathcal{L}_k$ and for any set of sequents $\mathbb{T} \cup \alpha$ of $\mathcal{L}_i$, we have $\mathbb{T} \underset{\mathcal{L}_i}{\vdash:} \alpha$ if and only if $\mathbb{T} \underset{\mathcal{L}_k}{\vdash:} \alpha$. It can be proved that resulting order relation is:

$$\begin{tikzcd}
 & & & \mathcal{L}_{int} \\
\mathcal{L}_0 \arrow[r, dash] & \mathcal{L}_{reg} \arrow[r, dash] & \mathcal{L}_{coh} \arrow[ur, dash] \arrow[dr, dash] & \\
 & & & \mathcal{L}_{bool}
\end{tikzcd}$$

From a given theory $\mathbb{T}$ in a formal system $\mathcal{L}_i$, $0 \leq i \leq 4$, a category of concepts $\mathcal{C}_{\mathbb{T}}$ can be built. The construction of $\mathcal{C}_{\mathbb{T}}$ always follows the same pattern. (It should be pointed out that Makkai and Reyes introduced a slightly different construction later on, although the result is equivalent. See [Antonius, 1975, 14–15]. In their book, Makkai and Reyes sketch a method that uses a completeness theorem proved earlier but claim that the construction is described in great detail in Dionne's thesis, whereas the latter proceeds in a purely syntactic manner. See [Makkai and Reyes, 1977, 241–242]. As we have already indicated, the objects of $\mathcal{C}_{\mathbb{T}}$ are equivalence classes of formulas, where the equivalence relation is defined by:

$$\phi(\vec{x}) \sim \psi(\vec{x}) \text{ if and only if } \phi(\vec{x}) \vdash_{\vec{x}} \psi(\vec{x}) \text{ and } \psi(\vec{x}) \vdash_{\vec{x}} \phi(\vec{x}) \text{ are derivable sequents of } \mathbb{T}.$$

The equivalence class of $\phi(\vec{x})$ is denoted by $[\phi(\vec{x})]$ and it is sometimes called a *formal set*.

A morphism of formal sets is a *formal function*. Informally, a formal function is given by a formula that defines, in each model of the theory $\mathbb{T}$, an actual morphism between the interpreted formal sets. In particular, when the theory is interpreted in the category of sets, it defines a set-theoretical function between sets. Specifically, let $[\phi(\vec{x})]$ and $[\psi(\vec{y})]$ be two formal sets such that the contexts $\vec{x}$ and $\vec{y}$ are disjoint and let ρ be a formula with free variables in the context $\vec{x}; \vec{y}$. Then we say that $\rho(\vec{x}, \vec{y})$ defines a formal function from $[\phi(\vec{x})]$ to $[\psi(\vec{y})]$, denoted by $\langle \vec{x} \mapsto \vec{y} : \rho \rangle : [\phi(\vec{x})] \longrightarrow [\psi(\vec{y})]$, if

1. $\rho(\vec{x}, \vec{y}) \vdash_{\vec{x},\vec{y}} (\varphi(\vec{x}) \wedge \psi(\vec{y}))$
2. $\rho(\vec{x}, \vec{y}) \wedge \rho(\vec{x}, \vec{z}) \vdash_{\vec{x},\vec{y},\vec{z}} \vec{y} = \vec{z}$
3. $\varphi(\vec{x}) \vdash_{\vec{x}} \exists \vec{y} \rho(\vec{x}, \vec{y}))$.

To finish the definition of morphisms, we identify formal functions that are provably equivalent: that is, whenever we have

$$\rho(\vec{x},\vec{y}) \vdash_{\vec{x},\vec{y}} o(\vec{x},\vec{y}) \text{ and } o(\vec{x},\vec{y}) \vdash_{\vec{x},\vec{y}} \rho(\vec{x},\vec{y})$$

where $(\vec{x},\vec{y},o)$ defines a function from $[\phi(\vec{x})]$ to $[\psi(\vec{y})]$, then $(\vec{x},\vec{y},\rho)$ and $(\vec{x},\vec{y},o)$ belong to the same equivalence class $\langle\vec{x}\mapsto\vec{y}:\rho\rangle : [\phi(\vec{x})] \longrightarrow [\psi(\vec{y})]$. In a nutshell, the morphisms of $\mathcal{C}_{\mathbb{T}}$ are equivalence classes of formulas that are provably equivalent functional relations of $\mathbb{T}$. Composition of morphisms is defined by following the same general strategy. Given formal sets and formal functions

$$[\phi(\vec{x})] \overset{\langle\vec{x}\mapsto\vec{y}:\rho\rangle}{\longrightarrow} [\psi(\vec{y})] \overset{\langle\vec{y}\mapsto\vec{z}:o\rangle}{\longrightarrow} [\theta(\vec{z})]$$

the composition $\langle\vec{y}\mapsto\vec{z}:o\rangle\circ\langle\vec{x}\mapsto\vec{y}:\rho\rangle : [\phi(\vec{x})] \longrightarrow [\theta(\vec{z})]$is given by the formal function $\langle\vec{x}\mapsto\vec{z}:\kappa\rangle$ where $\kappa = \exists\vec{y}(\rho\wedge o)$. The verifications that composition is associative and that the identity morphisms exist and satisfy the required properties are extremely tedious. The moral is that one obtains a genuine *small* category $\mathcal{C}_{\mathbb{T}}$ for each theory $\mathbb{T}$ in a given language. The results can be summarized with the following table:

If $\mathbb{T}$ is in	**then $\mathcal{C}_{\mathbb{T}}$ is a**
$\mathcal{L}_{reg}$	Regular category
$\mathcal{L}_{coh}$	Coherent category
$\mathcal{L}_{int}$	Heyting category
$\mathcal{L}_{Bool}$	Boolean category

The foregoing construction can be seen to be a direct generalization of the standard Lindenbaum–Tarski algebra for a propositional theory. Indeed, it can be seen that if $\mathbb{T}$ is a *propositional* coherent theory, then $\mathcal{C}_{\mathbb{T}}$ is a preorder, in fact a distributive lattice. In the general case of a coherent theory $\mathbb{T}$, the standard Lindenbaum-Tarski algebra is simply a part of $\mathcal{C}_{\mathbb{T}}$, that is the part restricted to sentences, e.g. formulas in the empty context. This provides further evidence to the claim that categorical logic is algebraic logic.

Notice, and this is an important *philosophical* point, that a category of concepts $\mathcal{C}_{\mathbb{T}}$ is *not* a category of structured sets and structure preserving functions: morphisms are *equivalence classes* of morphisms. Thus, one of the main tools of categorical logic is the construction of categories that are in a sense very different from the categories arising in the practice of mathematics, e.g. the category of groups or the category of vector spaces.

Whereas Dionne's thesis consists in a detailed description of the constructions of conceptual categories from theories and a proof, in each case, that the resulting category has the appropriate properties, Reyes 1974 starts right from the start with a description of the categories. Of course, as we have already emphasized, within the categorical community, it was assumed that replacing a theory $\mathbb{T}$ by its category of concepts $\mathcal{C}_{\mathbb{T}}$ was not only mathematically fruitful but also philosophically motivated. One of the goals was to give a purely categorical characterization of these categories and then work with these instead of the syntactical presentations of the theories.

4.5 Algebraic logic: from regular to Boolean categories

Regular categories are categories in which any morphism factors uniquely as a (regular) epimorphism followed by a monomorphism. In contrast with Volger's logical categories and elementary theories, any topos $\mathcal{E}$ turns out to be a regular category and, furthermore, first-order logic becomes completely integrated in the overall framework at the center of which sits the notion of topos.

In order to state the definition of a regular category, we need to recall a few simple notions. The pullback of a pair of equal morphisms $X \xrightarrow{f} Y \xleftarrow{f} X$, when it exists, is called the *kernel pair* of f. Given two parallel morphisms $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, a *coequalizer* of $\langle f, g\rangle$ is a morphism $q : Y \longrightarrow E$ such that $q \circ f = q \circ g$ and for any morphism $h : Y \longrightarrow Z$ such that $h \circ f = h \circ g$, there is a unique morphism $h' : E \longrightarrow Z$ such that $h' \circ q = h$. A morphism $f : X \longrightarrow Y$ is said to be a *regular epimorphism* when it is the coequalizer of a pair of arrows.

Definition: a category $\mathcal{C}$ is said to be *regular* if it satisfies the following properties:

1. It has all finite limits;
2. Coequalisers of kernel pairs exist;
3. Regular epimorphisms are stable under pullback.

The last condition simply means that if $f : X \longrightarrow Y$ is a regular epimorphism and $g : Z \longrightarrow Y$ is a morphism of $\mathcal{C}$, then the morphism $g' : X \times_Y Z \longrightarrow X$ obtained by pulling back f along g as in the following diagram

$$\begin{tikzcd} X \times_Y Z \arrow[r, "f'"] \arrow[d, "g'"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is also a regular epimorphism.

The category *Set* of sets is a regular category and, in fact, so is any elementary topos $\mathcal{E}$.

This is not the definition given by Reyes in Reyes [1974], but it is equivalent to it. In Reyes' paper, a regular category is a category with finite limits, finite sups of subobjects of a given object and images, which we will define. As we have already indicated, one of the main properties of a regular category is that any morphism $f : X \longrightarrow Y$ can be factored uniquely, up to isomorphism,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e"'] & & Y \\ & E \arrow[ur, "m"'] & \end{tikzcd}$$

where $e : X \longrightarrow E$ is a regular epimorphism and $m : E \longrightarrow Y$ is a monomorphism. The latter monomorphism is called the *direct image* of f and is denoted $\mathrm{Im}(f)$. $\mathrm{Im}(f)$ is defined only up to a unique isomorphism, but as such it determines a unique subobject of Y, which we denote $\exists_f(X)$. The choice of terminology is not a coincidence. Given any momomorphism $i : A \rightarrowtail X$, we can define $\exists_f(A) = \mathrm{Im}(f \circ i)$ and in this way $\exists_f$ is a well defined morphism from the subobjects of X to the subobjects of Y and it can be shown that, in fact, for any arrow $f : X \longrightarrow Y$ of a regular category $\mathcal{C}$, $\exists_f \dashv f^*$. Properties of the existential quantifier then follow immediately. In particular, what is now called the *Frobenius identity* holds in any regular category $\mathcal{C}$: for any morphism $f : X \longrightarrow Y$ and monomorphisms $i_A : A \rightarrowtail X$, $i_B : B \rightarrowtail Y$, $\exists_f(A \wedge f^*B) = \exists_f A \wedge B$, where both sides of the equality are subobjects of Y.

Notice how the latter characterization weaves together Lawvere's characterization of elementary theories and Daigneault's characterization of the condition for the existence of injective envelopes.

It is natural to consider *regular functors* between regular categories: a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between regular categories is said to be *regular* if it preserves finite limits and coequalizers of kernel pairs. We can therefore define the category *RegCat* with objects (small) regular categories and with morphisms regular functors. (We should point out that it was immediately clear that it is a 2-category.)

Joyal and Reyes then introduced *regular categories with stable* $\vee$, now called *coherent categories*, but they were also called *logical categories* from 1974 until 1977 inclusively. It should be emphasized that all these categories are regular categories with additional *properties* and not additional *structure*.

A category $\mathcal{C}$ is said to be a *coherent category* if:

1. It is a regular category;

2. For each object X of $\mathcal{C}$, $\mathrm{Sub}(X)$ has finite sups;

3. The inverse image morphism $f^* : \mathrm{Sub}(Y) \longrightarrow \mathrm{Sub}(X)$ preserves finite sups, i.e. finite sups are stable under pullbacks.

A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between coherent categories is said to be *coherent* if it is regular and if it preserves finite sups. Again, we can consider the category *Coh* of coherent categories and coherent functors between them.

A category $\mathcal{C}$ is said to be a *Heyting category* if:

1. It is a regular category;
2. For every morphism $f : X \longrightarrow Y$, the inverse image morphism $f^* : \mathrm{Sub}(Y) \longrightarrow \mathrm{Sub}(X)$ has a right adjoint $\forall_f : \mathrm{Sub}(X) \longrightarrow \mathrm{Sub}(Y)$;
3. For every object X of $\mathcal{C}$, $\mathrm{Sub}(X)$ is a lattice with smallest element.

It should be noted that the existence of the right adjoint $\forall_f$, for every f, implies the existence of the operation $\Rightarrow$ of implication in every lattice of subobjects. A *Heyting functor* between Heyting categories is a regular functor preserving $\forall_f$ and the lattice operations (in fact, supremums and implication suffice). The category of Heyting categories and Heyting functors will be denoted by Heyt.

A category $\mathcal{C}$ is said to be a *Boolean category* if

1. It is a Heyting category;
2. For every object X of $\mathcal{C}$, $\mathrm{Sub}(X)$ is a Boolean algebra.

A *Boolean functor* is a Heyting functor preserving all the lattice operations. The category of Boolean categories and Boolean functors will be denoted by Bool.

Thus, a theory $\mathbb{T}$ in a language $\mathcal{L}_i$ is now replaced by an i-category $\mathcal{C}_{\mathbb{T}}$. The goal is now to use concepts and methods of category theory to obtain results about logic. As usual, in a categorical framework, one first investigates the existence of adjoint functors to naturally given functors. Thus, there are obvious forgetful functors (in fact 2-functors):

$$\mathit{Bool} \longrightarrow \mathit{Heyt} \longrightarrow \mathit{Coh} \longrightarrow \mathit{Reg}.$$

Two questions arise immediately: 1. Are there adjoints to these natural forgetful functors? 2. How are these categories related to toposes? In other words, what are the functors between these categories and the category of toposes?

These questions are in part answered in Reyes 1974. Let us now summarize the content of that paper, since it reflects in part the knowledge of the field as of 1971-1972 approximately. The first section, written essentially by Joyal, defines the various foregoing categories and proves some of their important properties. The second section, written by Reyes, gives a definition of Grothendieck toposes, proves that every Grothendieck topos is a Heyting category and defines the category $\mathbb{T}\mathrm{op}$ of toposes with *geometric* morphisms as functors. Section three of the paper investigates the links between Grothendieck toposes and logic, in particular the existence of (left-)adjoints to forgetful functors from $\mathbb{T}\mathrm{op}$ to other logical categories. Reyes attributes to Joyal the observation that there is a left adjoint $Sh(-) : \mathit{Coh} \longrightarrow \mathbb{T}\mathrm{op}$ to the forgetful functor $\mathbb{T}\mathrm{op} \longrightarrow \mathit{Coh}$ defined by: for each coherent category $\mathcal{C}$, $Sh(\mathcal{C})$ is the topos of sheaves for the so-called finite cover topology and for any *regular* functor $\mathcal{C} \xrightarrow{u} \mathcal{D}$, we get a geometric morphism of toposes $\widehat{u}$

given by the pair of adjoints $Sh(\mathcal{C}) \underset{u_*}{\overset{u^*}{\rightleftarrows}} Sh(\mathcal{D})$. Furthermore, and this is the key observation that will lead to important concepts and results, although it is not emphasized nor put in perspective in the paper, the functor $Sh(-)$ satisfies the following universal property: for every *coherent* functor $u : \mathcal{C} \longrightarrow \mathcal{E}$ into a topos, there is a unique (up to isomorphism) morphism of topos $Sh(-) \underset{u_*}{\overset{u^*}{\rightleftarrows}} \mathcal{E}$ such that the triangle

$$\begin{tikzcd} \mathcal{C} \arrow[r, "h"] \arrow[dr, "u"'] & Sh(\mathcal{C}) \arrow[d, "u^*"] \\ & \mathcal{E} \end{tikzcd}$$

commutes, where the functor $h : \mathcal{C} \longrightarrow Sh(\mathcal{C})$ is the standard Yoneda embedding. As we will see, this result introduces without naming it what will be called the *classifying topos* of $\mathcal{C}$.

It is then shown in the next paragraph that there is no such left adjoint between the category of toposes and the category of Heyting categories. There is however a left adjoint to the forgetful functor from the category of Boolean toposes and the category of Boolean categories, provided that the functors in the latter category, that is the Boolean functors, preserve arbitrary sups.

In the last section of the paper, Reyes presents Joyal's completeness theorem for Boolean categories and explores links with models of non-standard analysis. We will only comment the completeness theorem.

Here is the formulation given in the paper: Let $\mathcal{C}$ be a small Boolean category. If X is distinct from the smallest object of $\wp(X)$, then there is a Boolean model $M : \mathcal{C} \longrightarrow Set$ such that $M(X) \neq \emptyset$. Three points have to be underlined. First, notice the similarity between this formulation and Daigneault's formulation of the completeness theorem and, thus, with Lawvere's formulation. Second, the proof of the completeness theorem given in the paper relies on what is essentially the same construction as the one stated in the second abstract on logic and foundations in the *Notices*, although there is no mention of pro-objects in the 1974 paper. This strongly suggests that Joyal had a proof of the completeness theorem by purely categorical means already in the fall of 1970. Third, although it is not worded in this manner in the paper, the theorem amounts to the following *representation* theorem: let $\mathcal{C}$ be a (small) coherent category. Then there is a *set* I and a *conservative* coherent functor $M : \mathcal{C} \longrightarrow Set^I$. Being "conservative" means, in this context, that M *reflects* isomorphisms, that is if $M(f)$ is an isomorphism in Set^I, then f is an isomorphism in $\mathcal{C}$. Notice here the similarity with the representation theorem for polyadic algebras, as formulated by Daigneault. It is striking that the set X in the latter formulation is replaced by the category Set of sets in this new formulation. Although the theorem is strictly speaking a representation theorem, it *implies* the completeness theorem, since the coherent category $\mathcal{C}$ is a conceptual category, i.e. a theory. Furthermore, since coherent functors between Boolean categories are in fact Boolean, a fact used explicitly by Joyal in his proof, the representation theorem for Boolean categories is a special case the theorem for coherent categories. Thus, one obtains directly the completeness theorem for classical first-order logic.

When compared to Volger's attempt at characterizing the doctrine of first-order logic, two elements standout in Joyal and Reyes's work. First, the elegance and simplicity of the hierarchy of categories defined, starting with regular categories, reveals a global conceptual coherence, no pun intended, of logical categories. Second, as we have already pointed out, regular categories also have an important status in category theory in general and thus their classification establishes important links between logical notions and categorical notions. Third, the approach seems to be at the right level of generality and flexibility, in contrast with Volger's work. In particular, it did not presuppose the existence of a subobject classifier, or an object of truth-value and still fitted in perfectly with toposes. The key was to find a way to articulate together fragments of first-order logic that would allow a categorical passage from one type of category to another and in such a way that one would not have to import additional structure, in the way Volger had done. It was certainly seen as an important gain that only properties were added to move from one level to the next. Finally, the connections with toposes made it possible to construct a dictionary between algebraic geometry and logic and one could hope that results and methods from both fields could interact and yield new and useful insights. Thus, a central piece of the puzzle of algebraic logic was now in place. The other parts were being added quickly. One of the pieces, already hinted at in Dionne's thesis, was to find a way to construct a theory $\mathbb{T}_\mathcal{C}$ from a category $\mathcal{C}$ in such a way that there is a canonical interpretation of $\mathbb{T}_\mathcal{C}$ in $\mathcal{C}$. But for the latter, one needs to have a general notion of interpretation of a theory in a category, which is the second piece. Once these have been defined appropriately, they fit together smoothly and the general conceptual picture emerging is of great beauty.

4.6 Constructing theories from categories

As early as 1971, the idea of interpreting a logical theory in a topos appeared in William Mitchell's work on the relation between Boolean toposes and set theory. In his paper, communicated by Mac Lane in October 1971 and published in 1972, he explicitly introduced a language $\mathcal{L}(\mathcal{E})$ for a topos $\mathcal{E}$ and two types of interpretations of $\mathcal{L}(\mathcal{E})$: an *external interpretation* and an *internal interpretation*. The internal interpretation would soon become the internal *language* of a topos and it would be used like any other deductive system to prove results *in* that specific theory, whereas the external interpretation would amount to using certain toposes as *models* of certain specified theories of sets. Both methods evolved quickly, but it seems clear that Bénabou, in his talk at Oberwolfach and Joyal in his talk in

Montreal, both in early 1973, found a way to define the internal language $\mathcal{L}(\mathcal{C})$ for a weaker category $\mathcal{C}$. Makkai and Reyes quickly used these methods in 1973 and 1974. Their first joint papers were presented in November 1974 and published in 1976.

Before we look at the internal language of a category $\mathcal{C}$, we have to consider the notion of an interpretation M of a many-sorted language $\mathcal{L}$ in a category $\mathcal{C}$ with finite limits. The basic idea was to lift to categories the traditional Tarskian semantics as defined with sets. The translation requires, as usual, that clauses stated in terms of elements and subsets be reformulated in terms of morphisms and subobjects.

Assume $\mathcal{C}$ is a category with finite limits and $\mathcal{L}_0$ is defined as above. A $\mathcal{C}$-*interpretation* M of $\mathcal{L}_0$ consists of:

1. an object MS of $\mathcal{C}$ for each sort S;
2. a morphism $Mf : MS_1 \times \cdots \times MS_n \longrightarrow MS$ for each function symbol $f : S_1 \times \cdots \times S_n \longrightarrow S$; in particular, when $n = 0$ a morphism $1 \xrightarrow{f} MS$;
3. a subobject $MR \rightarrowtail MS_1 \times \cdots \times MS_n$ for each relation symbol $R \rightarrowtail S_1 \times \cdots \times S_n$.

Notice immediately that when $\mathcal{C}$ is the category Set of sets, then a Set-interpretation is an ordinary many-sorted structure of similarity type $\mathcal{L}_0$. Homomorphisms $h : M \longrightarrow N$ between interpretations are defined in the obvious fashion, i.e. they preserve the interpretations of functions and relations. As usual, interpretations are extended to all terms and formulas of the language $\mathcal{L}_i$, where the value of i, $0 \leq i \leq 4$, will be clear from the context below. Let $\vec{x} = x_1, \ldots, x_n$ be a context with types $S_1, \ldots, S_n$ respectively, t a term of type S and $\vec{x} \cdot t$ a term-in-context over $\mathcal{L}_i$. To every such term-in-context $\vec{x} \cdot t$, we assign a morphism $[\![\vec{x} \cdot t]\!]_M : MS_1 \times \cdots \times MS_n \longrightarrow MS$ by recursion as follows:

1. If t is a variable, then it is a variable x_i of type S_i, and thus $[\![\vec{x} \cdot t]\!]_M = 1_{S_i} \circ \pi_i$ (we are being pedantic: a variable is interpreted as the appropriate identity morphism on the relevant type, the projection morphism π_i is there to get rid of the 'dummy' variables in the context);
2. If t is $f(t_1, \ldots, t_m)$, t_i terms of type T_i, then by induction we have that $[\![\vec{x} \cdot t_i]\!]_M : MS_1 \times \cdots \times MS_n \longrightarrow MT_i$ for each t_i, and $[\![\vec{x} \cdot f(t_1, \ldots, t_m)]\!]_M$ is the composite $MS_1 \times \cdots \times MS_n \xrightarrow{\langle [\![\vec{x} \cdot t_1]\!]_M, \ldots, [\![\vec{x} \cdot t_m]\!]_M \rangle} MT_1 \times \cdots \times MT_m \xrightarrow{Mf} MS$;
3. In particular, when t is a constant f, then $[\![\vec{x} \cdot t]\!]_M$ is the composite morphism $MS_1 \times \cdots \times MS_n \longrightarrow 1 \xrightarrow{Mf} MS$.

A formula-in-context $\vec{x} \cdot \phi$ is interpreted as a subobject $[\![\vec{x} \cdot \phi]\!]_M \rightarrowtail MS_1 \times \cdots \times MS_n$ as follows:

1. If ϕ is $(t_1 = t_2)$, and t_1 and t_2 are of type T, then $[\![\vec{x} \cdot \phi]\!]_M$ is the equalizer of $MS_1 \times \cdots \times MS_n \underset{[\![\vec{x} \cdot t_2]\!]}{\overset{[\![\vec{x} \cdot t_1]\!]}{\rightrightarrows}} MT$

2. If ϕ is $R(t_1, \ldots, t_m)$ with $t_1, \ldots, t_m$ of types $T_1, \ldots, T_m$ respectively, $[\![\vec{x} \cdot \phi]\!]_M$ is the pullback

$$\begin{tikzcd}
{[\![\vec{x} \cdot \phi]\!]} \arrow[r] \arrow[d, tail] & MR \arrow[d, tail] \\
MS_1 \times \cdots \times MS_n \arrow[r, "{<[\![\vec{x} \cdot t_1]\!], \ldots, [\![\vec{x} \cdot t_m]\!]>}"'] & MT_1 \times \cdots \times MT_m
\end{tikzcd}$$

3. If ϕ is $\top$, then $[\![\vec{x} \cdot \phi]\!]_M$ is $MS_1 \times \cdots \times MS_n$ or equivalently, the top element of $\mathrm{Sub}(MS_1 \times \cdots \times MS_n)$;

4. If ϕ is $(\psi \wedge \theta)$, then $[\![\vec{x} \cdot \phi]\!]_M$ is the pullback (in this case, since we are dealing with subobjects, it *is* the intersection or the inf)

$$\begin{tikzcd}
{[\![\vec{x} \cdot \phi]\!]} \arrow[r, tail] \arrow[d, tail] & {[\![\vec{x} \cdot \psi]\!]} \arrow[d, tail] \\
{[\![\vec{x} \cdot \theta]\!]} \arrow[r, tail] & MS_1 \times \cdots \times MS_n
\end{tikzcd}$$

5. If ϕ is $\perp$ and $\mathcal{C}$ is a coherent category, then $[\![\vec{x} \cdot \phi]\!]_M$ is the bottom element of $\mathrm{Sub}(MS_1 \times \cdots \times MS_n)$;

6. If ϕ is $(\psi \vee \theta)$ and $\mathcal{C}$ is a coherent category, then $[\![\vec{x} \cdot \phi]\!]_M$ is the union of the subobjects $[\![\vec{x} \cdot \psi]\!]_M$ and $[\![\vec{x} \cdot \theta]\!]_M$;

7. If ϕ is $(\psi \Rightarrow \theta)$ and $\mathcal{C}$ is a Heyting category, then $[\![\vec{x} \cdot \phi]\!]_M$ is the implication $[\![\vec{x} \cdot \psi]\!]_M \Rightarrow [\![\vec{x} \cdot \theta]\!]_M$ is the Heyting algebra $\mathrm{Sub}(MS_1 \times \cdots \times MS_n)$;

8. If ϕ is $\neg\psi$ (which is the same as $(\psi \Rightarrow \perp)$), and $\mathcal{C}$ is a Heyting category, then $[\![\vec{x} \cdot \phi]\!]_M$ is the Heyting negation $\neg[\![\vec{x} \cdot \psi]\!]_M$;

9. If ϕ is $(\exists y)\psi$, where y is a sort T, and $\mathcal{C}$ is a regular category, then $[\![\vec{x} \cdot \phi]\!]_M = \exists_\pi [\![\vec{x}, y \cdot \psi]\!]_M$ which is the image of the composite

$$[\![\vec{x}, y \cdot \psi]\!] \rightarrowtail MS_1 \times \cdots \times MS_n \times MT \xrightarrow{\pi_{\langle 1, \ldots, n \rangle}} MS_1 \times \cdots \times MS_n$$

where '$\vec{x} \cdot y$' denotes the context resulting from the juxtaposition of y to the list $\vec{x}$;

10. If ϕ is $(\forall y)\psi$, where y is a sort T, and $\mathcal{C}$ is a Heyting category, then $[\![\vec{x} \cdot \phi]\!]_M = \forall_\pi [\![\vec{x}, y \cdot \psi]\!]_M$ where π is the projection

$$MS_1 \times \cdots \times MS_n \times MT \xrightarrow{\pi\langle 1,\ldots,n\rangle} MS_1 \times \cdots \times MS_n.$$

Again, when $\mathcal{C}$ is the category *Set* of sets, then all these definitions translate into Tarski's usual semantics for a many-sorted first-order language. With all these definitions at hand, we define the notion of satisfaction and model as follows.

Let M be a $\mathcal{C}$-interpretation. We say that M is a *model* of a sequent $\sigma = (\phi \vdash_{\vec{x}} \psi)$ or that σ is *satisfied in* M, and write as usual $M \models_{\mathcal{C}} \sigma$ or simply $M \models \sigma$, if $[\![\vec{x} \cdot \phi]\!]_M \leq [\![\vec{x} \cdot \psi]\!]_M$ in $\mathrm{Sub}(MS_1 \times \cdots \times MS_n)$. We say that M is a *model of a theory* $\mathbb{T}$ if all the axioms of $\mathbb{T}$ are satisfied in M.

We denote the *category of models of* $\mathbb{T}$ *in* $\mathcal{C}$ by $\mathrm{Mod}_{\mathcal{C}}(\mathbb{T})$. An important feature of this framework is that it is possible to transfer models of $\mathbb{T}$ in $\mathcal{C}$ in an appropriate category $\mathcal{D}$ along appropriate functors. More specifically, if $\mathbb{T}$ is a (regular, coherent, etc.) theory, then any (regular, coherent, etc.) functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ induces a functor $\mathrm{Mod}(F) : \mathrm{Mod}_{\mathcal{C}}(\mathbb{T}) \longrightarrow \mathrm{Mod}_{\mathcal{D}}(\mathbb{T})$ in the obvious way.

Let $\mathcal{C}$ be a category with finite limits. Then the *internal language* $\mathcal{L}_{\mathcal{C}}$ of $\mathcal{C}$ is given as follows: for each object X of $\mathcal{C}$, there is a sort $\ulcorner X \urcorner$, the name of X; for each morphism $f : X_1 \times \cdots \times X_n \longrightarrow Y$ in $\mathcal{C}$, a function symbol $\ulcorner f \urcorner : \ulcorner X_1 \urcorner \times \cdots \times \ulcorner X_n \urcorner \longrightarrow \ulcorner Y \urcorner$ and for each subobject $R \rightarrowtail X_1 \times \cdots \times X_n$, a relation symbol $\ulcorner R \urcorner \rightarrowtail \ulcorner X_1 \urcorner \times \cdots \times \ulcorner X_n \urcorner$. (This is called the *extended internal language* in Makkai & Reyes 1977.) In other words, the language $\mathcal{L}_{\mathcal{C}}$ is nothing less than $\mathcal{C}$. There is a *canonical interpretation* of $\mathcal{L}_{\mathcal{C}}$ in $\mathcal{C}$, which is nothing less than the identity interpretation: each sort $\ulcorner X \urcorner$ is sent to the object X named by it, each function symbol $\ulcorner f \urcorner : \ulcorner X_1 \urcorner \times \cdots \times \ulcorner X_n \urcorner \longrightarrow \ulcorner Y \urcorner$ is sent to the morphism $f : X_1 \times \cdots \times X_n \longrightarrow Y$ it denotes and each relation symbol $\ulcorner R \urcorner \rightarrowtail \ulcorner X_1 \urcorner \times \cdots \times \ulcorner X_n \urcorner$ is sent to the subobject $R \rightarrowtail X_1 \times \cdots \times X_n$ it refers to. A sequent σ over $\mathcal{L}_{\mathcal{C}}$ will therefore have a canonical interpretation in $\mathcal{C}$. Whenever the sequent σ is verified in $\mathcal{C}$, we write $\mathcal{C} \models \sigma$. Thus, diagrams of $\mathcal{C}$ can be replaced by sequents over $\mathcal{L}_{\mathcal{C}}$. The *theory* $\mathbb{T}_{\mathcal{C}}$ *of* $\mathcal{C}$ is given by the collection of sequents of the internal language that are verified in $\mathcal{C}$ by the canonical interpretation. This is of course the purely model-theoretical point of view of a theory. The point of this apparently tautologous game is that it is possible to use the appropriate deductive system introduced above to prove properties of $\mathcal{C}$. More specifically, we can reproduce the foregoing system of theories as follows. Let $\mathcal{C}$ be a category. Then, the internal language $\mathcal{L}_{\mathcal{C}}$ is a special case of a primitive language $\mathcal{L}_0$. We can construct the theory $\mathbb{T}_{\mathcal{C}}$, denoted by $\mathbb{T}_0(\mathcal{C})$ by Boileau, by taking the axioms

i) $\top \vdash f(a) = a$ whenever $f : A \longrightarrow A$ is an identity morphism in $\mathcal{C}$;

ii) $\top \vdash g(f(a)) = h(a)$ whenever

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & C \\ & B \arrow[ur, "g"'] & \end{tikzcd}$$

is a commutative triangle in $\mathcal{C}$.

If $\mathcal{C}$ is a regular category, then, to the axioms of $\mathbb{T}_0(\mathcal{C})$, the axioms

iii) $\top \vdash \exists a(a = a)$ and $\top \vdash a_1 = a_2$ whenever A is a terminal object of $\mathcal{C}$;

iv) $\top \vdash \exists a(h(a) = b \wedge k(a) = c)$ and $h(a_1) = h(a_2) \wedge k(a_1) = k(a_2) \vdash a_1 = a_2$ whenever

$$C \xleftarrow{k} A \xrightarrow{h} B$$

is a product diagram in $\mathcal{C}$;

v) $f(b) = g(b) \vdash \exists a(h(a) = b)$, $\exists a(h(a) = b) \vdash f(b) = g(b)$ and $h(a_1) = h(a_2) \vdash a_1 = a_2$ whenever

$$A \xrightarrow{h} B \underset{g}{\overset{f}{\rightrightarrows}} C$$

is an equalizer diagram in $\mathcal{C}$

to obtain the theory $\mathbb{T}_1(\mathcal{C})$. By adding the appropriate axioms, we associate a specific theory $\mathbb{T}_i(\mathcal{C})$ to coherent, Heyting and Boolean categories.

Using the appropriate rules, together with the axioms of $\mathbb{T}_{\mathcal{C}}$, one can deduce formally results about $\mathcal{C}$. Proceeding in this manner, we obtain the following table:

If $\mathcal{C}$ is a	Then $\mathcal{L}_{\mathcal{C}}$ is	And $\mathbb{T}_{\mathcal{C}}$ is
Category	$\mathcal{L}_0(\mathcal{C})$	$\mathbb{T}_0(\mathcal{C})$
Regular category	$\mathcal{L}_1(\mathcal{C})$	$\mathbb{T}_1(\mathcal{C})$
Coherent category	$\mathcal{L}_2(\mathcal{C})$	$\mathbb{T}_2(\mathcal{C})$
Heyting category	$\mathcal{L}_3(\mathcal{C})$	$\mathbb{T}_3(\mathcal{C})$
Boolean category	$\mathcal{L}_4(\mathcal{C})$	$\mathbb{T}_4(\mathcal{C})$

It is straightforward to prove a soundness theorem for the various logical systems involved. More specifically, given a (regular, coherent, Heyting, Boolean) theory, M a model of $\mathbb{T}$ in a (regular, coherent, Heyting, Boolean) category $\mathcal{C}$, if σ is a sequent (in the appropriate fragment of the first-order language $\mathcal{L}$) which is provable in $\mathbb{T}$, then $M \models_{\mathcal{C}} \sigma$.

4.7 Building bridges

The resulting situation was somewhat surprising and to some, perhaps, uncomfortable. Whereas we saw Dionne constructing categories from logical theories in order to apply categorical methods and obtain in this manner results about the original theories, it became possible to use logical methods to obtain results about categories. In a sense, the frontier between logical theories as categories and categories as logical theories is thereby vanishing. When Mihály Makkai arrived in Montreal in the fall of 1973, he solved some of the problems considered by Joyal and Reyes by using purely logical methods. These results lead to two joint papers with Reyes submitted in the fall of 1974. The first paragraph of their second paper is revealing in this respect:

> Here, just as in [9] [their previous paper], our method consists in translating any given problem for categories into a purely logical problem via the mechanism described in Sec. 1 in [9], and then using familiar methods developed in logic. (. . .)
>
> Our point of view here is that categories provide an algebraic formulation of logic, in the spirit of equating theories with categories and models with certain ("logical") functors. We contend that much of logic (model theory) can and should be expressed this way. On the other hand, we freely use methods tied to the concrete "presentations", i.e., to primitive symbols, formulas, etc., usually employed in model theory; in fact, this is the novelty of our approach from the point of view of the category theorist. There is an obvious analogy to abstract groups versus groups defined by generators and relations. Just as in groups, the abstract formulation (categories) may (and does) point up examples of "theories" in branches of mathematics quite distant from logic. [Makkai and Reyes, 1976a, 385].)

Thus, we see again the claim that a category is the abstract algebraic expression of theory presented in a specific language with specific axioms. The main advantage underlined here, however, is that the abstract expression opens up unforeseen connections with other branches of mathematics. But the tension was certainly still present. In a paper written in 1975-76 while he was on sabbatical from Montreal, Reyes opens with the following claims:

> The goal of topos theory is to develop a language and an efficient method for the study of *concepts* of local character (as well as constructions on such concepts) that are found in diverse branches of mathematics : topology, algebraic geometry, analytic geometry, ...
>
> To the *geometric* aspect (or topological), which is the dominant aspect, another is dialectically opposed : the *logical* aspect.([Reyes, 1978, 156].)(our translation)

We are back to the early rhetoric with the emphasis, once again, on the geometric aspect, although at this stage it is clear that there is some sort of dictionary between the two. It is more as if the geometric and the logical are two faces of the same coin (but in fact, there are other faces as well, so it is more a cube than a coin).

But developing the logical machinery required to present Makkai and Reyes' results has its costs. Presenting all the preliminary details and the proofs contained in the first paper published in 1976 will constitute the bulk of the chapters 2, 3, 4, 5 and 6 (out of 9) of their book published in 1977. Chapters 7, 8 and 9 present the proofs of the second paper.

Here are the key results presented in the two papers presented in 1974 by Makkai and Reyes.

Having presented the foregoing notion of interpretation, of the internal language and the theory $\mathbb{T}_{\mathcal{C}}$ of a category $\mathcal{C}$, Makkai and Reyes proved the following completeness theorems for categories.

Theorem: i) let $\mathcal{C}$ be a small category with finite limits and with finite stable sups (i.e. each finite family of subobjects of an object has a stable sup). Then there is a complete Boolean algebra $\mathcal{B}$ and a functor

$$M : \mathcal{C} \longrightarrow Sh(\mathcal{B})$$

such that M preserves all finite limits, (finite or infinite) stable sups, stable images (finite or infinite), distributive infs, and distributive $\forall_f(C)$'s, and such that M is conservative.

ii) If $\mathcal{C}$ has finite limits only, the same conclusion holds without the clauses for infs and $\forall_f$'s.

The proof rests entirely on *logical* methods. In the paper, it is barely sketched. It proceeds by replacing $\mathcal{C}$ by $\mathbb{T}_{\mathcal{C}}$ and then using properties of the canonical interpretation adapting completeness results in the literature, e.g. Mansfield 1972 and an unpublished manuscript of Higgs.We should emphasize at this stage the importance of Higgs's (still) unpublished manuscript, widely read and extremely influential at the time, on a categorical version of Boolean-valued models.

The theorem has, among others, the following corollaries:

Corollary 1 (an improvement of Barr's theorem): every Grothendieck topos $\mathcal{E}$ has a surjective Boolean point, i.e. there is a complete Boolean algebra $\mathcal{B}$ and a geometric morphism $p : Sh(\mathcal{B}) \longrightarrow \mathcal{E}$ of toposes such that $p^* : \mathcal{E} \longrightarrow Sh(\mathcal{B})$ is faithful. Moreover, and this is the improvement, $\mathcal{B}$ and p can be chosen so that p^* preserves all (finite or infinite) distributive infs and all distributive $\forall_f(C)$'s in $\mathcal{E}$.

Corollary 2: (i) Let $\mathcal{C}$ be a coherent category. Then there is a set Y and a functor M

$$M : \mathcal{C} \longrightarrow Sh(\wp(Y)),$$

where $\wp(Y)$ denotes the Boolean algebra of subsets of Y, such that M is as in the foregoing theorem except that the sups and infs to be preserved by M are only the finite stable (distributive) ones.

The main theorem implies, together with a compactness argument, Deligne's theorem: every coherent topos $\mathcal{E}$ has a surjective Boolean point $p : Sh(\wp(Y)) \longrightarrow \mathcal{E}$ for some set Y. In chapter six of their book, the formulation of the theorem takes the more eloquent form: if $\mathcal{E}$ is a coherent topos, then there is a conservative model $M : \mathcal{E} \longrightarrow Set^I$ into a Boolean topos of the form Set^I, with I a set. In the third chapter, the theorem is stated for coherent categories as follows:

Let $\mathcal{C}$ be a (small) coherent category. Then:

1. If A and B are two subobjects of X in $\mathcal{C}$ such that $A \nleq B$, then there is a coherent functor $M : \mathcal{C} \longrightarrow Set$ such that $M(A) \nleq M(B)$ as subobjects of $M(X)$.

2. There is a set I and a faithful coherent functor $M : \mathcal{C} \longrightarrow Set^I$.

The foregoing results follow from the main theorem in which one considers *distributive infs* and *distributive* $\forall$'s. When one turns to models that preserve *all* infs and *all* $\forall$'s, then one obtains similar results for intuitionistic logic. The results become:

Theorem: For every Grothendieck topos $\mathcal{E}$ there is a complete Heyting algebra H and a surjective H-valued point $p : Sh(H) \longrightarrow \mathcal{E}$ such that $p^* : \mathcal{E} \longrightarrow Sh(H)$ preserves all (finite or infinite) infs and all $\forall_f(C)$'s in $\mathcal{E}$.

Theorem: If the topos $\mathcal{E}$ has a surjective Boolean point of the form

$$Sh(\wp(Y)) \longrightarrow \mathcal{E}$$

for a set Y, then the topos $Sh(H)$ can be taken to be the category of sheaves over a topological space.

These *are* completeness results for intuitionistic logic. Again the proofs use purely logical methods. (See chapter 6.3 of Makkai & Reyes 1977.) In this setting, Joyal's completeness theorem, closely related to Kripke's completeness theorem for intuitionistic logic, takes the following form:

Theorem (Joyal, as we saw, probably in 1970): Let $\mathcal{C}$ be a coherent category and let $Mod_{Set}(\mathcal{C})$ the category of *coherent* functors $\mathcal{C} \longrightarrow Set$, thus the category of set-models of $\mathcal{C}$. There is a small full subcategory P of $Mod_{Set}(\mathcal{C})$ such that the evaluation functor $ev : \mathcal{C} \longrightarrow Set^P$ is conservative and preserves all finite limits, stable finite sups, stable images and stable $\forall_f(C)$ existing in $\mathcal{C}$.

Some important remarks are in order. First, it is now possible to see that completeness theorems for certain logical theories are *equivalent* to representation theorems for categories. Thus, for instance, the completeness theorem for first-order classical theories is equivalent to the representation theorem for *coherent* categories. As we have already pointed out, the key construction for the implication from the representation theorem to the completeness theorem is the construction of the category of concepts of a theory. To prove the other direction, one uses the internal theory $\mathbb{T}_{\mathcal{C}}$ of a small coherent category $\mathcal{C}$. Thus, one can say that the completeness theorem and the representation theorem are translation-equivalent to one another. Second, Joyal's theorem has a remarkable feature: the link between a theory and its category of models is provided by a *canonically defined functor*, the evaluation functor. Given two categories $\mathcal{C}$ and $\mathcal{D}$, the evaluation functor $ev : \mathcal{C} \longrightarrow \mathcal{D}^{\left(\mathcal{D}^{\mathcal{C}}\right)}$ is *always* definable by $ev(X, F) = F(X)$ (and similarly for morphisms). Furthermore, the fact that, whenever $\mathcal{C}$ and $\mathcal{D}$ are coherent categories, ev is a coherent functor follows by abstract general nonsense. The specific observation is that it is also conservative.

In the early sixties, Lubkin-Freyd-Mitchell-Héron proved an important representation theorem for Abelian categories. The latter are categories satisfying certain abstract properties sufficient for the development of a large part of homological algebra. It is fair to say that Abelian categories together with the representation theorem just mentioned occupied a central position in the development of category theory in the sixties. Not only were they extremely powerful in their applications, but they also served as a model of the power of category theory itself. It should be mentioned at this point that the category Set is *not* an Abelian category. The typical example of an Abelian category is the category of Abelian groups or a category of modules over a (commutative) ring. Michael Barr, then at McGill University, was looking for non-additive versions of the Abelian notion as well as the corresponding representation theorem. As we have already mentioned, he succeeded in this attempt and introduced *regular* and *exact* categories in 1970 together with a representation theorem for exact categories, from which the known representation theorem for Abelian categories can be deduced. In contrast with the situation found with Abelian categories, the category Set is regular and exact. A fascinating fact is that a category is Abelian if and only if it is both additive and exact. It should be pointed out immediately that the notion of an exact category is simply a strengthening of the notion of a regular category and that only the latter property was used in Barr's proof of the representation theorem. Thus the category Set is regular (as well as coherent, Heyting, Boolean and a topos.) Barr's representation theorem for regular categories turns out to be equivalent to a completeness theorem for coherent logic when it is looked at from the proper angle.

First, the property of being conservative is crucial and captures a form of completeness. To understand this, it is imperative to see what the property of reflecting isomorphisms means in this particular case. The category Set is coherent and thus so is the category $Set^{Mod_{Set}(\mathcal{C})}$. Thus, the latter category inherits all its coherent properties from the category Set. Furthermore, the property of reflecting isomorphisms of the functor $e : \mathcal{C} \longrightarrow Set^{Mod_{Set}(\mathcal{C})}$ implies that whatever coherent property the category $Set^{Mod_{Set}(\mathcal{C})}$ has, so does the category $\mathcal{C}$. Hence the property of reflecting isomorphisms implies that any coherent category $\mathcal{C}$ has all coherent properties that Set has.

Second, the latter claim has to be compared with the Stone representation theorem for distributive lattices and Boolean algebras. The usual formulation of the Stone representation theorem is that any distributive lattice has an embedding into a power-set algebra. This statement can be shown to be equivalent to the claim that for any distributive lattice D, there is a 2-valued homomorphism $f : D \longrightarrow 2$

that reflects the order, i.e. if $f(x) \leq f(y)$, then $x \leq y$. This last statement can be interpreted as saying that any distributive lattice shares all the universal Horn properties of the 2-element lattice. Thus, moving from the propositional case to the first-order case requires that we replace the 2-element lattice by the category Set.

Finally, let us see how the representation theorem is equivalent to a completeness theorem. Assume the representation theorem for coherent categories and let $\mathcal{C}$ be the conceptual category $[\mathbb{T}]_{\text{coh}}$ for some coherent theory $\mathbb{T}$. Then the representation theorem yields automatically the completeness theorem. For the other direction, assume we have a coherent category $\mathcal{C}$ and let $\mathbb{T}_{\mathcal{C}}$ be its internal theory. Applying the classical completeness theorem for $\mathbb{T}_{\mathcal{C}}$, we get the representation theorem for $\mathcal{C}$, since the coherent functors $\mathcal{C} \longrightarrow Set$ are identical to the models of the internal theory $\mathbb{T}_{\mathcal{C}}$.

Let us come back to the evaluation functor $e : \mathcal{C} \longrightarrow Set^{Mod_{Set}(\mathcal{C})}$. Joyal showed that the functor e preserves all the existing Heyting structure that happens to exist in $Set^{Mod_{Set}(\mathcal{C})}$. In other words, if $\mathcal{C}$ is a Heyting category, then e is automatically a conservative Heyting functor. Hence, a representation theorem for Heyting categories is obtained for free: every small Heyting category has a conservative Heyting functor to a Heyting category of the form Set^P, where P is a small category (it has to be extracted from the large category $Mod_{Set}(\mathcal{C})$). Kripke's completeness theorem for first-order intuitionistic logic can be deduced from Joyal's theorem.

Let us close this section by coming back to Barr's representation theorem for regular categories. Joyal showed that Barr's theorem is equivalent to the claim that the evaluation functor $e : \mathcal{C} \longrightarrow Set^{Mod_{Set}(\mathcal{C})}$ is full and faithful for any small regular category $\mathcal{C}$ and where $Mod_{Set}(\mathcal{C})$ is the category of regular functors $\mathcal{C} \longrightarrow Set$. This is of course equivalent to a completeness result for first-order logic.

The second paper published in 1976 is important not only for the results it contains, but also for the *question* it asks. Indeed, one can argue that only a categorical formulation of logic allows the question to be stated precisely. Thus, in this case, it is not only the fact that we obtain a new result, but the very statement could not have been formulated without the categorical framework.

Since Lawvere's thesis, it was common to consider functors $I : \mathbb{T}_1 \longrightarrow \mathbb{T}_2$ between theories and the induced functors between models of theories

$$I^* : Mod(\mathbb{T}_2) \longrightarrow Mod(\mathbb{T}_1).$$

One of the questions that can be asked in this context is this: how are properties of I^* reflected in I? If the "logical" properties of I^* are indeed reflected in I, then certainly this is a kind of completeness. Since, T_1 and $\mathbb{T}_2$ are thought of as categories of concepts, it becomes natural to talk about a form of conceptual completeness whenever one can recover the properties of I from those of I^*. Let us now see how these are stated in Makkai and Reyes's paper.

First, they fix the categories they are working with: $\mathbb{T}_1$ and $\mathbb{T}_2$ are assumed to be small coherent categories and $I : \mathbb{T}_1 \longrightarrow \mathbb{T}_2$ is assumed to be a coherent functor. Furthermore, the category $Mod(\mathbb{T})$ is the category of all coherent functors $F : \mathbb{T} \longrightarrow Set$, it is a full subcategory of $Set^{\mathbb{T}}$. Thus we have the usual induced functor $I^* : Mod(\mathbb{T}_2) \longrightarrow Mod(\mathbb{T}_1)$. The following results, although interesting in themselves, are used to prove what will be call the conceptual completeness theorem. The first theorem is an application of the completeness theorem obtained in their first paper:

Theorem 1A: Assume I^* is surjective, that is for every model M in $Mod(\mathbb{T}_1)$ there is a model N in $Mod(\mathbb{T}_2)$ such that $I^*(N) \simeq M$. Then I is injective on subobjects, i.e. if $X_1 \rightarrowtail X$ and $X_2 \rightarrowtail X$ in $\mathbb{T}_1$ and $I(X_1) \leq I(X_2)$ (in the ordering of subobjects of $I(X)$), then $X_1 \leq X_2$. In particular, I is faithful.

Theorem 1B (Beth definability theorem): Assume that $\mathbb{T}_1$ and $\mathbb{T}_2$ are Boolean categories. Assume also that if $f : I^*(N_1) \longrightarrow I^*(N_2)$ is an isomorphism, then there is a morphism $g : N_1 \longrightarrow N_2$ such that $I^*(g) = f$. Then I is full with respect to subobjects, i.e. if $Y \rightarrowtail I(X)$ in $\mathbb{T}_2$, then there is $X_1 \rightarrowtail X$ in $\mathbb{T}_1$ such that $Y \simeq I(X_1)$.

Theorem 1.1: if I^* is full, then I is full with respect to subobjects. If in addition I^* is surjective, then I is full.

Whereas theorem 1B is closely related to the Beth definability theorem, theorem 1.1 comes naturally in a categorical framework and does not require that the theories be Boolean. (See [Makkai and Reyes, 1977, 195–197] to see how theorem 1B is related to the classical Beth definability theorem.)

The conceptual completeness theorem is formulated for *pretoposes*, a notion introduced by Grothendieck in the sixties. A *pretopos* is a small coherent category having finite disjoint sums (coproducts), and effective equivalence relations.

Theorem (conceptual completeness): Assume $\mathbb{T}_1$ is a pretopos, $\mathbb{T}_2$ is coherent and $I : \mathbb{T}_1 \longrightarrow \mathbb{T}_2$ is a coherent functor. Then, if I^* is an equivalence of categories, then so is I.

Two facets of the result have to be emphasized. First, the conceptual completeness theorem provides a characterization of pretoposes among coherent categories. Indeed, the result fails whenever $\mathbb{T}_1$ is not a pretopos. Second, the theorem can be read as follows: suppose that $I : \mathrm{T}_1 \longrightarrow \mathbb{T}_2$ is thought of an interpretation and that $\mathbb{T}_2$ is considered to be an extension of $\mathbb{T}_1$. One could say that I is *strongly conservative* whenever I^* is an equivalence. Then the theorem asserts that pretoposes and only pretoposes do not have proper strongly extensions. Still put differently, one could say that although $\mathbb{T}_2$ is an extension of $\mathbb{T}_1$, it has the same category of models. The result shows, and this was in fact part of the motivation underlying the original conjecture formulated by Reyes, that finite disjoint sums of formulas and quotients of a formula by a provable (in the theory) equivalence relation are coherent logical operations (although not representable in ordinary logic), *but there are no others*, whence the conceptual *completeness.* It should be emphasized, once more, that all the proofs rely on logical methods. Andrew Pitts gave purely categorical proofs of these results in the eighties. (See [Pitts, 1987; Pitts, 1989].)

The next theorem is Keisler's theorem, but now stated explicitly for coherent logic.

Theorem 2.1. Let K be an elementary class, i.e. defined by any set of sentences in $\mathcal{L}_2$. Then the following are equivalent:

1. K is the class of models of a coherent theory over $\mathcal{L}_2$.

2. Cat_K, the category generated by K, i.e. whose objects are the many-sorted structures of similarity type $\mathcal{L}_2$ and homomorphisms between them, is the dual of the category of points of some coherent topos.

3. K contains the colimit of any directed family taken from Cat_K.

It is pointed out that the equivalence between (1) and (3) was proven by Keisler. It is not mentioned that the equivalence with (2), although formulated differently and using different means, was announced by Joyal in 1970. Makkai and Reyes' proof relies on the notion of *classifying topos*.

4.8 Classifying topos and generic model of a theory

The notion of classifying topos has its own complex history. The first appearance of the concept is usually attributed to Monique Hakim in the sixties, but her work was published only in 1972. (See [Hakim, 1972].) The *term* explicitly appears in Giraud's contribution to the first meeting dedicated to elementary toposes in Dalhousie in the winter of 1971. In this paper, Giraud presents three different constructions of what he calls the classifying topos of a stack $\mathcal{C}$ and states a universal property characterizing classifying toposes in the bicategory of toposes. Needless to say, in this form, the notion has no explicit connection with logic. As we have already mentioned, an implicit version of the notion also appeared in Reyes 1974, this time having a direct connection to logic. Thus, the notion was implicitly used by Joyal and Reyes already in 1971-1972. Myles Tierney and Jean Bénabou also discovered a version of the notion in the spring of 1975, as reported by Tierney himself in 1976. (See [Tierney, 1976, 211, 216 and 217]. See also [Bénabou, 1975], where the notion is mentioned but not defined.) Johnstone and Wraith introduced particular cases of the notion independently also in 1974.

To introduce the concept of classifying topos, we digress and recall some important facts of ring theory. Up until the middle of the 19^{th} century, even people like Gauss, Galois and Abel took for granted that any polynomial with coefficients in a ring A had a zero *somewhere*, in an extension of A, although this had not been proved. Kronecker, following the lead of Cauchy, pointed out that the ring $A[X]/(p)$ contains the 'generic' zero $G = X + (p)$. Furthermore, this solution is universal in the sense that (taking $A = \mathbb{Z}$ to simplify), $G^* : RING(\mathbb{Z}[X]/(p), R) \simeq Zero_R(p)$, where $G^*(\varphi)$ is the obvious zero of p in R obtained from φ, namely $\varphi(G)$. Clearly, all of this can be generalized to ideals of polynomials, rather than single polynomials.

This result may be interpreted as showing that although the 'domain' A may fail to have a zero of a proper ideal $I \subset A[X_1, X_2, \ldots, X_n]$ there is an extension of this 'domain' namely $A[X_1, X_2, \ldots, X_n]/I$ which results from A by adding the generic zero G of I.

These ideas underline a far reaching analogy stressed by Joyal between rings and toposes and which can be summarized in the following table:

Ring theory	*Categorical logic*
Ring	Topos
Finitely presented ring	Coherent topos
$\mathbb{Z}$	*Set*
Ideal	Theory
Zero	Model
Proper ideal	Consistent theory
Generic zero	Generic model

Notice, once more, the algebraic inspiration underlying the analogy. The category *Set* plays a role analogous to the ring of integers $\mathbb{Z}$ in ring theory. The various constructions to be developed in topos theory are analogous to fundamental operations of ring theory, especially constructing from the data generic zero's to polynomials.

The classifying topos provided the context in which the notion of generic model could be defined and developed. Joyal and Reyes showed in the early seventies that for any coherent theory $\mathbb{T}$, there is a topos $\mathcal{B}(\mathbb{T})$ and a generic model G of $\mathbb{T}$ in $\mathcal{B}(\mathbb{T})$ satisfying the universal property: $G^* : \mathrm{TOP}(\mathcal{E}, \mathcal{B}(\mathbb{T})) \simeq \mathrm{Mod}_{\mathcal{E}}(\mathbb{T})$ where G^* is the functor which sends a geometric morphism (p^*, p_*) into a model $p^* \circ G$ of $\mathbb{T}$ in $\mathcal{E}$.

The topos $\mathcal{B}(\mathbb{T})$ is called the *classifying topos* of the theory $\mathbb{T}$ and it is a coherent topos in the sense of Grothendieck, that is roughly a topos defined by a topology having the property that every covering family is finite. The analogy with Kronecker's construction should be obvious. On the basis of this analogy we may view the classifying topos of $\mathbb{T}$ as the 'universe' $\mathcal{B}(\mathbb{T})$ which results from *Set* by forcing the existence of a generic model of $\mathbb{T}$.

One construction of $\mathcal{B}(\mathbb{T})$ proceeds as follows. First, given $\mathbb{T}$, construct the category of concepts $[\mathbb{T}]_{\mathrm{coh}}$. Define a Grothedieck topology J in $[\mathbb{T}]_{\mathrm{coh}}$ by taking all finite families $\{f_i : X_i \longrightarrow X : 1 \leq i \leq n\}$ such that the union of the images of the f_i is the whole of X. This yields a site $\left([\mathbb{T}]_{\mathrm{coh}}, J\right)$ and the topos of sheaves $\mathrm{Sh}\left([\mathbb{T}]_{\mathrm{coh}}, J\right)$. The latter is the classifying topos $\mathcal{B}(\mathbb{T})$ of $\mathbb{T}$. The Yoneda embedding $[\mathbb{T}]_{\mathrm{coh}} \longrightarrow \mathrm{Sh}\left([\mathbb{T}]_{\mathrm{coh}}, J\right)$ can be shown to be a coherent functor and it automatically yields the generic model G.

It turns out that every coherent topos is the classifying topos of a coherent theory (the details of this claim were worked out in Makkai & Reyes 1977). One of the first and important examples is the classifying topos of the theory of local rings, also called the Zariski topos. (This example goes back to Monique Hakim, although not from a logical point of view. See Hakim 1972.) Remember that the theory of local rings may be axiomatized by the following coherent theory. We add to the axioms for the theory of commutative rings with unity, the following two coherent axioms:

$$\begin{cases} \text{i. } 0 = 1 \vdash_{[]} \bot \\ \text{ii. } (\exists \mathrm{z})((x+y)z = 1) \vdash_{x,y} ((\exists z)(xz = 1) \vee (\exists z)(yz = 1)) \end{cases}$$

The topology of the site can be described explicitly and a description of the generic local ring can also be given explicitly.

A different but equally important example was given by Joyal. Joyal has shown that the topos of simplicial sets is the classifying topos of the theory of linear orders with different bottom and top elements. (See Mac Lane & Moerdijk 1994 for details.) Other examples have appeared in several investigations.

An important tool to find the coherent theory classified by a coherent topos is the conceptual completeness theorem mentioned above. In more detail, if $\mathbb{T}$ is a coherent theory which has a coherent model M in a coherent topos $\mathcal{E}$ such that there is an equivalence of categories given by the obvious functor $M^* : \mathrm{TOP}(\mathcal{E}, Set) \simeq \mathrm{Mod}_{Set}(\mathbb{T})$, then $\mathcal{E}$ is equivalent to $\mathcal{B}(\mathbb{T})$.

From the existence of the classifying topos and its 'converse', it follows that any coherent topos results from *Set* by adding a generic model of a suitable coherent theory in strict parallel to the fact that any finitely presented ring may be obtained as a quotient of a ring of polynomials with coefficients in $\mathbb{Z}$ divided by a suitable ideal.

The classifying topos theorem has an infinitary extension, as Makkai and Reyes have shown in their book. In fact, it turns out that every geometric theory has also a classifying topos and, furthermore, that every topos is the classifying topos of a geometric theory. Thus, every Grothendieck topos appears as the extension of *Set* obtained by adding a generic model of a suitable geometric theory; in complete analogy with the representation of an arbitrary ring as a ring of polynomials (in a possibly infinite number of indeterminates) modulo some ideal, showing how fruitful the analogy between topos theory and ring theory is. And it is far from clear that it has been fully exploited. For instance, is there any topos theoretical analog to Hilbert's celebrated Nullstellensatz?

All this work brought to the forefront the logical aspects of the work done by the Grothendieck School. For instance, as shown by Makkai and Reyes in their book, Deligne's theorem that coherent toposes have enough points turns out to be, modulo the theory of classifying toposes, equivalent to Gödel's completeness theorem for first order logic. Apart from Deligne's proof, there is a purely categorical proof given by Joyal that has strong similarities to Henkin's proof of the completeness theorem. Another result of Grothendieck about point of a topos being obtained as colimits of a set of points is a consequence of the downward Löwenheim-Skolem-Tarski theorem.

Here is the version presented by Makkai and Reyes in November 1974. Consider the functor category $\mathcal{E}^{\mathcal{E}_0}$ of all geometric functors $u^* : \mathcal{E}_0 \longrightarrow \mathcal{E}$, called the category of all $\mathcal{E}$-models of $\mathcal{E}_0$ and denoted by $Mod(\mathcal{E}_0, \mathcal{E})$. Let $\mathbb{T}$ be a coherent theory and $\mathcal{C}_{\mathbb{T}}$ its corresponding conceptual category. A $\mathcal{E}$-model of $\mathbb{T}$ in a topos $\mathcal{E}$ can be taken to be a coherent functor $M : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{E}$. Let $Mod(\mathbb{T}, \mathcal{E})$ denote the full subcategory of the functor category $\mathcal{E}^{\mathbb{T}}$ whose objects are the $\mathcal{E}$-models of $\mathbb{T}$. As usual, any $\mathcal{E}_0$-model $M_0 : \mathbb{T} \longrightarrow \mathcal{E}_0$ induces a functor $M_0^* : Mod(\mathcal{E}_0, \mathcal{E}) \longrightarrow \mathcal{E}^{\mathbb{T}}$. Now, $\mathcal{E}_0$ is a *classifying topos* of $\mathbb{T}$ with canonical model $M_0 : \mathbb{T} \longrightarrow \mathcal{E}_0$ if the induced functor $M_0^* : Mod(\mathcal{E}_0, \mathcal{E}) \longrightarrow Mod(\mathbb{T}, \mathcal{E})$ is an equivalence of categories, for any topos $\mathcal{E}$.

A slightly weaker definition of a classifying topos is given in their book: $\mathcal{E}_0$ is a *classifying topos* of $\mathbb{T}$ with canonical model $M_0 : \mathbb{T} \longrightarrow \mathcal{E}_0$ if for every $\mathcal{E}$-model $M : \mathbb{T} \longrightarrow \mathcal{E}$, for any topos $\mathcal{E}$, there is an $\mathcal{E}$-model of $\mathcal{E}_0$ and a unique (up to isomorphism) geometric functor $u^* : \mathcal{E}_0 \longrightarrow \mathcal{E}$ such that the diagram

$$\begin{tikzcd}
\mathbb{T} \arrow[r, "M_0"] \arrow[dr, "M"'] & \mathcal{E}_0 \arrow[d, "u^*"] \\
& \mathcal{E}
\end{tikzcd}$$

commutes. The main fact is that every coherent theory $\mathbb{T}$ has a classifying topos $\mathcal{E}_0$ together with a canonical model, also called a *generic model* in their book, $M : \mathbb{T} \longrightarrow \mathcal{E}_0$. In the paper, it is stated that the classifying topos $\mathcal{E}_0$ can be constructed in a purely syntactical manner. The first part of the construction consists in constructing the small coherent category $\mathcal{C}_{\mathbb{T}}$ from $\mathbb{T}$. One then constructs, by formal means, a pretopos $\mathcal{E}_{\mathbb{T}}$ and a coherent functor $I : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{E}_{\mathbb{T}}$ satisfying the universal property for pretoposes. It can then be shown that the pretopos $\mathcal{E}_{\mathbb{T}}$ is in fact a topos.

Two other important facts about classifying toposes are stated. First, if $\mathcal{E}$ is a coherent topos, as defined by Grothendieck, then there is a coherent theory $\mathbb{T}$ such that $\mathcal{E}_{\mathbb{T}} \simeq \mathcal{E}$. In other words, every coherent topos is the classifying topos of some coherent theory. It is for this reason that Makkai and Reyes claimed that the notion of *coherent* topos is a logical notion. Second, let $\mathcal{E}$ be a coherent topos and $Coh(\mathcal{E})$ be the pretopos of coherent objects of $\mathcal{E}$. (See, for instance, Makkai & Reyes 1977, p. 276 for a definition of coherent objects.) Let M be an interpretation of $\mathcal{L}_2$ in $Coh(\mathcal{E})$. Suppose that M induces an equivalence $\widehat{M} : Mod(\mathcal{E}, \mathcal{S}) \longrightarrow Mod(\mathbb{T}, \mathcal{S})$. Then M is a $Coh(\mathcal{E})$-model of $\mathbb{T}$ and $\mathcal{E}$ is the classifying topos for $\mathbb{T}$ with canonical model M. This result is usually interpreted as saying that points are enough for classifying.

Makkai and Reyes, following suggestions given by Mulvey and Lawvere, used the above results to prove results of Hakim, which amounts to the claim that the so-called Zariski topos is the classifying topos for the coherent theory of nontrivial local rings and that the Etale topos is the classifying topos for the coherent theory of separably closed local rings.

The second paper does not end there. The last section mentions that some of the results obtained have an obvious generalization to models preserving infinite sups, a generalization to certain infinitary sentences. It is worth mentioning that when Makkai arrived in Montreal, he was already an expert in infinitary logic, more specifically model theory of formulas with denumerably long disjunctions and denumerably long conjunctions. Furthermore, Reyes' thesis was partly dedicated to infinitary languages.[10] However, the full presentation of the generalization to what is now called *geometric logic* will only be introduced in their book. Their paper contains only one illustration of the generalization, a special case of the conceptual completeness theorem that we will not state here. (For the statement of the theorem and the complete proof, see Makkai & Reyes 1977, 217-236.)

Although it was not mentioned explicitly in their papers, the completeness result for coherent logic, in fact as we will see for geometric logic, had an interesting and potentially very useful metalogical consequence. Here is how Reyes formulated it later:

Let $\mathbb{T}$ be a coherent theory in a language $\mathcal{L}$ and σ the universal closure of a formula $\varphi \Rightarrow \psi$, where φ and ψ are coherent. Then if $\mathbb{T} \models \sigma$ in the classical sense (every set-model of $\mathbb{T}$ is a set-model of σ), then for any Grothendieck topos $\mathcal{E}$, σ is valid in any $\mathcal{E}$-model of $\mathbb{T}$. (Reyes, 1978, 166.)[our translation.]

The result means that to check that a *first-order coherent sequent* is valid intuitionistically, it is enough to check its validity in the topos of sets. Thus, the metatheorem allows direct and potentially interesting generalizations from a classical setting, where presumably the computations are easy and straightforward, to an arbitrary intuitionistic context. Reyes himself gives an application by proving that Swan's theorem is valid in any topos by giving a coherent formulation of it and verifying it in the topos *Set*. In fact, the metatheorem is true for *geometric logic* as well, to which we now turn.

4.9 Geometric logic

The notion of geometric logic appeared in print, although not as a full-blown formal system, in Reyes's paper on sheaves and logic in which it is presented as the logic of sheaves. Lawvere mentioned the idea in his conference in 1973 and in the published version of that conference. Underlying the idea of geometric logic is the concept of *geometric morphisms* between toposes introduced in the sixties by Grothendieck and was emphasized right from the start by Lawvere in 1970. In a nutshell, geometric logic is the logic preserved by a geometric morphism between toposes. Let us first give a precise definition of geometric logic.

Let $\mathcal{L}_0$ be as above. Then the class of *geometric formulas* $\mathcal{L}_{geo}$ over $\mathcal{L}_0$ is the class closed under those of coherent logic except that infinitary disjunctions $\vee_{i \in I}\varphi_i$ are now admitted. The infinitary rules are the obvious generalizations of the finite

[10] In a paper published in 1972, Reyes proved that every theory in $L_{\mu,\omega}$ is equivalent to a theory in $L_{\omega_1,\omega}$ in a specific technical sense that we won't clarify. Interestingly enough, Reyes used Joyal's Polyadic Spaces to prove his result. See [Reyes, 1972].

cases:

$\phi \vdash_{\vec{x}} \vee_{i\in I}\phi_i$ if $\phi \in \{\phi_i\}_{i\in I}$ and

$\dfrac{\phi \vdash_{\vec{x}} \theta \quad \vee_{i\in I}\phi_i \vdash_{\vec{x}} \theta}{\vee_{j\in J}\phi_j \vdash_{\vec{x}} \theta}$ where $\{\phi_j\}_{j\in J} = \{\phi\} \cup \{\phi_i\}_{i\in I}$.

A sequent $\phi \vdash_{\vec{x}} \psi$ is *geometric* if both ϕ and ψ are geometric formulas. A *geometric theory* T is a theory in which all sequents are geometric. Two interesting examples of geometric theories are the theory of torsion Abelian groups and the theory of finite sets.

Many of the results obtained by Joyal, Reyes and Makkai generalize to the case of geometric logic. Certainly the most important result is that any geometric theory T has a classifying topos $\mathcal{E}_{\mathrm{T}}$ together with a generic model $M_0 : \mathrm{T} \longrightarrow \mathcal{E}_{\mathrm{T}}$. Furthermore, every topos $\mathcal{E}$ is the classifying topos of a geometric theory, that is for any topos $\mathcal{E}$, there is a geometric theory $\mathrm{T}_{\mathcal{E}}$ such that $\mathcal{E} \simeq \mathcal{E}_{\mathrm{T}_{\mathcal{E}}}$.

5 HIGHER-ORDER LOGIC AND TOPOSES

As we have already mentioned, the very axioms of elementary toposes point towards a certain conception of sets or more naturally a type theory. Accordingly, connections between previous set theories and elementary toposes were explored immediately showing that it is possible to construct models of a weak set theory from various extensions of elementary toposes. For that purpose and already in 1971, W. Mitchell, Cole and Osius developed an *internal language* for toposes. Here is how Osius characterizes the situation in 1973 at the *Colloque sur l'algèbre des catégories* in Amiens in France.

> Concerning the relationship between set theory and elementary topoi we consider two problems :
>
> 1. Find a general procedure of proving set-theoretical results internally in a topos (internal aspect).
> 2. Characterize certain topoi as " the category of sets " arising from certain types of models for set theory, thus generalizing the well-known results of Cole, W. Mitchell, Osius (external aspect). ([Osius, 1974b, 157])

In his lectures on elementary toposes given in the fall of 1973, Gavin Wraith writes:

> There are certain developments, due to J. Bénabou, which I should have liked to have included. Until recently, when one wished to carry out a construction in an elementary topos that was well enough understood in S, the category of sets and functions, one had to wrestle with pullback diagrams and the like. Bénabou's formal language permits one to dispense with these problems, and to proceed directly to the

> construction from its formal description. I believe that these methods must displace the older, clumsier ones. ([Wraith, 1975, 115–116])

Thus Bénabou had also developed a method that allowed the use of a language to prove results about toposes. It goes without saying that Wraith's opinion that this new method should replace the older, clumsier one was polemical at the time. But, indeed, it appeared quickly that the use of logical *methods* could be of real value in the development of topos theory. In some cases, the geometrical methods were convoluted, long and difficult to follow and their logical counterpart were direct, simple and almost immediate. Thus, there was a practical dimension to the problem that seemed to be worth developing fully. Since an elementary topos $\mathcal{E}$ can be thought of as a universe of sets, although not the usual sets of ZF in general, it should be possible to treat it as such, that is to use a topos $\mathcal{E}$ in the same way that one uses the category of sets. In particular, one would like to treat objects of $\mathcal{E}$ as sets with "elements" and write down definitions just as in the category of sets with the usual set-theoretical language.

As we have said, various people gave a positive answer to this question. But Bénabou and his student Michel Coste, Michael Fourman and André Joyal in collaboration with André Boileau, all in 1973-1974, are usually credited with the introduction of a full, rigorous description of an appropriate formal system. The striking fact is that the resulting higher-order type theory is purely algebraic.

5.1 Interpreting higher-order logic in toposes

We will first present Boileau and Joyal's approach, based on Joyal's talk given in April 1974, expanded and developed by Boileau in his Ph.D. thesis submitted in 1976 and then in a joint paper received by the editors of the *Journal of Symbolic Logic* in july 1977 but published only in 1981. (See [Boileau and Joyal, 1981].)

A (higher-order) *similarity type* or *signature* $\mathcal{T}$ is given by

1. A set $\mathcal{S}$ of sorts (which does not include the symbol Ω). The set $\mathcal{T}$ of *types* is build from the set of sorts as follows:
   (a) if $S \in \mathcal{S}$, then $S \in \mathcal{T}$;
   (b) if $S_1, ..., S_n \in \mathcal{T}$, then $\Omega\,(S_1, \ldots, S_n) \in \mathcal{T}$, where n can be 0;
   (c) nothing else is a type.

The type $\Omega(\ )$ is denoted by Ω.

1. A set $\mathcal{F}$ of *function symbols*, to each function symbol f, we associate its *type* $S_1 \times \cdots \times S_n \times S$ (with the last sort having a distinguished status); we write $f : S_1 \times \cdots \times S_n \longrightarrow S$ to indicate the type of f. Whenever n is 0, f is a *constant* of type S and we write $1 \xrightarrow{f} X$. Furthermore, a functional symbol $f : S_1 \times \cdots \times S_n \longrightarrow \Omega$ with codomain Ω will be called a *relational symbol*.

Given a signature $\mathcal{T}$, the *language* $\mathcal{L}(\mathcal{T})$ consists of the signature, for each type S, a countable list of variables. Terms and formulas are defined by recursion as follows:

Definition of terms:

1. every variable of type S is a term of type S;
2. if $f : S_1 \times \cdots \times S_n \longrightarrow S$ is a function symbol and $t_1, \ldots, t_n$ are terms of type $S_1, \ldots, S_n$ respectively, then $f(t_1, \ldots, t_n)$ is a term of type S; (if f is a constant, we simply write f);
3. if ϕ is a formula and if $x_1, \ldots, x_n$ is a list of distinct variables of type $S_1, \ldots, S_n$, then $\{(x_1, \ldots, x_n) : \phi\}$ is a term of type $\Omega\,(S_1, \ldots, S_n)$;

Definition of formulas:

1. if t_1 and t_2 are terms of type S, then $t_1 = t_2$ is a formula;
2. if $t_1, \ldots, t_n$ are terms of type $S_1, \ldots, S_n$ and if t is a term of type $\Omega\,(S_1, \ldots, S_n)$ then $(t_1, \ldots, t_n) \in t$ is a formula;
3. $\top$ is a formula ('true' is a formula);
4. if ϕ and ψ are formulas, then $(\phi \wedge \psi)$ is a formula;

The reader will have noticed how the higher-order structure emerges: it is the recursive nature of 1.b for types that leads to higher types.

The remaining connectives can be *defined* as follows:

D1. $\forall x\phi \equiv (\{x : \phi\} = \top)$;
D2. $(\phi \Rightarrow \psi) \equiv ((\phi \wedge \psi) = \phi)$;
D3. $(\phi \vee \psi) \equiv \forall w\,[((\phi \Rightarrow w) \wedge (\psi \Rightarrow w) \Rightarrow w) = \top]$
D4. $\perp \equiv \forall w\,(w = \top)$;
D5. $\exists x\phi \equiv \forall w\,[\forall x\,((\phi \Rightarrow w) \Rightarrow w) = \top]$.

Notice that the foregoing definitions are all equational, whence the purely algebraic character of logical operations in this framework. A *sequent* over a signature $\mathcal{T}$ is defined as above, that is as a sequence of symbols of the form $(\phi \vdash_{\vec{x}} \psi)$, where ϕ and ψ are formulas over $\mathcal{T}$ and $\vec{x}$ is a context containing all the free variables of ϕ and ψ. The deductive system is again a sequent calculus, only shorter since there are fewer cases to cover.

1. Axiom schemas:

A1. $\varphi \vdash_{\vec{x}} \varphi$.
A2. $\varphi \vdash_{\vec{x}} \top$.
A3. $\top \vdash_{\vec{x}} x = x$.
A4. $\varphi \wedge (x = t) \vdash_{\vec{x}} \varphi(t/x)$ provided t is free for x in φ.
A5. $\varphi \vdash_{\vec{y}} (x_1, \ldots, x_n) \in \{(x_1, \ldots, x_n) : \varphi\}$.
A6. $(x_1, \ldots, x_n) \in \{(x_1, \ldots, x_n) : \varphi\} \vdash_{\vec{y}} \varphi$.

2. Rules of inference:

R1. $\frac{\varphi \vdash_{\vec{x}} \psi}{\varphi \vdash_{\vec{x},\vec{y}} \psi}$.

R2. $\dfrac{\varphi \vdash_{\vec{x}} \psi \quad \psi \vdash_{\vec{x}} \theta}{\varphi \vdash_{\vec{x}} \theta}$.

R3. $\dfrac{\varphi \vdash_{\vec{x}} \psi \quad \varphi \vdash_{\vec{x}} \theta}{\varphi \vdash_{\vec{x}} \psi \wedge \theta}$.

R4. $\frac{\varphi \vdash_{\vec{x}} \psi \wedge \theta}{\varphi \vdash_{\vec{x}} \psi}$ $\frac{\varphi \vdash_{\vec{x}} \psi \wedge \theta}{\varphi \vdash_{\vec{x}} \theta}$.

R5. $\frac{\varphi \vdash_{\vec{x}} \psi}{\varphi(t/x) \vdash_{\vec{y}} \psi(t/x)}$

where t is a term of the same type as $\vec{x}$, and $\vec{y}$ is any string of variables including all the variables occurring in the term t.

R6. $\frac{\phi \wedge (x_1,\ldots,x_n) \in t_1 \vdash_{\vec{x},\vec{y}} (x_1,\ldots,x_n) \in t_2 \ \ \phi \wedge (x_1,\ldots,x_n) \in t_2 \vdash_{\vec{x},\vec{y}} (x_1,\ldots,x_n) \in t_1}{\phi \vdash_{\vec{y}} t_1 = t_2}$

where the distinct variables $x_1, \ldots, x_n$ are not free in ϕ, t_1, t_2.

Axioms 5 and 6 are sometimes called "comprehension axioms" and R6 is sometimes referred to as "extensionality".

This presentation is admittedly somewhat awkward. For instance, it is far from obvious that we are dealing with an intuitionistic logic. But it was tailored in such a way that its interpretation in a topos becomes almost direct and, furthermore, the categorical manipulations to be done afterwards are few and almost trivial. It was shown by Boileau to be equivalent to the following, more standard, intuitionistic type theory (see [Kleene, 1952] or [Hatcher, 1968]);

Axioms:

1. $A \supset (B \supset A)$
2. $(A \supset B) \supset ((A \supset (B \supset C) \supset (A \supset C))$
3. $A \supset (A \vee B)$
4. $B \supset (A \vee B)$
5. $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
6. $A \supset (B \supset (A \wedge B))$
7. $(A \wedge B) \supset A$
8. $(A \wedge B) \supset B$
9. $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$
10. $\neg A \supset (A \supset B)$
11. $\forall x A(x) \supset A(t)$ provided that x is free in $A(x)$ and t is free for x in $A(x)$.
12. $A(x) \supset \exists x A(x)$ provided that $x$is free in $A(x)$
13. $x = x$
14. $x_1 = x_2 \supset (A(x_1) \supset A(x_2))$ provided that x_2 is free for x_1 in $A(x_1)$
15. $QxA \equiv Qx(A \wedge x = x)$ where $Q \in \{\forall, \exists\}$
16. $\forall x_1 \cdots \forall x_n((x_1, \ldots, x_n) \in X_1 \equiv (x_1, \ldots, x_n) \in X_2) \supset X_1 = X_2$
17. $A \equiv (x_1, \ldots, x_n) \in \{(x_1, \ldots, x_n) : A\}$.

Rules of inference:

R1. $\dfrac{A \quad A \supset B}{B}$ provided that all free variables in A are free in B.

R2. $\frac{C \supset A}{C \supset \forall x A}$ provided that $x$is not free in C.

R3. $\frac{A \supset C}{\exists x A \supset C}$ provided that x is not free in C.

The first fourteen axioms are the standard axioms for first-order intuitionistic logic. Axiom 15 together with the restriction on the rule of modus ponens reflects the fact that there may be empty types. The higher-order axioms are also standard. (See also [Osius, 1975] for what amounts to an equivalent proof of the same claim and [Fourman, 1977]. We will come back to the later in a short while.)

The interpretation of a language $\mathcal{L}(\mathcal{T})$ in a topos $\mathcal{E}$ is an extension of the interpretation of first-order language in a category of the appropriate type. Thus, a $\mathcal{T}$-*structure* or an *interpretation* M in $\mathcal{E}$ consists of:

1. an object MS of $\mathcal{E}$ for each sort S of $\mathcal{T}$; this is extended to types by stipulating that $M(\Omega(S_1, \ldots, S_n)) = P(MS_1 \times \cdots \times MS_n)$ where $P(X)$ denotes the power-object of X in $\mathcal{E}$;

2. a morphism $Mf : MS_1 \times \cdots \times MS_n \longrightarrow MS$ for each function symbol $f : S_1 \times \cdots \times S_n \longrightarrow S$ of $\mathcal{T}$; in particular, when $n = 0$ a morphism $1 \xrightarrow{f} MS$.

Variables-in-context, functional symbols-in-context and conjunction-in-context are interpreted as in the foregoing section.

1. If t is $\{(y_1, \ldots, y_m) : \phi\}$, then $[\![\vec{x} \cdot t]\!]_M$ is the morphism

$$MS_1 \times \cdots \times MS_n \longrightarrow P\left(MS_1 \times \cdots \times MS_n\right);$$

1. If ϕ is $t_1 = t_2$, and t_1 and t_2 are of type T, then $[\![\vec{x} \cdot \phi]\!]_M$ is the composite

$MS_1 \times \cdots \times MS_n \xrightarrow{\langle [\![\vec{x}\cdot t_1]\!], [\![\vec{x}\cdot t_2]\!]\rangle} MT \times MT \xrightarrow{\delta_{MT}} \Omega$, where the morphism $MT \times MT \xrightarrow{\delta_{MT}} \Omega$ is the characteristic morphism of the diagonal $MT \xrightarrow{\Delta_{MT}} MT \times MT$;

1. If ϕ is $(t_1, \ldots, t_m) \in t_{m+1}$, then $[\![\vec{x} \cdot \phi]\!]_M$ is the composite

$MS_1 \times \cdots \times MS_n \xrightarrow{\langle [\![\vec{x}\cdot t_1]\!], \ldots, [\![\vec{x}\cdot t_m]\!]\rangle} (MT_1 \times \cdots \times MT_m) \times P\left(MT_1 \times \cdots \times MT_m\right) \xrightarrow{e} \Omega;$

1. If ϕ is $\top$, then $[\![\vec{x} \cdot \phi]\!]_M$ is the composite

$$MS_1 \times \cdots \times MS_n \longrightarrow 1 \xrightarrow{\top} \Omega.$$

As is obvious from the definition, every formula in context is interpreted as a morphism into the subobject classifier Ω.

Let M be a $\mathcal{T}$-structure in a topos $\mathcal{E}$. We say that M is a *model* of a sequent $\sigma = (\phi \vdash_{\vec{x}} \psi)$ or that σ is *satisfied in* M, we write as usual $M \models_{\mathcal{E}} \sigma$ or simply $M \models \sigma$, if $[\![\vec{x} \cdot \phi]\!]_M \leq [\![\vec{x} \cdot \psi]\!]_M$ in $P(MS_1 \times \cdots \times MS_n)$. We say that M is a *model of a theory* $\mathbb{T}$ if all the axioms of $\mathbb{T}$ are satisfied in M.

As usual, a soundness theorem can now be proved: if $\mathbb{T} \vdash : (\varphi \vdash_{\vec{x}} \psi)$, then $\mathbb{T} \models (\varphi \vdash_{\vec{x}} \psi)$.

Given a theory $\mathbb{T}$ in the language $\mathcal{L}(\mathcal{T})$, it is once again possible to construct the conceptual category $\mathcal{C}_{\mathbb{T}}$ and show that it is in fact a topos, which we will denote by $\mathcal{E}_{\mathbb{T}}$. This yields the following:

Completeness Theorem: Let $\mathbb{T}$ be a higher-order type theory of similarity type $\mathcal{T}$. Then, $\mathbb{T}$ has a canonical model M_0, the canonical interpretation, in a topos $\mathcal{E}_{\mathbb{T}}$ such that $\mathbb{T} \vdash: (\varphi \vdash_{\vec{x}} \psi)$ if and only if $M_0 \models_{\mathcal{E}_{\mathbb{T}}} (\varphi \vdash_{\vec{x}} \psi)$.

Starting with an elementary topos $\mathcal{E}$, it is possible to define a higher-order signature $\mathcal{T}_{\mathcal{E}}$ from $\mathcal{E}$ by transposing the foregoing procedure appropriately and similarly the *internal language* $\mathcal{L}_{\mathcal{E}}$ and the type theory $\mathbb{T}_{\mathcal{E}}$ of $\mathcal{E}$ can be defined. The language $\mathcal{L}_{\mathcal{E}}$ is defined as follows:

1. To each object X of $\mathcal{E}$, we attribute a sort $\ulcorner X \urcorner$
2. To each morphism $f : X \longrightarrow Y$, we associate the function symbol $\ulcorner f \urcorner : \ulcorner X \urcorner \longrightarrow \ulcorner Y \urcorner$.
3. To each relation (or monomorphism) $\ulcorner R \urcorner \longrightarrow \ulcorner X \urcorner$.

(There are slight technical difficulties one has to take care of here, in particular with products, but we will skip these formal details.) There is then a canonical interpretation $M : \mathcal{L}_{\mathcal{E}} \longrightarrow \mathcal{E}$ defined in the obvious way, i.e. $M(\ulcorner X \urcorner) = X$, $M(\ulcorner f \urcorner : \ulcorner X \urcorner \longrightarrow \ulcorner Y \urcorner) = f : X \longrightarrow Y$, $M(\ulcorner R \urcorner \longrightarrow \ulcorner X \urcorner) = R \longrightarrow X$. The theory $\mathbb{T}_{\mathcal{E}}$ is also defined in the obvious way, i.e. it is the collection of sequents $\varphi \vdash_{\vec{x}} \psi$ such that $M \models \varphi \vdash_{\vec{x}} \psi$. One could be a little more specific and describe $\mathbb{T}_{\mathcal{E}}$ with explicitly definable axioms in such a way that its theorems are precisely the valid sequents of the canonical interpretation M. For instance, one could take the collection of sequents of $\mathcal{L}_{\mathcal{E}}$ obtained from the categorical properties of $\mathcal{E}$ with a dictionary that would include, for instance:

$X \xrightarrow{f} X$ is the identity morphism in $\mathcal{E}$ is translated into $\top \vdash_x \ulcorner f \urcorner(x) = x$, and so on, as we have seen above.

One can then verify that indeed $M \models \mathbb{T}_{\mathcal{E}}$.

It follows from the foregoing procedure, that given any topos $\mathcal{E}$, one can show that there is a theory, the internal theory of $\mathcal{E}$, such that $\mathcal{E} \models \mathbb{T}_{\mathcal{E}}$.

Joyal proved in 1974 that every elementary topos $\mathcal{E}$ arises from a higher-order theory (this is called the "théorème d'engendrement" in Boileau & Joyal 1981). More specifically, the statement is that for any elementary topos $\mathcal{E}$, there is a higher-order theory, naturally $\mathbb{T}_{\mathcal{E}}$ such that $\mathcal{E}$ is equivalent (as a category) to $\mathcal{C}_{\mathbb{T}_{\mathcal{E}}}$, the category of concepts of $\mathbb{T}_{\mathcal{E}}$. In this sense, a topos is nothing but a higher-order type theory. One can therefore use $\mathbb{T}_{\mathcal{E}}$ to prove properties of $\mathcal{E}$ and *working internally* means working in $\mathbb{T}_{\mathcal{E}}$. For instance, Boileau shows that any topos $\mathcal{E}$ has finite colimits, exponentiation from $\mathbb{T}_{\mathcal{E}}$. He also shows Cantor's theorem, i.e. if in a topos $\mathcal{E}$, there is an object X and an epimorphism $X \twoheadrightarrow \Omega^X$, then $\mathcal{E}$ is trivial, that is $1 \simeq 0$, from $\mathbb{T}_{\mathcal{E}}$.

The reader will have noticed that we have not included an object $\mathbb{N}$ of natural numbers in our theory. This constitutes an extension of the fundamental theory. Other extensions include being Boolean, being two-valued, or satisfying the axiom of choice, which can be formulated in different ways.

We can now see the unity underlying Joyal's work from regular categories to elementary toposes. Indeed, we start with the language $\mathcal{L}_0$ presented in section 4.4 and define four new higher-order languages: $\mathcal{L}_5$, $\mathcal{L}_6$, $\mathcal{L}_7$ and $\mathcal{L}_8$ from $\mathcal{L}_0$ as follows. First, for $\mathcal{L}_5$. We close the set of sorts with the operation $\Omega(-,\ldots,-)$ to obtain the set of types. Terms and formulas are defined as before, with the symbols $\exists$, $\vee$, $\perp$, $\supset$, $\forall$ to form new formulas. The language $\mathcal{L}_6$ is defined like $\mathcal{L}_5$, except that we now allow the negation operator $\neg$ in the construction of formulas. $\mathcal{L}_7$ is constructed from $\mathcal{L}_0$ by adding a new sort $\mathbb{N}$ and two new functional symbols, 0 and s. Types and formulas are constructed as in $\mathcal{L}_5$. To construct $\mathcal{L}_8$, we proceed as with $\mathcal{L}_7$, but by including all types and formulas of $\mathcal{L}_6$ instead of $\mathcal{L}_5$. The logical systems corresponding to these languages are then defined in the expected manner, i.e. by adding the appropriate sequents at each stage. Recall that $\mathcal{L}_0$ is purely equational logic, $\mathcal{L}_1$ is regular logic, $\mathcal{L}_2$ is coherent logic, $\mathcal{L}_3$ is first order intuitionistic logic, $\mathcal{L}_4$ is first order classical logic. As we have seen, $\mathcal{L}_5$ is intuitionistic type theory, $\mathcal{L}_6$ is classical type theory, $\mathcal{L}_7$ is intuitionistic type theory with $\mathbb{N}$ and $\mathcal{L}_8$ classical type theory with $\mathbb{N}$. One can then prove that:

$$\begin{tikzcd}
 & & & & \mathcal{L}_5 \\
 & & & \mathcal{L}_3 \arrow[ur, dash] \arrow[dr, dash] & \\
 & & & & \mathcal{L}_6 \\
\mathcal{L}_0 \arrow[r, dash] & \mathcal{L}_1 \arrow[r, dash] & \mathcal{L}_2 \arrow[uur, dash] \arrow[ddr, dash] & & \\
 & & & & \mathcal{L}_7 \\
 & & & \mathcal{L}_4 \arrow[ur, dash] \arrow[dr, dash] & \\
 & & & & \mathcal{L}_8
\end{tikzcd}$$

Needless to say, one can then move to conceptual categories and construct a diagram of functors between the respective categories thus obtained.

5.2 Fourman's approach

Let us now briefly consider Fourman's approach, since it differs from the foregoing in important and interesting ways.

Fourman starts with a primitive predicate, denoted by E, called the *existence predicate*. Thus, given a term t, one reads "Et" as "t exists". There is another primitive relation, namely a notion of *equivalence* between terms, i.e. $t \equiv s$. Equality of terms is then *defined* from the two primitive notions by $t = s \Leftrightarrow (t \equiv s) \wedge Et \wedge Es$. The definition of the formal system proceeds as follows. For convenience, another term is introduced, the *definite description* $Ix\varphi$, which is read "the unique x such that φ".

Definition: A *higher-order language* $\mathcal{L}$ is specified by the following data:

1. Two sets Sort and Const of sorts and constants.
2. A *power-type map* from $\bigcup_{n\in\omega} \mathrm{Sort}^n \longrightarrow \mathrm{Sort}$, written as $(S_1, \ldots, S_n) \mapsto [S_1, \ldots, S_n]$.
3. A map assigning a sort to each constant, $\# : \mathrm{Const} \longrightarrow \mathrm{Sort}$.

As in the foregoing case, for each sort S, there is a countable list of variables and each variable x has a sort $\#x$.

Definition of terms:

1. Every variable of sort S is a term of type S;
2. Every constant c of sort $\#c$ is a term;
3. If $x$is a variable of sort S and φ a formula, then $Ix\varphi$ is a term of sort S.

Definition of formulas:

1. If t is a term, then Et is a formula;
2. If t_1 and t_2 are terms of sort S, then $t_1 \equiv t_2$ is a formula;
3. If t is a term of sort $[S_1, \ldots, S_n]$ and $t_1, \ldots, t_n$ are of sorts $S_1, \ldots, S_n$ respectively, then $t\,(t_1, \ldots, t_n)$ is a formula;
4. If φ and ψ are formulas, then so are $(\varphi \wedge \psi)$ and $(\varphi \supset \psi)$;
5. If x is a variable and φ is a formula, then $\forall x\varphi$ is a formula.

The empty power sort $[]$ can be thought of as consisting of truth-values. Thus if t is a term of sort $[]$, then $t\,()$ is a formula and it asserts the proposition t.

The remaining connectives are also definable in this system, but we will skip the definitions.

1. Axioms

A1. $\varphi \supset (\psi \supset \varphi)$,
A2. $(\varphi \supset (\psi \supset \theta)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \theta))$,
A3. $(\varphi \wedge \psi) \supset \varphi$
A4. $(\varphi \wedge \psi) \supset \psi$
A5. $(\varphi \supset (\psi \supset (\varphi \wedge \psi)))$
A6. $(\varphi\,[y/x] \wedge y \equiv z) \supset \varphi\,[z/x]$ (Substituvity of equivalents),

A7. $\forall x(x \equiv y \Leftrightarrow x \equiv z) \supset y \equiv z$ (Extensionality),
A8. $(\forall x\varphi \wedge Ex) \supset \varphi$ (Universal instantiation),
A9. $\forall y(y \equiv Ix\varphi \Leftrightarrow \forall x(\varphi \Leftrightarrow x \equiv y))$ (Description),
A10. $EIy\forall\vec{x}(\varphi \Leftrightarrow y(\vec{x}))$ (Comprehension)
A11. $y(\vec{x}) \supset Ey \wedge \bigwedge_{i=1}^{n} Ex_i$ (Predication).
2. Rules of inference:
R1. $\frac{\varphi \quad \varphi \supset \psi}{\psi}$,
R2. $\frac{\varphi}{\varphi[t/x]}$,
R3. $\frac{(\psi \wedge Ex) \supset \varphi}{\psi \supset \forall x\varphi}$ where $x$is not free in ψ.

The notion of derivability is defined as usual: $\Gamma \vdash \varphi$ means that φ is derivable from the set Γ of formulas. A theory T is a set of formulas such that if $\mathrm{T} \vdash \varphi$, then $\varphi \in \mathrm{T}$. Also, a theory T has an underlying language denoted by $\mathcal{L}_{\mathrm{T}}$.

The reader will have noticed that types haven't been mentioned. They are in fact defined: a *type* is a term of the form $Iy\forall x(\varphi \Leftrightarrow y(x))$. If x is of sort X, then this term is of sort $[X]$. It is abbreviated as $\{x : \varphi\}$. Thus, types are seen as "subsets" of sorts. A type is *definable* if and only if it is a closed term.

Fourman proceeds to prove that the conceptual category $\mathcal{C}_{\mathrm{T}}$ of a theory T in the higher-order language $\mathcal{L}_{\mathrm{T}}$ is in fact a topos, denoted as usual $\mathcal{E}_{\mathrm{T}}$. He explicitly attributes the construction of $\mathcal{C}_{\mathrm{T}}$ to Joyal. He then defines the interpretation of a language $\mathcal{L}$ in a topos $\mathcal{E}$ and proves soundness and completeness theorems. Finally, given a topos $\mathcal{E}$, he defines the internal language $\mathcal{L}_{\mathcal{E}}$ of $\mathcal{E}$ and its theory $\mathrm{T}_{\mathcal{E}}$ and proves that $\mathcal{E} \simeq \mathcal{E}_{\mathrm{T}_{\mathcal{E}}}$. Various extensions, including toposes with a natural number object, are then considered.

5.3 Revising logic : the debate

An interesting debate between Fourman and Boileau & Joyal followed. We will simply summarize the main points raised in the debate.

Here are Fourman's arguments in favor of his system:

1. Fourman's approach contains an *original* component, namely an existence predicate;

2. Because their approach lacks this predicate, Boileau and Joyal are forced to make awkward restrictions on the rule of modus ponens;

3. In the words of Fourman's himself: "we *emphatically* do not agree that because of this [i.e. the modifications to the rule of modus ponens] "the traditional logical way of dealing with variables ... should be abandoned". The traditional use of variables has much to commend it and there are far less drastic remedies to hand." [Fourman, 1977, 1054])

In a nutshell: Fourman insists on the original component of his system, which allows him to avoid revising the way variables are used in deductions. Still in other words, the basic principles of deduction stay the same and there is an original component to the theory.

It seems only fair to bring in Dana Scott's voice to the debate, since he was Fourman's thesis advisor and did write on related issues around the same time. Scott's position finds its justification in a certain interpretation of intuitionism.

> Standard formulations of intuitionistic logic, whether by logicians or by category theorists, generally do not take into account partially defined elements. (For a recent reference see Makkai and Reyes 1977, esp. pp. 144-163.) Perhaps there is a simple psychological reason : we dislike talking of those things not already proved to exist. Certainly we should not *assume* that things exist without making this assumption explicit. In classical logic the problem is not important, because it is always possible to split the definition (or theorem) into cases according as the object in question does or does not exist. In intuitionistic logic this way is not open to us, and the circumstance complicates many constructions, the theory of descriptions, for example. Many people I find do not agree with me, but I should like to advocate in a mild way in this paper what I consider a simple extension of the usual formulation of logic allowing reference to partial elements. (...)
>
> Technically the idea is to permit a wider interpretation of *free* variables. All bound variables retain their usual existential import (when we say something exists it does exist), but free variables behave in a more " schematic " way. Thus there will be no restrictions on the use of *modus ponens* or on the rule of *substitution* involving free variables and their occurrences. The laws of quantifiers require some modification, however, to make the existential assumptions explicit. The modification is very straightforward, and I shall argue that what has to be done is simply what is done naturally in making a *relativization* of quantifiers from a larger domain to a subdomain.
>
> (...)
>
> The idea of schematic free variables is not new for classical logic, and the literature on " free " logic (or logic without existence assumptions) is extensive. (...) All I have done in this essay is to make what seems to me to be the obvious carryover to intuitionistic logic, because I think it is necessary and convenient. For those who do not like this formulation, some comfort can be taken from the fact that in topos theory both kinds of systems are completely equivalent, and the domains of partial elements can be *defined* at higher types (...). However, in first-order logic something is lost in not allowing partial elements, as I shall argue along the way.
>
> (...)
>
> Is the existence predicate E an illusion? Was the equality predicate

> an illusion? No. We shall find in the section, with a full statement of the laws of equality, that E can always be defined in terms of quantification :... . However, both in conception and in the models of (intuitionistic) logic we have in mind, the existence predicate is more basic than equality and prior to it. [Scott, 1979, 660–662])

The psychological reason can probably be dismissed altogether. From an anthropological point of view, it is hard to sustain the claim that "we dislike talking of those things not already proved to exist". In fact, the contrary seems to be the case: as a whole, we seem to enjoy very much talking of those things whose actual existence is completely unknown. Of course, Scott is here talking about *mathematicians*, a different cultural group if there is one and he might be formulating a deontological maxim for mathematicians than anything else. Notice that Scott acknowledges the fact that as far as higher-order intuitionistic logic is concerned, that is topos theory, the various systems presented can be shown to be formally equivalent. We should emphasize, in the same vein, the fact that Fourman also "regards topos theory as the "algebraic" form of this higher-order intuitionistic logic". (Fourman 1977, 1054.) First-order logic seems to be the real issue, although it is not so easy to separate the various logical systems in such a way.

Here is Boileau and Joyal's rebuttal.

1. First, as Fourman explicitly admits (see Fourman 1977, 1070), his system is an abstract codification of a category of H-sets, where H is a Heyting algebra, which was shown by Higgs to be equivalent to a topos of sheaves over the Heyting algebra of open sets of a topological space. Fourman is thus lead to consider a sheaf as a *set* with a complementary structure. Boileau and Joyal believe that the formal presentation should not present the objects as sets with a structure, but simply as sets with a specific logic.

2. The logical system developed by Boileau and Joyal is claimed to be more simple and natural than Fourman's system. One can work with the objects as if they were sets, provided that the logical rules, constructive in their workings if not in their spirit, are respected.

3. Fourman's system is perfectly legitimate for toposes, but the underlying intuition does not generalize to other cases. By adding the idea of the support of a sequent, one can treat logical systems in various categories.

4. The last point brings us to the underlying *unity* of Boileau and Joyal's approach. Although the notion of deduction is modified or restricted in contrast with the classical notion, one can treat various logical systems in a unified manner.

In a nutshell, Boileau and Joyal insists on the fact that Fourman's approach is somewhat *ad hoc* and that their approach is simpler and provides a unified framework to do and develop categorical logic in general. Furthermore, but this point is not underlined by the actors involved in the dispute, it is clear that Joyal's approach was right from the start motivated by the desire to minimize the distance between the syntax of categories and the syntax of logic. Evidence for this is easy to find: the choice of a Gentzen type presentation of logic, the choice of a multi-sorted system, the choice of rules which are simply rewriting of adjunctions between functors, and as we have seen, the higher-order type theory itself, which was chosen so that the interpretation of the connectives is simple and almost immediate in toposes. Stipulating the context of a deduction is nothing neither more nor less than the idea that a function *necessarily* has an underlying domain and codomain, although set theorists never conceived of functions that way. In contrast, Fourman and Scott are more preoccupied by purely *logical* issues and categories, in particular toposes, are seen as providing new metalogical resources in the study of intuitionistic and more generally constructive systems. An illustration of this position is found in their joint paper entitled *Sheaves and Logic,* also published in the proceedings of the Durham meeting and related to Scott's paper.

> Scott's paper in this volume describes a system of higher-order logic which may be interpreted in any topos. Here we describe the models of this logic given by sheaves over a complete Heyting algebra (cHa). These sheaf models subsume the more familiar Beth, Kripke and topological interpretations of intuitionistic logic, which correspond to interpretations in sheaves over the appropriate cHa of "truth values". They also provide a uniform way of extending these interpretations to higher-order logic and thus help to explain the models of analysis of Scott, Moschovakis and van Dalen. Once we go beyond first-order logic, these sheaf models are more general than Beth, Kripke or topological models. Models over a site (. . .) provide yet more generality. (. . .) Sites also arise naturally in first-order model theory once we take into account the comparisons between various models made possible by geometric morphisms. For a presentation of this theory see Makkai and Reyes 1977. However, we find a full-blown categorical presentation is often too abstract and results in very heavy machinery's being brought to bear on very simple problems. By restricting our attention to the special case of models over a cHa, we hope to make what is simple look simple. Models over cHa show clearly the link with traditional models for intuitionistic logic and are sufficient for many applications. ([Fourman and Scott, 1979, 303])

Thus, according to Fourman and Scott, logic has precedence over categorical ideas and methods. The latter are certainly useful, but there is no need to look for the most general result in that framework nor to find a unified framework that would be based on the categorical framework. Two specific rebuttals are in order. First, as Lawvere has emphasized in the eighties, one has to distinguish between *petit toposes* and *gros toposes.* Examples of petit toposes are precisely toposes of sheaves over a cHa (or a locale) and examples of gros toposes are toposes of all spaces of a certain type or the topos of simplicial sets. The point is that gros toposes have very different properties from petit toposes. Scott is here ignoring the class of gros toposes, which nonetheless are important not only in mathematics in general but for logic too. (See [Lawvere, 1986; Lawvere, 1989].) Second, for some applications, classifying toposes ("models over a site" in Scott's terminology) constitute a powerful tool. A classical example of this sort was given by Joyal in the seventies with his construction of the so-called formal reals for which local compactness fails. (See, [Ščedrov, 1984, section 4.2] and the whole paper for various examples or [Johnstone, 2002].)

6 THE METHOD OF FORCING IN TOPOSES: KRIPKE-JOYAL SEMANTICS

In the foregoing sections, the relationship between a topos $\mathcal{E}$ and its associated theory $\mathrm{T}_{\mathcal{E}}$ might seem to be, in some sense, too close for comfort. For $\mathcal{E}$ can be given in the form of $\mathrm{T}_{\mathcal{E}}$ and it is thus difficult to see how $\mathcal{E}$ could be useful in order to understand $\mathrm{T}_{\mathcal{E}}$. However, Grothendieck toposes were given and can be given independently of a theory and they are, in a precise technical sense that we will not clarify here, incompatible with *finitary* theories. Furthermore, Joyal quickly showed that Grothendieck toposes encompasses the usual semantics, at least for intuitionistic and classical logic: algebraic, topological and Kripke models. Joyal generalized Kripke's semantics for intuitionistic logic in the context of Grothendieck toposes. The interesting feature of the Kripke-Joyal semantics is that it can be manipulated classically. It should be emphasized immediately that Joyal's work was not a generalization for the sake of generalizing; it was seen and presented as an efficient *method* to prove results about toposes in the same way that Kripe semantics was used to prove results about intuitionistic logic.

Let $\mathcal{E}$ be an elementary topos, $\mathcal{L}$ an higher-order language and $M : \mathcal{L} \longrightarrow \mathcal{E}$ an interpretation of $\mathcal{L}$ in $\mathcal{E}$. Let $\mathcal{G}$ be a full sub-category of $\mathcal{E}$ such that

i) $\mathcal{G}$ is closed under subobjects, i.e. given $i : A \rightarrowtail X$ of $\mathcal{E}$ and X in $\mathcal{G}$, then i and therefore A is in $\mathcal{G}$;

ii) $|\mathcal{G}|$ is a set of generators for $\mathcal{E}$, that is if $X \underset{k}{\overset{h}{\rightrightarrows}} Y$ are such that $h \neq k$ in $\mathcal{E}$, then there is an object G in $\mathcal{G}$ and a morphism $A \xrightarrow{f} X$ such that $h \circ f \neq k \circ f$.

Definition (Kripke-Joyal semantics): Let G be an object of $\mathcal{G}$, $\varphi(x_1, \ldots, x_n)$ a formula of $\mathcal{L}$ and $G \xrightarrow{a_i} X_i$, $1 \leq i \leq n$, where $X_i = M($"the sort of x_i''). The morphism a_i is sometimes called a *generalized element* of X_i *at stage* G. We define the relation $G \Vdash \varphi(a_1, \ldots, a_n)$ and say that '$\varphi(a_1, \ldots, a_n)$ holds at stage G' or that 'G forces $\varphi(a_1, \ldots, a_n)$' as follows:

(1) $G \Vdash a_1 = a_2$ if and only if $a_1 = a_2 : G \longrightarrow X$;

(2) $G \Vdash a \in A$ if and only if the diagram

$$\begin{tikzcd} G \arrow[r, "!"] \arrow[d, "{(a,A)}"'] & 1 \arrow[d, "\top"] \\ X \times \Omega^X \arrow[r, "e_X"'] & \Omega \end{tikzcd}$$

commutes;

(3) $G \Vdash R(a_1, \dots, a_n)$ if and only if the triangle

$$\begin{tikzcd} G \arrow[r, "{(a_1,\dots,a_n)}"] \arrow[dr] & X_1 \times \cdots \times X_n \\ & M(R(x_1,\dots,x_n)) \arrow[u] \end{tikzcd}$$

commutes;

(4) $G \Vdash \varphi(\vec{a}) \wedge \psi(\vec{b})$ if and only if $G \Vdash \varphi(\vec{a})$ and $G \Vdash \psi(\vec{b})$;

(5) $G \Vdash \varphi(a_1, \dots, a_n) \vee \psi(b_1, \dots, b_m)$ if and only if there are morphisms $t_1 : G_1 \longrightarrow G$ and $t_2 : G_2 \longrightarrow G$ in $\mathcal{G}$ such that $t_1 + t_2 : G_1 + G_2 \twoheadrightarrow G$ is an epimorphism and $G_1 \Vdash \varphi(a_1t_1, \dots, a_nt_1)$ and $G_2 \Vdash \psi(b_1t_2, \dots, b_mt_2)$;

(6) $G \Vdash \varphi(a_1, \dots, a_n) \supset \psi(b_1, \dots, b_m)$ if and only if for all $t : G' \longrightarrow G$ in $\mathcal{G}$, if $G' \Vdash \varphi(a_1t, \dots, a_nt)$, then $G' \Vdash \psi(b_1t, \dots, b_mt)$;

(7) $G \Vdash \neg\varphi(\vec{a})$ if and only if for all $t : G' \longrightarrow G$ in $\mathcal{G}$, if $G' \Vdash \varphi(a_1t, \dots, a_nt)$, then $G' \simeq 0$;

(8) $G \Vdash \forall z\varphi(z, a_1, \dots, a_n)$ if and only if for all $t : G' \longrightarrow G$ in $\mathcal{G}$ and $b : G' \longrightarrow Z$, $G' \Vdash \varphi(b, a_1t, \dots, a_nt)$;

(9) $G \Vdash \exists z\varphi(z, a_1, \dots, a_n)$ there is $\left\{G_i \xrightarrow{t_i} G : i \in I\right\}$ in $\mathcal{G}$ and there is $\left\{G_i \xrightarrow{b_i} Z : i \in I\right\}$ such that

- $(\forall i \in I)(G_i \Vdash \varphi(b_i, a_1t_i, \dots, a_nt_i)$
- $\forall X$ in $\mathcal{E}\forall h, k \in Hom_{\mathcal{E}}(G, X)(h \neq k \supset \exists i \in I(ht_i \neq kt_i))$.

The full subcategory $\mathcal{G}$ comes from the case of a Grothendieck topos generated by a site, whose image in the topos is $\mathcal{G}$. One can also consider special cases of interest. First, for an arbitrary topos $\mathcal{E}$, it is always possible to let $\mathcal{G} = \mathcal{E}$. Two other cases worth mentioning are: 1) $\mathcal{G} = Sub(1)$, which corresponds to toposes of sheaves over a topological space and yields a particularly simple formulation of the semantics and 2) the case when $\mathcal{G} = 1$, that is when the terminal object is a generator. In the latter case, the topos $\mathcal{E}$ is Boolean and it can be shown that the semantics "is" the classical set-theoretical semantics (this is not quite precise, but can be clarified easily).

One natural question to ask is how the internal semantics of a topos $\mathcal{E}$ relates to the Kripke-Joyal semantics. In fact, they coincide in the following sense:

Theorem: Let $\mathcal{L}'$ be the language build from $\mathcal{L}$ by adding to the latter new sorts, one for each G of $\mathcal{G}$ and new functional symbols, one for each $G \xrightarrow{a} M(S)$, where S is a sort of $\mathcal{L}$, and let M' be the obvious extension of M to $\mathcal{L}'$. Then:

(i) $G \Vdash \varphi(a_1, \ldots, a_n)$ if and only if $M' \models \top \vdash_{\vec{a}} \varphi(a_1(g), \ldots, a_n(g))$

(ii) If φ is a sentence of $\mathcal{L}$, then $M \models \top \vdash \varphi$ if and only if $\forall G$ in $\mathcal{G}(G \Vdash \varphi)$.

7 FIBRED CATEGORIES AND LOGIC

Although this development found its full expression in the late eighties and early nineties, it has to be mentioned since it has become an important part of categorical logic and it has its roots in work done in the late sixties and mid-seventies. We will concentrate on the roots themselves and only point to certain presentations of the contemporary literature.

As we have mentioned, Lawvere and Lambek were both in Zurich in 1965-66 and at that time Lawvere was developing his ideas on categorical logic while, as we have seen, Lambek was generalizing results about posets to categories. Lawvere asked Lambek whether he had noticed that Tarski's fixed point theorem for posets generalized to categories and Lambek answered by showing him his manuscript of *A fixpoint theorem for complete categories*, published in 1968. (A result that is now better known by theoretical computer scientists than by category theorists, although the distinction between these two groups is nowadays more and more difficult to make.) Lawvere, as we have also mentioned, returned to the issue of fixed point theorems in 1968 and published his *Diagonal Arguments and Cartesian Closed Categories*, in the proceedings of the meeting of the Batelle Institute in 1969. This paper not only contains the definition of a Cartesian Closed Category (CCC), but it also suggests explicitly that a CCC is an algebraic version of type theory, on the one hand, and, on the other hand, that a theory gives rise to a category with finite products, now called a Cartesian category. There is, furthermore, an indication that Lawvere saw a connection between CCCs and Church's λ-calculus. Although there is no explicit mention of this connection in the paper, there is a clear indication that Lawvere did indeed see a connection. It is presented right from the start, in the very definition of a Cartesian Closed category. The key ingredient is in the definition of what Lawvere called the λ-transform of a morphism. Let $\mathcal{C}$ be a CCC and $h : X \longrightarrow Y^A$ be a morphism. Then h is said to be the λ-transform of f if and only if the diagram

$$\begin{tikzcd}
A \times X \arrow[d, "1\times h"'] \arrow[dr, "f"] & \\
A \times Y^A \arrow[r, "ev"'] & Y
\end{tikzcd}$$

is commutative. In particular, when $X = 1$, the terminal object, it follows that every $f : A \longrightarrow Y$ gives rise to a unique morphism $f : 1 \longrightarrow Y^A$, called the 'name of f'. Lambek was to use this connection systematically later.

At the same conference, Lambek presented his second paper on the relationships between categories and deductive systems and certainly took note of this connection, since it comes back in his third paper on the topic. Although these papers were not influential at first, they contain ideas that were to become important later.

Already in the mid-fifties, Lambek, together with his colleague Findley, had noticed a similarity between certain results in the theory of bimodules and deductive systems. Lambek and Findley developed a notation for certain concepts and wrote a paper on the subject, entitled *Calculus of Bimodules*, that was never published, since the results contained therein were supposedly subsumed by Cartan and Eilenberg's work that was about to appear in their book on homological algebra. (There is, however, a published trace of their work in a M.Sc. thesis submitted under Maranda's supervision at the Université de Montréal, done by Vaillancourt. See [Vaillancourt, 1968].) Lambek used the same notation in his first investigation of linguistics in a paper published in 1958. (See [Lambek, 1958].) Once he had learned category theory, Lambek realized that they were in fact looking at certain morphisms in specific concrete categories and that the results could be reformulated and developed in a categorical context. Lambek published three papers on deductive systems and categories, the first one in 1968, the second one in 1969 and the third one in 1972. Although the three papers have the same title and seem to be forming a complete whole, they are in fact independent from one another. They are simply on the same theme and constitute various attempts at developing the same ideas, which is formulated at the beginning of the very first paper:

> Roughly speaking, the situation is this: While deductive systems may be viewed as free algebra on the category of pre-ordered sets, they may also be used to construct free algebras on the category of all categories in some universe. The crucial step in the argument is the construction of a category whose objects are terms of a given deductive system and whose maps are equivalence classes of proofs. ([Lambek, 1968a, 287].)

In modern terminology, the main problem could be formulated as follows: how to construct all arrows $A \longrightarrow B$ in the free biclosed monoidal category generated by a graph and to decide when two such arrows are equal. The first problem, namely the construction of the free category of the right sort, was solved by generalizing Gentzen's cut-elimination theorem, first to substructural logic and then to certain structured categories. The second problem, deciding when two arrows are equal, was attacked in full generality in the second paper *Deductive Systems and Categories II. Standard Constructions and Closed Categories.* It also introduces explicitly the notion of a free biclosed monoidal category. However, the paper contains a certain number of mistakes. The satisfactory solution to the second problem was finally presented in a paper published in 1993. (See [Lambek, 1993].)

In his third paper, *Deductive systems and categories III*, presented at Dalhousie in January 1972, Lambek starts with Cartesian closed categories and tries to establish an equivalence with combinatory logic. More specifically, he shows how to go from combinatory logic, defined in the paper as *ontologies*, to Cartesian closed categories, but he was unable to verify that one could go in the opposite direction. This problem was solved later, in fact in the early eighties. It is here that Lambek developed systematically the connection between CCCs and typed λ-calculi. Modifying slightly the foregoing notation (and the formal context somewhat, but we won't give the details), given a morphism $\varphi(x) : 1 \longrightarrow Y$, there exists a unique morphism $f : A \longrightarrow Y$ such that $fx = \varphi(x)$, where $x : 1 \longrightarrow A$. The name of f can in this case be written as $\lambda_{x \in A}\varphi(x)$ in the corresponding λ-calculus. In collaboration with Phil Scott, they showed that the category of typed λ-calculi (with translations as morphisms) and the category of Cartesian closed categories with weak natural number objects (and Cartesian closed functors preserving weak natural numbers objects on the nose) are equivalent. ([Lambek and Scott, 1986, 79, Theorem 11.3].)

However, the main moral of all these papers is already stated in the foregoing quote: it is the correspondence between propositions of a deductive system with the objects of a corresponding category and the (equivalence classes of) proofs of the deductive system and the morphisms of the corresponding category. This correspondence is now summarized by the slogan *propositions-as-objects* and *proofs-as-morphisms*. This fundamental correspondence constitutes the first element of three essential ideas at the heart of developments of categorical logic that took part from the mid to the late eighties. The other two ideas were, in the first place, Lawvere's analysis of the comprehension principle and Bénabou's work on fibred categories.

Later in 1968, at the conference on applications of categorical algebra that took place in New York, Lawvere extended the connections between CCCs and logic in the direction of higher-order logic and type theory. The goal was to provide an analysis of equality and the comprehension schema as adjoint functors to elementary functors, thus including them in the algebraic approach.

Lawvere's basic strategy was to organize the logical data in a more complicated structure, now called an *hyperdoctrine*, given by the following data. First, one starts with a category $\mathcal{T}$ of types assumed to be Cartesian closed. This is, in a sense, the base category with its underlying logic. The morphisms of $\mathcal{T}$ are thought of as terms. Then, the additional structure is given "over" the types, so to speak. For each type X, there is a Cartesian closed category $P(X)$, whose morphisms are "deductions over X", and which is thought of as the "attributes of type X". Furthermore, for every term $f : X \longrightarrow Y$, there is a functor $f^* : P(Y) \longrightarrow P(X)$, called by Lawvere "substitution along f", such that $(gf)^* = f^*g^*$ (strictly speaking the last equality should be an isomorphism, but Lawvere notes that equality holds in all the examples he considers). Finally, the operation of substitution has two adjoints, left and right, which are, respectively, the existential and the universal quantification along f. Again, Lawvere emphasizes the fact that the concept of hyperdoctrine is purely equational.

Lawvere then gives a series of examples, some of which are barely sketched. For instance, he suggests that any multi-sorted higher-order theory, intuitionistic or classical, yields an hyperdoctrine. The objects are the basic types of the theory, closed under the operations of product and exponentiation and the morphisms are equivalence classes of (tuples) of terms from the theory. The equivalence class is not defined nor the construction presented. Given an object X, $P(X)$ consists of all formulas of the theory whose free variables correspond to the type X. Morphisms of $P(X)$ can be taken to be entailments and in this case $P(X)$ is a preordered set or "one may take suitable "homotopy classes" of deductions in the usual sense." [Lawvere, 1970a, 4] Needless to say, the last claim still has to be clarified. Lawvere himself cautiously writes that "one can write down an inductive definition of the "homotopy" relation, but the author does not understand well what results (some light is shed on this question by the work of Läuchli and Lambek cited above)." [Lawvere, 1970a, 4] Lambek's work referred to here is the work on coherence that was flawed. But Lawvere immediately moves on to add that "although such syntactically presented hyperdoctrines are quite important, it is fortunate for the intuition that there are also semantically-defined examples, as below." [Lawvere, 1970a, 4] And indeed, the examples involving sets are clear, interesting and support the basic intuition.

The simplest example, but perhaps not the most interesting, is the case when $\mathcal{T} = Set$, the category of (small) sets and mappings and, for each set X, $P(X) = 2^X$, the set of subsets of X, or equivalently the set of characteristic functions defined on X. In this case, f^* is simply the inverse image and the existential quantifier is the image. As such, this example is not surprising. More interesting perhaps is the case when $\mathcal{T}$ is the same but $P(X) = Set^X$. An attribute of X is then a family φ_x of sets indexed by the elements $x \in X$. A deduction $\varphi \longrightarrow \psi$ over X is any family of functions $\varphi_x \longrightarrow \psi_x$. In particular, $P(1)$ is the category Set of sets and can be thought of as the category of truth-values (an unusual many-valued logic). Lawvere conceives this example as "a kind of set-theoretical surrogate of proof theory" (Lawvere 1970, 4) and proceeds to speculate that "honest proof theory would presumably also yield a hyperdoctrine with nontrivial $P(X)$, but a syntactically-presented one." [Lawvere, 1970a, 4] It is precisely this insight that was about to become central to the developments in the eighties.

Going up on the ladder of abstraction, the next example consists in taking a category of small categories, which is Cartesian closed, and consider functor categories $P(\mathcal{C}) = Set^{\mathcal{C}}$ for the attributes over the type $\mathcal{C}$. Remember that Lawvere presented this material before the discovery of elementary toposes and he does not make an explicit connection with Grothendieck toposes. Needless to say, the connection has become important afterwards.

From a conceptual point of view, the important point is that hyperdoctrines are related to another categorical concept, although Lawvere did not exploit the connection. Indeed, he explicitly made a reference to fibred categories in the opening statement of his paper.

> The notion of hyperdoctrine was introduced (...) in an initial study of systems of categories connected by specific kinds of adjoints that arise in formal logic, proof theory, sheaf theory, and group-representation theory. .... Since then the author has noticed that yet another "logical operation", namely that which assigns to every formula φ its "extension" $\{x : \varphi(x)\}$ is characterized by adjointness, and that the "same" adjoint in a different hyperdoctrine leads to the notion of fibered category. [Lawvere, 1970a, 1]

What he says there is that the concept of hyperdoctrine leads to the notion of fibred category. Later in the same paper, Lawvere reports that "Gray, by introducing the appropriate notion of 2-dimensional adjointness, has shown that all the features of a hyperdoctrine, including our comprehension scheme, can be obtained by defining a type to be an arbitrary category and an attribute of type $\mathbf{B}$ to be any fibration over $\mathbf{B}$." [Lawvere, 1970a, 3] Thus, from a purely conceptual point of view, one could start with fibrations and use the later framework to develop the connection between type theory and categorical logic. What was lacking at first, perhaps, was a clear motivation and the right tools to do so. As early as 1974, Jean Bénabou pushed the idea that fibrations should be taken as the foundation of category theory itself and laid the groundwork that was to lead, when combined with Lawvere's ideas on hyperdoctrines, the groundwork done in the seventies on first-order and higher-order logics and results by Lambek, Scott, Seely, Hyland, Pitts and others, to the developments of categorical logic in this direction in the eighties. (The reference on the subject is [Jacobs, 1999].)

Bénabou had already given a series of talks on fibrations and logic in the summer of 1974 in Montréal. In the fall of 1975, he presented in Paris two related papers. In the first one, he describes, explicitly in the context of 2-categories, the hierarchy of categorical doctrines, e.g. Cartesian, regular, coherent, etc., the notion of a generic model of a theory and results on finitary objects in a topos. The second paper is about small and locally small fibrations and ends with a proof of a classifying topos for geometric theories. (See [Bénabou, 1975].) Three aspects are striking in Bénabou's papers. First, the papers are thoroughly categorical: all definitions and methods are categorical. For instance, the various types of categorical doctrines are defined not by specifying a formal first-order language, but by giving properties on diagrams, thus in the spirit of Ehresmann and of what are now called sketches. Second, all the results are expressed in the context of 2-categories and 2-functors, and some in bicategories. Bénabou himself had introduced earlier a study of bicategories, an important generalization of category theory that was to come back on the scene in the late eighties and nineties. At the time, even some category theorists were reluctant to "higher-dimensional" categories (although 2-categories are strict and not as such higher-dimensional in the way bicategories are). Third, the use of fibrations for the conceptual clarification of fundamental issues related to categories. We underline these three aspects to emphasize the originality, coherence and rigor of Bénabou's work at the time. We will now summarize some of the results on fibrations contained in the papers.

Bénabou presupposes the definition of a fibration in his papers. First, we introduce a notational but very suggestive convention. In the context of fibred categories, we use the vertical notation $\begin{matrix} \mathrm{E} \\ \downarrow p \\ \mathrm{B} \end{matrix}$ to denote a functor $p : \mathrm{E} \longrightarrow \mathrm{B}$. Given an object X in B, the *fibre* $p^{-1}(X)$ over X is the category whose objects are those objects A of E such that $p(A) = X$ and the morphisms are those $f : A \longrightarrow B$ in E such that $p(f) = 1_X$ in B. An object A in E such that $p(A) = X$ is said to be *above* X and, similarly, a morphism f in E such that $p(f) = u$ is said to be above u. The category B is called the *base category* and E the *total category*. This terminology goes back to the early 1940's when the notion of fibration was introduced in homotopy theory. Grothendieck extended it to categories in the early sixties.

Definition: Let $p : \mathrm{E} \longrightarrow \mathrm{B}$ be a functor.

1. A morphism $f : A \longrightarrow B$ in E is said to be *Cartesian over* $u : X \longrightarrow Y$ in B if $p(f) = u$ and if for every $g : C \longrightarrow B$ such that $p(g) = u \circ w$ for some $w : p(C) \longrightarrow X$, then there is a unique $h : C \longrightarrow A$ in E above w such that $f \circ h = g$. This situation is best depicted by the following diagram:

$$\begin{tikzcd} & C \arrow[d, dashed, "h"'] \arrow[dr, "g"] & \\ E \arrow[dd, "p"'] & A \arrow[r, "f"'] & B \\ & p(C) \arrow[d, "w"'] \arrow[dr, "u\circ w=p(g)"] & \\ B & X \arrow[r, "u"'] & Y \end{tikzcd}$$

2. The functor $p : \mathrm{E} \longrightarrow \mathrm{B}$ is a *fibration* if for every B in E and $u : X \longrightarrow p(B)$ in B, there is a Cartesian morphism $f : A \longrightarrow B$ in E above u. A fibration is also called a *fibred category*.

Bénabou gave conditions for a fibration to be small or locally small. These are important conceptual clarifications. Up to that day, category theorists, and in particular those working in topos theory, were assuming an ambient universe that allowed one to assume what being small and locally small meant. Thus, a category of sets was more or less assumed for certain foundational purposes, in particular for issues of size. What Bénabou showed, and developed later in a paper published in 1985 is that the concept of fibration was precisely what was needed to give a conceptually sound analysis of these problems. In a fibration $p : \mathrm{E} \longrightarrow \mathrm{B}$, the base category B provides the "universe" for the total category E. Thus, being locally small in E means that the Hom-sets of E can be represented as objects of B. As we have said, the most interesting result with respect to categorical logic announced by Bénabou is the proof of the existence of classifying toposes from the fact that certain fibrations are small. See [Bénabou, 1985].

In the eighties, categorical logicians picked up on the notion of hyperdoctrines and the conceptual framework provided by Bénabou, sometimes in terms of indexed categories, as a semantical framework and developed the syntactical aspects of the work to obtain more specific type theories and completeness theorems for them in this framework. (See [Seely, 1983; Seely, 1984; Seely, 1987; Lamarche, 1991; Moggi, 1991] and, of course, [Jacobs, 1999].)

8 THE DURHAM MEETING

It seems more than appropriate to end with a brief survey of the Durham meeting that took place from July 9 to July 21 in the summer of 1977. It constitutes in our mind the endpoint of the first period initiated by Lawvere in 1963. The title of the meeting was *Applications of Sheaves* and the subtitle was *Applications of Sheaf Theory to Logic, Algebra, and Analysis.* It was edited by M.P. Fourman, C. J. Mulvey and D. S. Scott. Among the participants, we find P. Aczel, B. Banaschewski, M. Barr, J. M. Beck, J. Bénabou, M. Bunge, J. C. Cole, M.-F. Coste, R. Diaconescu, E. J. Dubuc, M. P. Fourman, P. J. Freyd, J. W. Gray, R. J. Grayson, J. M. E. Hyland, J. Isbell, P. T. Johnstone, A. Joyal, G. M. Kelly, J. F. Kennison, A. Kock, J. Lambek, R. Lavendhomme, F. W. Lawvere, M. Makkai, R. B. Mansfield, C. J. Mulvey, G. E. Reyes, D. Schlomiuk, D. S. Scott, R. Seely, G. Takeuti, M. Tierney, D. van Dalen, H. Volger, G. C. Wraith and J. J. Zangwill. The list of talks is quite impressive but it does not correspond exactly to the list of papers published. For instance, G.E. Reyes gave two lectures on models in sheaves, but published a joint paper with Anders Kock on manifolds in formal differential geometry and a paper on Cramer's rule in the Zariski topos. Some of the talks were never transformed into papers. As usual, Bénabou and Joyal gave talks but did not submit papers. Bénabou talked about fibrations and Joyal gave a talk entitled *A topos as a space and a theory.* Makkai gave a talk on the syntactical constructions and basic properties of classifying toposes and Lambek gave a talk entitled *From λ-calculus to free topoi.* Peter Freyd presented results about complete higher-order theories. When one considers the published papers, one element stands: it is dominated by papers that are either about or in the context of sheaves over a topological space X, or equivalently H-sets, for H a complete Heyting algebra. Out of 33 papers (we exclude Gray's paper on the history of sheaf theory), at least 12 are clearly in the framework of a category of sheaves over a topological space X. The other striking feature is, as the title indicates, the search for concrete, precise *applications* of sheaf theory to various fields of mathematics. This second feature probably explains the first one. However, the underlying restriction, that is sheaves over a topological space or, more generally, sheaves over a complete Heyting algebra, certainly did not reflect the extent of the potential applications of the methods available. A more striking example, which we will simply mention, was given by Marie-Françoise Coste and Michel Coste shortly afterwards, using in an essential manner the classifying topos of a geometric theory. (See [Coste and Coste, 1979].) The search for applications continued for some time after and lead to interesting developments: models for synthetic differential geometry, the effective topos and Freyd's models for the independence of the axiom of choice, to mention but the most obvious. (See, for instance, [Kock, 1981; Moerdijk and Reyes, 1991] for synthetic differential geometry, [Hyland, 1982; Hyland, 1991; van Oosten, 2008] for the effective topos and [Freyd, 1980; Blass and Scedrov, 1989] for Freyd's models.) But by the early eighties, it seems that the mathematical community was shunning away from abstract and general methods.

With hindsight, two elements stand out when we look at the early history of categorical logic. First, a categorical analysis of certain of the key concepts of logic and set theory *is* not only possible – an idea that was at first seen as being absurd since it was thought that logical and set theoretical concepts were too primitive for such an analysis – , but fruitful in as much as they naturally fall under the central notions of the theory, e.g. adjoint functors. What this analysis amounted to was an adequate *algebraic* treatment of these notions. Although the analysis was associated with a distinctive ideological component, strongly political in the late sixties and early seventies, according to which logic and geometry were opposites, the fabric weaving together these facets was nonetheless thoroughly *algebraic* and this was acknowledged by all the parties involved and seen as a potentially powerful gain. Second, the development of categorical logic was not, in contrast for instance with Grothendieck's program in algebraic geometry which was carried on with the Weil conjectures as the final target, fueled by specific problems or conjectures deemed to be important by the community of logicians or mathematicians. A large part of the work done consisted, at first, in finding what one would consider being an adequate categorical "translation" of concepts and results of logic. The surprise was to see that so many notions coming from algebraic geometry and algebraic topology were mathematically equivalent to logical notions when they were formulated adequately in a categorical framework. This in itself opened new avenues of research, new ways of thinking about certain problems and theories.

For those who were not charmed by the beauty and elegance of categorical manners, who were convinced that category theorists had put the cart before the horses or had fallen prey to a complicated and terrifying new gadget whose purpose was far from being clear, the standard objection was simply: can you prove something by these means that cannot be proved by other, meaning more "traditional", means? If what one does in categorical logic is simply the same but in a different, more complicated and ultimately irrelevant guise, why bother learning this general abstract nonsense in the first place? If one does not care about a grand unifying picture of mathematics and its foundations, of unsuspected links between domains resulting in a complex and rich network of abstract structures, then is there any genuine value in the categorical toolbox?

It should be pointed out immediately that these complaints strangely resemble what mathematicians often ask logicians in general. What is original here is that it is a complaint coming from mathematical logicians towards categorical logicians. If logicians can or could reply to mathematicians that they were interested in a specific *kind* of analysis of mathematical notions, namely a *foundational* analysis, categorical logicians had a hard time convincing logicians that their analyses are genuinely foundational. Notice, however, that many logicians, including Tarski for instance, do want to use logic as a tool for mathematics and not specifically as a *foundational* tool. Be that as it may, in our specific case, the culprit is certainly related to the *algebraic* nature of a categorical analysis and algebraic logic has been seen, at least since Frege, as being foundationally misguided since it implicitly subsumes logic as a part of mathematics, when Frege and his successors assumed – not without reasons, it should be added – that the opposite must be true and we are back, once again, to an ideological, or perhaps some would prefer to say *philosophical*, debate. One possible answer to logicians, but that we will develop elsewhere, is that categorical logic *does* indeed offer a genuine foundational analysis of various mathematical concepts, no matter how logicians define the notion of "foundational analysis". One can start with various formal systems, whether propositional, first-order or higher-order, coherent, constructive or classical, develop them systematically, define semantics for these formal systems and prove various completeness or other kinds of theorems. Moreover, as we have already mentioned, categorical logic provides bridges between constructive and classical approaches, geometry and logic, topology and logic, to name but the most obvious links. The fate of categorical logic is presently intimately tied to theoretical computer science, and to a certain extent to the foundations of homotopy theory and its place in mathematics and mathematical physics. No matter how categorical logic develops, no matter what route it travels along, its algebraic roots will always reveal themselves.

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