A New Way Out of Galileo's Paradox

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Abstract

Galileo asked in his *Dialogue of the Two New Sciences* what relationship exists between the size of the set of all natural numbers and the size of the set of all square natural numbers. Although one is a proper subset of the other, suggesting that there are strictly *fewer* squares than natural numbers, the existence of a simple one-to-one correspondence between the two sets suggests that they have, in fact, the *same size*. Cantor famously based the modern notion of cardinality on the second intuition, but recent advances in mathematical logic (most notably, numerosity theory) have renewed an interest in the question whether Cantor's way out of Galileo's paradox was the only possible one. I present a new solution to Galileo's paradox and argue that it is a better alternative to the Cantorian solution than numerosity theory. In fact, I argue that it is the best possible way out of Galileo's Paradox that can be based on the "Euclidean" intuition that the whole is always greater than any of its proper parts.

1 Galileo's Paradox

How many natural numbers are there? Answering such a question requires us to extend our usual concepts of counting and size from finite to infinite collections. But many issues arise as soon as we undertake that task. Galileo offered a vivid presentation of one such issue in his *Dialogue of the Two New Sciences* [11], although, as thoroughly established by Mancosu [20], the problem itself has a much longer and richer history. On the first day of their dialogue, Salviati (Galileo's stand-in the Dialogue) is challenged by a perplexed Simplicio, who asks the following question: how is it possible that a line, containing infinitely many points, may however be a proper part of another line, who must therefore contain a greater infinity of points? Salviati replies:

> This is one of the difficulties which arise when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited; but this I think is wrong, for we cannot speak of infinite quantities as being the one greater or less than or equal to another. [11, p. 30]

Salviati argues for his position by giving another example. By a series of questions to Simplicio about the collection of all square numbers (i.e., numbers of the form n^2 for some natural number n) and the collection of all numbers, Salviati manages to make Simplicio contradict himself. On the one hand, since every square is a natural number, but the converse is false, there must be stricly more natural numbers than squares. On the other hand, since every square number can be mapped uniquely to its root, and every natural number is the root of some square, there must in fact be as many squares as there are roots of squares, and thus as many square numbers as there are natural numbers. Salviati draws the following conclusion from the situation:

> So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all numbers, nor the latter greater than the former; and finally the attributes "equal", "greater", and "less" are not applicable to infinite, but only to finite, quantities. [p. 31-32]

Galileo's solution to the paradox is therefore to retreat: it is simply impossible to extend coherently the concept of size in a way that would allow us to meaningfully compare infinite collections. Although this is certainly one way to avoid the paradox, it is not a particularly satisfactory one. Nonetheless, as I will explain below, I think there is some wisdom in Galileo's restraint.

Galileo's paradox highlights a tension between two principles which, following [20], I will call the *Bijection* and *Part-Whole* Principles.

Bijection Principle: For any two sets of natural numbers A, B, $size(A) = size(B)$ if and only if there is a one-to-one correspondence $f : A \to B$;

Part-Whole Principle: For any two sets of natural numbers A, B, if A is a proper part of B, then $size(A) < size(B)$.

Cantor famously solved the issue by endorsing the first of the two principles. The existence of a one-to-one correspondence between the set of natural numbers and the set of squares means that they have the same size. The other intuition, according to which a set can never have the same size as one of its proper subsets, remains valid in the realm of the finite, but must be abandoned when it comes to infinite sets. Cantor's choice led him to develop the modern notion of *cardinality*, which is often taken to be an extension of the notion of size to infinite sets.

The success of the Cantorian approach has led to a widespread belief that Cantor's was essentially the *only way* to extend the concept of size into the infinite. Gödel [12] famously argued for such a position when discussing the status of the Continuum Hypothesis in set theory. Such a view, however, is challenged by the development of alternative theories of size for infinite sets. Mancosu [20] lists several historical attempts to develop an arithmetic of the infinite based on the Part-Whole Principle rather than on Cantor's Bijection Principle. The idea traces back to Euclid's "common notion" that the whole is always greater than any of its proper parts, which is why theories of size trying to follow it are often called *Euclidean* theories. A prominent figure among such attempts in the Bohemian polymath Bernard Bolzano, who tried to develop such a "Calculation of the Infinite" in his *Paradoxes of the Infinite* [1, 27]. More recently, modern model-theoretic tools have been used to develop *numerosity theory* [2, 3, 4], which assigns sizes to infinite sets (and in particular to sets of natural numbers) in a way that is consistent with part-whole intuitions. A distinctive feature of numerosities is that the existence of a one-to-one correspondence between two sets is a necessary but not sufficient condition for their equinumerosity. As such, numerosities are a vastly more fine-grained notion of size than cardinality: while all infinite subsets of the natural numbers have the same cardinality, they have infinitely many distinct numerosities. Despite their sophistication, several arguments have been raised against the claim that numerosity theory offers a genuine alternative to the Cantorian path [24, 25, 32]. As we shall see below, I agree with some of these arguments, and I think numerosities do not provide a satisfactory alternative to the Cantorian notion of size based on part-whole intuitions. But I think such an alternative theory exists, and my goal in this paper is to present it and argue that it is, at least in the case of sets of natural numbers, a natural, well-motivated, and mathematical fruitful way out of Galileo's paradox. Before getting into this further, let me briefly outline what I take to be the relevance of this work, by highlighting how it relates to three debates in the philosophy of the mathematical infinite.

First, the original purpose of Gödel's argument is to show that the continuum hypothesis, which is strictly speaking an issue about cardinality, can also be understood as a basic question about the size of sets of real numbers. For Gödel, the fact that the continuum hypothesis can be presented as such a basic problem is evidence that it must, in fact, have a definite answer, since it asks an elementary question about our concept of set. Should cardinality, however, be one among several ways of extending our pre-theoretic notion of size into the infinite, this would also be significant for the debate in the philosophy of set theory regarding whether the continuum hypothesis is a definite problem [8, 14]. One may, for example, think that cardinality is a fascinating and mathematically fruitful notion to investigate when studying the infinite, without assuming that it corresponds to an "objective" property of infinite sets in the way that the size of a finite set is. In other words, cardinality is a technical notion worthy of mathematical investigation, but one that ultimately depends on our concept of set rather than on some objective features of the world. Such a view is therefore consistent with the idea that our concept of set might simply not be determined enough to settle questions such as the Continuum Hypothesis.

Second, the debate around Galileo's Paradox is also relevant for the foundation for mathematics, particularly for neologicists. Neologicists hold that Frege's goal of providing a foundation for arithmetic based solely on logic and analytically true principles can be achieved. One of the central tenets of the neologicist program [34] is the claim that Hume's Principle, from which the Dedekind-Peano axioms can be derived in second-order logic, is analytic. Hume's Principle states that the number of objects falling under a concept F is the same as the number of objects falling under another concept G if and only if there is a one-to-one correspondence between the Fs and the Gs. Neologicists hold that this is a conceptual truth, self-evident for anyone who grasps the concept of number. This view is consistent with the idea that Cantor's notion of cardinality is the one true way of extending the concept of size (and the closely related concept of number of elements) into the transfinite. However, as remarked in [15, 19], if one thinks that there are alternative ways of extending the concept of number from the finite to the infinite, including some that are consistent with part-whole ideas rather than with Cantorian ones, then the neologicist's claims become much harder to maintain. In particular, this leads to the "Good Company" objection to the neologicist program: since there are consistent principles based on part-whole intuitions that can deliver the Dedekind-Peano axioms yet are incompatible with Hume's Principle, on what grounds could the claim that the latter is a conceptual truth about the concept of number be based? If there are legitimate alternatives to the Cantorian definition of (finite and infinite) number, then Hume's Principle might still be true, but it is certainly not a *conceptual* truth about the concept of number.

Lastly, Euclidean ideas have recently been discussed in the context of probability theory. As it is well known, Kolmogorov probability theory faces some serious issues when dealing with uniform probability measures on infinite sample spaces. A famous example is *de Finetti's Lottery* [9] (also called "God's Lottery" in [23]), in which a natural number is chosen at random. It is easy to see that no probability function defined on all singletons could model such a situation. De Finetti's own solution was to give up Kolmogorov's axiom of countable additivity in favor of finite additivity and to argue that each finite subset of $\mathbb{N}$ has probability 0 of containing the winning ticket. Such a solution, however, is counter-intuitive in that it forces us to admit that some event that has probability 0 of occurring will in fact occur. Our pre-theoretic intuition of quantitative probability arguably respects the constraint that any possible event should receive a positive probability of occurring, an idea expressed by the following constraint:

Regularity: For any $A \in \mathbb{B}$, $A \neq 0$ implies $\mu(A) > 0$.

It is fairly straightforward to show that a finitely additive probability function on the powerset of the set of natural numbers that satisfies the Regularity constraint yields a Euclidean theory of size. The converse is also true, and in particular, numerosity theory has inspired the development of Non-Archimedean Probability theory [5, 6, 31]. But some of the criticisms raised against numerosity theory carry over to Non-Archimedean Probability theory [25, 26, 32]. Thus, although the development of a generalized probabilistic framework in which the existence of regular probability functions can be guaranteed is arguably a worthy goal to pursue [16, 18, 28], doing so in a convincing way arguably involves offering a satisfactory solution to Galileo's paradox based on Euclidean intuitions.

The rest of the paper is organized as follows. In Section 2, I introduce what I think is a helpful axiomatic framework for thinking about Galileo's paradox and describing what the logical space of solutions to the paradox looks like. This leads me to introduce in Section 3 what I call the Minimal Proposal, which I advocate for in the rest of the paper. In Section 4, I discuss the Density Intuition, an intuition about sizes of sets of numbers which, I argue, is often confused with part-whole considerations and is the true driving force behind numerosity theory. I also give an informal, intuitive presentation of numerosity theory that avoids getting into overly technical details. In Section 5, I argue that numerosities face a serious problem that I call the Invariance Problem and that, contrary to what has been argued before, the Invariance Problem comes from the Density Intuition and not from the desire to preserve part-whole intuitions. This leads me to introduce in Section 6 an alternative to numerosity theory called the Generic Approach and argue that this is the "best possible" Euclidean theory. As I will show, the Minimal Proposal and the Generic Approach actually yield the exact same theory, a result which I take to be strong evidence that my proposal is the best contender for an alternative to Cantor's theory of size based on part-whole considerations. Finally, in Section 7 I offer a brief summary of what I think the debate now looks like in light of the new proposal that I advocate for. Throughout the rest of this paper, I will sometimes refer to some formal results. Whenever those results are new, proofs have been included in Appendix A. The proofs themselves are all easy, and require no knowledge of set theory beyond, roughly, the ability to draw Venn diagrams and to understand notions such as injections and surjections.

2 An Axiomatic Framework for Size Relations

In this section, I focus on a particular way to address the issues behind Galileo's paradox, namely, an axiomatic approach toward size relations. This is a shift from the more common presentation of the problem in terms of number structures, but, as I explain below, I think it is a more illuminating perspective.

At its core, Galileo's paradox raises an issue regarding the following question: when does a set of natural numbers A contain "at most as many elements" as another set B? In the case of natural numbers and their squares, we saw that two seemingly intuitive principles contradict one another. Although those principles were phrased in terms of the size of a set, they can easily be rephrased in terms of the "at most as many elements" relation. According to the first one, the fact that we can map $\mathbb{N}$ into $\mathbb{N}^2$ in a one-to-one way (meaning that any two distinct natural numbers are mapped to two distinct squares) implies that there are at most as many natural numbers as squares. By contrast, according to the second intuition, the fact that the squares are a proper subset of the natural numbers implies that there must be strictly more natural numbers than squares. In other words, solving Galileo's paradox amounts to giving a precise characterization of a binary relation $\sqsubseteq$ between sets of natural numbers, where, for any two sets of natural numbers A and B, $A \sqsubseteq B$ if and only if there are "at most as many elements" in A as there are in B. This is what I will take to be the central problem that we need to address. It is related to, but not identical with, the issue of extending the structure of the natural numbers so as to assign a size to every set of natural numbers. A full treatment of this issue is beyond the scope of this paper,[1] but let me briefly highlight for now how the two relate to one another. If we have such a structure $\mathcal{N}$ and a function $size : \mathscr{P}(\mathbb{N}) \to \mathcal{N}$, then, assuming that the structure $\mathcal{N}$ is ordered by some relation $\preccurlyeq$, we can simply "pull back" the order to a relation on the powerset of $\mathbb{N}$. In other words, we could say that A has "at most as many elements" as B (i.e., $A \sqsubseteq B$) whenever the size of A is *less than or equal to* the size of B (i.e., whenever $size(A) \preccurlyeq size(B)$). Thus solving the "structure" problem also gives us a solution to the "relation" problem, at least under the assumption that the structure we construct is an ordered one. Importantly, the converse is also true, by a standard technique called a definition by abstraction. If we have a size relation $\sqsubseteq$ defined on the powerset of $\mathbb{N}$ that has certain properties (namely, being a quasi-order, which I define below), then we can define an *equivalence* relation $\equiv$ by letting $A \equiv B$ if and only if $A \sqsubseteq B$ and $B \sqsubseteq A$. Intuitively, A and B are *equivalent* with respect to their sizes if there are "at most as many elements" in A as there are in B, and vice versa. Once we have such an equivalence relation, it becomes easy to define what the size of a set A is: we can simply take it to be its equivalence class under the relation $\equiv$. In other words, for any set A, define $size(A) = \{B \subseteq \mathbb{N} \mid A \equiv B\}$. This is a standard construction in mathematics, particularly useful when introducing new number systems, which traces back (at least) to Cantor's construction of the cardinal numbers, Frege's definition of the natural numbers, or von Staudt's definition of the direction of a line [33]. Once we have such a definition, we can indeed take our structure extending the natural numbers to be given by the set $\{size(A) \mid A \in \mathscr{P}(\mathbb{N})\}$, and it can naturally be ordered by the relation $size(A) \preccurlyeq size(B)$ if and only if $A \sqsubseteq B$.

Determining what the relation $\sqsubseteq$ is and extending the natural numbers to an ordered structure of sizes for all sets of natural numbers are therefore, in a precise sense, equivalent problems. But it is easier to give a systematic description of the logical space of possibilities for the relation $\sqsubseteq$, which is what I will do in the rest of this section by presenting several properties (or "axioms") that we may want the relation $\sqsubseteq$ to satisfy. Let me start with three properties that I will take for granted.

QO: The relation $\sqsubseteq$ is a *quasi-order*, meaning that it is *reflexive* ($A \sqsubseteq A$ for any $A \subseteq \mathbb{N}$) and *transitive* ($A \sqsubseteq B$ and $B \sqsubseteq C$ together imply $A \sqsubseteq C$ for any $A, B, C \subseteq \mathbb{N}$).

IN: The relation $\sqsubseteq$ extends the inclusion ordering on sets of natural numbers, meaning that for any $A, B \subseteq \mathbb{N}$, $A \subseteq B$ implies $A \sqsubseteq B$.

DEC: The relation $\sqsubseteq$ satisfies the following "decomposition principle": whenever A_1, A_2, B_1 and B_2 are sets such that $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$, $A_1 \sqsubseteq B_1$ and $A_2 \sqsubseteq B_2$ together imply $A_1 \cup A_2 \sqsubseteq B_1 \cup B_2$.

Whenever $\sqsubseteq$ is a relation between sets of natural numbers that satisfies **QO**, **IN** and **DEC**, I will call $\sqsubseteq$ a *size relation*. Let me briefly comment on why I think those three are natural conditions to impose on the "at most as many elements" relation.

First, it seems really hard to deny that this relation is reflexive and transitive. Whatever having "at most as many elements" as another set may mean for a set of natural numbers, any set clearly has "at most as many elements" as itself. An intuitive justification for transitivity can be given in terms of some sort of monotonicity or stability of our counting procedure: whatever procedure we choose for establishing that A has at most as many elements as B, if it also establishes that B as at most as many elements as C, we should somehow be able to combine the

[1] For a recent survey of the matter that adopts the number structure approach, see [30].

two procedures so as to show that A has at most as many elements as C. Without transitivity, we may have some interesting things to say about "local" comparisons between sets, but it would be very difficult to have a global picture of the entire structure of sets of natural numbers under the "at most as many elements" relation. In fact, imposing reflexivity and transitivity on $\sqsubseteq$ is precisely what we need to ensure that the corresponding relation $\equiv$ is an equivalence relation, and thus to allow us to build a number structure out of our size relation.

The second criterion, **IN**, also seems fairly intuitive. Whatever "having at most as many elements" means in the context of sets of natural numbers, surely being contained as a subset is a sufficient condition. After all, if $A \subseteq B$, then *every* element of A is also an element of B. How could there not be at least as many elements in B as in A?

Finally, the third condition can be interpreted as a coherence condition imposed on judgments of size relationship. Whatever the correct way of determining when a set has "at most as many elements" as another may be, this procedure should be uniform for smaller sets and bigger sets. If we have established two size relationships $A_1 \sqsubseteq B_1$ and $A_2 \sqsubseteq B_2$ between two pairs of sets, then we should be able to lift this size relationship to one between their respective unions $A_1 \cup A_2$ and $B_1 \cup B_2$, under the assumption that these are disjoint unions, so as to avoid any "double counting". It might be helpful here to draw an analogy with probability theory. In most settings, probability functions are required to be (at least) finitely additive, meaning that the probability of the disjoint union of two events E_1 and E_2 should be exactly the sum of the probability of E_1 and the probability of E_2. This captures the idea that the probability of an event is determined by the probabilities of its subevents. One may think of **DEC** as a weak form of additivity for sizes of sets of natural numbers. Indeed, it is meant to capture the intuitive idea that the "number of elements" in a set A is determined by the "number of elements" of its subsets. If we can partition two sets A and B into subsets A_1 and A_2, and B_1 and B_2 respectively, in such a way that there are "at most as many elements" in A_1 as there are in B_1, and "at most as many elements" in A_2 as there are in B_2, then we should also conclude that there are "at most as many elements" in A as there are in B.

From now on, I will therefore assume that the "at most as many elements" relation is a size relation, meaning that it satisfies **QO**, **IN** and **DEC** as specified above. Note that, under one way of understanding Galileo's view, this is already asking for too much, since doing so commits one to asserting *some* size relationships between infinite sets of natural numbers. However, there is no risk of paradox here: the inclusion ordering on sets of natural numbers is an example of a size relation, and, in fact (by **IN**), the smallest possible such example. At the other end of the spectrum, there is a largest possible size relation: the total relation on the powerset of the natural numbers, according to which $A \sqsubseteq B$ for *any* $A, B \in \mathbb{N}$. Of course, neither option is a reasonable choice for a precise characterization of "at most as many elements". The inclusion order is too fine-grained, since two finite sets can have the exact same number of elements yet be incomparable with respect to the inclusion ordering: singletons are the simplest example of this. The total relation, on the other end, is too coarse-grained: clearly, there are not at most as many natural numbers as there are numbers in the set $\{0, 1\}$, to take one example among many.

This is where the two competing intuitions that lead to Galileo's paradox enter the picture. Both correspond to an additional axiom one can impose on the relation $\sqsubseteq$, although the two do not play the same role. I will call the first such axiom the Injection Principle (**IP**). According to **IP**, the "at most as many elements" relation between sets of natural numbers is completely characterized by the existence of a special kind of functions called injections. Recall that an injection (or one-to-one map) from a set S to a set T is a function $f : S \to T$ such that for any two distinct $s, s' \in S$, $f(s)$ is distinct from $f(s')$. In other words, an injection is a map that preserves non-identity between the elements of S. Mapping the passengers of a flight to their seat on the plane is an injection, while mapping them to their date of birth may not be. The Injection Principle states that the existence of an injection from a set A to a set B coincides with A having "at most as many elements" as B.

IP: For any $A, B \subseteq \mathbb{N}$, $A \sqsubseteq B$ if and only if there is an injection $f : A \to B$.

Importantly, **IP** gives necessary and sufficient conditions for a set to contain "at most as many elements" as another, and, as such, picks out exactly one size relation, which I will denote as $\sqsubseteq_{\mathcal{C}}$ (for cardinality). It is easy to verify that $\sqsubseteq_{\mathcal{C}}$ is a size relation. Whenever $A \subseteq B$, the identity is an injection from A to B, hence $A \sqsubseteq_{\mathcal{C}} B$. Moreover, the composition of two injections is also an injection, which implies that $\sqsubseteq_{\mathcal{C}}$ is transitive. Finally, if we have two injections $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$, we can simply take their union to obtain an injection $f : A_1 \cup A_2 \to B_1 \cup B_2$ (the fact that $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$ is a crucial assumption here). Moreover, the relation $\sqsubseteq_{\mathcal{C}}$ is precisely the one determined by the Cantorian notion of cardinality. For any sets A, B of natural numbers, we have that $card(A) = card(B)$ if and only if $A \equiv_{\mathcal{C}} B$. In other words, two sets have the same cardinality if and only if one can define an injection from either set into the other.[2]

The axiom **IP** completely determines what the "at most as many elements" relation is. The second axiom, by contrast, merely states a necessary condition for a set A to have "at most as many elements" as a set B: A cannot be a proper superset of B. This is the natural way of cashing out the intuition that "the whole is always greater than any of its proper parts", which is why I will refer to it as the Euclidean Constraint (**EC**).

EC: if A is a proper subset of B, then $B \not\sqsubseteq A$.

If one only considers finite collections, then **IP** and **EC** are not only compatible but true. The existence of a Dedekind-infinite set, however, is enough to derive a contradiction between the two. Indeed, recall that Dedekind [7] defined a set A as infinite if there is an injection $f : A \to A$ that is not a *surjection*, meaning that the range of f, $ran(f)$, is a *proper subset* of A. If A is such a set, then, according to **IP**, $A \sqsubseteq ran(f)$ (since f is an injection from A to its range), while, according to **EC**, $A \not\sqsubseteq ran(f)$ (since $ran(f)$ is a proper subset of A). So we must give up either **IP** or **EC**, exactly the situation of Galileo's paradox. Cantor's way out, of course, is to go with **IP**, which has the advantage of uniquely specifying what the relation $\sqsubseteq$ is. Let me now motivate an another option, and present an alternative to cardinality that satisfies the Euclidean Constraint.

3 The Minimal Proposal

Recall Galileo's original reaction to the paradox: we should tread very carefully when dealing with size relationships between sets. As I have argued above, Galileo is treading *too* carefully when he says that *nothing* can be said about size relationships between infinite sets. The relation $\sqsubseteq$ should at least be what I called a size relation. But I think there is something appealing to the idea that perhaps, in many cases, there is just nothing meaningful that can be said regarding whether there are more elements in one infinite set than in another. Let me now try and implement this idea in a precise way.

An obvious starting point is the inclusion ordering $\subseteq$. After all, it is the smallest size relation. What about the idea that it is also the *correct* size relation between sets of natural numbers? As

[2]Note that the "if" part of this equivalence is not trivial: the celebrated Cantor-Schröder-Bernstein theorem establishes that whenever there are injections $f : A \to B$ and $g : B \to A$, there is a one-to-one correspondence $h : A \to B$.

I mentioned above, there is a major issue here: the relation is too small or, equivalently, it allows for two many distinctions between sets. Clearly, finite sets with the same number of elements should have the same size. We want, after all, to extend finite arithmetic to infinite sets. To explain this further, let me first fix some notation. Whenever A is a finite set, I will write $|A|$ for its finite cardinality, i.e., the natural number that corresponds to how many elements are in A. We should not be spooked by the use the word "cardinality" here: as long as we consider finite sets, there is no disagreement regarding what their sizes are, and we can freely use the concept of finite cardinality to reason about such sizes. Our need to preserve finite arithmetic suggests adding something like the following to our list of axioms for the relation $\sqsubseteq$: whenever A is a finite set with n elements and B contains at least n elements, then $A \sqsubseteq B$. If B is a finite set, we can simply say that $A \sqsubseteq B$ whenever $|A| \leq |B|$. In the more general case, we can express this axiom in a rigorous way using injections:

FIN: For any finite set A and any $B \subseteq \mathbb{N}$, if there is an injection from A to B, then $A \sqsubseteq B$.

One may think of **FIN** as a way to extract the finitistic content from **IP**. At least when A is finite, the existence of an injection f from A to B is enough to conclude that there are "at most as many elements" in A as there are in B. This seems uncontroversial enough: after all, we could simply use f to count as many elements of B as there are in A. I will call a size relation *arithmetical* precisely when it satisfies **FIN**. Indeed, adding **FIN** to our list of axioms is exactly what is needed to ensure that finite arithmetic is preserved. Whenever A and B are finite sets of size n and m respectively, we will have that $A \sqsubseteq B$ whenever $n \leq m$. Whenever A is finite and B is infinite, we will also have that $A \sqsubseteq B$. But these are not the only cases that we should consider. To see this, recall first that the *set-theoretic difference* of two sets A and B of natural numbers is the set $A \setminus B = \{n \in \mathbb{N} \mid n \in A \text{ and } n \notin B\}$. Consider now the sets $A = 2\mathbb{N} \setminus \{0\}$ and $B = 2\mathbb{N} \setminus \{2\}$. The first set contains all even numbers except for 0, and the second one contains all even numbers except for 2. It is straightforward to see that for any size relation $\sqsubseteq$ that satisfies **FIN**, we have that $A \sqsubseteq B$. Indeed, we can partition A into the sets $A_1 = 2\mathbb{N} \setminus \{0, 2\}$ and $A_2 = \{2\}$, and B into the sets $B_1 = 2\mathbb{N} \setminus \{0, 2\}$ and $B_2 = \{0\}$. By reflexivity of $\sqsubseteq$, we have that $A_1 \sqsubseteq B_1$, and, by **FIN**, we have that $A_2 \sqsubseteq B_2$. By **DEC**, this means that $A \sqsubseteq B$. The same argument shows that $B \sqsubseteq A$, so that, in fact, $A \equiv B$. This, I think, is fairly intuitive: when we set aside the part that A and B have in common (i.e., their intersection $A \cap B$), we are left with two singletons. Even though A and B are both infinite, comparing their sizes ends up being a matter of finite arithmetic once we discard their intersection. Here is one way to turn this idea into a systematic definition:

MIN: For any $A, B \subseteq \mathbb{N}$, $A \sqsubseteq B$ if and only if there is a finite set $C \subseteq B \setminus A$ such that $|A \setminus B| \leq |C|$.

Let me make a few remarks about the definition above, which I call the *Minimal Proposal* for reasons that I will explain shortly. First, **MIN** really is a definition, in the same way that **IP** defines a particular size relation (namely, cardinality): it gives necessary and sufficient conditions for a set A to have "at most as many elements" as a set B. Accordingly, I will write $\sqsubseteq_{\mathcal{M}}$ for the relation defined by **MIN**. Second, although this might take a minute to notice, **MIN** does capture the intuition above. Indeed, whenever $A \sqsubseteq_{\mathcal{M}} B$, this is because $A \setminus B$ is finite, and that finite set contains at most as many elements as the set $B \setminus A$. In other words, once we "delete" from sets A and B their common part $A \cap B$, we're left with only finitely many elements in A, and at least as many elements in B, regardless of whether $B \setminus A$ is finite or infinite. Third, it is fairly straightforward to notice that $\sqsubseteq_{\mathcal{M}}$ is a relation that extends the inclusion ordering while preserving the Euclidean Constraint. Indeed, suppose that $A \subseteq B$. Then $A \setminus B$ is empty, so, clearly, there are at least as many elements in $B \setminus A$ as there are in $A \setminus B$, hence $A \sqsubseteq_{\mathcal{M}} B$. But if A is a proper subset of B, then $B \setminus A$ is not empty, which means that there is strictly more elements in $B \setminus A$ than there are in $A \setminus B$, hence $B \not\sqsubseteq_{\mathcal{M}} A$. Again, this is rather intuitive. If A is a proper subset of B, then, when we remove their common part $A \cap B$, there is nothing left of A, while some elements of B are left over. Lastly, let me explain why I call **MIN** the Minimal Proposal. It is simply because of the following result, the proof of which can be found in the Appendix (Theorem A.3.1).

Theorem 3.1. *The relation* $\sqsubseteq_{\mathcal{M}}$ *is the smallest arithmetical size relation on* $\mathscr{P}(\mathbb{N})$.

Note that there are two parts to the content of this result. First, $\sqsubseteq_{\mathcal{M}}$ is a relation that satisfies **QO**, **IN**, **DEC** and **FIN**. Second, it is the *smallest* such relation, meaning that, whenever $\sqsubseteq$ is a relation satisfying those four axioms, $A \sqsubseteq_{\mathcal{M}} B$ implies $A \sqsubseteq B$. In a sense, this means that $\sqsubseteq_{\mathcal{M}}$ captures well the Galilean idea that we should exert caution when assigning size relationships between infinite sets. Unlike Galileo, we do not want to say that the relation of having "at most as many elements" never holds between any two infinite sets, because we think that this relation is a size relation. We also cannot pick the smallest size relation, i.e., the inclusion ordering, because it is too fine-grained to preserve finite arithmetic. But $\sqsubseteq_{\mathcal{M}}$ is our best option for this minimalist or "modest" approach. By Theorem 3.1, the relation defined by the Minimal Proposal is exactly what we need if we want a size relation that preserves finite arithmetic. Any further assignment of size relationship is superfluous, unless of course there are further requirements that one wants to impose on the "at most as many elements" relation.

I think that there is a lot of appeal to the minimal approach. In fact, as will become clear in the rest of this paper, I think **MIN** is a legitimate alternative to the Cantorian definition of size via **IP**, and that $\sqsubseteq_{\mathcal{M}}$ is the correct way to give a precise meaning to the "at most as many elements" relation that respects the Euclidean Constraint. In other words, I think $\sqsubseteq_{\mathcal{M}}$ is the contender to the cardinality relation $\sqsubseteq_{\mathcal{C}}$ that proponents of a Euclidean theory of size have been searching for. I have already made a positive argument for $\sqsubseteq_{\mathcal{M}}$, by drawing from the Galilean idea that we should be careful in the way we determine size relationships between infinite sets. Let me now address two possible objections to the idea that $\sqsubseteq_{\mathcal{M}}$ is a legitimate contender for a notion of relative size for sets of natural numbers. Dealing with the first one will be easy, while dealing with the second one will take much longer.

The first objection one could raise is that the Minimal Proposal cannot really compete with the Cantorian notion of cardinality, because it presupposes it or some key parts of it. The argument, as I understand it, would run like the following. The Cantorian notion of cardinality only appeals to set-theoretic notions, like that of an injective function between two sets. By contrast, **MIN** appeals to notions such as finiteness, and finite cardinality. Worse, **MIN** "secretly" appeals to the Injection Principle, because it is equivalent to the following:

MIN': For any $A, B \subseteq \mathbb{N}$, $A \sqsubseteq B$ if and only if $A \setminus B$ is finite and $A \setminus B \sqsubseteq_{\mathcal{C}} B \setminus A$.

To be a genuine alternative to Cantorian cardinalities, the relation $\sqsubseteq_{\mathcal{M}}$ would have to be defined without any appeal to finiteness or cardinalities. Or so runs the argument.

I am not quite convinced by the argument. I think one could make the case that notions like "finiteness" are a common resource in this debate, and not the exclusive property of the Cantorian. Similarly, the fact that **MIN** can be reformulated using the relation $\sqsubseteq_{\mathcal{C}}$ is perhaps interesting (for example, it can be useful in establishing that $\sqsubseteq_{\mathcal{C}}$ extends $\sqsubseteq_{\mathcal{M}}$), but is not particularly relevant to the issue of whether **MIN** defines a size relation that is independent from cardinality. Being entirely unable to express **MIN** in a way that does not appeal to cardinality would probably be more of a problem. Fortunately however, one can express **MIN** in a way that does not appeal to cardinalities, nor even to finiteness, and is moreover well motivated.

The starting point is a cornerstone of finite combinatorics called the Pigeonhole Principle. In its simplest and most popular form, it states the following: whenever you are trying to fit a set of pigeons into a set of pigeonholes, if you have more pigeons than holes, then you must fit at least two pigeons in the same hole. There is a way to understand it as a statement about injections: no function from a set A into a set B with fewer elements can be an injection. Taking the contrapositive yields the "if" direction of **IP** in the finite case: if there is an injection $f : A \to B$, then there are at most as many elements in A as there are in B. But there is also a way of understanding the Pigeonhole Principle in terms of a relationship between injections and surjections (i.e., functions whose range is their entire codomain). Suppose that, instead of assigning pigeonholes to pigeons, we were assigning pigeons to pigeonholes, for example, by putting a pigeon's name on each pigeonhole. Since no pigeon needs two pigeonholes, this amounts to defining an injection from the set of pigeonholes to the set of pigeons. If there are more pigeons than pigeonholes, however, not every pigeon will have its name on a pigeonhole. In other words, the function is not surjective. By contraposition, this means that whenever every injection from B to A is also surjective, there are at most as many elements in A as there are in B. Just like in the purely injective account of the Pigeonhole Principle, we can therefore extract a sufficient condition for A to have at most as many elements as B from this more complex account. Note that, in the finite case again, this is also necessary: whenever there is an injection of B into A that is not surjective, then there are strictly more elements in A than elements in B.

The idea is now to take this version of the Pigeonhole Principle and extend it to infinite sets of natural numbers. We have to be a bit careful here. If A is a Dedekind-infinite set, then there is a function $A \to A$ that is injective but not surjective (this is exactly Dedekind's definition), but we certainly would not want to say that there are more elements in A than elements in A. A natural way to exclude this case is to consider only functions between sets that are disjoints. This motivates the following definition, which I call the Generalized Pigeonhole Principle:

GPP: For any $A, B \subseteq \mathbb{N}$, $A \sqsubseteq B$ if and only if any injection $f : B \setminus A \to A \setminus B$ is also surjective.

Again, the driving idea is similar to the one in the definition of $\sqsubseteq_{\mathcal{M}}$. Given two sets A and B, discard their common part $A \cap B$ and compare the two remaining sets $A \setminus B$ and $B \setminus A$. Then A has "at most as many elements" as B if and only if $A \setminus B$ has "at most many elements" as $B \setminus A$, which we can now understand via our alternative understanding of the Pigeonhole Principle: every injection from $B \setminus A$ into $A \setminus B$ is also surjective. Note that **GPP** does not appeal in any way to the Injection Principle, nor to any notion of finiteness. Nonetheless, we have the following, whose proof can be found in the Appendix (Theorem A.1).

Proposition 3.2. ***GPP*** *defines the relation* $\sqsubseteq_{\mathcal{M}}$. *In other words, for any sets* $A, B \subseteq \mathbb{N}$, $A \sqsubseteq_{\mathcal{M}} B$ *if and only if any injection* $f : B \setminus A \to A \setminus B$ *is also surjective.*

The Minimal Proposal can therefore be characterized exclusively in terms of set-theoretic notions such as injections and surjections. This definition, like the Injection Principle, can be motivated by considerations based on a natural principle in finite combinatorics, the Pigeonhole Principle. Depending on how one presents the principle, distinct ways of generalizing it to the infinite arise. I do not wish to claim that the account in terms of injections and surjections is "as natural" as the one in terms of injections alone. Nor do I think that I need to. All that matters is that the Minimal Proposal can be put on equal footing with the Injection Principle in terms of the kind of concepts that they rely on.

The second criticism that one could raise against the Minimal Proposal is that, just like Galileo's stance, it is too limited. There are instances in which we would like to ascribe size relationships between sets, yet the Minimal Proposal remains silent. As a matter of fact, it is very easy to see that many sets are incomparable with respect to their sizes under the Minimal Proposal. Whenever A and B are sets such that both $A \setminus B$ and $B \setminus A$ are infinite, we have neither $A \sqsubseteq_{\mathcal{M}} B$ nor $B \sqsubseteq_{\mathcal{M}} A$. For example, if A is the set of even numbers and B the set of odd numbers, then we cannot say whether there are more evens than odds, more odds than evens, or the same number of evens and odds. To put the point differently, $\sqsubseteq_{\mathcal{M}}$ fails to satisfy the following *Linearity* constraint:

LIN For any $A, B \subseteq \mathbb{N}$, either $A \sqsubseteq B$ or $B \sqsubseteq A$.

Ultimately, I think the failure of Linearity is a desirable feature of the theory, even though this is certainly a marked difference from the case of finite sets. In the realm of the finite, sets are linearly ordered with respect to their sizes. For any two finite sets A and B, either A has at most as many elements as B, or B has at most as many elements of A. Linearity is a feature of finite sets that is transferred to the whole powerset of the set of natural numbers by the Cantorian notion of cardinality, although in a rather trivial fashion: any set injects into any infinite set, so all infinite sets have the same size.[3] So how concerning is it that, under the Minimal Proposal, sets of natural numbers are not linearly ordered with respect to their sizes? In the next few sections, I will argue that this is not concerning at all and that, in fact, Linearity is *not* a desirable feature of a theory of size that wants to preserve the Euclidean Constraint. But doing so requires engaging with existing attempts to define a linear size relation that preserves the Euclidean Constraint (such as numerosity theory), and grappling with what I think is the (mistaken) intuitive motivation behind them, which I call the Density Intuition.

4 Numerosities and the Density Intuition

In this section, I present and discuss what I call the Density Intuition regarding sizes of sets of natural numbers. As we shall see, I think the idea has a rather strong intuitive pull, but that it is ultimately mistaken, for reasons that will become clear in the next section. Crucially, it has not been sufficiently distinguished from the Euclidean Constraint in the literature, which has led to some confusion regarding what Euclidean theories of size can be. I will start by a motivating example, before showing how standard numerosity theory implements the Density Intuition in a sophisticated way.

Suppose that we modify slightly the setup of Galileo's paradox as follows. Let $2^{\mathbb{N}}$ be the set $\{2^n \mid n \in \mathbb{N}\}$ of all powers of 2. That set is both infinite and coinfinite, so it is one of those "problematic" sets for which our pretheoretic Euclidean intuitions regarding size assignments are murky. What if we tried to compare $\mathbb{N}^2$ with $2^{\mathbb{N}}$ instead of $\mathbb{N}$? Are there more square numbers than powers of 2, or the other way around, or are their sizes perhaps incomparable?

For the Cantorian, the question is easily answered. Since both sets are countably infinite, they have the same size. In other words, $2^{\mathbb{N}} \equiv_{\mathcal{C}} \mathbb{N}^2$. In fact, there is a particularly nice bijection that one can define from $2^{\mathbb{N}}$ to $\mathbb{N}^2$: simply map 2^n to n^2 for any $n \in \mathbb{N}$! By contrast, the Euclidean Constraint stays silent regarding the relationship between $2^{\mathbb{N}}$ and $\mathbb{N}^2$, since neither set is a subset of the other. In fact, since both $\mathbb{N}^2 \setminus 2^{\mathbb{N}}$ and $2^{\mathbb{N}} \setminus \mathbb{N}^2$ are infinite, the sizes of the two sets are also incomparable according to the minimal relation $\sqsubseteq_{\mathcal{M}}$: $\mathbb{N}^2 \not\sqsubseteq_{\mathcal{M}} 2^{\mathbb{N}}$, and $\mathbb{N}^2 \not\sqsubseteq_{\mathcal{M}} 2^{\mathbb{N}}$. As I will argue below, I think this is a welcome feature of the theory. But I also think that there

[3] It is worth mentioning that the linearity of cardinals, by contrast with the countable case, is far from trivial, and is in fact equivalent of Zermelo-Fraenkel set theory to the Axiom of Choice.

is some intuitive pull towards the idea that there *should be* some relationship between the size of $2^{\mathbb{N}}$ and that of $\mathbb{N}^2$. Let me now explain what the idea is.

Suppose that we start enumerating the elements of $\mathbb{N}^2$ and those of $2^{\mathbb{N}}$:

$$\mathbb{N}^2 : 0, 1, 4, 9, 16, 25, 36, 49, \ldots$$
$$2^{\mathbb{N}} : 1, 2, 4, 8, 16, 32, 64, 128, \ldots$$

Then we might quickly notice a pattern emerging between the two enumerations. The values in the sequence of powers of 2 "grow much faster" than those in the sequence of squares: already, the eighth power of 2, 128, is more than twice larger than the eighth square number, 49. Moreover, this pattern is easily seen to be more pronounced as we move further along the sequence of natural numbers: the 101^{st} square number, 10000, is vastly smaller than the 101^{st} power of 2, 2^{100}, whose decimal notation contains 31 digits. From this fact, one may get the sense that powers of 2 are more *scattered* among the natural numbers than squares. Even though both sets are infinite, we will typically run into square numbers *faster* than we would run into powers of 2. What I call the *Density Intuition* is the idea that this very fact tells us something about the relative sizes of $\mathbb{N}^2$ and $2^{\mathbb{N}}$: there are *more* squares than powers of 2, because the distribution of squares along the sequence of natural numbers is *more dense* than the distribution of powers of 2.

In short, the following is exactly what I take to be the core principle of the Density Intuition: the size of a set A of natural numbers is determined by the distribution of its elements along the sequence of natural numbers with their usual ordering. The more scattered a set is, the smaller its size. Admittedly, the phrasing that I have used so far is rather vague, and I think that, ultimately, a precise formulation of what follows from the Density Intuition is an extremely difficult task. But there are several examples that one can give to sharpen the pretheoretic intuition. The example of the set of squares versus the set of powers of 2 is such an example: because the elements of the latter are more scattered along the sequence of natural numbers, the former set has larger size. Arguably, one could also think that the distribution of the elements of a set of natural numbers could sometimes even determine precise *ratios*. For example, the set $2\mathbb{N} = \{2n \mid n \in \mathbb{N}\}$ of even numbers is precisely *half* the size of $\mathbb{N}$, because every pair of two consecutive natural numbers contains exactly one even number. Similarly, one could argue that, for any $n > 0$, the size of the set $n\mathbb{N}$ of all multiples of n is precisely $\frac{1}{n}$ of the size of $\mathbb{N}$, because any sequence of n consecutive natural numbers contains exactly one multiple of n.

Although I have not given it a precise mathematical formulation, I think there is nonetheless something quite robust about the Density Intuition. First of all, it arguably played a role in the development of many non-Cantorian intuitions about the infinite. For example, the Islamic philosopher Ibn Qurra, one of the very first to defend the idea that the infinite comes in many different sizes, argued in the ninth century that the set of natural numbers was twice bigger than the set of even numbers, three times bigger than the set of multiples of three, and so on [20].

Second, there exists already a mathematically precise (and fruitful) way to make sense of the Density Intuition, via the notion of *asymptotic density*. For any set of natural number A, its asymptotic density is defined as the real number

$$\lim_{n \to \infty} \frac{|A \cap \{0, ..., n\}|}{|\{0, ..., n\}|},$$

whenever such a limit is defined. Asymptotic density is a central notion in modern number theory, and it plays a key role in some of its most celebrated results, such as Szemerédi's theorem or the prime number theorem. For sets of the form $n\mathbb{N}$, their asymptotic density can easily be computed to be $\frac{1}{n}$. But every finite set, no matter how large, has asymptotic density 0. In fact, many *infinite* sets (including prime numbers, squares, or powers of 2) also have asymptotic density 0, because their distribution along the sequence of natural numbers is not bounded below by any positive rational number. Perhaps even more troublesome is the fact that not every set of natural numbers has an asymptotic density, a direct consequence of the fact that not every sequence of rationals converges to a real value.

Asymptotic Density is therefore far from a perfect tool for fleshing out a theory of size for sets of natural numbers from the Density Intuition. Because it assigns value 0 to any finite set and to many infinite sets, it does not extend finite arithmetic nor does it satisfy the Euclidean Constraint. But one might think that the issue here has nothing to do with the Density Intuition itself, and rather comes from the fact that one tries to assign a *real* value to sets of natural numbers. In short, the real line is too *coarse* a structure to make sense of the subtle differences existing between finite sets, or between infinite sets whose distribution along the sequence of natural numbers becomes unboundedly sparse. This is, in a way, the key idea behind *numerosity theory*, arguably the most sophisticated mathematical endeavor to assign sizes to set of natural numbers in a way that is coherent with the Euclidean Constraint. The original presentation of Benci and di Nasso's ideas [3] is quite technical, and uses fairly advanced set-theoretic and model-theoretic tools. Instead of following their presentation, I will therefore introduce the main features of the theory thanks to a useful thought experiment.

Suppose that the task of determining size relationships between sets of natural numbers is given to an ideal agent, called a Surveyor, with the following characteristics. First, any Surveyor has a complete and effortless mastery of finite arithmetic. She is able to count the number of elements in any finite set of natural numbers and to determine ratios between the sizes of any two finite sets. Second, any Surveyor has tremendous determination and stamina: she is capable of carrying out such processes of counting and comparing finite sets as long as she wishes, including an infinite number of times. She also has perfect memory, meaning that she remembers every operation she has ever performed. However, she has one major limitation: she cannot determine the size of a set, or count its number of elements, when that set is infinite. She is, so to speak, "finite-sighted": her capacity to perform any act of counting or size comparing is limited by the fact that she can only consider finitely many elements at once. How would such a Surveyor proceed to determine size relationships between two sets of natural numbers A and B, regardless of whether they are finite or infinite? Intuitively, all she can do is approach the problem by comparing finite subsets of A and B, hoping that this gives her a good approximation of the true size relationship between the two sets.

Within this conceptual framework, Benci and di Nasso's idea can now be presented as follows. Given a (possibly infinite) set A, a Surveyor should proceed *sequentially*. For any natural number n, she should look at the first n natural numbers $\{0, ..., n-1\}$, and count how many elements from A appear in that set. As she progresses along the sequence of natural numbers, she looks at longer and longer initial segments of $\mathbb{N}$, and this allows her to approximate the size of A with ever increasing accuracy. Clearly, if A is a finite set, then our sequential Surveyor's computation of the size of A will become constant from some point onwards: as soon as she starts looking at initial segments of the natural numbers that contain all the elements of A, her approximation of the size of A will not increase anymore, and will be perfectly accurate. If A is infinite, on the other hand, then her approximation of A will never stabilize, and keep increasing as she considers larger and larger initial segments of A. In this case, there is no size that the Surveyor can safely assign to the set A. But this does not mean that all infinite sets will eventually "converge" towards the same behavior. Because she has perfect memory, our Sequential Surveyor is always aware of the way in which her approximation of the size of A changes. If A is very big, say, a cofinite set, then the Surveyor's approximation of the size of A changes almost all the time, except for finitely many values. By contrast, if A is infinite and coinfinite, then the Surveyor's approximation of the size A will be a sequence that increases infinitely many times, but also repeats itself infinitely many times.

What about determining relative size relationships between two sets of a natural numbers A and B? Clearly, a sequential Surveyor can apply the same strategy. She can go through every initial segment $\{0, ..., n-1\}$ of the natural numbers and check every time whether there are more elements of A in that set than elements of B, or vice versa, or whether there are exactly as many elements from either set. Once again, if either A or B is finite, the answer she will get is constant from some point onwards. If A and B are both finite, she will be able to determine the true size relationship between these two sets once she considers an initial segment of the natural numbers large enough to include their union $A \cup B$. If A is finite and B infinite, then such a Surveyor will also notice that, from some point onwards, the answer she gets is always the same: once she reaches an initial segment of the natural numbers that contains A as well as strictly more elements from B, she will notice that there are always more elements from B than elements from A. What if A and B are both infinite? Several situations may arise. For example, if A is the set $2\mathbb{N}$ of even numbers and B is the set of odd numbers, the answers that she will get oscillates between "A has more elements than B" (whenever the initial segment she considers has an odd number of elements) and "A has exactly as many elements as B" (whenever the initial segment has an even number of elements). Crucially, however, this is not the only possibility. Whenever A is a proper subset of B, the answer will be constant from some point onwards. Once a sequential Surveyor has reached an initial segment that contains a natural number that is in B but not in A, she will notice that, from that point onwards, there will always be strictly more elements from B than elements from A in any longer initial segment. In that sense, a sequential Surveyor is very sympathetic to part-whole considerations. But a similar situation arises in the case of $\mathbb{N}^2$ and $2^{\mathbb{N}}$: from some point onwards (say, $n = 40$), she will notice that there are consistently more square numbers than powers of 2 in all the initial segments of the natural numbers that she considers. This is precisely because, as we noted above, powers of 2 are more scattered along the sequence of natural numbers than squares.

Based on how I have described sequential Surveyors so far, we are in the following situation. Given a pair A, B of sets of natural numbers, a Surveyor can associate to any natural number n exactly one of the following three possibilities:

- $A \prec B$ (there are strictly fewer elements in A than elements in B);
- $A \succ B$ (there are strictly more elements in A than elements in B);
- $A \sim B$ (there are exactly as many elements in A as elements in B).

In fact, any natural number n determines a full size relation for sets of natural numbers, based on all the answers given by a Surveyor to questions of the form "What is the size relationship between sets A and B?" once she focuses on the initial segment of the natural numbers up to n. Of course, such a theory depends on the natural number n, and is only of limited interest. It is, after all, only an approximation of the "correct" theory, up to the first n natural numbers. What is left to do for a Surveyor is to build her theory of size by extracting a definitive answer from the multitude of all these partial ones. In order to do this, she must decide when a particular answer appears "often enough" in the sequence of natural numbers to be the correct one. This is where distinct sequential Surveyors may differ, and therefore yield different size relations. In

Benci and di Nasso's numerosity theory, a Surveyor must determine what "often enough" means in a way that guarantees the following three properties:

F1 Cofiniteness: Any answer that always appears from some point onwards (i.e., for any natural number m greater than some natural number n) appears "often enough";

F2 Coherence: Two answers appear "often enough" if and only if their conjunction also appears "often enough";

F3 Decisiveness: For any question regarding the size relationship between two sets A and B, exactly one answer appears "often enough".

Whenever a sequential Surveyor S fixes a notion of what counts as an answer appearing "often enough" that satisfies those three properties, one can define a relation $\sqsubseteq_{\mathcal{D}}$ on $\mathscr{P}(\mathbb{N})$ by letting $A \sqsubseteq_{\mathcal{D}} B$ if and only if the answer $A \prec B$ or $A \sim B$ appears "often enough" according to S. The resulting relation has the following properties:

- $\sqsubseteq_{\mathcal{D}}$ is a size relation: it satisfies **QO**, **IN** and **DEC**;
- $\sqsubseteq_{\mathcal{D}}$ is Euclidean: it satisfies **EC**;
- $\sqsubseteq_{\mathcal{D}}$ is arithmetical: it satisfies **FIN**;
- $\sqsubseteq_{\mathcal{D}}$ is linear: it satisfies **LIN**.

The last property, Linearity, is a crucial difference between $\sqsubseteq_{\mathcal{D}}$ and $\sqsubseteq_{\mathcal{M}}$. Unlike the relation $\sqsubseteq_{\mathcal{M}}$, $\sqsubseteq_{\mathcal{D}}$ is a linear order, so any two sets, including two infinite and coinfinite sets, are comparable. In other words, a sequential Surveyor always gives a definitive answer to the question "are there more elements in A than in B, or fewer, or do they have the same number of elements"? This is, of course, because we required the Surveyor to be "decisive" when determining what counts as "often enough". Arguably, any sequential Surveyor also implements fairly well the Density Intuition. In the case of $\mathbb{N}^2$ and $2^{\mathbb{N}}$, she gives the answer that one would expect: $2^{\mathbb{N}} \sqsubseteq_{\mathcal{D}} \mathbb{N}^2$, since $2^{\mathbb{N}} \prec \mathbb{N}^2$ is the answer that the sequential Surveyor obtains cofinitely many times when she considers initial segments of the natural numbers. What about other aspects of the Density Intuition, such as the idea that there are exactly as many even numbers as odd numbers? In general, this will depend on what the Surveyor counts as "often enough". By imposing more conditions, one can choose a sequential Surveyor that implements more of the Density Intuition. For example, if one requires her to consider that, for any $n > 0$, the set of multiples on n counts as "often enough", then one makes sure that the sequential Surveyor thinks that congruence classes modulo some natural number m all have the same size. In other words, she will think that $A \equiv_{\mathcal{D}} B$ whenever A and B are two sets of the form $\{mn + i \mid n \in \mathbb{N}\}$ for some natural number m and some $i < m$.[4]

As I have said above, the Density Intuition is open-ended. One could always add more requirements regarding size relationships to those already imposed. Whenever the distribution of the elements of a set A along the sequence of natural numbers is consistently more scattered than that of the elements in a set B from some point onwards, any sequential Surveyor will conclude that there are strictly fewer elements in A than in B. For sets with a periodic distribution such as congruence classes modulo some natural number n, additional constraints can be put on what the Surveyor thinks count as "often enough". But these arguably do not exhaust the

[4]This is simply because, for any $m > 0$ and any natural number n, whenever n is a multiple of m, the set $\{0, ..., n-1\}$ is partitioned by congruence modulo m into exactly m classes, all of which have the same size, exactly for the reason given by Ibn Qurra in the quote above.

Density Intuition itself. However, there are very strong reasons to believe that the Density Intuition on its own cannot specify a unique sequential Surveyor satisfying Benci and di Nasso's constraints. This is because the existence of any such sequential Surveyor is equivalent (over Zermelo-Fraenkel set theory) to the existence of a particular kind of set-theoretic object called a *free ultrafilter* on $\mathbb{N}$ [3].[5] Such free ultrafilters are well-known objects in mathematical logic, and have wide-ranging applications in model theory, infinite combinatorics, set theory, and much more [13]. But they are also highly complex objects, whose existence can only be guaranteed by assuming a strong fragment of the Axiom of Choice. As such, free ultrafilters cannot be described (or, as set theorists would put it, they are not *definable*). In our context, this means that there is no way for us to explicitly specify how a sequential Surveyor satisfying Benci and di Nasso's constraints would determine what counts as an answer appearing "often enough". The existence of such sequential Surveyors can only be guaranteed by non-constructive means. As such, the way in which they operate remains shrouded in mystery: there is no way for us to have a fully explicit description of it. Assuming they exist, we can select among them those that satisfy additional criteria that correspond to the Density Intuition. But we will never be able to select precisely one of them in that way. This is the price we have to pay in order to get the linearity of the relation $\sqsubseteq_{\mathcal{D}}$. In a sense, this shows that the Density Intuition itself does not deliver the linearity of the structure of sizes of sets of natural numbers. If we want any two sets to have comparable sizes, we must choose a sequential Surveyor whose notion of what counts as "often enough" is compatible with the Density Intuition, but also goes beyond it, and trust her judgments. This obviously raises some serious issues regarding how well-motivated the predictions of numerosity theory could be. Once we go beyond the Density Intuition, as we know we must do if we want our size relation to be linear, on what grounds could such size relationship judgements be motivated? I think this problem is serious enough, and a significant reason why the claim that numerosity theory is a viable alternative to the Cantorian notion of size has been met with skepticism. But I also think that focusing on this issue obfuscates a bigger issue having to do with the Density Intuition itself.[6] As I will argue in the next section, the problem is not that the Density Intuition is too weak to deliver a linear size relation on its own. Rather, the issue is that the Density Intuition is misleading, because it leads us to making assignments of size relationship that cannot possibly be well motivated.

5 The Invariance Problem

In this section, I discuss what I take to be the main problem of numerosity theory, which I call the Invariance problem. In short, the point is that there is one additional requirement that one would arguably want to impose on the size relation $\sqsubseteq$, having to do with the notion of a permutation of $\mathbb{N}$. A *permutation* of a set S is simply a one-to-one correspondence $\pi : S \to S$. One could think of a permutation of $\mathbb{N}$ as a mere relabeling of its elements. Permutations are important for sets of natural numbers, because any permutation $\pi : \mathbb{N} \to \mathbb{N}$ induces another one-to-one correspondence $\pi_* : \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$, given by $\pi_*(A) = \{\pi(a) \mid a \in A\}$ for every $A \subseteq \mathbb{N}$. In other words, given a permutation $\pi : \mathbb{N} \to \mathbb{N}$, we can associate to any set $A \subseteq \mathbb{N}$ the set of all images of its elements under π.

[5]Benci and di Nasso's original framework uses free ultrafilters that actually have an extra property called the Ramsey property. But I am leaving out this extra feature for the sake of simplicity.

[6]I should also mention here a recent proposal by Trlifajova [29], who defended a partial theory of size that still obeys the Density Intuition. Unlike numerosity theory, Trlifajova's proposal is constructive and can be explicitly described. In a nutshell, I think trading linearity for definability is a step in the right direction. But Trlifajova's approach still tries to build on the Density Intuition and, as such, it does not evade the criticism that I will raise against numerosity theory in the next section.

There is a widespread belief in the literature that sizes should be preserved in some way by permutations [24, 25, 32]. Parker [24] in particular has raised a forceful challenge to Euclidean theories of size by presenting some arguments based on permutations, to which I will return below. For now, let me just mention that giving a precise formulation to the idea in terms of an additional axiom to impose on the relation $\sqsubseteq$ is a delicate affair. I think the correct way to do so is via the following constraint, which I call Relative Invariance (**RI**):

RI: For any $A, B \subseteq \mathbb{N}$ and any permutation $\pi : \mathbb{N} \to \mathbb{N}$, if $A \sqsubseteq B$, then $\pi_*(A) \sqsubseteq \pi_*(B)$.

The conceptual motivation behind Relative Invariance is simply that permutations cannot change size relations between sets of natural numbers. In other words, if there are "at most as many elements" in a set A as there are in a set B, then one cannot change that fact by merely looking at the images of A and B under a permutation. Note that, because every permutation has an inverse, the conditional in **RI** is really a biconditional: permutations cannot "destroy" size relations between sets, but they also cannot "create" any new ones.

Relative Invariance is easily seen to be compatible with the Euclidean Constraint. Indeed, the inclusion ordering is an example of a Euclidean size relation that is also relatively invariant, since whenever $\pi : X \to X$ is a permutation of the elements of a set X and $A \subseteq B \subseteq X$, we also have that $\pi_*(A) \subseteq \pi_*(B)$. Nonetheless, it is easy to see that Benci and di Nasso's numerosities are not relatively invariant. One could come up with many counterexamples having to do with the fact that numerosities are linearly ordered,[7] but I think the following is a more telling example. Recall that, according to any sequential Surveyor, there are "strictly more" squares than powers of 2, i.e., $2^{\mathbb{N}} \sqsubseteq_{\mathcal{D}} \mathbb{N}^2$ and $\mathbb{N}^2 \not\sqsubseteq_{\mathcal{D}} 2^{\mathbb{N}}$. At the same time, both $2^{\mathbb{N}} \setminus \mathbb{N}^2$ and $\mathbb{N}^2 \setminus 2^{\mathbb{N}}$ are infinite sets. So we may fix a bijection $f : 2^{\mathbb{N}} \setminus \mathbb{N}^2 \to \mathbb{N}^2 \setminus 2^{\mathbb{N}}$ and use it to define a permutation π that swaps n with its image under f whenever $n \in 2^{\mathbb{N}} \setminus \mathbb{N}^2$, and leaves every other natural number untouched. It is easy to see that, given such a permutation π, $\pi_*(2^{\mathbb{N}}) = \mathbb{N}^2$ and $\pi_*(\mathbb{N}^2) = 2^{\mathbb{N}}$, which means that $\pi_*(2^{\mathbb{N}}) \not\sqsubseteq_{\mathcal{D}} \pi_*(\mathbb{N}^2)$, contradicting Relative Invariance. Now, the idea that there were strictly more squares than natural numbers traces all the way back to the Density Intuition and is not a mere quirk of its implementation via numerosity theory. So the situation is really that Relative Invariance is incompatible with the Density Intuition and, *a fortiori*, with any theory that tries to implement it, including numerosity theory or Trlifajova's recent proposal [29].

There are, however, at least two very strong reasons to take **RI** to be an axiom that any suitable size relation should satisfy. The first has to do with the applicability of such a size relation beyond sets of natural numbers, while the second has to do with the very concept of a size relation for sets. Regarding the first issue, the point is that, although everything I have said so far was about relative size assignments between sets of natural numbers, we would presumably want our approach to be applicable to a wider context, say all countable sets. After all, finite arithmetic is useful because it applies to any collection of finite sets, not just to sets of natural numbers. Given a countable set S, there is a natural way to use a size relation $\sqsubseteq$ on sets of natural numbers to determine size relationships between subsets of S: label all the elements of S with the natural numbers and determine size relationships between to sets $A, B \subseteq S$ by looking at the two sets of natural numbers that correspond to the labels of the elements of A and the labels of the elements of B respectively. Formally, this amounts to defining a relation $\sqsubseteq^*$ on $\mathscr{P}(S)$ by fixing a bijection $f : \mathbb{N} \to S$, and then letting $A \sqsubseteq^* B$ iff $f^{-1}[A] \sqsubseteq f^{-1}[B]$, where, for any $C \subseteq S$, $f^{-1}[C] = \{n \in \mathbb{N} \mid f(n) \in C\}$. But which bijection f should we choose? The point is that this matters if and only if the relation $\sqsubseteq$ is not relatively invariant.[8] In other words, if $\sqsubseteq$ is relatively

[7] Parker [24] gives such an example.

[8] In essence, this is because composing a labeling $f_1 : \mathbb{N} \to S$ with the inverse of another labeling $f_2 : \mathbb{N} \to S$ induces a permutation of $\mathbb{N}$.

invariant, we can transfer this size relation to a size relation on the powerset of any countable set S in a unique way, regardless of the way in which we label the elements of S with natural numbers. If the relation $\sqsubseteq$ is not relatively invariant, however, different labelings of the elements of S will induce different size relations on $\mathscr{P}(S)$, with no general way to determine a particular one as the "correct one". A non-relatively invariant size relation on $\mathscr{P}(\mathbb{N})$ has therefore limited applicability, because trying to apply it to the subsets of a countable set S involves an irreducible element of arbitrariness, namely the labeling of the elements of S by natural numbers.

This first issue, I think, points towards a more conceptual one. Recall that our starting point is the idea that the relation of having "at most as many elements" as another set is a *coarsening* of the inclusion ordering on the powerset of the natural numbers. Whereas the inclusion ordering distinguishes between sets with respect to their membership, size relations are meant to abstract away from the exact list of elements of a set of natural numbers and consider only "how many" numbers appear in the set. The inclusion ordering however, as I have noted above, is relatively invariant. If the relation of having "at most as many elements" were not relatively invariant, this would yield a paradoxical situation in which permutations somehow preserve the *finer* relation (the inclusion ordering), but do not preserve the *coarser* one. In other words, this would mean that "at most as many elements" relation tracks a feature of sets that is not captured by the subset relation. But sets are precisely *unstructured* entities, completely determined by their elements. I think that one could therefore legitimately argue that a relation $\sqsubseteq$ on sets of natural numbers that is not relatively invariant might have some interest as a size relation for *structured sets* of some kind, but not as a size relation for sets understood as completely unstructured entities. The point can be made perhaps even more forcefully by considering number structures rather than size relations. Recall, that, given a size relation $\sqsubseteq$, we can define a corresponding structure of sizes $(\mathcal{N}, \preccurlyeq)$ by considering the set of equivalence classes $\{A^{\cdot} \mid A \subseteq \mathbb{N}\}$, where $A^{\cdot} = \{B \subseteq \mathscr{N} \mid A \equiv B\}$, ordered by letting $A^{\cdot} \preccurlyeq B^{\cdot}$ if and only if $A \sqsubseteq B$. We may now wonder how permutations of the natural numbers act on such a structure $(\mathcal{N}, \preccurlyeq)$. Here, we find that there is a stark contrast between size relations that satisfy **RI** and those who don't. Indeed, if $\sqsubseteq$ is relatively invariant, then every permutation π induces an automorphism π^* of $(\mathcal{N}, \preccurlyeq)$ (i.e., an isomorphism from $(\mathcal{N}, \preccurlyeq)$ into itself), given by letting $\pi^*(A^{\cdot}) = (\pi_*(A))^{\cdot}$ for any $A \subseteq \mathbb{N}$. Intuitively, if one takes $A^{\cdot}$ to stand for the *size* of A, this means that any permutation defines a structure-preserving map on the structure of sizes by mapping the size of a set A to the size of its permutation. By contrast, if $\sqsubseteq$ is not relatively invariant, then there is no guarantee that the map π_* is an automorphism. In fact, it might not be a function, since the failure of Relative Invariance allows for the case of two equivalent sets A and B such that $\pi_*(A) \not\equiv \pi_*(B)$. In other words, sets that have the same size might suddenly have different sizes once we define a permutation of the natural numbers. Whatever property of natural numbers is being tracked by such a relation, it is hard to view such a high sensitivity to permutations as a characteristic of size.

Let me conclude this section by briefly discussing Parker's criticism of Euclidean theories. Roughly, Parker argues that Euclidean theory of size must either be arbitrary, because they fail to satisfy certain criteria of invariance under permutations, or too weak to be fruitful. In short, I agree with some of Parker's points, but I think his arguments overreach in at least two ways. First, Parker does not make, like I do, a distinction between the Euclidean Constraint and the Density Intuition. This leads him to regularly attribute to the part-whole intuitions certain claims that, I think, could only be motivated by considerations inspired from the Density Intuition (such as the idea that there are as many positive integers as there are negative integers). As I will argue in the next section, there is a well-motivated Euclidean size relation that is also relatively invariant and that it is far from being "too weak". Second, Parker argues that any theory size for infinite sets should satisfy a stronger version of invariance, which I will call Absolute Invariance **(AI)**:

AI: For any $A \subseteq \mathbb{N}$ and any permutation $\pi : \mathbb{N} \to \mathbb{N}$, $A \sqsubseteq \pi_*(A)$.

Intuitively, Absolute Invariance states that permutations of $\mathbb{N}$ cannot "shrink" the size of a set of natural numbers. There might be some intuitive pull to the idea, but one can quickly see that it is incompatible with the Euclidean Constraint. Indeed, for any infinite and coinfinite set A, one can define a permutation π such that $\pi_*(A)$ is a proper subset of A.[9] Parker takes this to be evidence that the Euclidean Constraint should be disregarded, because the "at most as many elements" relation should be absolutely invariant. A full discussion of Parker's arguments is beyond the scope of this paper, since I am more interested in finding the best possible Euclidean theory of size rather than in defending the mere possibility of such a theory. Therefore, I'll limit myself to the following two comments. First, one could try to argue for **AI** by coming up with versions of the two arguments I gave for **RI**. For example, if one uses a size relation $\sqsubseteq$ that is relatively invariant but not absolutely invariant to determine size relations for subsets of a countable set S, one could have a case of two distinct labelings $f, g : \mathbb{N} \to S$ such that $f^{-1}[A] \not\equiv g^{-1}[A]$ for some $A \subseteq S$. In other words, unsurprisingly, there is no arbitrariness in determining *relative* size relationships between subsets of S, but there could be some arbitrariness in determining the *absolute* size of subsets of S if $\sqsubseteq$ is not absolutely invariant. Regarding my second argument for **RI**, one could also argue that permutations should not only preserve the structure of sizes, but, in fact, should not affect sizes at all! This amounts to saying that, given a permutation π, the induced automorphism π^* of the structure $(\mathcal{N}, \preccurlyeq)$ should actually be the identity. I think those arguments raise interesting questions regarding how exactly one should think about "sizes" according to a Euclidean size relation, but that none of them amounts to the devastating objection Parker is after.

Second, I think the following result, whose proof can be found in the Appendix A (Lemma A.4), sheds an interesting light on the relationship between relative and Absolute Invariance.

Lemma 5.1. *An arithmetical size relation $\sqsubseteq$ is absolutely invariant if and only if it is relatively invariant and linear.*

In short, this means that the intuition behind Absolute Invariance can be analysed as a combination of Relative Invariance and Linearity. If we understand things that way, then requiring Absolute Invariance instead of mere Relative Invariance appears like a way to sneak **LIN** back into the picture. As we will now see, abandoning Linearity, however, is the key to reconciling invariance with the Euclidean Constraint.

6 The Generic Approach

As I have argued in the previous section, there is something deeply wrong with the Density Intuition, because it forces us to make certain determinations of size relationships that are not relatively invariant under permutations of the set of natural numbers. In hindsight, this should be far from surprising. After all, the key idea behind the Density Intuition is that size relationships between sets of natural numbers are determined by the distributions of their respective elements along the sequence of natural numbers. But any permutation of the natural numbers will somewhat disturb the sequence of natural numbers. A permutation that swaps only finitely many natural numbers may not disturb it enough so as to change what the Density

[9] Here is a simple way to do this. Given A, define first a bijection f between $\mathbb{N}$ and the integers $\mathbb{Z}$ as follows: map all natural numbers in A onto the non-negative integers, and all natural numbers not in A onto the negative integers. Then define the permutation π by $\pi(n) = f^{-1}(f(n) + 1)$, i.e., by shifting the image of n to the right and taking the preimage of the result. It is then easy to check that $\pi_*(A) = A \setminus f^{-1}(0)$, so $\pi_*(A)$ is a proper subset of A.

Intuition takes size relationships between sets of natural numbers to be, but a more radical permutation surely will. If one wants to define a size relation that satisfies **RI**, one must therefore abandon the Density Intuition, and with it the idea that size relationships between infinite sets can be approximated by looking at initial segments of the natural numbers. Let me now present a way to do so by going back to our thought experiment of the Surveyor.

Recall that a sequential Surveyor's way of determining whether there were "at most as many" elements in a set A as in a set B is to obtain partial approximations of the correct answer by focusing on finite sets of the form $\{0, ..., n-1\}$ for some natural number n. Ultimately, the reason why a sequential Surveyor implements the Density Intuition is because she focuses solely on initial segments of the natural numbers. But what if, instead, we allowed her to consider *any* finite set of natural numbers for her approximations of the correct size relationship between two infinite sets A and B? In other words, for any sets of natural number A and B and any finite set C, we now let our Surveyor determine whether there are "at most as many elements" in A as there are in B from the point of view of the set C by computing whether $|A \cap C| \leq |B \cap C|$. The idea here is that, in her assessment of size relationships, a sequential Surveyor is *biased* by the fact that she only considers initial segments of the natural numbers for her partial measurements. By contrast, our new Surveyor thinks that such measurements should be performed whenever possible, i.e., relative to any arbitrary finite set of natural numbers.

The picture is now the following. Given two sets A and B, our new Surveyor determines a function that assigns to any finite set of natural numbers an answer to the question "does A contain at most as many elements as B?". Here again, whenever one of A or B is finite, there is a sense in which the answer will be constant "from some point onwards". For example, if both A and B are finite, then as soon as a Surveyor considers a finite set C that contains their union $A \cup B$, she will always obtain the same answer. The same holds whenever A is a proper subset of B, regardless of whether or not either set is finite. Indeed, for any finite set C, the Surveyor will always know that there are at most as many elements in A as there are in B relative to C and, as soon as C contains at least one element of the set $B \setminus A$, she will also know that B contains strictly more elements than A. What about the example used before to motivate the Density Intuition, namely squares and powers of 2? Recall that, from some point onwards, a sequential Surveyor always obtained that there were strictly more squares than powers of 2. Predictably, the same does not occur for our new version of the Surveyor. Indeed, there are finite sets of arbitrarily large size relative to which there are at most as many squares as there are powers of 2: for example, just take any finite subset of $2^{\mathbb{N}}$! In fact, more is true: given any finite set C, we can always find a larger set $D \supseteq C$ such that, relative to D, there are more powers of 2 than squares: simply add to C more elements of $2^{\mathbb{N}} \setminus \mathbb{N}^2$ than there are elements in C, which you can always do, since $2^{\mathbb{N}} \setminus \mathbb{N}^2$ is infinite.

Once again, our Surveyor is in a situation in which she has infinitely many answers to the question "are there at most as many elements in A as there are in B?", and she must find a way to extract from them a definite answer. Faced with the same issue, the sequential Surveyor of Benci and di Nasso's chooses what counts as "often enough" in a way that satisfies the three constraints of Cofiniteness, Coherence and Decisiveness. But recall that the existence of such a sequential Surveyor required strong non-constructive principles and, with, them some unavoidable element of arbitrariness in the choice of sequential Surveyor. By contrast, we can specify a unique Surveyor with the following definition of what counts as "often enough":

G1 Cofinality: An answer of the form "A has at most as many elements as B" appears "often enough" if and only if there is a finite set C such that, for any finite set $D \supseteq C$, there are at most as many elements in $A \cap D$ as there are elements in $B \cap D$.

I will call the Surveyor determined by such a notion of "often enough" the Generic Surveyor, and I will denote the relation that she defines on $\mathscr{P}(\mathbb{N})$ by $\sqsubseteq_{\mathcal{G}}$. Note that this relation can be explicitly described as follows.

GEN For any $A, B \subseteq \mathbb{N}$, $A \sqsubseteq B$ if and only if there is a finite set $C \subseteq \mathbb{N}$ such that for all finite $D \supseteq C$, $|A \cap D| \leq |B \cap D|$.

I think there are strong reasons to believe that what counts as "often enough" for the Generic Surveyor is indeed often enough. After all, Cofinality means that the Surveyor needs only to consider a particular finite set C of natural numbers to be sure of her answer. As soon as she considers that finite set, plus maybe more natural numbers, her answer will not change. Note that this means also that, no matter what finite set D she starts from, she will always "converge" to the same answer, as soon as she reaches the finite set $C \cup D$. So Cofinality certainly isn't too weak a requirement to impose on what counts as "often enough". But is it too strong? Could the Generic Surveyor be a bit more liberal in what she counts as "often enough"? The following result arguably shows that, in fact, this is the best she can do if she wants to remain both Euclidean and well-motivated.

Theorem 6.1. *The relation $\sqsubseteq_{\mathcal{G}}$ is the largest Euclidean and relatively invariant size relation.*

Again, note that there are two parts to this theorem. First, $\sqsubseteq_{\mathcal{G}}$ is a size relation that is both Euclidean and relatively invariant. Second, it is the *largest* such relation: any size relation $\sqsubseteq$ for which there are sets A and B such that $A \not\sqsubseteq_{\mathcal{G}} B$, yet $A \sqsubseteq B$ must violate either **EC** or **RI**. As we have seen in the previous section, there is some tension between the Euclidean property and invariance. More invariance, such as Absolute Invariance, means that the Euclidean property must be abandoned, Similarly, some Euclidean theories such as numerosity theory are not even relatively invariant. What Theorem 6.1 shows is that $\sqsubseteq_{\mathcal{G}}$ is, in a sense, the optimal equilibrium point between the two: no size relation can make strictly more assignments than $\sqsubseteq_{\mathcal{G}}$ without losing at least one of the two properties.

Let's take stock. We started from the Minimal Proposal, with a relation $\sqsubseteq_{\mathcal{M}}$ that was the smallest possible size relation preserving finite arithmetic. We worried that $\sqsubseteq_{\mathcal{M}}$ was too small a relation, in particular because it was not linear, and we considered whether the Density Intuition could help in finding a Euclidean and linear size relation. As I have argued, however, any size relation based on the Density Intuition should be abandoned, because it is incompatible with the criterion of Relative Invariance. But we have now obtained a theory, the Generic Approach, that determines a Euclidean size relation $\sqsubseteq_{\mathcal{G}}$ that is also relatively invariant, and is in fact the largest such size relation. Although it is not linear, it could still appear as an improvement on the Minimal Proposal. How should we choose between the two, i.e., between the size relations $\sqsubseteq_{\mathcal{M}}$ and $\sqsubseteq_{\mathcal{G}}$? The good news is that, in fact, we do not have to choose.

Theorem 6.2. *For any set $A, B \subseteq \mathbb{N}$, $A \sqsubseteq_{\mathcal{M}} B$ if and only if $A \sqsubseteq_{\mathcal{G}} B$. In other words, the relations $\sqsubseteq_{\mathcal{M}}$ and $\sqsubseteq_{\mathcal{G}}$ are identical.*

This result, whose proof can be found in the Appendix (Theorem A.1), is strong evidence that our initial worry about the Minimal Proposal can be dispelled. As long as we want a Euclidean theory of size that is relatively invariant under permutations, the Minimal Proposal is in fact also a maximal one: no other Euclidean and relatively invariant size relation can do better. Note that an immediate corollary of Theorems 3.1, 6.1 and 6.2 is the following unique characterization of $\sqsubseteq_{\mathcal{G}}$.

Theorem 6.3. *The relation $\sqsubseteq_{\mathcal{G}}$ is the unique arithmetical, Euclidean and relatively invariant size relation on $\mathscr{P}(\mathbb{N})$.*

Again, I think this result should make us very confident about the Minimal Proposal or, equivalently, about the Generic Approach. We tried to approximate what a Euclidean version of the "at most as many elements" relation might be "from below", by heeding the Galilean call for modesty, and we ended up with the Minimal Proposal. We tried to approximate it "from above", by looking for a size relation that extracts the non-arbitrary kernel from the Density Intuition, and we ended up with the Generic Approach as encapsulated by the generic Surveyor, whose judgments coincide with the Minimal Proposal. As a consequence, we now have six different characterizations of the generic relation $\sqsubseteq_{\mathcal{G}}$, all with their own intuitive motivation.

1. $A \sqsubseteq_{\mathcal{G}} B$ if and only if there is $C \subseteq B \setminus A$ such that $|A \setminus B| \leq |B \setminus A|$;
2. $\sqsubseteq_{\mathcal{G}}$ is the smallest arithmetical size relation;
3. $A \sqsubseteq_{\mathcal{G}} B$ if and only if any injection $f : B \setminus A \to A \setminus B$ is also surjective;
4. $A \sqsubseteq_{\mathcal{G}} B$ if and only if there is a finite set $C \subseteq \mathbb{N}$ such that for any finite $D \supseteq C$, $|A \cap D| \leq |B \cap D|$;
5. $\sqsubseteq_{\mathcal{G}}$ is the largest Euclidean and relatively invariant size relation;
6. $\sqsubseteq_{\mathcal{G}}$ is the unique arithmetical, Euclidean and relatively invariant size relation.

Once again, this is in stark contrast to numerosity theory, according to which the correct size relation on $\mathscr{P}(\mathbb{N})$ cannot be specified explicitly, and must instead be chosen from a multitude of non-constructive options. This leads me to think that $\sqsubseteq_{\mathcal{G}}$ is a better alternative to the Cantorian notion of size for sets of natural numbers. In fact, given its characterization as the *unique* arithmetical, Euclidean and relatively invariant size relation, I think $\sqsubseteq_{\mathcal{G}}$ is the *best possible* candidate for such a Euclidean challenge to the notion of cardinality.

7 Conclusion

The axiomatic approach I have embraced here brings, I think, a lot of clarity to the tension that is at the root of Galileo's Paradox. The results I have presented entail that no size relation on the powerset of $\mathbb{N}$ can be linear, Euclidean and relatively invariant. The joint incompatibility between those three requirements gives rise to a trilemma, which, as shown in Table 1, gives rise to three possible roads for the "at most as many elements" relation:

- Adopt Linearity and Relative Invariance, which leads to Cantor's way out of the paradox and to the relation $\sqsubseteq_{\mathcal{C}}$;
- Adopt Linearity and the Euclidean Constraint, which is the road followed by adepts of the Density Intuition and leads to many sequential Surveyors and many possibilities for the relation $\sqsubseteq_{\mathcal{D}}$;
- Adopt Relative Invariance and the Euclidean Constraint, which leads to the generic relation $\sqsubseteq_{\mathcal{G}}$.

As I have argued at length, I think the third path is superior to the second one. Granted, this requires giving up the ability to always determine which of any two sets of natural numbers has more elements than the other, but I think the sacrifice is worth it. Assignments of size relationships under the Generic Approach are always grounded in some elementary facts that have a strong element of *finiteness* to them. To be sure of her answer, the generic Surveyor

	EC	**RI**	**LIN**
$\sqsubseteq_{\mathcal{C}}$	×	✓	✓
$\sqsubseteq_{\mathcal{D}}$	✓	×	✓
$\sqsubseteq_{\mathcal{M}}$ / $\sqsubseteq_{\mathcal{G}}$	✓	✓	×

Table 1: Galileo's trilemma

must be able to determine that it is the correct answer by looking merely at a finite set. When we consider sets of natural numbers as pure sets (thus forgetting about the standard order on the sequence of natural numbers), there are many sets for which no size relationship can be determined on the sole basis of such finitary facts. But, precisely for this reason, it is hard to see what kind of motivation could be given for determining size relationships between such sets, unless one adopts the Cantorian idea that they all have the same size.

Despite this motivation for embracing the partiality of the generic relation, I would like to conclude with a few words of solace for the friends of linearity. Arguably, giving up on linearity feels like a high cost to pay because of a strong historical and conceptual link between the notion of a quantity and linearity, in the sense that any two quantities of the same kind should always be comparable to one another.[10] In fact, I tend to think that there might be, after all, a way to save some form of linearity even within the framework that I presented here. In short, starting from the Galilean perspective of the Minimal Proposal, one can try to achieve linearity in several ways, not all of which are equally good. Numerosity theory enforces linearity by giving into arbitrariness and undefinability. Any two sets are comparable with respect to their sizes, but very few such comparisons are well-motivated, as the failure of Relative Invariance indicates. Cantor's notion of cardinality, on the other hand, delivers linearity for sets of natural numbers, but, arguably, at the price of triviality: any two infinite sets of natural numbers are comparable, because they have the same size. It is also worth mentioning that, beyond the realm of the countable, the linearity of cardinalities comes at a heavy price again, as it requires the Axiom of Choice. Crucially, both approaches extend the relation $\sqsubseteq_{\mathcal{M}}$ until it becomes a linear order. The solution that I have in mind, by contrast, embraces its partial nature, but changes the meaning of what it takes for a relation to be linear. Roughly speaking, the idea is that the statement of linearity is disjunctive, and that disjunctions may sometimes hold of certain mathematical structures not in virtue of one of their disjuncts holding determinately, but rather because these structures exhibit a certain blend of definiteness and partiality. Under such a view, there is no need to extend the generic relation, but we may instead view it as a partial relation that nonetheless *behaves* linearly. Interestingly, the same strategy can be applied to the number structure induced by $\sqsubseteq_{\mathcal{G}}$, which can then be embedded into a larger structure that has the same first-order theory as the natural numbers. Fleshing out the details of this "semantic" road to linearity involves discussing an alternative to the standard Tarskian semantics of first-order logic known as possibility semantics, which is largely beyond the scope of this paper.[11] For now, I will therefore limit myself to mentioning this as a promising way to reconcile size relationships between infinite sets with the ideal of linearity, thereby offering perhaps the most convincing way out of Galileo's Paradox.

[10] Forti [10] seems to be making such a similar point when advocating for numerosity theory.

[11] The reader may consult [17, 21] for more on this semantic framework, and in particular [22, Chap.8], where the very idea sketched here is presented in more detail.

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A Appendix

Theorem A.1. *Let $A, B \subseteq \mathbb{N}$. The following are equivalent:*

1. *There is a finite $C \subseteq \mathbb{N}$ such that for all finite $D \supseteq C$, $|A \cap D| \leq |B \cap D|$;*
2. *Every one-to-one function $f : B \setminus A \to A \setminus B$ is onto;*
3. *There is a finite $C \subseteq B \setminus A$ such that $|A \setminus B| \leq |C|$.*

Proof. We show the following chain of implications: $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

$1 \Rightarrow 2$ Suppose that there is a one-to-one function $f : B \setminus A \to A \setminus B$ which is not onto, and let C be a finite subset of $\mathbb{N}$. We will show that there is a finite $D \supseteq C$ such that $|B \cap D| < |A \cap D|$. Since f is not onto, $A \setminus B$ is non-empty, so fix some $n \in A \setminus B$. Moreover, let $C' = C \cap B \setminus A$, and $D' = \{f(x) \mid x \in C'\}$. Since f is one-to-one, we have that $|D'| = |C'|$. Now let $D = C \cup D' \cup \{n\}$. Since D', $\{n\}$ and $A \cap B \cap C$ are all disjoint, we have that $|A \cap D| \geq |D'| + |\{n\}| + |A \cap B \cap C|$. Moreover, since $B \cap (D' \cup \{n\}) = \emptyset$ and $C' \cap A \cap B \cap C = \emptyset$, we also have that $|B \cap D| = |C'| + |A \cap B \cap C|$. But then we have the following chain of inequalities:

$$|B \cap D| = |C'| + |A \cap B \cap C| = |D'| + |A \cap B \cap C| < |D'| + |\{n\}| + |A \cap B \cap C| \leq |A \cap D|.$$

Hence for any finite set C, there is $D \supseteq C$ such that $|B \cap D| < |A \cap D|$. By contraposition, this means that if there is a finite $C \subseteq \mathbb{N}$ such that for all finite $D \supseteq C$, $|A \cap D| \leq |B \cap D|$, then every one-to-one function $f : B \setminus A \to A \setminus B$ is onto.

$2 \Rightarrow 3$ Suppose that, whenever $C \subseteq B \setminus A$ is finite, $A \setminus B$ is strictly greater in cardinality than C. Clearly, if $B \setminus A$ is finite, this means that $|B \setminus A| < |A \setminus B|$, which means that there is a one-to-one function from $B \setminus A$ to $A \setminus B$ which is not onto. So we may assume that $B \setminus A$ is infinite. But this means that $A \setminus B$ is also infinite, for, otherwise, we could find a finite subset of $B \setminus A$ that is greater than or equal to $A \setminus B$ in cardinality. So $A \setminus B$ and $B \setminus A$ are two infinite sets, which means that we can find a one-to-one map from $B \setminus A$ to $A \setminus B$ that is not onto. By contraposition, if every one-to-one function $f : B \setminus A \to A \setminus B$ is onto, then there is a finite $C \subseteq B \setminus A$ such that $|A \setminus B| \leq |C|$.

$3 \Rightarrow 1$ Let $C \subseteq B \setminus A$ be a finite set such that $|A \setminus B| \leq |C|$, and let $D \subseteq \mathbb{N}$ be a finite set such that $C \subseteq D$. Let $D_1 = D \cap A \setminus B$ and $D_2 = D \cap A \cap B$. Then we have the following equalities:

$$|A \cap D| = |A \cap C| + |D_1| + |D_2|$$
$$|B \cap D| = |C| + |D_2|.$$

Since $C \subseteq B \setminus A$, $|A \cap C| = 0$, so $|A \cap D| \leq |B \cap D|$ if and only if $|D_1| \leq |C|$. But $D_1 \subseteq A \setminus B$, hence $|D_1| \leq |A \setminus B| \leq |C|$ by assumption on C. Hence $|A \cap D| \leq |B \cap D|$, which completes the proof. □

Lemma A.2. *The relation $\sqsubseteq_{\mathcal{G}}$ on $\mathscr{P}(\mathbb{N})$ has the following properties:*

1. ***QO**: $\sqsubseteq_{\mathcal{G}}$ is a quasi-order, i.e., it is reflexive and transitive.*

2. ***IN**: $\sqsubseteq_{\mathcal{G}}$ extends the inclusion ordering, i.e., $A \subseteq B$ implies $A \sqsubseteq_{\mathcal{G}} B$.*

3. ***DEC**: if A_1, A_2, B_1 and B_2 are sets such that $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$, then $A_1 \sqsubseteq_{\mathcal{G}} B_1$ and $A_2 \sqsubseteq_{\mathcal{G}} B_2$ together imply $A_1 \cup A_2 \sqsubseteq_{\mathcal{G}} B_1 \cup B_2$.*

4. ***FIN**: for any finite set A and any $B \subseteq \mathbb{N}$, if there is an injection from A to B, then $A \sqsubseteq_{\mathcal{G}} B$.*

5. ***EC**: for any $A \subsetneq B$, $B \not\sqsubseteq_{\mathcal{G}} A$.*

6. ***RI**: for any $A, B \subseteq \mathbb{N}$ and any permutation $\pi : \mathbb{N} \to \mathbb{N}$, $A \sqsubseteq_{\mathcal{G}} B$ implies $\pi_*(A) \sqsubseteq_{\mathcal{G}} \pi_*(B)$.*

Proof. We prove all items in turn. We will use several equivalent characterizations of $\sqsubseteq_{\mathcal{G}}$.

1. Clearly, for any $A \subseteq \mathbb{N}$ and any finite $C \subseteq \mathbb{N}$, we have that $|A \cap C| \leq |A \cap C|$. This shows that $\sqsubseteq_{\mathcal{G}}$ is reflexive. For transitivity, suppose that $A \sqsubseteq_{\mathcal{G}} B \sqsubseteq_{\mathcal{G}} C$. Then we have finite sets D_1 and D_2 such that for any $D \supseteq D_1$, $|A \cap D| \leq |B \cap D|$ and for any $D' \supseteq D_2$, $|B \cap D'| \leq |C \cap D'|$. Let $E = D_1 \cup D_2$. Then for any finite $D \supseteq E$, we have that $|A \cap D| \leq |B \cap D| \leq |C \cap D|$, since $E \supseteq D_1, D_2$. But this means that $|A \cap D| \leq |C \cap D|$ for any finite $D \supseteq E$, hence $A \sqsubseteq_{\mathcal{G}} C$.

2. Suppose $A \subseteq B$. Then for any finite $C \subseteq \mathbb{N}$, $|A \cap C| \leq |B \cap C|$. Hence $A \sqsubseteq_{\mathcal{G}} B$.

3. Let A_1, A_2, B_1, B_2 be subsets of $\mathbb{N}$ such that $A_1 \sqsubseteq_{\mathcal{G}} B_1$, $A_2 \sqsubseteq_{\mathcal{G}} B_2$ and $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$. Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. Fix finite sets C_1, C_2 such that for any finite set D, $D \supseteq C_1$ implies $|A_1 \cap D| \leq |B_1 \cap D|$ and $D \supseteq C_2$ implies $|A_2 \cap D| \leq |B_2 \cap D|$. Let $C = C_1 \cup C_2$, and let $D \supseteq C$ be a finite set. Since $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$, we have the following two equalities:

$$|A \cap D| = |A_1 \cap D| + |A_2 \cap D|$$
$$|B \cap D| = |B_2 \cap D| + |B_2 \cap D|$$

   Now since $D \supseteq C_1 \cup C_2$, we have that $|A_1 \cap D| \leq |B_1 \cap D|$ and $|A_2 \cap D| \leq |B \cap D|$. Hence

$$|A \cap D| = |A_1 \cap D| + |A_2 \cap D| \leq |B_1 \cap D| + |B_2 \cap D| = |B \cap D|,$$

   which shows that $A \sqsubseteq_{\mathcal{G}} B$.

4. Let A be a finite set and $B \subseteq \mathbb{N}$ such that there is an injection $f : A \to B$. Let $C \subseteq B$ be the range of f, and note that, since f is injective, we have that $|A| = |C|$. Now let $D \supseteq C$. Since $C \subseteq B \cap D$, we have that $|A \cap D| \leq |A| = |C| \leq |B \cap D|$. But this means that $A \sqsubseteq_{\mathcal{G}} B$.

5. Suppose that $A \subsetneq B$, and let C be a finite set. To show that $B \not\sqsubseteq_{\mathcal{G}} A$, it is enough to find $D \supseteq C$ such that $|B \cap D| \not\leq |A \cap D|$. Let $n \in B \setminus A$, and let $D = C \cup \{n\}$. Then $A \cap D$ is a proper subset of $B \cap D$, from which it follows that $|A \cap D| < |B \cap D|$. This shows that $B \not\sqsubseteq_{\mathcal{G}} A$.

6. Let $A, B \subseteq \mathbb{N}$ and let $\pi : \mathbb{N} \to \mathbb{N}$ be a permutation. If $A \sqsubseteq_{\mathcal{G}} B$, then there is a finite $C \subseteq B \setminus A$ such that $|A \setminus B| \leq |C|$. Now notice the following simple facts about the map $\pi_* : \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ for any $C, D \subseteq \mathbb{N}$:

- $C \subseteq D$ implies $\pi_*(C) \subseteq \pi_*(D)$;
- $\pi_*(C \setminus D) = \pi_*(C) \setminus \pi_*(D)$;
- If C is a finite set, then $|\pi_*(C)| = |C|$.

Hence $\pi_*(C) \subseteq \pi_*(B \setminus A) = \pi_*(B) \setminus \pi_*(A)$, and we have the following chain of inequalities:

$$|\pi_*(A) \setminus \pi_*(B)| \leq |\pi_*(A \setminus B)| = |A \setminus B| \leq |C| = |\pi_*(C)|.$$

Hence $\pi_*(A) \sqsubseteq_{\mathcal{G}} \pi_*(B)$. □

Theorem A.3. *The following completely characterize the relation $\sqsubseteq_{\mathcal{G}}$ among the set of size relations on $\mathscr{P}(\mathbb{N})$:*

1. *$\sqsubseteq_{\mathcal{G}}$ is the smallest size relation on $\mathscr{P}(\mathbb{N})$ that satisfies **FIN**.*
2. *$\sqsubseteq_{\mathcal{G}}$ is the largest size relation on $\mathscr{P}(\mathbb{N})$ that satisfies **EC** and **RI**.*
3. *$\sqsubseteq_{\mathcal{G}}$ is the unique size relation on $\mathscr{P}(\mathbb{N})$ that satisfies **FIN**, **EC** and **RI**.*

Proof. We prove all three items of the theorem in turn.

1. Note first that $\sqsubseteq_{\mathcal{G}}$ satisfies **FIN** by Lemma A.2. To show that it is the smallest such size relation, suppose that $\sqsubseteq$ is a size relation that satisfies **FIN**, and let $A, B \subseteq \mathbb{N}$ such that $A \sqsubseteq_{\mathcal{G}} B$. We must show that $A \sqsubseteq B$. Since $A \sqsubseteq_{\mathcal{G}} B$, there is a finite $C \subseteq B \setminus A$ such that $|A \setminus B| \leq |C|$. In particular, $A \setminus B$ is finite and injects into $B \setminus A$. Since $\sqsubseteq$ satisfies **FIN**, this means that $A \setminus B \sqsubseteq B \setminus A$. Moreover, since $\sqsubseteq$ is a size relation, we have that $A \cap B \sqsubseteq A \cap B$. Now clearly A can be decomposed into $A \setminus B$ and $A \cap B$, and B can be decomposed into $B \setminus A$ and $A \cap B$. By **DEC**, it follows that $A \sqsubseteq B$.

2. Here again, note that $\sqsubseteq_{\mathcal{G}}$ satisfies **EC** and **RI** by Lemma A.2. To show that it is the largest such size relation, suppose that $\sqsubseteq$ is a size relation that satisfies **EC** and **RI**, and let $A, B \subseteq \mathbb{N}$ such that $A \sqsubseteq B$. We must show that $A \sqsubseteq_{\mathcal{G}} B$. Suppose, towards a contradiction, that $A \sqsubseteq B$ but $A \not\sqsubseteq_{\mathcal{G}} B$. By Theorem A.1, we have $f : B \setminus A \to A \setminus B$ such that f is one-to-one but not onto. Now define $\pi : \mathbb{N} \to \mathbb{N}$ by letting $\pi(a) = b$, for b the unique element of $B \setminus A$ such that $f(b) = a$, if $a \in ran(f)$, $\pi(b) = f(b)$ for any $b \in B \setminus A$, and $\pi(x) = x$ for any $x \notin B \setminus A \cup ran(f)$. In other words, π is a permutation of $\mathbb{N}$ that swaps the domain of f with its range and leaves every other natural number undisturbed. Now we make the following observations. First, since $\sqsubseteq$ satisfies **RI** and $A \sqsubseteq B$, we have that $\pi_*(A) \sqsubseteq \pi_*(B)$. Moreover, $B \subseteq \pi_*(A)$. Indeed, if $b \in A \cap B$, then $\pi(b) = b \in A$, and if $b \in B \setminus A$, then $\pi(b) = f(b) \in A$. Since $\sqsubseteq$ satisfies **IN**, this means that we have:

$$A \sqsubseteq B \sqsubseteq \pi_*(A) \sqsubseteq \pi_*(B),$$

whence, by transitivity of $\sqsubseteq$, we have that $A \sqsubseteq \pi_*(B)$. But notice also that $\pi_*(B) \subsetneq A$. Indeed, if $\pi(b) \in B$, then either $\pi(b) \in B \cap A$, which can only happen if $\pi(b) = b$ and hence if $b \in A$, or $\pi(b) \in B \setminus A$, in which case $f(\pi(b)) = b$, which means that $b \in A \setminus B$. Moreover, since f is not onto, there is some $a \in A \setminus B$ such that $\pi(a) = a$, which means that $a \in A \setminus \pi_*(B)$. Hence we have that $\pi_*(B) \subsetneq A$ and $A \sqsubseteq \pi_*(B)$, which contradicts the assumption that $\sqsubseteq$ satisfies **EC**. This completes the proof.

3. By Lemma A.2, $\sqsubseteq_{\mathcal{G}}$ satisfies **FIN**, **EC** and **RI**. To show that this uniquely characterizes $\sqsubseteq_{\mathcal{G}}$, suppose that $\sqsubseteq$ is a size relation that satisfies **FIN**, **EC** and **RI**. By part 1, $A \sqsubseteq_{\mathcal{G}} B$ implies $A \sqsubseteq B$, since $\sqsubseteq$ satisfies **FIN**. By part 2, $A \sqsubseteq B$ implies $A \sqsubseteq_{\mathcal{G}} B$, since $\sqsubseteq$ satisfies **EC** and **RI**. Hence $A \sqsubseteq B$ if and only if $A \sqsubseteq_{\mathcal{G}} B$, which means that $\sqsubseteq_{\mathcal{G}}$ is the unique size relation satisfying **FIN**, **EC** and **RI**. □

Lemma A.4. *An arithmetical size relation $\sqsubseteq$ is absolutely invariant if and only if it is relatively invariant and linear.*

Proof. Fix a size relation $\sqsubseteq$. Suppose first that $\sqsubseteq$ is absolutely invariant. Then $\sqsubseteq$ is clearly relatively invariant, since, if π is a permutation and $A, B \subseteq \mathbb{N}$, we have that $A \sqsubseteq B$ implies that $\pi_*(A) \sqsubseteq A \sqsubseteq B \sqsubseteq \pi_*(B)$. To see that it is linear, let $A, B \subseteq \mathbb{N}$ and consider the sets $A \setminus B$, $B \setminus A$. If one of them is finite, then by **FIN** we have that either $A \setminus B \sqsubseteq B \setminus A$, or $B \setminus A \sqsubseteq A \setminus B$. But then it follows from **DEC** that we have either $A \sqsubseteq B$ or $B \sqsubseteq A$. If both $A \setminus B$ and $B \setminus A$ are infinite, then we can fix a bijection f between those two sets, which gives rise to a permutation π such that $\pi_*(A) = B$. But then, by Absolute Invariance, we have that $A \sqsubseteq \pi_*(A) = B$.

For the converse direction, suppose now that $\sqsubseteq$ is linear and relatively invariant. Note first that this implies that, whenever $A, B \subseteq \mathbb{N}$ are such that $B \setminus A$ and $A \setminus B$ are infinite, we have that $A \sqsubseteq B$. Indeed, suppose A and B are such sets. By linearity, we have either $A \sqsubseteq B$ or $B \sqsubseteq A$. Suppose the latter holds. Since both $A \setminus B$ and $B \setminus A$ are infinite, there is a bijection between these two sets, which can be lifted to a permutation π such that $\pi_*(A) = B$ and $\pi_*(B) = A$. But then, by Relative Invariance together with $A \sqsubseteq B$, it follows that $A = \pi_*(B) \sqsubseteq \pi_*(A) = B$. Now suppose, towards a contradiction, that there is a set A and a permutation π such that $A \not\sqsubseteq \pi_*(A)$. Without loss of generality, we may assume that A is infinite and coinfinite (otherwise, we can easily derive the contradiction by using **FIN** and **DEC**). We distinguish two cases. If both $A \setminus \pi_*(A)$ and $\pi_*(A) \setminus A$ are infinite, then it follows from the observation made above that $A \sqsubseteq \pi_*(A)$, which is the contradiction we were after. Otherwise, without loss of generality, we have that $\pi_*(A) \setminus A$ is finite. But then $A \cup \pi_*(A) = A \cup \pi_*(A) \setminus A$ is coinfinite. So fix some infinite set B such that $B \cap (A \cup \pi_*(A)) = \emptyset$. It follows that $B \setminus A$, $A \setminus B$, $\pi_*(A) \setminus B$ and $B \setminus \pi_*(A)$ are all infinite, so, by the observation above again, we have that $A \sqsubseteq B \sqsubseteq \pi_*(A)$. But this contradicts the transitivity of $\sqsubseteq$. □

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