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B-frame duality

Guillaume Massas

University of California, Berkeley, 970 Evans Hall, Berkeley, CA 94720, United States of America

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ABSTRACT

This paper introduces the category of b-frames as a new tool in the study of complete lattices. B-frames can be seen as a generalization of posets, which play an important role in the representation theory of Heyting algebras, but also in the study of complete Boolean algebras in forcing. This paper combines ideas from the two traditions in order to generalize some techniques and results to the wider context of complete lattices. In particular, we lift a representation theorem of Allwein and MacCaull to a duality between complete lattices and b-frames, and we derive alternative characterizations of several classes of complete lattices from this duality. This framework is then used to obtain new results in the theory of complete Heyting algebras and the semantics of intuitionistic propositional logic.

1. Introduction

Topological dualities have become a standard tool in the representation of lattices and in the semantics of non-classical logics. Stone famously established a duality between Boolean algebras and Stone spaces [61] which he later generalized to a duality between distributive lattices and spectral spaces [62]. Priestley [52] presented an alternative duality between distributive lattices and Priestley spaces, while Esakia's work [18,19] yields dualities for Heyting and bi-Heyting algebras. In the general case of bounded lattices, several dualities have been proposed. Urquhart [63] gave a topological representation of bounded lattices that directly generalizes Stone and Priestley's theorems and which was later lifted by Hartung [34] to a duality for bounded lattices and surjective lattice morphisms. Other dualities for lattices and various lattice expansions have been proposed by Allwein, Dunn and Hartonas [1,29,31,32], as well as by Jipsen and Moshier [50] and Gehrke and van Gool [24]. Although these topological dualities can be used to give representations of com-

E-mail address: gmassas@berkeley.edu.

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plete lattices, there is also a long tradition of discrete, purely relational representations of complete lattices. This tradition originates with Tarski's duality between sets and complete and atomic Boolean algebras, which was later expanded to Boolean algebras with operators (BAOs) and used to provide a semantics for modal logic. Tarski's duality was also generalized to a duality between posets and superalgebraic locales [15], also known as completely join-prime generated complete lattices [53,54]. In set theory, an alternative representation of complete Boolean algebras as the regular open sets of a poset has also become a cornerstone of forcing [40,45] and has recently been used to provide an alternative semantics for modal logic known as possibility semantics [3,35,36,38,39]. This latest representation of complete Boolean algebras is also related to a more general representation of complete lattices obtained by Allwein and MacCaull in [2].

In this paper, we lift Allwein and MacCaull's representation theorem to a full duality. This is achieved by establishing first an idempotent adjunction between the category **cLat** of all complete lattices and a category **Bos** of bi-preordered sets (*bosets* for short). Bi-ordered sets already played a role in Urquhart's representation theorem, although the Allwein-MacCaull dual bosets we consider differ from Urquhart's, and they have also been discussed in connection with the representation of complete Heyting algebras [6,7,48]. As shown in [37], there is also a strong connection between representations of complete lattices via bi-ordered sets and via polarities [10,13,21,32,33]. We use our b-frame duality to provide discrete representations of various classes of complete lattices and use these alternative characterizations to obtain some results in the theory of complete Heyting algebras and the semantics of intermediate logics.

The paper is organized as follows. In Section 2, we introduce bosets and the relevant notion of morphism between them, and we lift the Allwein-MacCaull representation of complete lattices to an idempotent adjunction between the category of bosets and the category of complete lattices. In order to restrict this adjunction to a duality, we generalize the notion of dense embeddings from forcing posets to the setting of bosets, and we use this to characterize the fixpoints of the adjunction. This allows us to define the category **bF** of b-frames, dual to the category **cLat** of complete lattices. We conclude the section by comparing b-frame duality to some existing discrete and topological representations of lattices.

In the following two sections, we develop this framework further by establishing a correspondence between algebraic properties of complete lattices and first-order properties of b-frames. This allows us to obtain alternative representations of complete distributive lattices, complete Heyting algebras and complete Boolean algebras in Section 3, while in Section 4 the duality obtained for complete Heyting algebras is further restricted to obtain geometric, amalgamation-like characterizations of the duals of spatial and superalgebraic locales.

The last two sections are devoted to new applications of this framework to the theory of complete Heyting algebras and the semantics of intermediate logics. In Section 5, the notion of a coproduct of two bosets is defined and used to prove the following decomposition theorem for complete bi-Heyting algebras:

Theorem 5.14. *Let L be a complete bi-Heyting algebra. Then L is a complete subdirect product of $L_1 \times L_2$ in* **cLat***, where L_1 is a completely join-prime generated locale and L_2 is locale with no completely join-prime element.*

A related result in the theory of Boolean algebras [25] states that any complete Boolean algebra is the product of an atomic and an atomless Boolean algebra, although Theorem 5.14 is a result about complete bi-Heyting algebras in **cLat**, rather than in the category of complete bi-Heyting algebras and complete bi-Heyting morphisms, which is not a full subcategory of **cLat**.

Finally, Section 6 discusses some applications of this framework to the semantics of intuitionistic logic. We introduce boset semantics, a semantics as general as locale semantics for intuitionistic logic and show how semantics that are equivalent to Kripke and topological semantics arise as natural restrictions imposed on boset semantics. As a consequence, boset semantics provides a uniform presentation of most of the semantic hierarchy for intuitionistic logic introduced in [6]. We conclude with an application to the incompleteness problem for intermediate logics:

Theorem 6.14. *The intermediate logic SL, originally proved by Shehtman [60] to be Kripke-incomplete, is in fact incomplete with respect to the larger class of all complete bi-Heyting algebras.*

A similar result has recently and independently been obtained by Bezhanishvili, Gabelaia and Jibladze in [5], via Esakia duality and through a fairly intricate argument. By contrast, our proof is a straightforward adaptation of Shehtman's original argument, which we take as evidence that boset semantics can be a fruitful framework for the study of intermediate logics.

2. B-frame duality

In this section, we introduce the category **bF** of b-frames and prove that it is dual to the category of complete lattices **cLat**. This is done in two steps. First, we introduce a category **Bos** of bi-preordered sets and establish an idempotent adjunction between **Bos** and **cLat**. As was already noted by Allwein and McCaul in their representation theorem for complete lattices obtained in [2], all complete lattices are fixpoints of this adjunction. This means that we only need to restrict **Bos** to a full subcategory of fixpoints in order to obtain a category dual to **cLat**. We call such fixpoints *b-frames* and show that they are completely characterized by certain properties of bi-preordered sets. Finally, in Section 2.5, we connect this adjunction to well-known discrete dualities for complete lattices, showing in particular how it generalizes Tarski's duality between **CABA** and **Sets**, Raney's duality between superalgebraic lattices and posets and the forcing duality between complete Boolean algebras and separative posets. We also discuss connections with several existing dualities for lattices, including Urquhart-Hartung duality [34,63], Allwein-Hartonas duality [29] and Hartonas-Dunn duality [31–33].

2.1. Bosets and B-morphisms

Our starting point is the notion of a bi-preordered set, which will be called *bosets* for short. In other words, a boset is a tuple $(X, \leq_1, \leq_2)$ such that $\leq_1$ and $\leq_2$ are preorders on X. Bi-ordered sets have been used before in the representation theory of bounded lattices, in particular by Urquhart [63], Hartung [34] and in various ways by Allwein, MacCaull, Hartonas and Dunn [1,2,32,33]. We refer the reader to Section 2.5 for a comparison of our approach to this literature. More recently, bi-ordered sets have also been discussed in connection with the representation of complete Heyting algebras in [6,7,48]. This connection will be explored further in Section 6. For now, we introduce the notion of morphism between bosets that will be relevant for our purposes:

Definition 2.1. Let $(X, \leq_1^X, \leq_2^X)$ and $(Y, \leq_1^Y, \leq_2^Y)$ be two bosets. A map $f : X \to Y$ is a *boset morphism* (b-morphism) if the following are true:

1. f is monotone in both orderings, i.e., for any $x, y \in X$, if $x \leq_i^X y$, then $f(x) \leq_i^Y f(y)$ for $i \in \{1, 2\}$;
2. $\forall x \in X\, \forall y \geq_2^Y f(x)\, \exists z \geq_2^X x : f(z) \geq_1^Y y$;
3. $\forall x \in X\, \forall y \geq_1^Y f(x)\, \exists z \geq_1^X x : f(z) \geq_2^Y y$.

It is straightforward to verify that the composition of two b-morphisms is still a b-morphism.

Lemma 2.2. *Let* $f : (X, \leq_1^X, \leq_2^X) \to (Y, \leq_1^Y, \leq_2^Y)$ *and* $g : (Y, \leq_1^Y, \leq_2^Y) \to (Z, \leq_1^Z, \leq_2^Z)$ *be two b-morphisms. Then* $g \circ f : (X, \leq_1^X, \leq_2^X) \to (Z, \leq_1^Z, \leq_2^Z)$ *is a b-morphism.*

Proof. Monotonicity is clear. Suppose $x \in X$ and $z \geq_2^Z gf(x)$. Then since g is a b-morphism there is $y \geq_2^Y f(x)$ such that $g(y) \geq_1^Z z$. But since f is a b-morphism, this implies that there is $x' \geq_2^X x$ such that $f(x') \geq_1^Y y$. Thus $gf(x') \geq_1^Z g(y) \geq_1^Z z$. Hence $g \circ f$ satisfies condition 2. The proof that $g \circ f$ also satisfies condition 3 is completely similar. □

Therefore bosets and b-morphisms form a category **Bos**. Our main goal is to understand how this category relates to **cLat**, the category of complete lattices and complete lattice morphisms between them. Throughout this paper, given a poset $(P, \leq)$ and $A \subseteq P$, we will write $\uparrow A$ and $\downarrow A$ for the sets $\{p \in P \mid \exists q \in A : q \leq p\}$ and $\{p \in P \mid \exists q \in A : p \leq q\}$ respectively.

Example 2.3. Any preordered set $\mathbb{P} = (P, \leq)$ may be viewed as a boset in two different ways: either as a *Kripke boset* $\mathbb{P}_F = (P, \leq, \geq)$, i.e., by letting the second ordering be the converse of the ordering on P, or as *forcing boset* $\mathbb{P}_B = (P, \geq, \geq)$, obtained by letting the two orderings be the converse ordering.[1] It is straightforward to verify that a b-morphism between Kripke bosets is precisely a monotone map between the underlying preordered sets, while a b-morphism between forcing bosets is a monotone map $f : (P, \leq_P) \to (Q, \leq_Q)$ that is also *weakly dense*, i.e., f is such that $f[\downarrow p]$ is dense (in the sense of the downset topology induced by the ordering) in $\downarrow f[p]$ for every $p \in P$.

Any boset $\mathscr{X} = (X, \leq_1^X, \leq_2^X)$ can be regarded as a bi-topological space, by letting τ_1 and τ_2 be the upset topologies induced by the orders $\leq_1$ and $\leq_2$ respectively. We write C_1 and C_2 for the corresponding closure operators. We can then consider the complete lattices $\mathscr{O}_1$ and $\mathscr{O}_2$ of open sets in τ_1 and τ_2 respectively and define two antitone maps: $\neg_1 : \mathscr{O}_2 \to \mathscr{O}_1$ and $\neg_2 : \mathscr{O}_1 \to \mathscr{O}_2$ by letting $\neg_i U = X - C_i(U)$ for any $U \in \mathscr{O}_j$ and $i \neq j \in \{1, 2\}$. Now clearly for any $U \in \mathscr{O}_1$ and $V \in \mathscr{O}_2$,

$$U \subseteq \neg_1 V \text{ iff } U \subseteq X - V,$$

and

$$V \subseteq \neg_2 U \text{ iff } V \subseteq X - U.$$

So $\neg_1$ and $\neg_2$ form a Galois connection, which means that the composite map $\neg_1\neg_2$ is a closure operator on $\mathscr{O}_1$. A fixpoint of $\neg_1\neg_2$ is called *regular open*. Notice in particular that if $\tau_1 = \tau_2$, this definition coincides with the usual notion of a regular open subset of a topological space. It is useful to observe that a set $U \subseteq X$ is regular open if and only if for any $x \in X$:

$$x \in U \text{ iff } \forall y \geq_1^X x \, \exists z \geq_2^X y : z \in U.$$

As the fixpoints of a closure operator on a complete lattice always form a complete lattice [59, Thm. 5.2], it follows that the fixpoints $\mathsf{RO}_{12}(\mathscr{X})$ form a complete lattice. It is straightforward to verify that for any collection $\{U_i\}_{i \in I}$ of sets in $\mathsf{RO}_{12}(\mathscr{X})$, $\bigwedge_{i \in I} U_i = \bigcap_{i \in I} U_i$ and $\bigvee_{i \in I} U_i = \neg_1\neg_2(\bigcup_{i \in I} U_i)$. Regular open sets in bitopological spaces have been studied before in the context of duality theory for lattices, in particular in the Pairwise Stone duality for distributive lattices developed in [8] and in using Priestley and Esakia duality to give a topological characterization of MacNeille completions of Heyting algebras [27]. The next lemma shows that the inverse image of a b-morphism maps regular opens to regular opens.

[1] The reason for flipping the order is simply historical: in the forcing literature, one typically works with regular open *downsets*, while Kripke semantics is typically defined in terms of *upsets*. Since we will be working with upsets, yet several notions defined below are generalizations of notions about forcing posets, flipping the order when representing forcing posets as bosets will help avoid any confusion.

Lemma 2.4. *Let $f : (X, \leq_1^X, \leq_2^X) \to (Y \ \leq_1^Y, \leq_2^Y)$ be a b-morphism. Then for any 1-upset $U \subseteq Y$, $f^{-1}[\neg_1\neg_2(U)] = \neg_1\neg_2 f^{-1}[U]$.*

Proof. We claim that $C_i(f^{-1}[U]) = f^{-1}[C_i(U)]$ for any $U \in \mathscr{O}_j$, $i \neq j \in \{1, 2\}$. This is clearly enough to establish that $f^{-1}[\neg_1\neg_2(U)] = \neg_1\neg_2 f^{-1}[U]$ for any $U \in \mathscr{O}_1$. For the proof of the claim, suppose $f(y) \in U$ for some $y \geq_i^X x$. Since $f(y) \geq_i^X f(x)$, by the b-morphism conditions, there is some $z \geq_i^X x$ such that $f(z) \geq_j^Y f(y)$, for $j \neq i$. Since U is j-open, $f(z) \in U$, so $f(x) \in C_i(U)$. This shows that $C_i(f^{-1}[U]) \subseteq f^{-1}[C_i(U)]$. Conversely, if $f(x) \leq_i^Y y$ for some $y \in U$, then by the b-morphism conditions again there is some $z \geq_i^X x$ such that $f(z) \geq_j y$. Once again, since U is j-open this implies that $z \in f^{-1}[U]$, and thus $x \in C_i(f^{-1}[U])$. □

This allows us to define a contravariant *regular open functor* $\rho : \mathbf{Bos} \to \mathbf{cLat}$:

- For any boset $\mathscr{X} = (X, \leq_1, \leq_2)$, $\rho(\mathscr{X}) = \mathsf{RO}_{12}(\mathscr{X})$, i.e., the complete lattice of fixpoints of the $\neg_1\neg_2$ closure operator on the 1-upward closed sets of $\mathscr{X}$.
- Given a b-morphism $f : (X, \leq_1^X, \leq_2^X) \to (Y, \leq_1^Y, \leq_2^Y)$, $\rho(f) : \mathsf{RO}_{12}(Y) \to \mathsf{RO}_{12}(X)$ is defined as the restriction to $\mathsf{RO}_{12}(Y)$ of the preimage function f^{-1}.

Since $\mathsf{RO}_{12}(\mathscr{X})$ is a complete lattice for any boset $\mathscr{X}$, ρ is well-defined on objects. To see that it is well defined on morphisms, suppose $f : \mathscr{X} \to \mathscr{Y}$ is a b-morphism. Then by Lemma 2.4 $\rho(f)$ is a map from $\mathsf{RO}_{12}(\mathscr{Y}) \to \mathsf{RO}_{12}(\mathscr{X})$. Moreover, let $\{U_i\}_{i\in I}$ be any collection of sets in $\mathsf{RO}_{12}(\mathscr{Y})$. It is routine to check that $\bigcap_{i\in I} f^{-1}[U_i] = f^{-1}[\bigcap_{i\in I} U_i]$, and by Lemma 2.4, we have that

$$f^{-1}[\neg_1\neg_2(\bigcup_{i\in I} U_i)] = \neg_1\neg_2(f^{-1}[\bigcup_{i\in I} U_i]) = \neg_1\neg_2(\bigcup_{i\in I} f^{-1}[U_i]).$$

Thus $\rho(f)$ is a complete lattice morphism from $\mathsf{RO}_{12}(\mathscr{Y})$ to $\mathsf{RO}_{12}(\mathscr{X})$.

2.2. From lattices to bosets

Having constructed the first half of the adjunction, let us now define a functor going from complete lattices to bosets. The construction on objects was already introduced in [2], although Allwein and MacCaull do not extend their representation theorem to a full duality.

Definition 2.5. Let L be a complete lattice. The *dual Allwein-MacCaull boset* of L is the boset $(P_L, \leq_1^L, \leq_2^L)$ such that:

- $P_L = \{(a, b) \in L \mid a \not\leq b\}$;
- $(a, b) \leq_1^L (c, d)$ iff $a \geq_L c$;
- $(a, b) \leq_2^L (c, d)$ iff $b \leq_L d$.

Let $f : L \to M$ be a complete lattice homomorphism. By the adjoint functor theorem, f has a left adjoint $\cdot^f$ and a right adjoint $\cdot_f$, where for any $a \in M$, $a^f = \bigwedge\{c \in L \mid f(c) \geq a\}$ and $a_f = \bigvee\{c \in L \mid f(c) \leq a\}$. The following lemma shows how to use the existence of those adjoints to construct a b-morphism from f.

Lemma 2.6. *Let $f : L \to M$ be a complete lattice homomorphism. The map $\alpha(f) : (P_M, \leq_1^M, \leq_2^M) \to (P_L, \leq_1^L, \leq_2^L)$ defined by $\alpha(f)(a, b) = (a^f, b_f)$ is a b-morphism.*

Proof. • Showing that $\alpha(f)$ is well defined amounts to proving that for any $a, b \in M$, $a^f \leq b_f$ implies that $a \leq b$. But clearly as $\cdot^f$ and $\cdot_f$ are left and right adjoint to f respectively, we have that $a \leq f(a^f)$ and $f(b_f) \leq b$, so by monotonocity of f, $a^f \leq b_f$ implies that $a \leq f(a^f) \leq f(b_f) \leq b$.
- Monotonicity of $\alpha(f)$ in the two orderings is straightforward.
- Let $(a, b) \in P_M$ and suppose $\alpha(f)(a, b) \leq_2^L (c, d)$ for some $c \not\leq d \in L$. We claim that the pair $(f(c), b)$ is in P_M. To see this, note that if $f(c) \leq b$, then $c \leq b_f$. But since $\alpha(f)(a, b) = (a^f, b_f) \leq_2^L (c, d)$, we have that $c \leq d$, a contradiction. Thus $f(c) \not\leq b$. Therefore $(a, b) \leq_2^M (f(c), b)$ and $(c, d) \leq_1^L (f(c)^f, b_f)$.
- Let $(a, b) \in P_M$ and suppose $\alpha(f)(a, b) \leq_1^L (c, d)$ for some $c \not\leq d \in L$. We claim that the pair $(a, f(d))$ is in P_M. To see this, note that if $a \leq f(d)$, then $a^f \leq d$. But since $\alpha(f)(a, b) = (a^f, b_f) \leq_1^L (c, d)$, we have that $c \leq d$, a contradiction. Thus $a \not\leq f(d)$. Therefore $(a, b) \leq_1^M (a, f(d))$ and $(c, d) \leq_2^L (a^f, f(d)_f) = \alpha(f)(a, f(d))$. □

The contravariant $\alpha : \mathbf{cLat} \to \mathbf{Bos}$ is defined as follows:

- for any complete lattice L, $\alpha(L) = (P_L, \leq_1^L, \leq_2^L)$, the dual Allwein-MacCaull boset of L.
- for any complete lattice morphism $f : L \to M$, $\alpha(f) : (P_M, \leq_1^M, \leq_2^M) \to (P_L, \leq_1^L, \leq_2^L)$ is defined as the map $(a, b) \mapsto (a^f, b_f)$.

Remark 2.7. A careful look at the definition of the functor α reveals that it could easily be extended to the category of all lattices and morphisms that have both a left and right adjoint. However, in the absence of the adjoint functor theorem, this condition on morphisms is fairly cumbersome. We therefore limit ourselves to discussing morphisms between complete lattices, for which having a left and a right adjoint is equivalent to being a complete lattice homomorphism.

We are now in a position to provide a representation theorem for all complete lattices. As mentioned in the introduction, this result was already obtained in [2, Thm. 4.2.9]. However, Allwein and MacCaull use some notation introduced by Urquhart [63], which differs quite significantly from ours. A more similar proof to the one we give below can be found in [37], although Holliday works with downsets while we work with upsets. Moreover, none of the works mentioned above presents their result from a categorical viewpoint, while we are also in a position to establish the naturality of the isomorphism between L and the regular opens of its dual boset, a key step in proving the idempotent adjunction we are after.

Lemma 2.8. *For any complete lattice L, L is isomorphic to $\rho\alpha(L)$ naturally in L.*

Proof. Let L be a complete lattice with dual boset $\alpha(L) = (P_L, \leq_1^L, \leq_2^L)$. We claim that the map $\phi_L : a \mapsto {\uparrow_1}(a, 0)$ is a complete lattice isomorphism natural in L between L and $\mathsf{RO}_{12}(P_L)$.

- ϕ_L is well defined: let $(c, d) \in P_L$ such that $c \not\leq a$. Then the pair (c, a) is in P_L, which implies that $(c, d) \notin \neg_1\neg_2({\uparrow_1}(a, 0))$. Thus $\neg_1\neg_2({\uparrow_1}(a, 0)) = {\uparrow_1}(a, 0)$.
- ϕ_L is order-preserving and order-reflecting: suppose $a \leq_L b$. Then $(b, 0) \leq_1^L (a, 0)$, which implies that ${\uparrow_1}(a, 0) \subseteq {\uparrow_1}(b, 0)$. Conversely, if $a \not\leq b$, then $(b, 0) \notin {\uparrow_1}(a, 0)$.
- ϕ_L is surjective: Suppose $U \subseteq P_L$ such that $\neg_1\neg_2(U) = U$. We claim that $U = \phi(a)$, where $a = \bigvee\{c \mid \phi(c) \subseteq U\}$. Suppose that $(c, d) \in U$ for some $c \not\leq d \in U$. Then since U is a 1-upset, ${\uparrow_1}(c, d) = {\uparrow_1}(c, 0) \subseteq U$, so $c \leq a$. Since ϕ_L is order-preserving, this implies that ${\uparrow_1}(c, 0) = \phi_L(c) \subseteq \phi_L(a)$, and therefore $U \subseteq \phi_L(a)$. For the converse, let $(c, d) \geq_1^L (a, 0)$. We claim that there is $b \in L$ such that $\phi_L(b) \subseteq U$ and $b \not\leq d$. To see this, note that, otherwise, d is an upper bound of the set $\{b \in L \mid \phi_L(b) \subseteq U\}$, which implies that $a \leq d$. But $c \leq a$, and therefore $c \leq d$, a contradiction. Thus $(c, d) \leq_2^L (b, d)$ for some b such that $\phi_L(b) \subseteq U$, and therefore $(a, 0) \in \neg_1\neg_2(U) = U$. This completes the proof that ϕ_L is a complete lattice isomorphism.

- For naturality in L, suppose we have a complete lattice morphism $f : L \to M$. We want to show that $\phi_M(f)(a) = \rho\alpha(f)(\phi_L(a))$ for any $a \in L$. Note that $\phi_M(f)(a) = \uparrow_1(f(a), 0)$. Then we compute:

$$\begin{aligned}\rho\alpha(f)(\phi_L(a)) &= \rho\alpha(f)(\uparrow_1(a,0)) \\ &= \{(c,d) \in \alpha(M) \mid \alpha(f)(c,d) \in \uparrow_1(a,0)\} \\ &= \{(c,d) \in \alpha(M) \mid (c^f, d_f) \geq_1^L (a,0)\} \\ &= \{(c,d) \in \alpha(M) \mid c^f \leq_1^L a\} \\ &= \{(c,d) \in \alpha(M) \mid c \leq_1^L f(a)\} = \uparrow_1(f(a),0).\end{aligned}$$

This completes the proof. □

This result yields a representation of complete lattices as regular opens of some boset. For our purposes however, it also allows us to establish that all complete lattices are fixpoints of a contravariant adjunction. The existence of this adjunction is the main theorem of this section:

Theorem 2.9. *The functors $\alpha : \mathbf{cLat} \to \mathbf{Bos}$ and $\rho : \mathbf{Bos} \to \mathbf{cLat}$ form a contravariant adjunction.*

Proof. Let L be a complete lattice and $\mathscr{X} = (X, \leq_1, \leq_2)$ a boset. We define bijections between $\mathrm{Hom}_{\mathbf{cLat}}(L, \rho(\mathscr{X}))$ and $\mathrm{Hom}_{\mathbf{Bos}}(\mathscr{X}, \alpha(L))$, natural in both L and $\mathscr{X}$.

For any $f : L \to \rho(\mathscr{X})$ and any $x \in X$, let $x^f = \bigwedge\{a \in L \mid x \in f(a)\}$ and $x_f = \bigvee\{b \in L \mid x \in \neg_2(f(b))\}$. Let $\overline{f} : \mathscr{X} \to \alpha(L)$ be defined as $\overline{f}(x) = (x^f, x_f)$. We claim that $\bar{\cdot} : \mathrm{Hom}_{\mathbf{cLat}}(L, \rho(\mathscr{X})) \to \mathrm{Hom}_{\mathbf{Bos}}(\mathscr{X}, \alpha(L))$ is an isomorphism natural in L and $\mathscr{X}$.

- $\overline{f}$ is a b-morphism:
  - Note first that $\overline{f}$ is well defined: since f is a complete lattice morphism, for any $x \in X$, $x \in f(x^f)$ and $x \notin f(x_f)$. Thus $\overline{f}(x) \in P_L$.
  - For monotonicity, notice that $x \leq_1 y$ implies that if $x \in f(a)$, then $y \in f(a)$ for any $a \in L$. Therefore $x^f \geq y^f$, and therefore $\overline{f}(x) \leq_1^L \overline{f}(y)$. Similarly, if $x \leq_2 y$, then $x \in \neg_2(f(b))$ implies $y \in \neg_2(f(b)$, and thus $x_f \leq y_f$. Therefore $\overline{f}(x) \leq_2^L \overline{f}(y)$.
  - Suppose that $(c,d) \geq_2^L \overline{f}(x)$ for some $x \in X$, $c, d \in L$. We claim that there is $y \geq_2 x$ such that $y \in f(c)$. Otherwise, $x \in \neg_2(f(c))$, and therefore $c \leq x_f$. But this implies that $c \leq d$, a contradiction. Thus such a $y \geq_2 x$ exists. But since $y \in f(c)$, it follows that $\overline{f}(y) \geq_1^L (c,d)$.
  - Suppose now that $(c,d) \geq_1^L \overline{f}(x)$. We claim that there is $y \geq_1 x$ such that $y \in \neg_2(f(d))$. Otherwise, $x \in \neg_1\neg_2(f(d)) = f(d)$, and thus $c \leq x^f \leq d$, a contradiction. Now since $y \in \neg_2(f(d))$, we have that $d \leq f_y$, and thus $(c,d) \leq_2^L \overline{f}(y)$.
- $\bar{\cdot}$ is injective: let $f_1, f_2 : L \to \rho(\mathscr{X})$ such that $f_1 \neq f_2$. Without loss of generality, there is some $a \in L$ such that $f_1(a) \nsubseteq f_2(a)$. Let $x \in f_1(a)$ such that $x \notin f_2(a)$. Then there is $y \geq_1 x$ such that $y \in f_1(a)$ and $y \in \neg_2(f_2(a))$. This implies that $y^{f_1} \leq a \leq y_{f_2}$. As $y^{f_1} \nleq y_{f_1}$, this means that $y_{f_1} \neq y_{f_2}$, and therefore $\overline{f_1}(y) \neq \overline{f_2}(y)$.
- $\bar{\cdot}$ is surjective: let $g : \mathscr{X} \to \alpha(L)$ and consider the map $f : L \to \rho(\mathscr{X})$ defined by $f(a) = g^{-1}[\uparrow_1(a,0)]$. We claim that $g = \overline{f}$. Indeed, for any $x \in \mathscr{X}$ such that $g(x) = (c,d)$ and any $a \in L$, we have that $g(x) \in f(a)$ iff $c \leq_1 a$, and $g(x) \in \neg_2(f(a))$ iff $a \leq d$. Thus $\overline{f}(x) = (c,d)$.

Finally, it remains to verify that $\sim$ is natural in both L and $\mathscr{X}$. This means that for any $M \in \mathbf{cLat}$, $\mathscr{Y} \in \mathbf{Bos}$, $g : M \to L$ and $h : \mathscr{Y} \to \mathscr{X}$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{cLat}}(L,\rho(\mathscr{X})) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbf{Bos}}(\mathscr{X},\alpha(L)) \\ \downarrow{\scriptstyle \mathrm{Hom}_{\mathbf{cLat}}(g,\rho(h))} & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathbf{Bos}}(\alpha(g),h)} \\ \mathrm{Hom}_{\mathbf{cLat}}(M,\rho(\mathscr{Y})) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbf{Bos}}(\mathscr{Y},\alpha(M)) \end{array}$$

i.e., $\overline{\rho(h)\circ f\circ g} = \alpha(g)\circ \overline{f}\circ h$ for any $f : L \to \rho(\mathscr{X})$. Now let $y \in \mathscr{Y}$ and compute that:

$$\begin{aligned}\alpha(g)(\overline{f})(h)(y) &= \alpha(g)((h(y)^f, (h(y))_f)) \\ &= (\bigwedge\{a \in M \mid g(a) \geq (h(y))^f\}, \bigvee\{a \in M \mid g(a) \leq (h(y))_f\})\end{aligned}$$

and

$$\begin{aligned}\overline{\rho(h)\circ f\circ g}(y) &= (\bigwedge\{a \in M \mid y \in \rho(h)(f)(g(a))\}, \bigvee\{a \in M \mid y \in \neg_2(\rho(h)(f)(g(a)))\}) \\ &= (\bigwedge\{a \in M \mid y \in h^{-1}[f(g(a))]\}, \bigvee\{a \in M \mid y \in \neg_2 h^{-1}[f(g(a))]\}).\end{aligned}$$

Thus it is enough to show for any $a \in M$ that:

$$g(a) \geq (h(y))^f \Leftrightarrow h(y) \in f(g(a)) \tag{1a}$$

$$g(a) \leq (h(y))_f \Leftrightarrow y \in \neg_2 h^{-1}[f(g(a))] \tag{1b}$$

Now (1a) follows directly from the definition of $(h(y))^f$. For (1b), note that $g(a) \leq (h(y))_f$ iff $h(y) \in \neg_2 f(g(a))$ iff $y \in h^{-1}[\neg_2 f(g(a))]$. But since h is a b-morphism, we have that $h^{-1}[\neg_2 f(g(a))] = \neg_2 h^{-1}[f(g(a))]$, which completes the proof. □

For the sake of clarity, we will sometimes refer to this adjunction as a *covariant* adjunction between **cLat** and $\mathbf{Bos}^{op}$. If we think of α and ρ as covariant functors, it then follows from the previous theorem that α is left-adjoint to ρ. It therefore makes sense to talk about the unit and counit of this adjunction as natural transformations $\eta : Id_{\mathbf{cLat}} \to \rho\alpha$ and $\epsilon : Id_{\mathbf{bF}} \to \alpha\rho$.

Remark 2.10. Closer inspection of the proof of Theorem 2.9 shows that the counit of the adjunction is given by the map $\epsilon_{\mathscr{X}} : \mathscr{X} \to \alpha\rho(\mathscr{X})$, defined as $\epsilon_{\mathscr{X}}(x) = (U^x, V_x)$, where for any $x \in \mathscr{X}$, $U^x = \neg_1\neg_2(\uparrow_1 x)$ and $V_x = \{z \mid \neg\exists y : y \geq_2 x \wedge y \geq_1 z\}$. To see that V_x is regular open, note first that it is clearly a 1-upset. Now suppose that $y \notin V_x$. This means that there is $z \in \mathscr{X}$ such that $z \geq_2 x$ and $z \geq_1 y$. But then for any $w \geq_2 z$, $w \geq_2 x$, and therefore $y \notin \neg_1\neg_2(V_x)$. Hence $\neg_1\neg_2(V_x) \subseteq V_x$, which implies that V_x is regular open.

The Allwein-MacCaull representation theorem (Lemma 2.8), coupled with Theorem 2.9, implies the corollary mentioned above.

Corollary 2.11. *The functors α and ρ form an idempotent contravariant adjunction.*

Indeed, to establish that the adjunction is idempotent, it is enough to show that the unit of the adjunction is a natural isomorphism. Now for any $\mathscr{X} \in \mathbf{Bos}$, $L \in \mathbf{cLat}$, $g \in \mathrm{Hom}_{\mathbf{Bos}}(\mathscr{X}, \alpha(L))$, and $a \in L$, $\overline{g}^{-1}(a) = g^{-1}[\uparrow_1(a, 0)] = \rho(g)(\phi_L(a))$. Thus ϕ_L is the unit of the adjunction between α and ρ. Moreover, since by Lemma 2.6 ϕ_L is an isomorphism natural in L, $\alpha(\phi_L)$ is an isomorphism natural in $\alpha(L)$.

It is a general categorical fact that the fixpoints of an idempotent adjunction induce an equivalence of categories. Therefore the following definition is natural.

Definition 2.12. A *b-frame* is a boset $\mathscr{X}$ such that $\epsilon_{\mathscr{X}} : \mathscr{X} \to \alpha\rho(\mathscr{X})$ is an isomorphism. Let **bF** be the full subcategory of **Bos** of all b-frames.

As an immediate consequence of Corollary 2.11, we obtain the following:

Theorem 2.13. *The categories* **cLat** *and* **bF** *are dually equivalent.*

This result, however, only amounts to an abstract characterization of the duals of complete lattices. A more useful characterization would identify precisely which properties of a boset $\mathscr{X}$ guarantee that $\epsilon_{\mathscr{X}}$ is an isomorphism. In the next part of this section, a special class of b-morphisms, which generalize the notion of a dense embedding in the forcing literature, is introduced. We then show that for any boset $\mathscr{X}$, $\epsilon_{\mathscr{X}}$ is such a dense embedding. Finally, in the last part, we will show that imposing some natural conditions on bosets allows us to strengthen this dense embedding to an isomorphism, thus obtaining a more concrete characterization of b-frames.

2.3. Dense embeddings

We begin by introducing the following notation which will be used extensively:

Definition 2.14. Let $\mathscr{X} := (X, \leq_1, \leq_2)$ be a boset. For any $x, y \in X$ and $k, j \in \mathscr{P}(\{1,2\}) - \{\emptyset\}$, we introduce the following notation:

$$x\ {}_j\!\perp_k y \text{ iff } \neg\exists z : y \leq_s z \text{ for all } s \in j \text{ and } x \leq_t z \text{ for all } t \in k.$$

In particular, we say that x is *independent from* y whenever $x_2\!\perp_1 y$.

It is straightforward to note that for any poset $\mathbb{P} = (P, \leq)$, if we view $\mathbb{P}$ as a Kripke boset $(P, \leq, \geq)$, we have that $x_2\!\perp_1 y$ iff $y \not\leq x$, while if we view $\mathbb{P}$ as a forcing boset $(P, \geq, \geq)$, we have that $x_2\!\perp_1 y$ iff $x \perp y$, where $\perp$ is the standard incompatibility relation in the forcing literature. More generally, following the notation introduced in Remark 2.10, we have in any boset $\mathscr{X}$ that $x_2\!\perp_1 y$ iff $x \notin U^y$ iff $y \in V_x$.

In Allwein-MacCaull bosets, i.e., bosets of the form $\alpha(L)$ for some complete lattice L, independence can be seen as a purely graph-theoretic way of capturing the order on L:

Lemma 2.15. *Let* $(X, \leq_1, \leq_2)$ *be* $\alpha(L)$ *for some complete lattice* L. *Then for any* $x = (f_x, i_x)$ *and any* $y = (f_y, i_y)$, *we have that:*

1. $x_2\!\perp_1 y$ *iff* $f_y \leq i_x$*;*
2. $x\ {}_{12}\!\perp_2 y$ *iff* $f_x \leq i_x \vee i_y$*;*
3. $x\ {}_{12}\!\perp_1 y$ *iff* $f_x \wedge f_y \leq i_y$*.*

Proof. All three items follow immediately from the fact that for any $a, b \in L$, $a \not\leq b$ iff the pair $(a, b) \in \alpha(L)$. $\square$

Let us now focus on a specific class of b-morphisms, which generalize in a natural way the notion of a dense embedding between forcing posets.[2]

Definition 2.16. Let $\mathscr{X} = (X, \leq_1^X, \leq_2^X)$ and $\mathscr{Y} = (Y, \leq_1^Y, \leq_2^Y)$ be two bosets and $f : \mathscr{X} \to \mathscr{Y}$ a b-morphism. Then:

[2] For an overview of the basic notions and techniques in forcing, see for example [45].

- f is *dense* if for any $y \in \mathscr{Y}$ there is $x \in \mathscr{X}$ such that $y \leq_{12}^{Y} f(x)$.
- f is an *embedding* if for any $x, y \in \mathscr{X}$, we have that $x_1 \bot_2 y$ iff $f(x)_1 \bot_2 f(y)$.

The next two lemmas show that dense b-morphisms and embeddings are dual to injective and surjective lattice morphisms respectively.

Lemma 2.17. *Let $f : \mathscr{X} = (X, \leq_1^X, \leq_2^X) \to \mathscr{Y} = (Y, \leq_1^Y, \leq_2^Y)$ be a b-morphism. Then:*

1. *f is dense iff $\rho(f)$ is injective;*
2. *f is an embedding iff $\rho(f)$ is surjective.*

Proof. 1. Suppose f is dense, and let $U, V \in \rho(\mathscr{Y})$ such that $U \neq V$. Without loss of generality, there is $y \in U \cap \neg_2 V$, and since f is dense, there must be $x \in \mathscr{X}$ such that $f(x) \geq_{12}^{Y} y$. But then $f(x) \in U \cap \neg_2 V$, which means that $x \in \rho(f)(U) - \rho(f)(V)$. Hence $\rho(f)$ is injective.
Conversely, suppose there is $y \in \mathscr{Y}$ such that for all $x \in X$, $f(x) \ngeq_{12}^{Y} y$. Let $U = \neg_1\neg_2(\uparrow_1 y)$ and $V = \{z \mid y_2 \bot_1 z\}$. Clearly, $U \nsubseteq V$, but we claim that $f^{-1}[U] \subseteq f^{-1}[V]$. Note that this implies that $U \cap V \neq U$ but $f^{-1}[U] = f^{-1}[U \cap V] = f^{-1}[U] \cap f^{-1}[V]$ and thus that $\rho(f)$ is not injective. For the proof of the claim, suppose towards a contradiction that there is $x \in f^{-1}[U] - f^{-1}[V]$. Since both $f^{-1}[U]$ and $f^{-1}[V]$ are regular open, without loss of generality we may assume that $x \in \neg_2 f^{-1}[V]$. Moreover, note that, since $x \in f^{-1}[U]$, then there is $q \geq_1^Y y$ such that $f(x) \leq_2^Y q$. But this means that there is $z \geq_2^X x$ such that $f(z) \geq_1^Y q \geq_1^Y y$. Now since $x \in \neg_2 f^{-1}[V]$, this means that $f(z) \notin V$, and hence there is $q' \geq_1^Y f(z)$ such that $q' \geq_2^Y y$. Hence there is $z' \geq_1^X z$ such that $f(z') \geq_2^Y q' \geq_2^Y y$. But since also $f(z') \geq_1^Y f(z) \geq_1^Y y$, we have that $f(z') \geq_{12}^Y y$, contradicting our assumption. This completes the proof.
2. Suppose f is an embedding, and let $U \in \rho(\mathscr{X})$. We claim that $f^{-1}f[U] = U$. To see this, assume that $x \in f^{-1}f[U]$. Then $f(x) = f(y)$ for some $y \in U$. Now for any $z \geq_1 x$, this implies that $f(z) \geq_1 f(y)$, and thus $\neg f(y)_1 \bot_2 f(z)$. Hence $\neg y_1 \bot_2 z$, which implies that $z \in C_2(U)$, and hence $x \in \neg_1\neg_2(U) = U$. Thus $f^{-1}f[U] \subseteq U$, and the converse direction is obvious. Now let $V = \neg_1\neg_2 f[U]$, and note that we have that

$$\rho(f)(V) = f^{-1}(\neg_1\neg_2 f[U]) = \neg_1\neg_2(f^{-1}f[U]) = \neg_1\neg_2(U) = U.$$

Thus $\rho(f)$ is surjective.
Conversely, assume there are $x, y \in \mathscr{X}$ such that $x_1 \bot_2 y$ but there is $z \in \mathscr{Y}$ such that $f(x) \leq_1 z$ and $f(y) \leq_2 z$. Note that this implies that there is $y' \geq_2 y$ such that $z \leq_1 f(y')$. We claim that for any $U \in \rho(\mathscr{Y})$, if $f(x) \in U$, then $f(y') \in U$. Since $y' \notin \neg_1\neg_2(\uparrow_1 x)$, this will imply that $\rho(f)$ is not surjective. For the proof of the claim, it is enough to notice that $f(y') \geq_1 z \geq_1 f(x)$, since any $U \in \rho(\mathscr{Y})$ is a 1-upset. This completes the proof. □

Lemma 2.18. *Let $f : L \to M$ be a lattice homomorphism. Then:*

1. *f is injective iff $\alpha(f)$ is dense;*
2. *f is surjective iff $\alpha(f)$ is an embedding.*

Proof. 1. Note that by lemma 1.11, we have that f is injective iff $\rho\alpha(f)$ is injective. But by the previous lemma, we have that $\rho\alpha(f)$ is injective iff $\alpha(f)$ is dense.
2. Similarly, we have that f is surjective iff $\rho\alpha(f)$ is surjective, which by the previous lemma is equivalent to $\alpha(f)$ being an embedding. □

Dense embeddings will be of crucial relevance later on, as we will use extensively the fact that a dense embedding between two bosets induces an isomorphism of the dual complete lattices. Once again, this can be seen as a generalization of the well-known result that two posets are forcing equivalent iff there is a dense embedding between them. In particular, if $\mathscr{X} = (X, \leq_1^X, \leq_2^X)$ is a boset, then a *dense sub-boset* of $\mathscr{X}$ is a boset $\mathscr{Y} = (Y, \leq_1^X|Y, \leq_2^X|Y)$, where $Y \subseteq X$ and for any $x \in X$ there is $y \in Y$ such that $x \leq_{12} y$. The proof of the following lemma is immediate when one realizes that if $\mathscr{Y}$ is a dense sub-boset of $\mathscr{X}$, then the inclusion map $\iota : \mathscr{Y} \to \mathscr{X}$ is a dense embedding.

Lemma 2.19. *Let $\mathscr{Y}$ be a dense sub-boset of $\mathscr{X}$. Then $\rho(\mathscr{X})$ is isomorphic to $\rho(\mathscr{Y})$.*

Moreover, as shown in Lemma 2.6, the unit η_L of the adjunction $\alpha \dashv \rho$ is an isomorphism for any complete lattice L. A similar result holds for the counit $\epsilon_{\mathscr{X}}$.

Lemma 2.20. *For any boset $\mathscr{X}$, the map $\epsilon_{\mathscr{X}} : \mathscr{X} \to \alpha\rho(\mathscr{X})$ is a dense embedding.*

Proof. Suppose we have that $x_2 \perp_1 y$. Then $y \in V_x$, which implies that $U^y \subseteq V_x$. Hence $(U^x, V_x)_2 \perp_1 (U^y, V_y)$, which means that $\epsilon_{\mathscr{X}}$ is an embedding. For density, assume $U, V \in \rho(\mathscr{X})$ are such that $U \nsubseteq V$. Then since both U and V are regular open there is $y \in \mathscr{X}$ such that $y \in U \cap \neg_2 V$. But this implies that $U^y \subseteq U$ and that $V \subseteq V_y$, and hence $(U, V) \leq_{12} (U^y, V_y)$. □

However, it is easy to verify that dense embeddings are not isomorphisms in the category of bosets: since b-morphisms are maps sending points to points, any b-morphism with an inverse must be bijective. In order to characterize b-frames, we must therefore impose some extra conditions on a boset $\mathscr{X}$ that guarantee that the dense embedding $\epsilon_{\mathscr{X}}$ is an isomorphism.

2.4. Characterizing B-frames

The following definition generalizes the notion of a separative poset in forcing:

Definition 2.21. A boset $\mathscr{X} = (X, \leq_1, \leq_2)$ is *separative* if it satisfies the following three properties:

- $\leq_1 \cap \leq_2$ is anti-symmetric;
- for any $x, y \in \mathscr{X}$, $x \leq_1 y \Leftrightarrow \forall z(z_2 \perp_1 x \to z_2 \perp_1 y)$ (1-*separativity*);
- for any $x, y \in \mathscr{X}$, $x \leq_2 y \Leftrightarrow \forall z(x_2 \perp_1 z \to y_2 \perp_1 z)$ (2-*separativity*).

In particular, it is straightforward to verify that any poset $(X, \leq)$ is separative iff the corresponding forcing boset $(X, \geq, \geq)$ is separative.

In order to characterize b-frames, we will also need a second property.

Definition 2.22. A boset $\mathscr{X} = (X, \leq_1, \leq_2)$ is *complete* if for any $U, V \in \rho(\mathscr{X})$ such that $U \nsubseteq V$, there is $z \in \mathscr{X}$ such that $U = U^z$ and $V = V_z$.

Unlike separativity, this property requires (monadic) second-order quantification to be expressed. We will show later on (Lemma 3.24) that this requirement is necessary, i.e., that there is no possible first-order axiomatization of b-frames.

We can now establish that separativity and completeness entirely characterize b-frames. Let us start by observing that the regular open sets of a complete separative boset $\mathscr{X}$ have a very concrete characterization: they are precisely the principal 1-upsets of $\mathscr{X}$.

Lemma 2.23. *Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be a complete separative boset. Then for any non-empty $U \subseteq X$, $U \in \rho(\mathscr{X})$ iff $U = \uparrow_1 x$ for some $x \in X$.*

Proof. We first claim that for any $x \in X$, $\neg_1\neg_2(\uparrow_1 x) = \uparrow_1 x$. To see this, note that it suffices to show the left-to-right direction since $\uparrow_1 x$ is 1-upward closed. By separativity, if $x \nleq_1 y$ for some $y \in X$, then there is $z \in X$ such that $z_2\perp_1 x$ but $\neg z_2\perp_1 y$. Let $z' \geq_1 y$ such that $z' \geq_2 z$. Clearly, $z'{}_2\perp_1 x$, for otherwise we would have $\neg z_2\perp_1 x$. Hence $z' \in \neg_2(\uparrow_1 x)$, which implies that $y \notin \neg_1\neg_2(\uparrow_1 x)$, which concludes the proof of the claim. Hence for any $x \in X$, $\uparrow_1 x \in \rho(\mathscr{X})$. Now let U be a non-empty subset in $\rho(\mathscr{X})$. Then as $U \nsubseteq \emptyset$, there is some $x \in X$ such that $U = U_x = \neg_1\neg_2(\uparrow_1 x) = \uparrow_1 x$. Thus any non-empty $U \in \rho(\mathscr{X})$ is $\uparrow_1 x$ for some $x \in X$. □

The next two lemmas establish the characterization of b-frames mentioned above.

Lemma 2.24. *Every b-frame is separative and complete.*

Proof. It is enough to show that $\alpha(L)$ is separative and complete for any complete lattice L. Note first that it is clear from the definition of $\alpha(L)$ that $\leq_1 \cap \leq_2$ is antisymmetric. For 1-separativity, suppose $(a, b) \nleq_1 (c, d)$ for some $a, b, c, d \in L$. This means that $c \nleq a$, and thus $(c, a) \in \alpha(L)$. But clearly $(c, a)_2\perp_1(a, b)$ yet $\neg(c, a)_2\perp_1(c, d)$. The converse direction is trivial. For 2-separativity, suppose $(a, b) \nleq_2 (c, d)$. Then $b \nleq d$, which means that $(b, d) \in \alpha(L)$. But then $(a, b)_2\perp_1(b, d)$, yet $\neg(c, d)_2\perp_1(b, d)$. Hence $\alpha(L)$ is separative. For completeness, recall first that $\eta_L : L \to \rho\alpha(L)$ is an isomorphism. For any $U \nsubseteq V \in \rho\alpha(L)$, let $a = \eta_L^{-1}(U)$ and $b = \eta_L^{-1}(V)$ be elements of L, and note that we have that $(a, b) \in \alpha(L)$. Since $U = \eta_L(a) = \uparrow_1(a, 0) = \uparrow_1(a, b)$, we have that $U = U^{(a,b)}$. Moreover, for any $(c, d) \in \alpha(L)$, we have that $(a, b)_2\perp_1(c, d)$ iff $c \leq b$ iff $(b, 0) \leq_1 (c, d)$ iff $(c, d) \in \eta_L(b) = V$. Thus $V = V_{(a,b)}$, which completes the proof. □

Coupled with Lemma 2.20, this lemma generalizes to bosets the standard result that any poset is forcing equivalent to a separative poset.

Lemma 2.25. *Every complete separative boset is a b-frame.*

Proof. Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be a complete separative boset. We have to show that the map $\epsilon_{\mathscr{X}} = \mathscr{X} \to \alpha\rho(\mathscr{X})$ is an isomorphism, i.e., that it is bijective and reflects both preorders.

- Note first that since $\leq_1 \cap \leq_2$ is antisymmetric, to prove injectivity it is enough to show that both preorders are reflected by $\epsilon_{\mathscr{X}}$. Let $x, y \in \mathscr{X}$, and assume $x \nleq_1 y$. Then $\uparrow_1 y \nsubseteq \uparrow_1 x$, which since $\mathscr{X}$ is separative implies that $U^y \nsubseteq U^x$ and hence that

$$\epsilon_{\mathscr{X}}(x) = (U^x, V_x) \nleq_1 (U^y, V_y) = \epsilon_{\mathscr{X}}(y).$$

  Similarly, if $x \nleq_2 y$, by separativity there is $z \in \mathscr{X}$ such that $x_2\perp_1 z$ but $\neg y_2\perp_1 z$. But this implies that $z \in V_x$ yet $z \notin V_y$. Hence $\epsilon_{\mathscr{X}}(x) \nleq_2 \epsilon_{\mathscr{X}}(y)$.
- Finally, surjectivity is an immediate consequence of $\mathscr{X}$ being complete, since points in $\alpha\rho(\mathscr{X})$ are precisely pairs (U, V) of elements of $\rho(\mathscr{X})$ such that $U \nsubseteq V$. □

Putting the previous two lemmas together, we obtain the last result of this section.

Theorem 2.26. *A boset is a b-frame iff it is separative and complete.*

2.5. B-frame duality and lattice representations

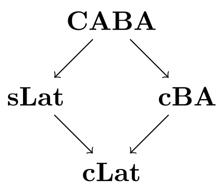
Let us conclude this section by comparing the results obtained above with known results in the literature. There exist, of course, many adjunctions and dualities between categories of lattices and concrete categories, which have various advantages and drawbacks. Representations of lattices as upsets of certain posets typically involve adding some further structure, either in the form of a topology as in Priestley and Esakia duality, or in the form of a second relation. Since our b-frame duality is of the latter kind, we first discuss how it relates to some classical discrete representations in the literature, before comparing it to some well-known dualities of the former kind.

2.5.1. Discrete representations

As mentioned in the introduction, the duality exposed here can be seen as a generalization both of the duality between posets and completely join-prime generated or superalgebraic locales, which itself generalizes Tarski duality between sets and complete atomic Boolean algebras, and of the representation of complete Boolean algebras as regular open sets of separative posets which lies at the heart of some classical results in forcing. Although these representations are well known, they are not always presented from a categorical perspective. We therefore briefly present them below in some detail and somewhat more systematically than what is commonly found in the literature, as this will illuminate the sense in which the b-frame duality presented here generalizes those results.

Definition 2.27. Let **cBA** and **CABA** be the full subcategories of **cLat** whose objects are complete Boolean algebras and complete and atomic Boolean algebras respectively. Let **sLat** be the subcategory of **cLat** whose objects are superalgebraic complete lattices (i.e., completely join-prime generated complete lattices) and whose morphisms are complete lattice homomorphisms.

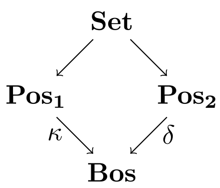
Thus we obtain the following diagram of inclusions of categories:



On the geometric side of these dualities, we have the category of sets and two categories of posets:

Definition 2.28. Let **Set** be the category of all sets and functions between them, $\mathbf{Pos_1}$ the category of posets and monotone maps between them and $\mathbf{Pos_2}$ the category of posets and weakly-dense maps between them.

As mentioned in Example 2.3 above, a poset $(P, \leq_P)$ can be viewed as the boset $(P, \leq_P, \geq_P)$, or as the boset $(P, \geq_P, \geq_P)$. It is straightforward to verify that both constructions lift to two full embedding functors $\kappa : \mathbf{Pos_1} \to \mathbf{Bos}$ and $\delta : \mathbf{Pos_2} \to \mathbf{Bos}$, from which we obtain the following commuting diagram:



We now briefly recall the various correspondences that our result aims to generalize:

Theorem 2.29 *(Tarski).* **CABA** *and* **Set** *are dual to one another:*

- *The functor* $At : \mathbf{CABA} \to \mathbf{Set}$ *maps any complete atomic Boolean algebra to the set of its atoms and any complete Boolean homomorphism* $h : B \to C$ *to the restriction of its left adjoint* $h^* : C \to B$ *to the atoms of* B *and* C*.*
- *The functor* $\mathscr{P} : \mathbf{Set} \to \mathbf{CABA}$ *maps any set* S *to its powerset* $\mathscr{P}(S)$ *and any function* $f : S \to T$ *to the inverse image map* $f^{-1} : \mathscr{P}(T) \to \mathscr{P}(S)$*.*

An early reference for the following result is [54]:

Theorem 2.30 *(Raney).* **sLat** *and* $\mathbf{Pos_1}$ *are dual to one another:*

- *The functor* $\gamma : \mathbf{sLat} \to \mathbf{Pos_1}$ *maps any superalgebraic locale* L *to the poset of its completely join-prime elements with the reverse order on* L *and any complete lattice homomorphism* $h : L \to M$ *maps to the restriction of its left adjoint* $h^* : M \to L$ *to the completely join-prime elements of* M *and* L*.*
- *The functor* $\tau : \mathbf{Pos_1} \to \mathbf{sLat}$ *maps any poset* $(P, \leq_P)$ *to its complete lattice of upsets* $Up(P)$ *and any monotone map* $f : (P, \leq_P) \to (Q, \leq_Q)$ *to the inverse image map* $f^{-1} : Up(Q) \to Up(P)$*.*
- *This* γ*-*τ *duality restricts precisely to Tarski duality between* **CABA** *and* **Set***.*

Let us also note that De Jongh and Troelstra [15] observed that the Raney dual of a complete lattice homomorphism h is a p-morphism if and only if h is also a complete Heyting morphism, meaning that it also preserves the right-adjoint of the meet operation, which exists in any superalgebraic lattice. This yields a restriction of Raney duality to De Jongh-Troesltra duality between the category of superalgebraic locales and Heyting morphisms between them, and the category of posets and p-morphisms, which can also be shown to be generalized by our b-frame duality.

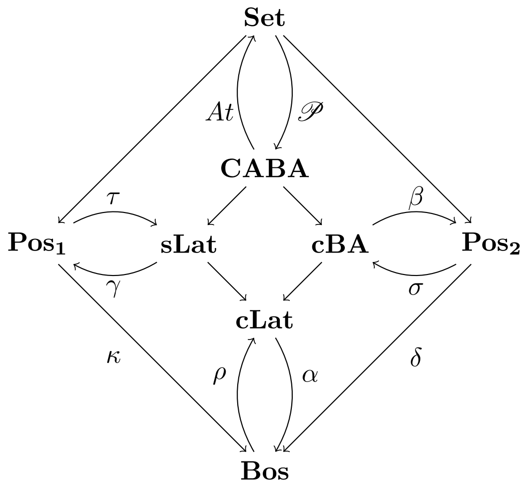
Finally, the following definition is needed in order to express the last one of our theorems:

Definition 2.31. A poset $(P, \leq_P)$ is *separative* if for any $x, y \in P$ such that $x \nleq_P y$, there is $z \in P$ such that $z \leq_P x$ and for all $w \in P$, if $w \leq_P z$ then $w \nleq_P y$. A poset $(P, \leq_P)$ is *complete* if for every non-empty regular open subset U of $(P, \leq_P)$ there is $p \in P$ such that $U = \downarrow p$.

Theorem 2.32 *([40, Chap. 14]). There is an idempotent contravariant adjunction between* **cBA** *and* $\mathbf{Pos_2}$*:*

- *The functor* $\beta : \mathbf{cBA} \to \mathbf{Pos_2}$ *maps any complete Boolean algebra* B *to the poset* $(B_+, \leq_B |B_+)$*, where* $B_+ = B \setminus 0$ *and any complete Boolean homomorphism* $h : B \to C$ *to the restriction of its left-adjoint* $h^* : C_+ \to B_+$*.*
- *The functor* $\sigma : \mathbf{Pos_2} \to \mathbf{cBA}$ *maps any poset to its Boolean algebra of regular open downsets* $RO(P)$ *and any weakly dense map* $f : (P, \leq_P) \to (Q, \leq_Q)$ *to the inverse image map* $f^{-1} : RO(Q) \to RO(P)$*.*
- *The functors* σ *and* β *restrict to a duality between the full subcategories of fixpoints of* $\sigma\beta$ *and* $\beta\sigma$*, i.e., between* **cBA** *and the full subcategory of* $\mathbf{Pos_2}$ *of complete separative posets.*
- *If* B *is a complete atomic Boolean algebra, then* $At(B)$ *with the discrete order is a dense subposet of* $\beta(B)$*. Conversely, if* S *is a set, then* $\sigma(S, \Delta_S)$ *is isomorphic to* $\mathscr{P}(S)$*.*

Combining these results with our adjunction between complete lattices and bosets, we obtain the following diagram of categories:



Note that not all squares in the diagram above commute, not even up to isomorphisms. For example, if B is a complete and atomic Boolean algebra, then $At(B)$ is a discrete poset, while the order on $\beta(B)$ is the restriction of the order on B. Similarly, for a superalgebraic lattice L, the second order on $\kappa(L)$ is the converse of the first one, while this is not the case for $\alpha(L)$. Nonetheless, we have the following result, which gives a precise meaning to the claim that our α-ρ adjunction generalizes both the γ-τ duality and the β-σ adjunction (the obvious inclusion functors have been omitted):

Theorem 2.33.

1. *There is a natural isomorphism between the functors $\rho\kappa$ and τ and between the functors $\rho\delta$ and σ.*
2. *There are natural transformations $\eta^1 : \kappa\gamma \to \alpha$ and $\eta^2 : \delta\beta \to \alpha$, such that each component is a dense embedding.*

Proof. 1. Let $\mathbb{P} = (P, \leq_P)$ be a poset. Then $\kappa(\mathbb{P}) = (P, \leq_P, \Delta_P)$. Since the second ordering on P is discrete, the regular opens of $\kappa(\mathbb{P})$ are clearly the upsets of $(\mathbb{P})$, hence $\rho\kappa(\mathbb{P})$ is isomorphic to $\tau(\mathbb{P})$. The naturality condition is straightforward. Similarly, if $\mathbb{Q} = (Q, \leq_Q)$ is a poset, then $\delta(\mathbb{Q}) = (Q, \geq_Q, \geq_Q)$. Clearly, the regular opens of $\delta(\mathbb{Q})$ are precisely the regular open downsets of $\mathbb{Q}$, hence $\rho\delta(\mathbb{Q})$ is isomorphic to $\sigma(\mathbb{Q})$. Again, the naturality condition is straightforward.

2. Let L be superalgebraic, with $\leq$ the order on L. Then $\gamma(L) = (J(L), \geq |J(L))$, where $J(L)$ is the set of completely join-prime elements of L and $\kappa\gamma(L) = (J(L), \geq |J(L), \Delta_{J(L)})$. Now given a completely-join prime element $p \in J(L)$, let $\eta^1_L(p) = (p, p^\delta)$, where $p^\delta = \bigvee\{c \in L \mid p \nleq c\}$. It is straightforward to check that $\eta^1_L : \kappa\gamma(L) \to \alpha(L)$ is well-defined and is a b-morphism. Moreover, for any $(a, b) \in \alpha(L)$, we have that $a \nleq b$ and hence, since L is superalgebraic, there is $p \in J(L)$ such that $p \leq a$ but $p \nleq b$. But this at once implies that $(a, b) \leq_{12} \eta^1_L(p)$, which establishes that η^L_1 is dense. Finally, to see that it is an embedding, note that for any $p, q \in J(L)$, $p\,{}_2{\perp}_1 q$ iff $p \nleq q$ iff $q \leq p^\delta$, from which it follows that $(p, p^\delta)\,{}_2{\perp}_1(q, q^\delta)$. Hence each component of $\eta^1 : \kappa\gamma \to \alpha$ is a dense embedding. The naturality condition on η^1 is left to the reader.

   Similarly, let B be a complete Boolean algebra, with $\leq$ the order on B. Then $\beta(B) = (B_+, \leq |B_+)$ and $\delta\beta(B) = (B_+, \geq |B_+, \geq |B_+)$. Given $b \in B_+$, let $\eta^2_B(b) = (b, \neg b)$. Once again it is straightfoward to check that $\eta^2_B : \delta\beta(B) \to \alpha(B)$ is well-defined and a b-morphism. Moreover, for any $(a, b) \in \alpha(B)$, we have that $a \nleq b$, hence $a \wedge \neg b \in B_+$. But clearly $(a, b) \leq_{12} \eta^2_B(a \wedge \neg b)$, which shows that η^2_B is dense. Finally, to check that it is also an embedding, note that, for any $a, b \in B_+$, $a\,{}_2{\perp}_1 b$ iff $a \wedge b \leq 0$ iff $b \leq \neg a$ iff $\eta^2_B(a)\,{}_2{\perp}_1\eta^2_B(b)$. Hence each component of $\eta^2 : \delta\beta \to \alpha$ is a dense embedding. Once again, the naturality condition is left to the reader. □

Finally, let us conclude by mentioning once again that the representation of any complete lattice as the regular opens of some boset was already proved in [2]. However, Allwein and MacCaull do not offer a treatment of morphisms, nor do they identify the duals of complete lattices. By contrast, the notion of a dense embedding, which is a generalization of a standard tool in forcing, plays a central role in our characterization of b-frames and will also prove itself very useful in establishing correspondences between lattice equations and first-order properties of bosets.

2.5.2. Topological dualities

The discrete dualities presented above only offer representations for complete lattices. As is well known, extending such dualities to categories of (possibly incomplete) lattices typically requires one to topologize the dual geometric structures. The celebrated examples are, of course, Stone's duality between Boolean algebras and Stone spaces [61], Priestley's duality between bounded distributive lattices and Priestley spaces [52] and Esakia's duality between Heyting algebras and Esakia spaces [18].

For bounded lattices, several dualities have been developed. Urquhart [63] developed a topological representation for bounded lattices in which the points in the dual space of a bounded lattice are pairs of a filter and an ideal which are maximal with respect to one another. This representation, which appeals to Zorn's Lemma in an essential way, was later lifted to a duality by Hartung [34] and generalizes Stone and Priestley's dualities, in the sense that the restriction to distributive lattices and Boolean algebras yields Priestley and Stone spaces. However, the morphisms covered by the Urquhart-Hartung duality are only the surjective lattice homomorphisms, and the duality is often seen as more cumbersome to work with than Priestley or Stone's. As a consequence, a number of alternative dualities for bounded lattices have been proposed over the years. Gehrke and van Gool [24] have recently developed a duality closely related to Urquhart-Hartung duality, in which however the morphisms between lattices considered are not the usual lattice morphisms. Dualities based on spaces of filters rather than maximal filters have also been offered by Hartonas [29] and Jipsen and Moshier [50]. These dualities do not immediately generalize Stone duality for Boolean algebras or Priestley duality for distributive lattices (even though Jipsen and Moshier's approach is closely related to Stone's duality for distributive lattices via spectral spaces [62]), but still involve defining only one topology on the space of proper filters of the dual lattice and rely on the axiom of choice.

Finally, the existing dualities closest to b-frame duality are filter-ideal based dualities, such as the duality between bounded lattices and enhanced L-spaces presented by Allwein and Hartonas in [1] and the duality with L-frames introduced by Hartonas and Dunn in [32] and subsequently developed more recently by Hartonas in [31,33]. The Allwein-Hartonas representation of a bounded lattice L is obtained by considering pairs of a filter and an ideal on L that do not intersect and using the inclusion orderings on filters and ideals to define order-closed sets that generate a topology. Morphisms are defined as continuous functions that preserve both orderings and satisfy a condition similar to that imposed on b-morphisms. The Hartonas-Dunn duality, by contrast, is inspired from the theory of polarities and has close ties with the theory of generalized Kripke frames of Gehrke [21]. A bounded lattice L is mapped to the triple $(X, \perp, Y)$, where X and Y are the posets of filters and ideals of L endowed with a Stone-like topology, and $\perp$ is a relation on $X \times Y$. Such triples are called L-frames, and a lattice L can then be recovered as the clopen sets of X that are also fixpoints of the Galois connection between $\mathscr{P}(X)$ and $\mathscr{P}(Y)$ induced by $\perp$. Morphisms between two L-frames $(X, \perp, Y)$ and $(X', \perp', Y')$ are pairs of continuous maps between X and X' and between Y and Y' that commute with the closure operators generated by $\perp$ and $\perp'$. Since points in our b-frame representation of a complete lattice L are pairs of elements of the original lattice, and we work with two orderings, it is natural to see this latest duality as giving rise to a "topologized" version of our b-frame duality, just like Stone duality topologizes Tarski duality, or the more recent duality between Boolean algebras and UV-spaces presented in [9] topologizes the duality between complete Boolean algebras and complete separative posets. Instead of being a triple composed of two spaces and a relation between them, the duals of lattices in such a topologized version of b-frame duality would rather be single bitopological spaces of filter-ideal pairs, and fixpoints of the Galois connection induced by the relation $\perp$ would be replaced regular open sets induced by the two topologies. The details of such a duality and of its exact relationship to the Hartonas-Dunn one are left for future work. For a systematic comparison of the representation of complete lattices via polarities and bi-ordered sets, we refer the reader to [37].

3. Correspondence theory

In the previous section, we established an idempotent adjunction between complete lattices and bosets and showed how to restrict it to a duality between complete lattices and b-frames. In this section, we will see how this duality restricts further to specific classes of complete lattices. The goal is to identify properties of b-frames which correspond to properties of complete lattices, in the precise sense that a b-frame $\mathscr{X}$ has a property P if and only if $\rho(\mathscr{X})$ is in a certain class K of complete lattices. As it will become apparent later on, once we find such a characterizing condition on b-frames, we can always extend our result to a correspondence between *bosets* and complete lattices. In this section, we restrict ourselves to equationally definable classes and focus on characterizing the duals of complete distributive lattices, Heyting algebras and Boolean algebras. Our approach for Heyting algebras is also straightforwardly adapted in Section 5 to provide characterizations of complete co-Heyting and bi-Heyting algebras. As mentioned in the previous section, there is a well-established duality theory for such structures, originating with Stone duality for Boolean algebras [61]. The Stone duals of complete Boolean algebras are *extremally disconnected* Stone spaces, in which the closure of every open set is open. Building on this characterization, Priestley [52] identifies the Priestley duals of complete distributive lattices as those Priestley spaces in which the smallest closed upset containing S is open for every open upset S. An equivalent characterization in terms regular opens being clopens also exists in the bitopological duality for distributive lattices of [8, Thm. 6.25]. Finally, the topological representations of MacNeille completions of Heyting, co-Heyting and bi-Heyting algebras via Esakia duality obtained in [27] also yield characterizations of the Esakia duals of complete Heyting, co-Heyting and bi-Heyting algebras.

In the context of the study of semantics for non-classical logics based on complete lattices, we see two advantages of the discrete approach we develop here over the standard topological approach. First, discrete, graph-theoretic semantics allow for simple geometric arguments that are sometimes harder to adapt in a topological setting. Of course, there is always the option to "discretize" a topological representation. For example, one can forget about the topology on the dual Priestley space of a distributive lattice L and focus instead on the lattice of upsets of the resulting poset. But the obvious drawback is that this lattice will not be isomorphic to L, but only to its canonical extension,[3] which is always a superalgebraic locale. Furthermore, all characterizations of particular classes of complete lattices mentioned above require imposing second-order conditions on the dual topological spaces. By contrast, the dual b-frames of the kind of complete lattices considered in this section and the next two can be straightforwardly given first-order, geometrically intuitive characterizations in the language of bosets, even though the corresponding characterization for bosets must be second-order. To sum up, there is a necessary trade-off between generality and concreteness when giving representations of lattices, and we believe that the discrete representation of complete lattices developed here is a suitable equilibrium point for our purposes.

3.1. Distributive lattices

We start by characterizing the duals of distributive lattices. It is well known that the variety of distributive lattices, unlike the varieties of Heyting and Boolean algebras, is not closed under MacNeille completions. A

[3] See [17,22,23,42,43] for some literature on canonical extensions.

similar phenomenon manifests itself here: the characterization of the dual b-frames of complete distributive lattices is more intricate and uses the duality in an essential way.

We start by identifying a property of b-frames that are the duals of distributive lattices.

Lemma 3.1. *Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be a b-frame such that $\rho(\mathscr{X})$ is distributive. Then $\mathscr{X}$ satisfies the following property:*

$$\forall x, y, z((x\,{}_{12}{\perp}_1 y \wedge x\,{}_{12}{\perp}_2 z) \rightarrow \exists w(y\,{}_1{\perp}_2 w \wedge w\,{}_1{\perp}_2 z)). \tag{2}$$

Proof. Let $x = (f_x, i_x), y = (f_y, i_y), z = (f_z, i_z)$ such that $x\,{}_{12}{\perp}_1 y$ and $x\,{}_{12}{\perp}_2 z$. Then $f_x \wedge f_y \leq i_x$ and $f_x \leq i_z \vee i_x$. We claim that this implies that $i_z \not\leq f_y$. Note that if this is true, then there is $w = (i_z, f_y)$ such that $y\,{}_1{\perp}_2 w$ and $w\,{}_1{\perp}_2 z$. For the proof of the claim, assume towards a contradiction that $i_z \leq f_y$. Then

$$f_x = f_x \wedge (i_z \vee i_x) \leq f_x \wedge (f_y \vee i_x) \leq (f_x \wedge f_y) \vee (f_x \wedge i_x) \leq i_x \vee (f_x \wedge i_x) = i_x,$$

a contradiction. □

It is also straightforward to see that this property is also sufficient for the dual lattice of a b-frame to be distributive:

Lemma 3.2. *Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be the dual b-frame of some complete lattice L. Then if $\mathscr{X}$ satisfies (2), L is distributive.*

Proof. Recall that a lattice L is distributive iff for any $a, b, c \in L$, $a \wedge c \leq b$ and $a \leq b \vee c$ implies that $a \leq b$. So assume towards a contradiction that there are $a, b, c \in L$ such that $a \not\leq b$, but $a \wedge c \leq b$ and $a \leq b \vee c$. Since this implies that $1 \not\leq c \not\leq 0$, consider the points $x = (a, b)$, $y = (c, 0)$ and $z = (1, c)$. Note that, by assumption, we have that $x\,{}_{12}{\perp}_1 y$ and $x\,{}_{12}{\perp}_2 z$, so since (2) holds there is some $w = (f_w, i_w)$ such that $y\,{}_1{\perp}_2 w$ and $w\,{}_1{\perp}_2 z$. But the former implies that $c \leq i_w$ and the latter implies that $f_w \leq c$, and therefore $f_w \leq i_w$, a contradiction. □

In light of the previous two lemmas, we may define a *distributive* boset to be a boset $\mathscr{X}$ satisfying (2). Distributive b-frames (i.e., distributive bosets that are also b-frames) and b-morphisms between them form a category **DbF**. The following theorem is an immediate consequence of the two previous lemmas.

Theorem 3.3. *The duality between* **cLat** *and* **bF** *restricts to a duality between* **cDL** *and* **DbF**.

We therefore obtain a first-order characterization of the dual b-frames of distributive lattices. Moreover, using results from the previous section, we can also obtain a second-order characterization of bosets $\mathscr{X}$ such that $\rho(\mathscr{X})$ is distributive as follows:

Lemma 3.4. *For any boset $\mathscr{X}$, $\rho(\mathscr{X})$ is distributive iff $\mathscr{X}$ densely embeds into a distributive b-frame.*

Proof. For the left to right direction, recall that $\epsilon_{\mathscr{X}} : \mathscr{X} \rightarrow \alpha\rho(\mathscr{X})$ is a dense embedding. Moreover, by the previous lemma, $\alpha\rho(\mathscr{X})$ is distributive if $\rho(\mathscr{X})$ is distributive. For the converse direction, recall that if $f : \mathscr{X} \rightarrow \mathscr{Y}$ is a dense embedding, then $\rho(f)$ is an isomorphism. Thus if $\mathscr{X}$ densely embeds into a distributive b-frame, $\rho(\mathscr{X})$ must be distributive. □

3.2. Heyting algebras

Let us now move on to the case of Heyting algebras. We will first isolate a property of certain points in a boset, called Heyting points and show that the existence of enough such points in a boset $\mathscr{X}$ guarantees that $\rho(X)$ is a Heyting algebra. As we will see, for an arbitrary boset $\mathscr{X}$, the existence of enough Heyting points in $\mathscr{X}$ is not necessary for $\rho(\mathscr{X})$ to be a cHA, but we will show that it is in the case of b-frames. This will give us a complete, first-order characterization of the dual b-frames of cHA's, which can then be extended to bosets in a straightforward way. A key notion in this characterization is that of a *nucleus* on a complete lattice. Nuclei play an important role in pointfree topology [41,51], where they provide an algebraic generalization of the notion of subspace of a topological space. Nuclei on complete Heyting algebras have also been used to provide alternative semantics for intuitionistic logic [6,7]. The connection with nuclear semantics for intuitionistic logic will be further explored in Section 6.

Definition 3.5. Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be a boset. A *Heyting point* of $\mathscr{X}$ is a point x^* such that $\forall y \in \mathscr{X}$, $x^*_{12}\bot_1 y$ iff $x^*{}_2\bot_1 y$.

Note that, in this definition, the right-to-left direction is satisfied by any point x in a boset $\mathscr{X}$: for any two $x, y \in \mathscr{X}$, if there is no 2-successor of x that is also a 1-successor of y, then in particular there is no 1-and-2-successor of x that is also a 1 successor of y. The converse direction, however, does not hold in general. Thus Heyting points are those for which their independence from any other point is equivalent to a weaker condition.

Definition 3.6. A *Heyting boset* is a boset $\mathscr{X}$ such that the Heyting points of $\mathscr{X}$ are dense, i.e., the following holds:

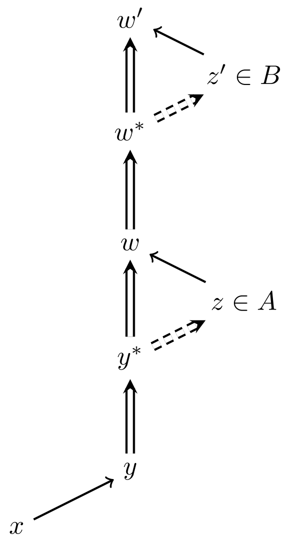
$$\forall x \exists x^* \geq_{12} x \forall y (x^*_{12}\bot_1 y \leftrightarrow x^*_2\bot_1 y). \tag{3}$$

Equivalently, Heyting bosets are bosets in which the sub-boset of Heyting points is dense. In that sense, we may think of Heyting bosets as bosets in which there are "enough" Heyting points. The importance of Heyting points is established by the next lemma.

Lemma 3.7. *Let $(X, \leq_1, \leq_2)$ be a Heyting boset. Then $\neg_1\neg_2$ is a nucleus on $\mathscr{O}_1$.*

Proof. Recall first that a nucleus on a complete lattice L is a closure operator j such that $j(a \wedge b) = j(a) \wedge j(b)$ for any $a, b \in L$. Since $\neg_1\neg_2$ is always a closure operator on $\mathscr{O}_1$, we only need to check that for any $A, B \in \mathscr{O}_1$, $\neg_1\neg_2(X) \cap \neg_1\neg_2(Y) \subseteq \neg_1\neg_2(X \cap Y)$. Suppose $x \in \neg_1\neg_2(X) \cap \neg_1\neg_2(Y)$, and let $y \geq_1 x$. Fix some Heyting point $y^* \geq_{12} y$, and note that $y^* \geq_1 x$, which means that there is $z \geq_2 y^*$ such that $z \in A$. Since $\neg\,(y^*{}_2\bot_1 z)$, we also have $\neg\,(y^*{}_{12}\bot_1 z)$, so let $w \geq_{12} y^*$ such that $z \leq_1 w$, and fix a Heyting point $w^* \geq_{12} w$. Since A is a 1-upset, we have that $w^* \in A$. Moreover, since $x \leq_1 y \leq_1 y^* \leq_1 w^*$, there is $z' \geq_2 w^*$ such that $z' \in B$. Since $\neg\,(w^*{}_2\bot_1 z')$, we also have $\neg\,(w^*{}_{12}\bot_1 z')$, so let $w' \geq_{12} w^*$ such that $w' \geq_1 z'$. Since both A and B are 1-upsets, we have that $w' \in A \cap B$. Moreover, since $y \leq_2 y^* \leq_2 w^* \leq_2 w'$, it follows that $y \in C_2(A \cap B)$. The entire argument is summarized by the following diagram, where single lines represent the first ordering, dashed double lines the second ordering and full double lines the intersection of the two orderings[4]:

[4] From now on, we will use this convention to denote the various orderings diagrammatically.



Thus $\neg_1\neg_2(A) \cap \neg_1\neg_2(B) \subseteq \neg_1\neg_2(A \cap B)$, which establishes that $\neg_1\neg_2$ is a nucleus. □

The fixpoints of a nucleus on a complete Heyting algebra always form a complete Heyting algebra [16, p. 71]. Thus the previous lemma implies that the regular opens of any Heyting boset always form a cHA. On the other hand, it is easy to see that the converse fails: a boset $\mathscr{X}$ need not be Heyting for $\rho(\mathscr{X})$ to be a cHA.

Example 3.8. Suppose $\mathbb{P} = (P, \leq_P)$ is a poset such that $Up(\mathbb{P})$ is not a complete Boolean algebra (for example $P = \omega$ with the usual order). Note that this implies that there must be some $A \in Up(\mathbb{P})$ such that $A \cup I(P - A) \subsetneq P$, where I is the interior operator induced by the upset topology. This in turn means that $P = I(A \cup P{-}A) \nsubseteq A \cup I(P - A)$, so $I(U \cup V) \neq I(U) \cup I(V)$ in general. Taking complements, this means that the topological closure C induced by the upset topology on $\mathbb{P}$ is not a nucleus. However, the downsets of any poset always form a cHA. Thus, if we think of C as a closure operator on the lattice of open sets of P when P is endowed with the discrete topology, this gives us an example of a closure operator k on the lattice of upsets of a poset which is not a nucleus even though the fixpoints of k form a cHA.[5] But it is now easy to turn this into an example of a non-Heyting boset whose dual lattice is a cHA: letting $\mathscr{P} = (P, \Delta_P, \leq_P)$, we have that $\neg_1\neg_2$ on $\mathscr{P}$ is precisely the closure operator C above.

Thus for an arbitrary boset $\mathscr{X}$, the existence of densely many Heyting points is not necessary for $\rho(\mathscr{X})$ to be a cHA. On the other hand, the next lemma shows that the dual b-frame of a cHA is always Heyting.

Lemma 3.9. *Let A be a cHA and $\alpha(A) := (X, \leq_1, \leq_2)$ its dual b-frame. Then $\alpha(A)$ is a Heyting b-frame.*

Proof. Let $x = (f_x, i_x) \in X$ and consider the point $x^* = (f_x, f_x \to i_x)$. Clearly, $x \leq_{12} x^*$. Now for any $y \in X$, we have that $x^*\ {}_{12}{\perp}_1 y$ iff $f_x \wedge f_y \leq f_x \to i_x$ iff $f_x \wedge f_y \leq i_x$ iff $f_y \leq f_x \to i_x$ iff $x^*\ {}_2{\perp}_1 y$. □

Moreover, by Lemma 3.7 and the fact that $\eta_L : L \to \rho\alpha(L)$ is an isomorphism, the converse also holds:

Lemma 3.10. *Let L be a complete lattice such that $\alpha(L)$ is Heyting. Then L is a Heyting algebra.*

As an immediate consequence of the previous results, we obtain the following corollary.

[5] More involved examples of such posets are also given in [7] and [16].

Corollary 3.11. *Let L be a lattice. Then L is a Heyting algebra iff $\alpha(L)$ is Heyting.*

Thus we obtain a complete characterization of the dual b-frames of complete Heyting algebras. Once again, using results established in the previous section, we can now give necessary and sufficient conditions for when the regular opens of any boset form a complete Heyting algebra.

Lemma 3.12. *For any boset $\mathscr{X}$, $\rho(\mathscr{X})$ is a Heyting algebra iff $\mathscr{X}$ densely embeds into a Heyting b-frame.*

Proof. From left to right, if $\rho(\mathscr{X})$ is a Heyting algebra, then $\epsilon_{\mathscr{X}} : \mathscr{X} \to \alpha\rho(\mathscr{X})$ is a dense embedding into a Heyting b-frame.
Conversely, if $\mathscr{X}$ densely embeds into a Heyting b-frame $\mathscr{Y}$, then $\rho(\mathscr{X})$ is isomorphic to $\rho(\mathscr{Y})$ and thus is a Heyting algebra by Lemma 3.7. □

Finally, recall that morphisms of cHA's are complete lattice homomorphisms which also preserve the Heyting implication. In order to identify the duals of such morphisms, we need the following strenghtening of the definition of a b-morphism:

Definition 3.13. Let $\mathscr{X} = (X, \leq_1^X, \leq_2^X)$ and $\mathscr{Y} = (Y, \leq_1^Y, \leq_2^Y)$ be two bosets. A *Heyting b-morphism* (h-morphism) from $\mathscr{X}$ and $\mathscr{Y}$ is a b-morphism satisfying the following strengthening of condition 3:

3'. $\forall x \in X\, \forall y \geq_1^Y f(x)\, \exists z \geq_1^X x : f(z) \geq_{12}^Y y$.

The next lemma shows that if $f : \mathscr{X} \to \mathscr{Y}$ is an h-morphism of Heyting bosets, then $\rho(f)$ preserves Heyting implications.

Lemma 3.14. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a h-morphism. Then for any $A, B \in \mathsf{RO}_{12}(Y)$, we have $f^{-1}[I_1((Y - A) \cup B)] = I_1(f^{-1}[Y - A] \cup f^{-1}[B])$.*

Proof. Note that the left-to-right inclusion is an immediate consequence of f being 1-monotone. For the converse, assume that for all $y \geq_1^X x$, $f(y) \notin A$ or $f(y) \in B$, and let $y \geq_1^Y f(x)$ be in A. We claim that $y \in B$. To see this, let $z \geq_1^Y y$. By condition $3'$ of an h-morphism, there is $z' \geq_1^X x$ such that $z \geq_{12}^Y f(z')$. Since $x \leq_1^X z'$, by assumption we have that $f(z') \notin A$ or $f(z') \in B$. But since $y \leq_1^Y z \leq_{12}^Y f(z')$ and A is a 1-upset, we have that $f(z') \in B$. Thus $y \in \neg_1\neg_2(B) = B$. □

It follows that if $f : \mathscr{X} \to \mathscr{Y}$ is a h-morphism between Heyting bosets, then the dual $\rho(f) : \rho(\mathscr{Y}) \to \rho(\mathscr{X})$ is a complete HA-homomorphism. Conversely:

Lemma 3.15. *Let L, M be two complete Heyting algebras, and let $f : L \to M$ be a complete HA-homomorphism. Then $\alpha(f) : \alpha(M) \to \alpha(L)$ is a h-morphism.*

Proof. Recall that for any $(a, b) \in \alpha(M)$,

$$\alpha(f)(a, b) = (a^f, b_f).$$

Since $\alpha(f)$ is a b-morphism, we only have to check that condition $3'$ holds. So assume $(c, d) \geq_1^L \alpha(f)(a, b)$ for some $(a, b) \in \alpha(M)$. We claim that $(a \wedge f(c), f(d)) \in \alpha(M)$. To see this, note that, otherwise, $a \wedge f(c) \leq f(d)$, and hence $a \leq f(c) \to f(d) = f(c \to d)$. But then $c \leq a^f \leq c \to d$, which implies that $c \leq d$, a contradiction. Thus $(a \wedge f(c), f(d)) \in \alpha(M)$ and clearly $(a, b) \leq_1^M (a \wedge f(c), f(d))$. Moreover, since $(a \wedge f(c))^f \leq c$ and $d \leq (f(d))^f$, it follows that $(c, d) \leq_{12}^L \alpha(f)(a \wedge f(c), f(d))$. Thus $\alpha(f)$ is an h-morphism. □

We may therefore form the category **HbF** of Heyting b-frames and h-morphisms between them. The previous results readily imply the following theorem:

Theorem 3.16. *The duality between* **cLat** *and* **bF** *restricts to a duality between* **cHA** *and* **HbF**.

3.3. Boolean algebras

Finally, let us consider the case of Boolean algebras. Here we will follow a similar pattern as in the case of Heyting algebras. We start with the definition of a Boolean point.

Definition 3.17. Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be a boset. A *Boolean point* of $\mathscr{X}$ is a point $x^* \in X$ such that for any $y \in X$, $x^*_1 \bot_{12} y \leftrightarrow x^*_2 \bot_{12} y$.

Similarly to the definition of a Heyting boset as a boset having "enough" Heyting points, we may define a *Boolean boset* as a boset $\mathscr{X}$ such that the Boolean points of $\mathscr{X}$ are dense, i.e., the following holds:

$$\forall x \exists x^* \geq_{12} x \forall y (x^*{}_1 \bot_{12} y \leftrightarrow x^*{}_2 \bot_{12} y)). \tag{4}$$

The existence of a dense set of Boolean points in a boset $\mathscr{X}$ has some important consequences for the operator $\neg_1 \neg_2$.

Lemma 3.18. *Let $(X, \leq_1, \leq_2)$ be a Boolean boset. Then $\neg_1 \neg_2$ is the double negation nucleus on $\mathscr{O}_1$.*

Proof. We first show that $\neg_1 \neg_2(A) \subseteq \neg_1 \neg_1(A)$ for any $A \in \mathscr{O}_1$. Let $y \geq_1 x$ for some $x \in \neg_1 \neg_2(A)$. Then since $x \leq_1 y^*$, there is $z \geq_2 y^*$ such that $z \in A$. This implies that $\neg\, y^*{}_2 \bot_{12} z$, and thus also $\neg y^*{}_1 \bot_{12} z$. But this implies that $y \in C_1(A)$. Thus $x \in \neg_1 \neg_1(A)$.

We now show the converse, i.e., that $\neg_1 \neg_1(A) \subseteq \neg_1 \neg_2(A)$. Let $y \geq_1 x$ for some $x \in \neg_1 \neg_1(A)$. Since $x \leq_1 y^*$, there is $z \geq_1 y^*$ such that $z \in A$. Since this implies that $\neg y^*{}_1 \bot_{12} z$, it follows that $\neg y^*{}_2 \bot_{12} z$. But this implies that $y \in C_2(A)$, and therefore $x \in \neg_1 \neg_2(A)$. □

Since the regular open sets of any topological space always form a complete Boolean algebra, the previous lemma clearly implies:

Lemma 3.19. *Let L be a lattice such that $\alpha(L) = (P_L, \leq_1^L, \leq_2^L)$ is Boolean. Then L is a Boolean algebra.*

Moreover, the converse holds as well:

Lemma 3.20. *Let $\alpha(L) := (X, \leq_1, \leq_2)$ be the dual b-frame of a Boolean algebra L. Then $\alpha(L)$ is Boolean.*

Proof. Given $x = (f_x, i_x)$, let $x^* = (f_x \wedge \neg i_x, \neg f_x \vee i_x)$. Note that

$$(f_x \wedge \neg i_x) \to (\neg f_x \vee i_x) = \neg f_x \vee i_x = f_x \to i_x \neq 1,$$

thus x^* is well defined. Moreover, for any $y = (f_y, i_y)$, we have that $(f_x \wedge \neg i_x) \wedge f_y \leq i_y$ iff $f_y \leq (f_x \wedge \neg i_x) \to i_y$ iff $f_y \leq (\neg f_x \vee i_x) \vee i_y$. □

Corollary 3.21. *A complete lattice L is a Boolean algebra iff its dual b-frame is Boolean.*

Note that, once again, this first-order characterization of the b-frames that are dual to a complete Boolean algebra extends to a characterization of bosets $\mathscr{X}$ for which $\rho(\mathscr{X})$ is a Boolean algebra.

Corollary 3.22. *For any boset $\mathscr{X}$, $\rho(\mathscr{X})$ is a Boolean algebra iff $\mathscr{X}$ densely embeds into a Boolean b-frame.*

Proof. From left to right, if $\rho(\mathscr{X})$ is a Boolean algebra, then $\epsilon_{\mathscr{X}} : \mathscr{X} \to \alpha\rho(\mathscr{X})$ is a dense embedding into a Boolean b-frame. Conversely, if $\mathscr{X}$ densely embeds into a Boolean b-frame $\mathscr{Y}$, then $\rho(\mathscr{X})$ is isomorphic to $\rho(\mathscr{Y})$ and thus is a Boolean algebra by Lemma 3.19. □

Finally, since complete lattice homomorphisms between complete Boolean algebras are complete Boolean homomorphisms, we obtain the following duality:

Theorem 3.23. *Boolean b-frames and b-morphisms form a category* **BbF** *dual to the category* **cBA** *of complete Boolean algebras and complete Boolean homomorphisms.*

Before discussing other classes of complete lattices, let us derive a straightforward application of this characterization of the dual b-frames of Boolean algebras.

Lemma 3.24. *The class of b-frames is not first-order definable. In particular, completeness is not first-order definable in the language of bosets.*

Proof. Suppose that completeness is equivalent to some set Φ of first-order formulas in the language of bosets (i.e., pure first-order logic with two relation symbols $\leq_1$ and $\leq_2$). Let $\alpha(C)$ be the dual b-frame of the Cohen algebra C, i.e., the MacNeille completion of the countable atomless Boolean algebra [40, Chap. 30]. Since C has size $2^{\aleph_0}$, and points in $\alpha(C)$ are pairs of elements in C, $\alpha(C)$ also has size continuum. Let M be a countable elementary substructure of $\alpha(C)$, which exists by the downward Löwenheim-Skolem theorem. Now since separativity is a first-order condition and so is completeness by assumption, it follows that M is a b-frame, hence M is isomorphic to $\alpha(L)$ for some complete lattice L. Moreover, since the property of having a dense set of Boolean points is also first-order, M is a Boolean b-frame, and therefore L is a complete Boolean algebra. But M is countable, hence $\alpha(L)$ is also countable. Since there is a surjection $\pi : \alpha(L) \to L \setminus \{0\}$ defined by $(a, 0) \mapsto a$, it follows that L is countable. But there is no countable complete Boolean algebra. Thus the property of completeness is not first-order definable. □

4. Spatial and superalgebraic locales

In this section, we focus on two classes of complete Heyting algebras that are of particular relevance in the literature on semantics for intuitionistic logic: spatial and superalgebraic locales. Both classes have been extensively studied in the literature. Spatiality is a key notion in pointfree topology [41,51], as spatial locales are precisely those locales that can be represented as the lattice of open sets of a topological space. Superalgebraic or completely join-prime generated locales on the other hand have long been known to be precisely the lattices that arise as the collection of downward- or upward-closed sets of a poset [15,54]. Our goal here is to offer alternative representations of both spatial and superalgebraic locales by restricting the duality between Heyting b-frames and complete Heyting algebras obtained in Section 3.2. These results are then used in Section 6 to provide a unified framework for Kripke, topological and nuclear semantics for intuitionistic logic. We start by recalling the following definitions.

Definition 4.1. Let L be a cHA.

- L is *spatial* iff L is isomorphic to the lattice of open sets $\Omega(\mathscr{X})$ for some topological space $\mathscr{X} = (X, \tau)$.

- L is *superalgebraic* iff L is isomorphic to the lattice of upward-closed sets $Up(\mathscr{X})$ of a poset $\mathscr{X} = (X, \leq)$.[6]

Our goal in this section is to characterize b-frames whose dual lattices are spatial and superalgebraic locales. Our strategy will be the same for both classes of cHA's: first, we recall that spatial and superalgebraic locales are characterized by having certain algebraic "separation properties": any two distinct elements of a spatial locale can be separated by a meet-prime element, while any two distinct elements of a superalgebraic locale can be separated by a completely join-prime element. We then translate these algebraic properties into graph-theoretic properties of b-frames and prove that those properties do characterize the duals of spatial and superalgebraic locales. We conclude this section by an immediate application of these results: a new, purely b-frame-theoretic proof that any spatial Boolean locale is also superalgebraic.

4.1. Spatial locales

Recall that, given a lattice L, an element $c \in L$ is *meet-prime* if for any $a, b \in L$, $a \wedge b \leq c$ iff $a \leq c$ or $b \leq c$. It is *completely join-prime* if for any $A \subseteq L$, $c \leq \bigvee A$ iff $c \leq a$ for some $a \in A$. The following is a basic result of pointfree topology.

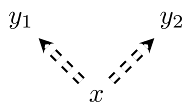
Lemma 4.2 *([51, Prop. II.5.3]). A locale L is spatial iff for any $a \not\leq b \in L$, there is a meet-prime element $c \in L$ such that $a \not\leq c$ and $b \leq c$.*

Identifying the points in a b-frame that correspond to meet-prime elements in the dual lattice is therefore an essential step in characterizing the duals of spatial locales. This is the role of the following definition:

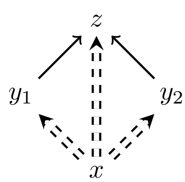
Definition 4.3. Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be a boset. A *spatial point* of $\mathscr{X}$ is a point $x \in X$ such that the following holds:

$$\forall y_1 y_2 (x \leq_2 y_1 \wedge x \leq_2 y_2 \rightarrow \exists z (y_1 \leq_1 z \wedge y_2 \leq_1 z \wedge x \leq_2 z)).$$

Spatial points can be understood as having a certain amalgamation property. Indeed, by simply spelling out the previous definition, we may notice that a point $x \in \mathscr{X}$ is spatial precisely if any diagram of the form



can be completed as follows:



The next two lemmas highlight the relevance of spatial points in identifying the duals of spatial locales.

[6] This terminology is used by Picado and Pultr in [51], who first define superalgebraic locales as join-prime generated locales, before proving the equivalence with the definition given here.

Lemma 4.4. *Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be a boset such that every point in $\mathscr{X}$ is spatial. Then $\rho(\mathscr{X})$ is a spatial locale.*

Proof. Suppose that every point in $\mathscr{X}$ is spatial. Note that since $\neg_1$ and $\neg_2$ form a Galois connection, the regular opens $\rho(\mathscr{X})$ are isomorphic to the regular closed sets of $\mathscr{X}$, i.e., the lattice of sets $U \subseteq X$ such that $U = C_2 I_1(U)$ or, equivalently, $-U = \neg_2\neg_1(-U)$. We claim that the regular closed sets of $\mathscr{X}$ form a topology on X. Clearly for any family $\{U_i\}_{i\in I}$ of regular closed sets, we have that $C_2I_1(U_i) \subseteq C_2I_1(\bigcup_{i\in I} U_i)$ for any $i \in I$, and therefore

$$\bigcup_{i\in I} U_i = \bigcup_{i\in I} C_2I_1(U_i) \subseteq C_2I_1(\bigcup_{i\in I} U_i).$$

Since C_2I_1 is a kernel operator on the 2-downsets of $\mathscr{X}$, this implies that the regular closed sets of $\mathscr{X}$ are closed under arbitrary unions. Therefore we only have to check that they are also closed under finite intersection. Suppose U_1, U_2 are regular closed. Clearly $U_1 \cap U_2$ is also a 2-downset, and hence $C_2I_1(U_1 \cap U_2) \subseteq U_1 \cap U_2$. For the converse, suppose $x \in U_1 \cap U_2$. Since both U_1 and U_2 are regular closed, this means that there is $y_1 \in I_1(U_1)$ and $y_2 \in I_1(U_2)$ such that $x \leq_2 y_1$ and $x \leq_2 y_2$. Since by assumption x is spatial, this means that there is $z \geq_2 x$ such that $z \geq_1 y_1, y_2$. But this implies that $z \in I_1(U_1) \cap I_1(U_2) = I_1(U_1 \cap U_2)$. Hence $x \in C_2I_1(U_1 \cap U_2)$ and $U_1 \cap U_2$ is regular closed, which completes the proof that the regular closed sets form a topology on X. Therefore $\rho(X)$ is spatial. □

Lemma 4.5. *Let L be a spatial locale. Then the set of spatial points of $\alpha(L)$ is dense.*

Proof. Suppose L is spatial and $(a, b) \in \alpha(L)$. Since L is spatial, there is a meet prime $c \in L$ such that $a \nleq c$ and $b \leq c$. Hence the point $(a, c) \in \alpha(L)$, and we have that $(a, b) \leq_{12} (a, c)$. We claim that (a, c) is a spatial point of $\alpha(L)$. To see this, suppose that $(a, c) \leq_2 (x_1, y_1)$ and $(a, c) \leq_2 (x_2, y_2)$. Since $x_i \nleq y_i$ and $c \leq y_i$, we have that $x_i \nleq c$ for $i \in \{1, 2\}$. Since c is meet-prime, this means that $x_1 \wedge x_2 \nleq c$. But then $(x_1 \wedge x_2, c)$ is the required point. □

As a straightforward consequence, we obtain the following characterization of the duals of spatial locales:

Theorem 4.6.

1. *A locale L is spatial iff the set of spatial points of $\alpha(L)$ is dense.*
2. *For any boset $\mathscr{X}$, $\rho(\mathscr{X})$ is spatial iff $\mathscr{X}$ densely embeds into a b-frame $\mathscr{Y}$ with densely many spatial points.*

Proof.
1. The left-to-right direction follows from the previous lemma. For the converse, if the spatial points of $\alpha(L)$ are dense, then letting $\mathscr{X}$ be the dense subframe of $\alpha(L)$ induced by its spatial points, we have by Lemma 4.4 that $\rho(\mathscr{X})$ is spatial and by Lemma 2.19 that L is isomorphic to $\rho(\mathscr{X})$, hence also spatial.
2. This follows directly from the first part. □

Let us now move on to superalgebraic locales, for which we apply a similar method.

4.2. Splitting locales

As mentioned above, superalgebraic locales are precisely those locales in which any two distinct elements can be separated by a completely join-prime one. Our characterization of the dual b-frames of superalgebraic locales essentially uses this fact, but the following property will be easier to work with:

Definition 4.7. Let L be a lattice. Given $a, b \in L$ such that $a \nleq b$, a *splitting pair* for the pair (a, b) is a pair (c, d) of elements of L such that $c \nleq d$, $c \leq a$, $b \leq d$ and for any $x \in L$, $c \leq x$ or $x \leq d$.

A locale L is *splitting* if for any $a \nleq b \in L$, there is a splitting pair (c, d) for the pair (a, b).

Splittings in lattices have a long history, going back to Whitman [64]. Splitting locales are a special kind of separated locales, the study of which originates with Raney [54]. While separated locales coincide with supercontinuous locales and are precisely the complete homomorphic images of frames of downsets of posets (or, equivalently, completely distributive complete lattices [51, Prop. VII.8.5.1]), splitting locales coincide with superalgebraic locales, as is well-known.

Lemma 4.8. *A locale L is superalgebraic iff L is splitting.*

Proof. For the left-to-right direction, assume without loss of generality that $L = Up(\mathscr{X})$ for some poset $\mathscr{X} = (X, \leq)$. Given $U \nsubseteq V \in \mathscr{X}$, let $x \in U - V$, and let $U' = \uparrow x$ and $V' = X - \downarrow x$. Then clearly $U' \nsubseteq V'$, $U' \subseteq U$ and $V \subseteq V'$, and moreover for any $Y \in Up(\mathscr{X})$, since either $x \in Y$ or $x \notin Y$, we must have that $U' \subseteq Y$ or $Y \subseteq V'$.

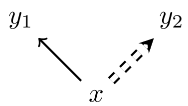
For the converse direction, as superalgebraic locales are precisely the completely join-prime generated locales (see for example [51, Prop. VII.8.3]), it is enough to observe that for any splitting pair $(c, d) \in L$, c is completely join-prime. But this is a well-known argument [49, Remark. 4.1]. □

We now define the boset counterpart of splitting pairs.

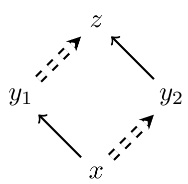
Definition 4.9. Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be a boset. A *splitting point* of $\mathscr{X}$ is a point $x \in X$ such that the following holds:

$$\forall y_1 y_2 (x \leq_1 y_1 \wedge x \leq_2 y_2 \rightarrow \exists z (y_1 \leq_2 z \wedge y_2 \leq_1 z)).$$

Similarly to spatial points, splitting points exhibit a certain amalgamation property. Indeed, a point $x \in \mathscr{X}$ is splitting precisely if any diagram of the form



can be completed as follows:



The next two lemmas establish the equivalence between separation by splitting pairs in lattices and density of splitting points in bosets:

Lemma 4.10. *Let $\mathscr{X} = (X, \leq_1, \leq_2)$ be a boset such that the splitting points of $\mathscr{X}$ are dense. Then $\rho(\mathscr{X})$ is splitting.*

Proof. Let $U, V \in \rho(\mathscr{X})$ such that $U \nsubseteq V$. This means that there is $x \in \mathscr{X}$ such that $x \in U \cap \neg_2 V$. Let $x' \geq_{12} x$ be a splitting point, and notice that $x' \in U \cap \neg_2 V$. We claim that $(U^{x'}, V_{x'})$ is a splitting pair for (U, V). By Lemma 4.8, this implies that $\rho(\mathscr{X})$ is superalgebraic. For the proof of the claim, it is clear that $U^{x'} \subseteq U$, $V \subseteq V_{x'}$ and that $U^{x'} \nsubseteq V_{x'}$. Now let T be any regular open set. If $x' \in T$, then $U^{x'} \subseteq T$. Otherwise, if $x' \notin T$, there is $y_1 \geq_1 x'$ such that $y_1 \in \neg_2 T$. But then for any $w \in T$, if $\neg x' {}_2\bot_1 w$, there must be some $y_2 \geq_2 x$ such that $y_2 \geq_1 w$. Since by assumption x' is splitting, there is $z \in \mathscr{X}$ such that $z \geq_2 y_1$ and $z \geq_1 y_2$. But this is a contradiction, since $z \geq_1 y_2 \geq_1 w$ implies that $z \in T$, while $z \geq_2 y_1$ implies that $z \notin T$ since $y_1 \in \neg_2 T$. Hence for any $w \in T$, $x' {}_2\bot_1 w$, which means that $T \subseteq V_{x'}$. This completes the proof. □

Lemma 4.11. *Let L be superalgebraic. Then the splitting points of $\alpha(L)$ are dense.*

Proof. Let $(a, b) \in \alpha(L)$. Since L is superalgebraic, by Lemma 4.2, there is a splitting pair (c, d) for the pair (a, b). Note that by the definition of a splitting pair, we have that $(a, b) \leq_{12} (c, d)$. We claim that (c, d) is a splitting point of $\alpha(L)$. Suppose $(c, d) \leq_1 (x_1, y_1)$ and $(c, d) \leq_2 (x_2, y_2)$. Now $x_1 \leq c$ yet $x_1 \nleq y_1$, which means that $c \nleq y_1$, and hence $y_1 \leq d$ since (c, d) is a splitting pair. Similarly $d \leq y_2$ yet $x_2 \nleq y_2$, which implies that $x_2 \nleq d$, and therefore that $c \leq x_2$. Hence $(x_1, y_1) \leq_2 (c, d)$, and $(x_2, y_2) \leq_1 (c, d)$, which shows that $\neg(x_1, y_1){}_2\bot_1(x_2, y_2)$ and establishes that (c, d) is a splitting point. □

As a consequence, we obtain the following characterization of b-frames that are dual to superalgebraic locales:

Theorem 4.12.

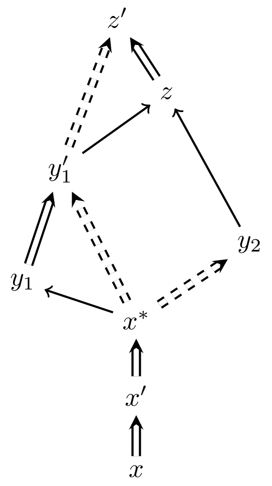
1. *A locale L is superalgebraic iff the set of splitting points of $\alpha(L)$ is dense.*
2. *For any boset $\mathscr{X}$, $\rho(\mathscr{X})$ is splitting iff $\mathscr{X}$ densely embeds into a b-frame $\mathscr{Y}$ with densely many splitting points.*

Proof. 1. The left-to-right direction follows from the previous lemma. For the converse, if the splitting points of $\alpha(L)$ are dense, then by Lemma 4.10 $\rho\alpha(L)$ is superalgebraic. But since L is isomorphic to $\rho\alpha(L)$, it is also superalgebraic.
2. This follows readily from the first part. □

As an immediate application of the results of this section, we can now use b-frames to prove the following well-known fact about Boolean locales (see [51, Section II.5.4] for a standard proof):

Corollary 4.13. *Any spatial Boolean locale is superalgebraic.*

Proof. Let B be a spatial Boolean locale. We claim that $\alpha(B)$ is a splitting b-frame. To see this, let $x \in \alpha(B)$. Since B is spatial, $\alpha(B)$ is a spatial b-frame, which means that there is some spatial point $x' \geq_{12} x$. Since B is also Boolean, there is a Boolean point $x^* \geq_{12} x'$. We claim that x^* is a splitting point. Indeed, suppose $y_1 \geq_1 x^*$ and $y_2 \geq_2 x^*$. Since x^* is Boolean and $\neg y_1 {}_1\bot_{12} x^*$, there is some $y_1' \geq_{12} y_1$ such that $x^* \leq_2 y_1$, and note that we may assume that y_1' is Boolean. Hence we have that $x' \leq_2 y_2$ and $x' \leq_2 y_1'$, so since x' is spatial we have some $z \geq_1 y_2, y_1'$. Now since y_1' is Boolean and $\neg y_1' {}_1\bot_{12} z$, there must be some $z' \geq_{12} z$ such that $z' \geq_2 y_1'$. Thus $z' \geq_2 y_1' \geq_2 y_1$, and $z' \geq_1 z \geq_1 y_2$. Hence x^* is a splitting point. The argument is summarized by the diagram below:



□

5. A decomposition theorem for bi-Heyting algebras

In this section, we apply elements of our b-frame duality to prove a new result regarding complete bi-Heyting algebras. The motivation for our result is the following theorem about complete Boolean algebras[7]:

Lemma 5.1. *For any complete Boolean algebra B, there are complete Boolean algebras C_1 and C_2 such that C_1 is atomic, C_2 is atomless, and $B = C_1 \times C_2$.*

In our setting, atomic Boolean algebras must be generalized to superalgebraic locales (notice that Boolean superalgebraic locales are precisely the atomic Boolean algebras). This is in line with the fact that completely join-prime elements are usually taken to be the relevant generalization of atoms for cHA's. Accordingly, we propose as a relevant generalization of atomless Boolean algebras the following definition:

Definition 5.2. A complete lattice is *anti-algebraic* if it has no completely join-prime element.

We will use our b-frame duality to show that any complete Heyting algebra is, in the category of complete lattices, a subdirect product of a superalgebraic locale and an anti-algebraic locale. As will be made explicit below, this decomposition theorem holds in the category **cLat** of complete lattices and complete lattice homomorphisms, but not in the category of complete bi-Heyting algebras and complete bi-Heyting homomorphisms between them, which is not a full subcategory of **cLat**. Of course, the issue does not arise in the Boolean case, since **cBA** is a full subcategory of **cLat**.

5.1. Coproducts of bosets

We start by defining the coproduct of two bosets. By duality, this induces a boset representation of the product of two complete lattices. This is an adaptation of the standard correspondence between products and disjoint unions.

Definition 5.3. Let $\mathscr{X} = (X, \leq_1^X, \leq_2^X)$ and $\mathscr{Y} = (Y, \leq_1^Y, \leq_2^Y)$ be two bosets. The *disjoint sum* of $\mathscr{X}$ and $\mathscr{Y}$, written as $\mathscr{X} \sqcup \mathscr{Y}$, is the boset $(Z, \leq_1^Z, \leq_2^Z)$, where $Z = X \sqcup Y$, $\leq_1^Z = \leq_1^X \sqcup \leq_1^Y$, and $\leq_2^Z = \leq_2^X \sqcup \leq_2^Y$.

Lemma 5.4. *For any two bosets $\mathscr{X}$ and $\mathscr{Y}$, $\mathscr{X} \sqcup \mathscr{Y}$ is the coproduct of $\mathscr{X}$ and $\mathscr{Y}$ in the category of bosets.*

[7] See for example [25], p. 227.

Proof. Note first that we have two obvious inclusion b-morphisms $\lambda_1 : \mathscr{X} \to \mathscr{X} \sqcup \mathscr{Y}$ and $\lambda_2 : \mathscr{Y} \to \mathscr{X} \sqcup \mathscr{Y}$. Moreover, if $\mathscr{T}$ is any boset such that there are b-morphisms $\tau_1 : \mathscr{X} \to \mathscr{T}$ and $\tau_2 : \mathscr{Y} \to \mathscr{T}$, then it is routine to check that the map $h : \mathscr{X} \sqcup \mathscr{Y} \to \mathscr{T}$ given by $h(z) = \tau_1(z)$ if $z \in \mathscr{X}$, and $h(z) = \tau_2(z)$ if $z \in \mathscr{Y}$ witnesses the universal property of the coproduct. □

Lemma 5.5. *Let $\mathscr{X}$, $\mathscr{Y}$ be two bosets. Then $\rho(\mathscr{X}) \times \rho(\mathscr{Y}) = \rho(\mathscr{X} \sqcup \mathscr{Y})$.*

Proof. Recall that, as a covariant functor from $\mathbf{bF}^{op}$ into **cLat**, ρ has a left adjoint α. This means that ρ preserves limits. Since $\mathscr{X} \sqcup \mathscr{Y}$ is the coproduct of $\mathscr{X}$ and $\mathscr{Y}$ in **bF**, it is their product in $\mathbf{bF}^{op}$, and thus $\rho(\mathscr{X} \sqcup \mathscr{Y}) = \rho(\mathscr{X}) \times \rho(\mathscr{Y})$. □

5.2. Characterizing co- and bi-Heyting algebras

Next, we extend the characterization of the dual b-frames of Heyting algebras obtained in Section 3 to co- and bi-Heyting algebras. Recall that a co-Heyting algebra is a distributive lattice in which the join $\vee$ has a left adjoint $-\!\!<$, and that a bi-Heyting algebra is a Heyting algebra that is also a co-Heyting algebra. Bi-Heyting algebras and their representation theory were extensively studied by Rauszer [55–58].

Definition 5.6. Let $\mathscr{X} = (X, \leq_1^X, \leq_2^X)$ be a boset.

- A point $x^* \in \mathscr{X}$ is *co-Heyting* if for all $y \in \mathscr{X}$, $x^* {}_{12}\!\perp_1 y$ iff $x^* {}_1\!\perp_2 y$.
- $\mathscr{X}$ is a *co-Heyting boset* if the co-Heyting points of $\mathscr{X}$ are dense.
- A b-morphism $f : \mathscr{X} \to \mathscr{Y}$ is a *co-Heyting morphism* (denoted coh-morphism) if it satisfies the following strengthening of condition 2:
  $2'$ $\forall x \in X \, \forall y \geq_2^Y f(x) \, \exists z \geq_2^X x f(x) \geq_{12}^Y y$.

Lemma 5.7.

1. *If $\mathscr{X}$ is a co-Heyting boset, then $\neg_2\neg_1$ is a nucleus on $Up_2(\mathscr{X})$, and consequently $\rho(\mathscr{X})$ is a co-Heyting algebra.*
2. *If L is a complete co-Heyting algebra, then $\alpha(L)$ is a co-Heyting b-fame.*
3. *For any boset $\mathscr{X}$, $\rho(\mathscr{X})$ is a co-Heyting algebra iff $\mathscr{X}$ densely embeds into a co-Heyting b-frame.*

Proof. 1 Similar to Lemma 3.7.
2 Similar to Lemma 3.9. Given a pair $(a, b) \in \alpha(L)$, $a \not\leq b$ implies that $a -\!\!< b \neq b$, and thus $(a -\!\!< b, b) \in \alpha(L)$. It is routine to check that this is a co-Heyting point of $\alpha(L)$.
3 Similar to Lemma 3.12. □

Lemma 5.8. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a b-morphism.*

1. *If f is a coh-morphism, then $\rho(f) : \rho(\mathscr{Y}) \to \rho(\mathscr{X})$ is a co-Heyting homomorphism.*
2. *If $h : L \to M$ is a co-Heyting homomorphism of co-Heyting algebras then the map $\alpha(h) : \alpha(M) \to \alpha(L)$ is a coh-morphism.*

Proof. 1. Similar to Lemma 3.14.
2. Similar to Lemma 3.15. □

We therefore obtain a description of the dual of the category of co-Heyting algebras and co-Heyting homomorphisms:

Theorem 5.9. *Co-Heyting b-frames and coh-morphisms form a category* **coHbF** *dual to the category* **cocHA** *of complete co-Heyting algebras and co-Heyting homomorphisms.*

Standard dualities for complete co-Heyting and bi-Heyting algebras can also be obtained via Esakia duality [4]. In our setting, we also need to identify the dual b-frames of complete bi-Heyting algebras. Given a boset $\mathscr{X}$, let us define a *bi-Heyting point* of $\mathscr{X}$ as a point $x^* \in \mathscr{X}$ that is both a Heyting and a co-Heyting point. Establishing the existence of bi-Heyting points in dual b-frames of bi-Heyting algebras requires a technical lemma.

Lemma 5.10. *Let L be a complete bi-Heyting algebra. Then for any $a, b, c, d \in L$:*

1. $(a \prec b) \wedge c \leq a \to b$ *iff* $c \leq a \to b$*;*
2. $a \prec b \leq (a \to b) \vee d$ *iff* $a \prec b \leq d$*.*

Proof. 1. Note that:

$$\begin{aligned} & a \prec b \wedge c \leq a \to b \\ \text{iff } & (a \wedge c) \wedge (a \prec b) \leq b \\ \text{iff } & ((a \wedge c) \vee b) \wedge ((a \prec b) \vee b) \leq b \\ \text{iff } & (a \vee b) \wedge (c \vee b) \wedge (a \vee b) \leq b \\ \text{iff } & (a \wedge c) \vee b \leq b \\ \text{iff } & c \leq a \to b. \end{aligned}$$

2. This follows from 1 applied to L^δ, the dual bi-Heyting algebra to L. □

Theorem 5.11. *Let L be a complete lattice. Then L is a bi-Heyting algebra iff the bi-Heyting points of $\alpha(L)$ are dense.*

Proof. The right-to-left direction follows immediately from Lemmas 3.9 and 5.8. For the left-to-right direction, suppose L is a bi-Heyting algebra and $(a, b) \in \alpha(L)$. We claim that $(a \prec b, a \to b)$ is a bi-Heyting point of $\alpha(L)$. That $(a \prec b, a \to b) \in \alpha(L)$ follows from the previous lemma, with $c = 1$. Moreover, for any $(c, d) \in \alpha(L)$ we have by the previous lemma (item 1) that $(a \prec b, a \to b)_2 \perp_1 (c, d)$ iff $c \leq a \to b$ iff $a \prec b \wedge c \leq a \to b$ iff $(a \prec b, a \to b)_{12} \perp_1 (c, d)$. Hence $(a \prec b, a \to b)$ is a Heyting point. Similarly, by item 2 in the previous lemma, we have that $(a \prec b, a \to b)_1 \perp_2 (c, d)$ iff $a \prec b \leq d$ iff $a \prec b \leq a \to b \vee d$ iff $(a \prec b, a \to b)_{12} \perp_2 (c, d)$. Hence $(a \prec b, a \to b)$ is a bi-Heyting point, and it is immediate that $(a, b) \leq_{12} (a \prec b, a \to b)$. Therefore the bi-Heyting points of $\alpha(L)$ are dense. □

5.3. Subdirect product representation of bi-HAs

We are now in a position to prove our main result about complete bi-Heyting algebras.

Recall first that if $\{B_i\}_{i \in I}$ is a family of complete lattices, then a complete lattice A is a *subdirect product* of $\{B_i\}_{i \in I}$ if there is an injective homomorphism $e : A \to \Pi_{i \in I} B_i$ such that for any $i \in I$, $\pi_i \circ e$ is surjective.

We start by defining a *maximal* point of a boset $\mathscr{X}$ as a maximal point in the 1-and-2 ordering, that is a point $x \in \mathscr{X}$ such that for any $y \in \mathscr{X}$, $y \geq_{12} x$ implies that $y = x$. If $\mathscr{X}$ is a distributive b-frame, then maximal points in $\mathscr{X}$ correspond to very specific pairs of elements of the dual lattice:

Lemma 5.12. *Let L be a complete distributive lattice and $(c, d) \in \alpha(L)$. The following are equivalent:*

1. *(c, d) is maximal;*
2. *(c, d) is a splitting pair of L;*
3. *c is completely join prime, d is completely meet-prime, $d = \bigvee\{f \in L \mid c \not\leq f\}$ and $c = \bigwedge\{e \in L \mid e \not\leq d\}$;*
4. *for any $(a, b) \in \alpha(L)$, if $(c, d) \leq_1 (a, b)$, then $(a, b) \leq_2 (c, d)$, and if $(c, d) \leq_2 (a, b)$, then $(a, b) \leq_1 (c, d)$.*

Proof. $1 \Rightarrow 2$ Suppose that there is some $k \in L$ such that $c \not\leq k$ and $k \not\leq d$. Since L is distributive and $c \not\leq d$, either $c \wedge k \not\leq d$ or $c \not\leq k \vee d$. Either way, we have a pair $(c', d') \neq (c, d)$ such that $(c, d) \leq_{12} (c', d')$, contradicting maximality.

$2 \Rightarrow 3$ The equivalence between 2 and 3 is well known [49]. We include the argument for the left-to-right direction for the sake of completeness. Let $F \subseteq L$. If $c \not\leq f$ for all $f \in F$, then, since (c, d) is a splitting pair, $f \leq d$ for all $f \in F$, from which it follows that $\bigvee F \leq d$ and therefore $c \not\leq \bigvee F$. Similarly, if $f \not\leq d$ for all $f \in F$, then $c \leq f$ for all $f \in F$, hence $c \leq \bigwedge F$ and therefore $\bigwedge F \not\leq d$. Thus c and d are completely join-prime and completely meet-prime respectively. Finally, note that for any $f \in L$, $f \leq d$ iff $c \not\leq f$, from which it follows that $\bigvee\{f \mid c \not\leq f\} \leq d$ and $c \leq \bigwedge\{e \mid e \not\leq d\}$. Since $c \not\leq d$, we conclude that $\bigvee\{f \mid c \not\leq f\} = d$ and $c = \bigwedge\{e \mid e \not\leq d\}$.

$3 \Rightarrow 4$ Suppose that $(c, d) \leq_1 (a, b)$. Then $c \not\leq b$, hence $b \leq d$, which implies that $(a, b) \leq_2 (c, d)$. Similarly, if $(c, d) \leq_2 (a, b)$, then $a \not\leq d$, hence $c \leq a$, and $(a, b) \leq_1 (c, d)$.

$4 \Rightarrow 1$ Suppose $(c, d) \leq_{12} (a, b)$. Then since 4 holds we have that $(a, b) \leq_{12} (c, d)$, so $c = a$ and $b = d$. □

The next lemma relates maximal points in separative bosets and anti-algebraic locales:

Lemma 5.13. *Let $\mathscr{X}$ be a separative boset with no maximal point. Then $\rho(\mathscr{X})$ is anti-algebraic.*

Proof. We show that there are no splitting pairs in $\rho(\mathscr{X})$. Let $U \not\subseteq V \in \rho(\mathscr{X})$, and suppose $x \in U \cap \neg_2 V$. Since x is not a maximal point, there is $y \geq_{12} x$ such that $y \not\leq_1 x$ or $y \not\leq_2 x$. We distinguish two cases:

- $y \not\leq_1 x$: By separativity $\uparrow_1 y = U^y \in \rho(\mathscr{X})$, and since $x \notin \uparrow_1 y$ and $y \in \neg_2 V$, we have that $U \not\subseteq U^y$ and $U^y \not\subseteq V$.
- $y \not\leq_2 x$: By separativity there is $z \in \mathscr{X}$ such that $x_2 \perp_1 z$ and $\neg y_2 \perp_1 z$, which implies that $z \in V_y \setminus V$, so that $V_y \not\subseteq V$. On the other hand, since $y \in U$, we have that $U \not\subseteq V_y$.

Hence (U, V) is not a splitting pair. But this in turn implies that U is not completely join-prime and therefore that L is anti-algebraic. □

We can now prove the main theorem of this section. As will become clear below, we are considering bi-Heyting algebras as complete lattices in the category **cLat**, meaning that the morphisms considered here need not preserve the Heyting or co-Heyting implication.

Theorem 5.14. *Let L be a complete bi-Heyting algebra. Then L is a subdirect product of $L_1 \times L_2$ in* **cLat***, where L_1 is superalgebraic and L_2 is anti-algebraic.*

Proof. Let $\mathscr{X}$ be the subframe of $\alpha(L)$ induced by the set of all maximal points in $\alpha(L)$, and let $\mathscr{Y}$ be the subframe of $\alpha(L)$ induced by the set of all bi-Heyting points $y \in \alpha(L)$ such that for any $z \geq_{12} y$, z is not a maximal point. Note that, by duality, it is enough to show that there are embeddings $\nu_1 : \mathscr{X} \to \alpha(L)$ and $\nu_2 : \mathscr{Y} \to \alpha(L)$ such that the induced b-morphism $\nu : \mathscr{X} \sqcup \mathscr{Y} \to \alpha(L)$ is dense, since this will imply that $\rho(\nu) : L \to \rho(\mathscr{X}) \times \rho(\mathscr{Y})$ is injective and that $\rho(\nu_1) = \rho(\nu) \circ \rho(\lambda_1) : L \to \rho(\mathscr{X})$ and $\rho(\nu_2) = \rho(\nu) \circ \rho(\lambda_2) : L \to \rho(\mathscr{Y})$ are surjective.

- For any $(c, d) \in \mathscr{X}$, define $\nu_1(c, d) = (c, d)$. We claim that $\nu_1 : \mathscr{X} \to \alpha(L)$ is an embedding. Monotonicity is clear. If $\nu_1(c, d) \leq_1 (a, b)$ for some $(a, b) \in \alpha(L)$, then, since (c, d) is maximal, we have that $(a, b) \leq_2 \nu_1(c, d)$, which means that ν_1 satisfies condition 2 of a b-morphism. Similarly, if $\nu_1(c, d) \leq_2 (a, b)$, we have that $(a, b) \leq_1 \nu_1(c, d)$, and thus ν_1 is a b-morphism. Finally, to see that it is an embedding, suppose that $\neg\nu_1(c, d)_2\bot_1\nu_1(c', d')$. Then there is some $(a, b) \in \alpha(L)$ such that $(c, d) \leq_2 (a, b)$ and $(c', d') \leq_1 (a, b)$. But this in turn implies that $(c', d') \leq_1 (a, b) \leq_1 (c, d)$, so $\neg(c, d)_2\bot_1(c', d')$.
- For any $(a, b) \in \mathscr{Y}$, define $\nu_2(a, b) = (a, b)$. We claim that $\nu_2 : \mathscr{X} \to \alpha(L)$ is an embedding. Once again, monotonicity is clear. To see that ν_2 satisfies conditions 2 and 3 of b-morphism, note first that for any $(a, b) \in \mathscr{Y}$ and any bi-Heyting $(a', b') \in \alpha(L)$, if $(a, b) \leq_{12} (a', b')$, then $(a', b') \in \mathscr{Y}$. Now fix some $(a, b) \in \mathscr{Y}$ and assume that $\nu_2(a, b) \leq_1 (c, d)$ for some $(c, d) \in \alpha(L)$. Since (a, b) is bi-Heyting, we have that $(a, b) \leq_{12} (a', b')$ for some bi-Heyting $(a', b') \geq_2 (c, d)$. But then $(a', b') \in \mathscr{Y}$, which shows that ν_2 satisfies property 2. Similarly, assume that $\nu_2(a, b) \leq_2 (c, d)$ for some $(c, d) \in \alpha(L)$. Since (a, b) is bi-Heyting, we have some bi-Heyting $(a', b') \geq_{12} (a, b)$ such that $(a', b') \geq_1 (c, d)$. But then $(a', b') \in \mathscr{Y}$, so ν_2 satisfies property 3 of a b-morphism. Finally, to see that ν_2 is an embedding, assume $\neg\nu_2(a, b)_2\bot_1\nu_2(a', b')$ for some $(a, b), (a', b') \in \mathscr{Y}$. Then there is some $(c, d) \in \alpha(L)$ such that $(a, b) \leq_2 (c, d)$ and $(a', b') \leq_1 (c, d)$. As (a, b) is bi-Heyting, there is some bi-Heyting point $(a^*, b^*) \geq_{12} (a, b)$ such that $(c, d) \leq_1 (a^*, b^*)$. But then $(a', b') \leq_1 (a^*, b^*)$ and $(a^*, b^*) \in \mathscr{Y}$, which implies that $\neg(a, b)_2\bot_1(a', b')$.
- Finally, by the universal property of the coproduct, the map $\nu : \mathscr{X} \sqcup \mathscr{Y} \to \alpha(L)$, defined by $\nu(a, b) = (a, b)$ for any $(a, b) \in \mathscr{X} \sqcup \mathscr{Y}$, is a b-morphism. Moreover, we claim that it is dense. Suppose $(a, b) \in \alpha(L)$. There are two possible cases:
  - $(a, b) \leq_{12} (c, d)$, for some maximal point (c, d). Then $(c, d) \in \mathscr{X}$.
  - $(a, b) \not\leq_{12} (c, d)$ for any maximal point (c, d). Then since L is a bi-Heyting algebra, $(a, b) \leq_{12} (a', b')$ for some bi-Heyting point (a', b') such that $(a', b') \not\leq_{12} (c, d)$ for any maximal point (c, d), which implies that $(a', b') \in \mathscr{Y}$.

  Hence for any $(a, b) \in \alpha(L)$ there is some $(c, d) \in \mathscr{X} \sqcup \mathscr{Y}$ such that $(a, b) \leq_{12} \nu(c, d)$, and hence ν is dense.

Thus, in **cLat**, L is a subdirect product of $\rho(\mathscr{X})$ and $\rho(\mathscr{Y})$. It remains to be shown that $\mathscr{X}$ is superalgebraic and that $\mathscr{Y}$ is anti-algebraic.

- Since all points in $\mathscr{X}$ are maximal, they are also splitting points: if $(c, d) \leq_1 (c_1, d_1)$ and $(c, d) \leq_2 (c_2, d_2)$, for some $(c, d), (c_1, d_1), (c_2, d_2) \in \mathscr{X}$, then $(c_1, d_1) \leq_2 (c, d)$ and $(c_2, d_2) \leq_1 (c, d)$, and thus $\neg(c_1, d_1)_2\bot_1(c_2, d_2)$. Hence $\rho(\mathscr{X})$ is superalgebraic.
- Clearly, by construction, $\mathscr{Y}$ has no maximal points. So it is enough to show that $\mathscr{Y}$ is separative in order to establish that $\rho(\mathscr{Y})$ is anti-algebraic. Suppose $(a, b) \not\leq_1 (a', b')$ for some $(a, b), (a', b') \in \mathscr{Y}$. Then since $\alpha(L)$ is separative, there is some $(a'', b'') \geq_1 (a', b')$ such that $(a'', b'')_2\bot_1(a, b)$. Since (a', b') is bi-Heyting, there is some $(a^*, b^*) \geq_{12} (a', b')$ such that $(a^*, b^*) \geq_2 (a'', b'')$. But then $(a^*, b^*) \in \mathscr{Y}$ and $(a^*, b^*)_2\bot_1(a, b)$. This shows that $\mathscr{Y}$ is 1-separative. The argument for 2-separativity is completely similar. Thus $\mathscr{Y}$ is separative and has no maximal points, from which it follows that $\rho(\mathscr{Y})$ is anti-algebraic.

This completes the proof of the theorem. □

Let us conclude this section with some remarks on the theorem obtained in this section. First, the proof of this theorem does not simply rely on the Allwein-MacCaull representation of complete lattices, but requires the full power of b-frame duality. Moreover, the main idea of the proof uses the fact that bosets can be "split" in a fairly simple way, because they are discrete structures.[8]

Furthermore, it is worth emphasizing that this result only holds in **cLat**, i.e., the morphisms under consideration here are complete lattice homomorphisms and not Heyting or bi-Heyting homomorphisms. Indeed, as was pointed out by an anonymous referee, the following is an example of a subdirectly irreducible complete bi-Heyting algebra that is neither superalgebraic nor anti-algebraic:

Example 5.15. Consider the chain $A = \mathbb{N} \oplus [0, 1] \oplus \top$, where $\mathbb{N}$ and $[0, 1]$ have the usual order. Every element in $\mathbb{N}$ is completely join-prime, while no element of $[0, 1]$ is completely join-prime. Moreover, A is a complete bi-Heyting algebra with a second least and a second greatest element, which means that it is subdirectly irreducible in the category of bi-Heyting algebras and bi-Heyting homomorphisms. But clearly, A is neither superalgebraic nor anti-algebraic and thus cannot be written as a subdirect product of a superalgebraic locale and an anti-algebraic locale.

However, the standard decomposition result about Boolean algebras follows directly from Theorem 5.14, once one recalls that **cBA** is a full subcategory of **cLat** and that join-prime generated elements in Boolean algebras coincide with co-atoms.

Finally, while the definition of anti-algebraic locales does not seem to appear anywhere in the literature, it is arguably a straightforward generalization of the notion of a complete atomless Boolean algebra. Moreover, existentially-closed Heyting algebras (in the sense of model theory) have recently been axiomatized by Darnière in [14] as those Heyting algebras A satisfying the two "strong order" axioms of Density and Splitting, as well as a countable set of formulas expressing the fact that the complete theory of A eliminates quantifiers. Since, as is well known [11, p. 194], atomless Boolean algebras are precisely the existentially-closed Boolean algebras, one may wonder whether the anti-algebraic locales we define here satisfy Darnière's axioms. For now, we leave this as an open problem and move on to discussing applications of bosets to the semantics of intuitionistic logic.

6. Semantics for IPC

In this final section, we outline some applications of the results obtained above to the semantics of intuitionistic propositional logic. As shown in [6], the algebraic approach to a semantics $\mathcal{S}$ for IPC associates $\mathcal{S}$ to the class $\mathcal{H}_{\mathcal{S}}$ of Heyting algebras represented by the models of $\mathcal{S}$. Given two semantics $\mathcal{S}$ and $\mathcal{S}'$, $\mathcal{S}$ is *more general* than $\mathcal{S}'$ (denoted $\mathcal{S}' \leq \mathcal{S}$) whenever every Heyting algebra in $\mathcal{H}_{\mathcal{S}'}$ is isomorphic to a Heyting algebra in $\mathcal{H}_{\mathcal{S}}$. Under this ordering, it can be shown that Kripke semantics is strictly less general than topological semantics, which is itself strictly less general than nuclear semantics such as Dragalin [16] and Fairtlough-Mendler [20] semantics. Indeed, the Heyting algebras that arise as the upset of a Kripke frame are precisely superalgebraic locales and those arising as open sets of a topological space are spatial locales. In nuclear semantics, a nucleus is defined on the upset of a poset $(P, \leq)$, for example by endowing this poset with a function $D : P \to \mathscr{P}(\mathscr{P}(P))$ satisfying certain conditions (as is done in Dragalin semantics), or by adding a second ordering $\preccurlyeq$ on P such that $\preccurlyeq \subseteq \leq$ (as is the case in FM semantics). Formulas of IPC are then evaluated as upsets of $(P, \leq)$ that are also fixpoints of the nucleus thus defined. Building on a result of Dragalin [16, pp. 75-76], Bezhanishvili and Holliday [7] proved that any locale arises as the fixpoints of such a nucleus and that both Dragalin and FM-semantics are as general a semantics for intuitionistic logic as locale semantics.

[8] I thank an anonymous referee for pointing out that one can also follow a similar strategy and prove this result using the more standard techniques of Priestley and Esakia duality.

This semantic hierarchy is particularly relevant for the study of the incompleteness phenomenon for intermediate logics. Indeed, if $\mathcal{S}' \leq \mathcal{S}$, then every $\mathcal{S}'$-complete intermediate logic is also $\mathcal{S}$-complete, but the converse may fail to be true. However, in contrast with the situation in modal logic [38], little is known about Kripke, topological or locale incompleteness for intermediate logics. One possible explanation for this phenomenon is the fact that IPC is a much less expressive language than modal propositional logic. Moreover, the standard representation theorems that underlie each of these semantics do not fit neatly in a hierarchy that immediately witnesses the increase of generality between them. Dragalin's representation of any locale as the fixpoints of a nuclear algebra does not restrict to the $\Omega - pt$ representation of spatial locales of pointfree topology, which itself does not restrict to the de Jongh-Troelstra representation of superalgebraic locales.

Our goal in this section is to provide a uniform framework for comparing Kripke, topological and nuclear semantics for intuitionistic logic. We first show how Heyting bosets can be used to provide a semantics for IPC that is as general as nuclear semantics and thus equivalent to FM and Dragalin semantics. We then show how the characterizations of spatial and superalgebraic locales obtained in Section 4 allow us to restrict boset semantics to semantics that are equivalent to topological and Kripke semantics. Finally, our main result is a strengthening of one of the only known results regarding Kripke incompleteness of intermediate logics. Using boset semantics, we show that a logic shown in [60] to be Kripke incomplete is in fact incomplete with respect to all complete bi-Heyting algebras. As mentioned in Section 1, a similar result has recently been obtained independently by Bezhanishvili, Gabelaia and Jibladze [5], using Esakia duality.

6.1. Boset semantics

As is standard, we let Var be a countable set of propositional variables and Fml be the set of all formulas of IPC over this set of propositional variables and proceed to define valuations inductively. However, it is useful to define a relation of refutation of a formula at a point, on top of the usual definition of satisfaction. Refutation systems for propositional and modal logic have a long history [26], going back to Łukasiewicz [47]. Refutation relations have also recently been used in the context of generalized Kripke semantics for non-classical logics [12,13,21,28,30]. The introduction of such a relation alongside a satisfaction relation is motivated by the two-sorted nature of these generalized Kripke frames, itself a consequence of the underlying representation of complete lattices via polarity relations of [32] mentioned in Section 2.5.

Definition 6.1. A *boset model* for IPC is a structure $(X, \leq_1, \leq_2, V)$ in which the underlying domain $\mathscr{X} = (X, \leq_1, \leq_2)$ is a Heyting boset and V is a map from Var to $\rho(\mathscr{X})$.

Definition 6.2. Let $(\mathscr{X}, V)$ be a boset model. We define the relations $\Vdash^+$ (*satisfaction*) and $\Vdash^-$ (*refutation*) on $X \times Fml$ inductively as follows:

- $x \Vdash^+ p$ iff $x \in V(p)$;
- $x \Vdash^- p$ iff $x \in \neg_2 V(p)$;
- $x \Vdash^+ \phi \wedge \psi$ iff $x \Vdash^+ \phi$ and $x \Vdash^+ \psi$;
- $x \Vdash^- \phi \wedge \psi$ iff $\forall y \geq_2 x \exists z \geq_1 y : z \Vdash^- \phi$ or $z \Vdash^- \psi$;
- $x \Vdash^+ \phi \vee \psi$ iff $\forall y \geq_1 x \exists z \geq_2 y : z \Vdash^+ \phi$ or $z \Vdash^+ \psi$;
- $x \Vdash^- \phi \vee \psi$ iff $x \Vdash^- \phi$ and $x \Vdash^- \psi$;
- $x \Vdash^+ \phi \to \psi$ iff $\forall y \geq_1 x : y \Vdash^+ \phi$ implies $y \Vdash^+ \psi$;
- $x \Vdash^- \phi \to \psi$ iff $\forall y \geq_2 x \exists z \geq_1 y : y \Vdash^+ \phi$ and $y \Vdash^- \psi$.

For any formula ϕ, we write the sets $\{x \in X : x \Vdash^+ \phi\}$ and $\{x \in X : x \Vdash^- \phi\}$ as $V^+(\phi)$ and $V^-(\phi)$ respectively.

This definition ensures that the semantic value of any formula is always a regular open set. Indeed, a simple induction on the complexity of formulas establishes the following:

Lemma 6.3. *For any formula ϕ:*

- $V^-(\phi) = \neg_2(V^+(\phi))$, *and* $V^+(\phi) = \neg_1(V^-(\phi))$;
- $\neg_1\neg_2 V^+(\phi) = V^+(\phi)$ *and* $\neg_2\neg_1 V^-(\phi) = V^-(\phi)$.
- V^+ *is a homomorphism from the Lindenbaum-Tarski algebra of* IPC *into* $\rho(\mathscr{X})$.

Next, we define validity in the standard way:

Definition 6.4. Let $\mathscr{X}$ be a Heyting boset. A formula ϕ is valid on a boset model $(\mathscr{X}, V)$ if $V^+(\phi) = X$, and ϕ is valid on $\mathscr{X}$ if it is valid on $(\mathscr{X}, V)$ for any valuation V.

This allows for the following soundness and completeness theorem:

Theorem 6.5. *IPC is sound and complete with respect to boset semantics. Moreover, boset semantics is as general as FM and Dragalin semantics.*

Proof. Soundness follows directly from Lemma 6.3. For completeness, note first that a model in any semantics for IPC is characterized by a HA-homomorphism from the free Heyting algebra on countably many generators (also called the Lindenbaum-Tarski algebra of IPC) into a Heyting algebra. Thus any isomorphism between Heyting algebras also induces an isomorphism between corresponding models. By Lemma 2.8, any complete Heyting algebra can be represented as the regular opens of some Heyting boset. Since the Lindenbaum-Tarski algebra of IPC embeds into its MacNeille completion, the completeness of IPC with respect to boset semantics follows. Moreover, since the regular opens of any Heyting boset always form a cHA, and any cHA can also be represented as the cHA of fixpoints of an FM or Dragalin frame (see [6]), it follows that boset semantics is as general as FM and Dragalin semantics. □

Let us conclude by remarking that the dual b-frame of a locale L is closely related to the *canonical FM-frame* introduced in [6, Def. 4.32], since the latter can be obtained from the former by defining the second ordering as the intersection of the two orderings on $\alpha(L)$. The regular open sets of an FM-frame $(X, \leq, \preccurlyeq)$ are guaranteed to form a complete Heyting algebra because of the requirement that $\preccurlyeq$ be a subrelation of $\leq$. As discussed in Section 3.2, this condition is not necessary for the regular opens of a boset to be a complete Heyting algebra, unlike the characterization presented in Lemma 3.12.

6.2. Spatial and splitting semantics

Bosets semantics provides a uniform framework for semantics for IPC. Indeed, now that we have established that boset semantics is as general a semantics based on complete lattices as possible, we can also use our characterization of spatial and superalgebraic locales to define more stringent semantics which are easily seen to be equivalent to topological and Kripke semantics respectively.

Definition 6.6. Let $(\mathscr{X}, V)$ be a boset model.

- $(\mathscr{X}, V)$ is a *spatial model* if for any $x \in X$ and any formulas ϕ, ψ, $x \Vdash^- \phi \wedge \psi$ iff $x \Vdash^- \phi$ or $\Vdash^- \psi$.

- $(\mathscr{X}, V)$ is a *splitting model* if for any $x \in X$ and any formula ϕ, $x \Vdash^+ \phi$ or $x \Vdash^- \phi$.

Note that every splitting model is also spatial: suppose $(\mathscr{X}, V)$ is splitting and let $x \in \mathscr{X}$ and ϕ, ψ be two formulas. Then if $x \nVdash^- \phi$ and $x \nVdash^- \psi$, this implies that $x \Vdash^+ \phi$ and $x \Vdash^+ \psi$, and thus $x \Vdash^+ \phi \wedge \psi$.

Next, we show how spatial and splitting models relate to spatial and splitting points in a boset:

Lemma 6.7. *Let $\mathscr{X}$ be a Heyting boset.*

1. *A point x in $\mathscr{X}$ is spatial iff for any boset model $(\mathscr{X}, V)$ and any formulas ϕ and ψ, $x \Vdash^- \phi \wedge \psi$ iff $x \Vdash^- \phi$ or $\Vdash^- \psi$.*
2. *A point x in $\mathscr{X}$ is splitting iff for any boset model $(\mathscr{X}, V)$ and any formula ϕ, either $x \Vdash^+ \phi$ or $x \Vdash^- \phi$.*

Proof. 1. For the left-to-right direction, assume $x \nVdash^- \phi$ and $x \nVdash^- \psi$. Then we have $y_1, y_2 \geq_2 x$ such that $y_1 \Vdash^+ \phi$ and $y_2 \Vdash^+ \psi$. If x is spatial, we can complete the diagram with a point $z \geq_2 x$ such that $z \geq_1 y_1, y_2$. But this implies that $z \Vdash^+ \phi$ and $z \Vdash^+ \psi$, so $x \nVdash^- \phi \wedge \psi$. Thus $x \Vdash^- \phi \wedge \psi$ implies that $x \Vdash^- \phi$ or $x \Vdash^- \psi$, and the converse direction is always true.

For the right-to-left direction, suppose x is not spatial and we have $y_1, y_2 \geq_2 x$ such that for any $z \geq_1 y_1, y_2$, $z \ngeq_2 x$. Let $V(p) = U^{y_1}$ and $V(q) = U^{y_2}$. Then $y_1 \Vdash^+ p$ and $y_2 \Vdash^+ q$, which means that $x \nVdash^- p$ and $x \nVdash^- q$. On the other hand, since $\mathscr{X}$ is Heyting, we have that $V^+(p \wedge q) = U^{y_1} \cap U^{y_2} = \neg_1\neg_2(\uparrow_1 y_1) \cap \neg_1\neg_2(\uparrow_1 y_2) = \neg_1\neg_2(\uparrow_1 y_1 \cap \uparrow_1 y_2)$. Since $x \in \neg_2(\uparrow_1 y_1 \cap \uparrow_1 y_2)$, this implies that $x \Vdash^- p \wedge q$.

2. For the left-to-right direction, assume $x \nVdash^+ \phi$ and $x \nVdash^- \phi$. Then there are $y_1 \geq_1 x$ and $y_2 \geq_2 x$ such that $y_1 \Vdash^- \phi$ and $y_2 \Vdash^+ \phi$. But then, if $z \geq_2 y_1$ and $z \geq_1 y_2$, we have that $z \Vdash^+ \phi$ and $z \Vdash^- \phi$, a contradiction. Thus, by contraposition, if x is a splitting point, we have that $x \Vdash^+ \phi$ or $x \Vdash^- \phi$.

Conversely, assume x is not a splitting point and let $y_1 \geq_1 x$, $y_2 \geq_2 x$ such that $y_1 \,{}_2{\perp}_1 y_2$. Define $V(p) = U^{y_2}$. Then clearly $y_2 \in V^+(p)$ and $y_1 \in V^-(p)$, which in turn implies that $x \nVdash^- \phi$ and $x \nVdash^+ \phi$. □

Recall that, as we have shown in Section 4, any superalgebraic locale is isomorphic to a boset in which all points are splitting, and any spatial locale is isomorphic to a boset in which all points are spatial. Together with the previous result, this implies the following corollary:

Corollary 6.8.

1. *An intermediate logic L is Kripke complete iff it is complete with respect to a class of Heyting bosets $\mathfrak{C}$ such that for any $\mathscr{X} \in \mathfrak{C}$, any model $(\mathscr{X}, V)$ is splitting.*
2. *An intermediate logic L is topologically complete iff it is complete with respect to a class of Heyting bosets $\mathfrak{C}$ such that for any $\mathscr{X} \in \mathfrak{C}$, any model $(\mathscr{X}, V)$ is spatial.*

Finally, let us conclude by showing how spatial and splitting models can be respectively turned into topological and Kripke models on the same set:

Lemma 6.9. *Let $\mathscr{X}$ be a Heyting boset and V a spatial valuation on $\mathscr{X}$. Then there is a topology τ on $\mathscr{X}$ and a topological valuation V^* such that for any $x \in \mathscr{X}$, $x, V \nVdash^- \phi$ iff $x, V^* \vDash \phi$.*

Proof. Let $(\mathscr{X}, V)$ be a spatial model, and let τ be generated by the sets $\{[\phi] \mid \phi \in Fml\}$, where for any formula ϕ, $[\phi] = \{x \in X \mid x \nVdash^- \phi\}$. Note that for any ϕ, ψ, we have that

$$[\phi \wedge \psi] = \{x \in X \mid x \nVdash^- \phi \wedge \psi\} = \{x \in X \mid x \nVdash^- \phi \text{ and } x \nVdash^- \psi\} = [\phi] \wedge [\psi],$$

where the third equality holds because $(\mathscr{X}, V)$ is a spatial model. This implies that the sets of the form $[\phi]$ form a basis for τ. Similarly, we have that

$$[\phi \vee \psi] = \{x \in X \mid x \nVdash^- \phi \vee \psi\} = \{x \in X \mid x \nVdash^- \phi \text{ or } x \nVdash^- \psi\} = [\phi] \cup [\psi].$$

Moreover, we claim that $[\phi \to \psi] = I_\tau(-[\phi] \cup [\psi])$. Assume first that $x \in [\phi \to \psi]$. Then, $x \nVdash^- \phi \to \psi$. Now if $x \nVdash^- \phi$, we have that $x \nVdash^- \phi \wedge (\phi \to \psi)$, which implies that $x \nVdash^- \psi$. Thus $x \in -[\phi] \cup [\psi]$. Conversely, assume $x \in I_\tau(-[\phi] \cup [\psi])$. Since sets of the form $[\phi]$ form a basis for τ, this means that $x \in [\chi]$ for some formula χ such that $[\chi] \subseteq -[\phi] \cup [\psi]$. Now suppose that $x \Vdash^- \phi \to \psi$. Since $x \nVdash^- \chi$, there is $y \geq_2 x$ such that $y \Vdash^+ \chi$ and $y \Vdash^- \phi \to \psi$. Thus there is $z \geq_1 y$ such that $z \Vdash^+ \chi$, $z \Vdash^+ \phi$ and $z \Vdash^- \psi$. But then $z \in [\chi] \cap ([\phi] - [\psi])$, a contradiction. Thus $x \in [\phi \to \psi]$, which establishes that $[\cdot]$ defines a valuation V^* on (X, τ). Clearly, for any $x \in \mathscr{X}$, we have that $x, V \nVdash^- \phi$ iff $x, V^* \vDash \phi$ for any formula ϕ. □

Lemma 6.10. *Let $(X, \leq_1, \leq_2, V)$ be a splitting model. Then there is a Kripke valuation V^* on $(X, \leq_1)$ such that for any $x \in X$ and any formula ϕ, $x, V \Vdash^+ \phi$ iff $x, V^* \vDash \phi$.*

Proof. Let $(X, \leq_1, \leq_2, V)$ be a splitting model, and consider the Kripke model $(X, \leq_1, V^*)$, where for any formula ϕ, $V^*(\phi) = \{x \in X \mid x \Vdash^+ \phi\}$. Note that it is enough to show that V^* is a well-defined valuation in order to complete the proof. To see this, note that for any $x \in X$ and any formulas ϕ, ψ, we have that $x \in V^*(\phi \wedge \psi)$ iff $x \Vdash^+ \phi \wedge \psi$ iff $x \Vdash^+ \phi$ and $x \Vdash^+ \psi$ iff $x \in V^*(\phi) \cap V^*(\psi)$. Similarly, we have that $V^*(\phi \to \psi) = -{\downarrow}(V^*(\phi) - V^*(\psi))$. Finally, since $(X, \leq_1, \leq_2, V)$ is splitting, for any x, ϕ and ψ, we have that

$$\begin{aligned} x \Vdash^+ \phi \vee \psi &\text{ iff } x \nVdash^- \phi \vee \psi \\ &\text{ iff } x \nVdash^- \phi \text{ or } x \nVdash^- \phi \\ &\text{ iff } x \Vdash^+ \phi \text{ or } x \Vdash^+ \psi. \end{aligned}$$

But this implies at once that for any formulas ϕ, ψ, $V^*(\phi \vee \psi) = V^*(\phi) \cup V^*(\psi)$, which completes the proof. □

Dragalin [16] showed that the open sets of any topological space are isomorphic to the fixpoint of some Dragalin frame. Similarly, Kripke [44] showed that the upsets of any poset are isomorphic to the fixpoints of a certain kind of nuclear frame known as a Beth frame (see [6] for more details on Beth semantics). The previous two lemmas can be seen as relating nuclear semantics to Kripke and topological semantics in a similar fashion, although there are two notable differences. First, Dragalin's and Kripke's results in the literature go from less general to more general semantics, while Lemmas 6.9 and 6.10 go from boset models to Kripke and topological semantics, so from a more general semantics to less general ones. Moreover, while the results mentioned above show how to turn Heyting algebras arising from some semantics into Heyting algebras arising from another one, our results are in some sense more fine-grained, as they show how to turn valuations into valuations, i.e., how to turn Heyting homomorphisms from the Lindenbaum-Tarski algebra of *IPC* into a complete Heyting algebra into Heyting homomorphisms that arise as valuations in some alternative semantics.

6.3. Complete bi-Heyting algebras and the Shehtman logic

Finally, we conclude this section with a generalization of an important result in the literature on intermediate logics. Consider the following inference rule schema, which we call *Litak's Rule*, where ϵ is some uniform substitution:

$$\frac{\begin{array}{c}(1)\ \ (\psi \vee (\psi \to \epsilon(\chi))) \to \chi \\ (2)\ \ \psi \leftrightarrow (\sigma \to \tau) \\ (3)\ \ (\sigma \vee \tau) \to \epsilon(\sigma) \wedge \epsilon(\tau) \\ (4)\ \ \chi \leftrightarrow (\psi \vee \epsilon(\tau))\end{array}}{\chi}$$

Proofs of (variants of) the following theorems can be found in [60] and [46]:

Theorem 6.11. *Let L be an intermediate logic in which Litak's Rule is not admissible. Then for every class $\mathfrak{C}$ of Kripke frames adequate for L, there is a frame $F \in \mathfrak{C}$ and a point in F which refutes the Gabbay-de Jongh bounded branching axiom bb_2.*[9]

Theorem 6.12. *There exists an intermediate logic SL such that $SL \vdash bb_2$ and Litak's rule R is not admissible in SL.*

As a corollary, the Shehtman logic SL is Kripke-incomplete. We strengthen this result as follows:

Theorem 6.13. *The Shehtman logic SL is incomplete with respect to all complete bi-Heyting algebras.*

This is established via the following generalization of Theorem 6.11:

Theorem 6.14. *Let L be an intermediate logic in which Litak's Rule is not admissible. Then for every class $\mathfrak{C}$ of b-frames dual to complete bi-Heyting algebras, if $\mathfrak{C}$ is adequate for L, then there is a b-frame $\mathscr{X} \in \mathfrak{C}$ such that the Gabbay-de Jongh bounded branching axiom bb_2 is refuted at some point in $\mathscr{X}$.*

The proof will take several lemmas. Suppose first that $\mathfrak{C}$ is a class of b-frames dual to a bi-Heyting locale, and notice that this implies that for any $\mathscr{X} \in \mathfrak{X}$, the bi-Heyting points of $\mathscr{X}$ are dense. Assume that $\mathfrak{C}$ is adequate for L. Then since L is valid on any b-frame in $\mathfrak{C}$, the following holds:

Lemma 6.15. *Let $\mathscr{X} \in \mathfrak{C}$ and V be a valuation on $\mathscr{X}$. The following are true for any $x \in X$ and $n \in \omega$:*

1. *$x, V \Vdash^+ \epsilon^n(\sigma \vee \tau)$ implies $x, V \Vdash^+ \epsilon^i(\sigma) \wedge \epsilon^j(\tau)$ for all $i, j \geq n$;*
2. *$x, V \Vdash^+ \epsilon^n(\chi)$ implies $x, V \Vdash^+ \epsilon^i(\chi)$ for any $i \leq n$; moreover, $x, V \Vdash^+ \epsilon^n(\psi)$ implies $x, V \Vdash^+ \epsilon^i(\chi)$ for all $i \leq n$.*
3. *$x, V \Vdash^+ \epsilon^n(\sigma)$ implies $x, V \Vdash^+ \epsilon^m(\chi)$ for all $m \in \omega$;*
4. *$x, V \Vdash^+ \epsilon^n(\sigma)$ and $x, V \Vdash^- \epsilon^n(\tau)$ together imply that $x, V \Vdash^- \epsilon^n(\psi)$.*
5. *$x, V \Vdash^- \epsilon^n(\chi)$ implies that there exists $y, z \geq_1 x$ such that $y, V \Vdash^+ \epsilon^n(\sigma)$, $y, V \Vdash^- \epsilon^n(\tau)$, $z, V \Vdash^+ \epsilon^n(\psi)$ and $z, V \Vdash^- \epsilon^{n+1}(\chi)$.*

Proof. 1. By a repeated use of axiom (3).
2. By a repeated use of axiom (1).
3. Fix $n, m \in \omega$, and let $k = max\{n, m\}$. By 1 above, $x \Vdash^+ \epsilon^n(\sigma)$ implies $x \Vdash^+ \epsilon^{k+1}(\tau)$. By axiom (4), this in turn implies that $x \Vdash^+ \epsilon^k(\chi)$. But then from 2 above it follows that $x \Vdash^+ \epsilon^m(\chi)$.
4. Assume $y \geq_2 x$. Then $y \Vdash^+ \epsilon^n(\sigma)$ and $y \nVdash^+ \epsilon^n(\tau)$, from which it follows that $y \nVdash^+ \epsilon^n(\psi)$. Hence $x \Vdash^- \epsilon^n(\psi)$.
5. Assume $x \Vdash^- \epsilon^n(\chi)$. Then, by axiom (1), $x \Vdash^- \epsilon^n(\psi)$, hence, (by axiom (2)) we have $x \Vdash^- \epsilon^n(\sigma) \to \epsilon^n(\tau)$ and $x \Vdash^- \epsilon^n(\psi) \to \epsilon^{n+1}(\chi))$. This means that there is $y \geq_1 x$ such that $y \Vdash^+ \epsilon^n(\sigma)$ and $y \Vdash^- \epsilon^n(\tau)$, and there is $z \geq_1 x$ such that $z \Vdash^+ \epsilon^n(\psi)$ and $z \Vdash^- \epsilon^{n+1}(\chi)$. □

[9] See below for the definition of bb_2.

Now since Litak's Rule is not admissible in L, there is some $\mathscr{X} = (X, \leq_1, \leq_2) \in \mathfrak{C}$ and a valuation V on $\mathscr{X}$ such that all the premises of Litak's rule are true at all points in X and there is $x \in X$ such that $x \Vdash^- \chi$.

In what follows, for $i \in \{0, 1, 2\}$, we write $i+1$ and $i+2$ for $i+1$ and $i+2$ *mod* 3 respectively. Recall that bb_2 is the axiom:

$$\bigwedge_{i \in \{0,1,2\}} (p_i \to (p_{i+1} \vee p_{i+2}) \to (p_{i+1} \vee p_{i+2})) \to (p_0 \vee p_1 \vee p_2).$$

Definition 6.16. For every $n < \omega$, let $S_n = \bigcap_{m \neq n < \omega} V(\epsilon^m(\psi)) - V(\epsilon^n(\psi))$.

It is easy to see that for any $n < \omega$, $\bigcap_{m<\omega} V(\epsilon^m(\psi)) \cup S_n$ is a $\leq_1$-upset (this is because if $x \leq_1 y$ and $x \in S_n$ for some $n < \omega$, then since $y \in V(\epsilon^n(\psi)) \cup (X - V(\epsilon^n(\psi)))$, we have that either $y \in \bigcap_{n<\omega} V(\epsilon^n(\psi))$ or $y \in S_n$).

Now let p_0, p_1, p_2 be three fresh propositional variables and define a valuation V' as follows:

- $V'(q) = V(q)$ for any propositional variable $q \in L_{IPC}$ such that $q \neq p_i$ for $i \in \{0, 1, 2\}$;
- $V'(p_i) = \neg_1 \neg_2 (\bigcap_{n<\omega} V(\epsilon^n(\psi)) \cup \bigcup_{n<\omega} S_{3n+i})$ for $i \in \{0, 1, 2\}$.

We will now need to prove three lemmas that will give the key to the proof. The general idea is the following: We first prove that any x that refutes χ must also refute the disjunction $p_0 \vee p_1 \vee p_2$. We then show that any point that refutes one of the antecedent of bb_2 must be the root of an analogue of the Beth comb[10] in the setting of b-frames. Finally, showing that the teeth of such a Beth comb must satisfy precisely one of $\{p_0, p_1, p_2\}$ will imply, by contradiction, that x must also satisfy the antecedent of bb_2.

We start with the refutation of the consequent of bb_2.

Lemma 6.17. *For all $x \in X$, if $x \Vdash^- \chi$, then $x \Vdash^- p_i$ for $i \in \{0, 1, 2\}$, which implies that $x \Vdash^- p_0 \vee (p_1 \vee p_2)$.*

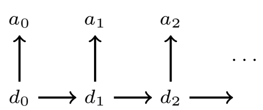
Proof. Assume $x \Vdash^- \chi$, and let $y \geq_2 x$. We claim that $y \notin V'(p_i)$ for $i \in \{0, 1, 2\}$. To see this, let $z \geq_2 y$. Note first that since $z \Vdash^- \chi$, we must have that $z \Vdash^- \epsilon^n(\psi)$ for any $n < \omega$ by Lemma 6.15.2. But this implies that for all $z \geq_2 y$, $z \notin \bigcap_{n<\omega} V(\epsilon^n(\psi)) \cup \bigcup_{n<\omega} S_{3n+i}$. Hence $y \notin V'(p_i)$ for $i \in \{0, 1, 2\}$. From this it follows that $x \Vdash^- p_i$ for $i \in \{0, 1, 2\}$, and hence $x \Vdash^- p_0 \vee (p_1 \vee p_2)$. □

Let us now move on to the lemmas that will be used to prove that x must also satisfy the antecedent of bb_2.

Lemma 6.18. *For any $x \in X$, if $x \Vdash^- p_0 \vee (p_1 \vee p_2)$, then there is $n < \omega$ such that $x \nVdash^+ \epsilon^n(\chi)$.*

Proof. Assume $x_0 \Vdash^+ \epsilon^n(\chi)$ for all $n < \omega$, and $x_0 \Vdash^- p_0 \vee (p_1 \vee p_2)$. Let $x \geq_{12} x_0$ be a bi-Heyting point, and note that this implies that $x \Vdash^+ \epsilon^n(\chi)$ for all $n < \omega$, and $x \Vdash^- p_0 \vee (p_1 \vee p_2)$. This implies that $x \nVdash^+ \epsilon^n(\psi)$ for some $n < \omega$, for otherwise $x \in \bigcap_{n<\omega} V(\epsilon^n(\psi))$, which means that $x \Vdash^+ p_i$ for all $i \in \{0, 1, 2\}$ (since $\bigcap_{n<\omega} V(\epsilon^n(\psi))$ is a $\leq_1$-upset). Let j be the smallest number n such that $x \nVdash^+ \epsilon^n(\psi)$. Then there is some $y_0 \geq_1 x$ such that $y_0 \Vdash^- \epsilon^j(\psi)$, and since x is bi-Heyting, $y_0 \leq_2 y$ for some bi-Heyting point $y \geq_{12} x$. Now

[10] Recall that the Beth comb is the set $\{a_n\}_{n\in\omega} \cup \{d_n\}_{n\in\omega}$ endowed with the following structure:



.

since $x \Vdash^+ \epsilon^j(\chi)$, we also have that $y \Vdash^+ \epsilon^j(\chi)$, so by axiom (4) we have that $y \Vdash^+ \epsilon^j(\psi) \vee \epsilon^{j+1}(\tau)$. But this implies that there is $z_0 \geq_2 y$ such that $z \Vdash^+ \epsilon^{j+1}(\tau)$, and since y is a bi-Heyting point, this implies that there is some $z \geq_1 z_0$ such that $y \leq_{12} z$. Thus $z \Vdash^+ \epsilon^{j+1}(\tau)$, and hence, since j is the smallest number n such that $x_0 \nVdash^+ \epsilon^n(\psi)$, we have that $z \in \bigcap_{n<\omega} V(\epsilon^n(\psi)) \cup S_j$, and therefore $z \Vdash^+ p_i$ for $i = j \mod 3$. But this contradicts the fact that $x \Vdash^- p_0 \vee (p_1 \vee p_2)$, since $x \leq_{12} y \leq_{12} z$. Therefore for any $x \in X$, if $x \Vdash^- p_0 \vee (p_1 \vee p_2)$, then there is $n < \omega$ such that $x \nVdash^+ \epsilon^n(\chi)$. □

The previous lemma used the fact that the bi-Heyting points of $\mathscr{X}$ are dense. It is straightfoward to verify that this is the only place where this fact is used in the proof of Theorem 6.14.

Lemma 6.19. *For all $x \in X$, if there is $n \in \omega$ such that $x \Vdash^+ \bigwedge_{j<n} \epsilon^j(\psi) \wedge \epsilon^n(\sigma)$ and $x \Vdash^- \epsilon^n(\tau)$, then $x \Vdash^+ p_i$ and $x \Vdash^- p_{i+1} \vee p_{i+2}$ for $i \in \{0,1,2\}$ such that $n = i \mod 3$.*

Proof. Assume $x \Vdash^+ \bigwedge_{j<n} \epsilon^j(\psi) \wedge \epsilon^n(\sigma)$ and $x \Vdash^- \epsilon^n(\tau)$. Note that this implies that $x \in S_n$, and therefore $x \Vdash^+ p_i$ for $i = n \mod 3$. Moreover, let $y \geq_2 x$. Then we have that $y \nVdash^+ \epsilon^n(\psi)$, and therefore $y \notin \bigcap_{m<\omega} V(\epsilon^m(\psi)) \cup S_k$ for any $k \neq n$. Hence $y \nVdash^+ p_j$ for any $j \neq i \in \{0,1,2\}$. Hence $x \Vdash^- p_j$ for $j \neq i \in \{0,1,2\}$. □

We have now gathered all the ingredients for the proof of Theorem 6.14:

Proof. Recall that there is $x \in X$ such that $x \Vdash^- \chi$. We will prove that axiom bb_2 is refuted at x, i.e., we prove that $x \Vdash^+ (p_i \to (p_{i+1} \vee p_{i+2})) \to (p_{i+1} \vee p_{i+2})$ for all $i \in \{0,1,2\}$ and that $x \Vdash^- p_0 \vee (p_1 \vee p_2)$. To see this, note first that the latter follows immediately from Lemma 6.17. Moreover, assume that for some $i \in \{0,1,2\}$, $x \nVdash^+ (p_i \to (p_{i+1} \vee p_{i+2})) \to (p_{i+1} \vee p_{i+2})$. This means that there is $y \geq_1 x$ such that $y \Vdash^+ p_i \to (p_{i+1} \vee p_{i+2})$ and $y \Vdash^- p_{i+1} \vee p_{i+2}$. Note that this implies that $y \Vdash^- p_0 \vee (p_{i+1} \vee p_{i+2})$. Now by Lemma 6.18, this means that $y \nVdash^+ \epsilon^n(\chi)$ for some $n \in \omega$ and hence that there is $z \geq_1 y$ such that $z \Vdash^- \epsilon^n(\chi)$. Let n be the smallest number such that $z \Vdash^- \epsilon^n(\chi)$. This means that $z \Vdash^+ \bigwedge_{j<n}(\epsilon^j(\psi) \vee \epsilon^{j+1}(\tau))$. Now since $z \Vdash^+ \epsilon^j(\tau) \to \epsilon^n(\tau)$ for any $j \leq n$, this implies that $z \Vdash^+ \bigwedge_{j<n}(\epsilon^j(\psi) \vee \epsilon^n(\tau))$, i.e., $z \Vdash^+ \bigwedge_{j<n} \epsilon^j(\psi) \vee \epsilon^n(\tau)$. This means that there is $z' \geq_2 z$ such that $z' \Vdash^+ \bigwedge_{j<n} \epsilon^j(\psi)$ or $z' \Vdash^+ \epsilon^n(\tau)$. But the latter is impossible, since $z \Vdash^- \epsilon^n(\chi)$. Hence $z' \Vdash^+ \bigwedge_{j<n} \epsilon^j(\psi)$. But then, by repeated use of Lemma 6.15.5, there must be $z'' \geq_1 z' \geq_1 z \geq_1 y$ such that $z'' \Vdash^+ \bigwedge_{j<m} \epsilon^j(\psi) \wedge \epsilon^m \sigma$ and $z'' \Vdash^- \epsilon^m(\tau)$ for $m \geq n$ such that $m = i \mod 3$. By Lemma 6.19, this implies that $z'' \Vdash^+ p_i$ and $z'' \Vdash^- p_{i+1} \vee p_{i+2}$, contradicting the fact that $y \Vdash^+ p_i \to (p_{i+1} \vee p_{i+2})$. Hence $x \Vdash^+ (p_i \to (p_{i+1} \vee p_{i+2})) \to (p_{i+1} \vee p_{i+2})$ for all $i \in \{0,1,2\}$, which completes the proof that x refutes bb_2. □

A similar example of an intermediate logic that is incomplete with respect to complete bi-Heyting algebras has recently and independently been obtained by G. Bezhanishvili, D. Gabelaia and M. Jibladze in [5]. It is worth mentioning that the proof presented here is but a minor variation on Litak's proof for Kripke incompleteness, while the proof in [5] requires a significantly different argument. This fact can be seen as an additional reason to believe that boset semantics might offer a generalization of Kripke semantics that still retains many of its attractive features. We should also note that it seems unlikely that the same proof could be generalized any further. Indeed, from an algebraic perspective, the proof appears to be exploiting in a key way the fact that complete bi-Heyting algebras satisfy the Meet Infinite Distributive Law (i.e., arbitrary meets distribute over finite joins). Since complete bi-Heyting algebras are the largest class of cHA's satisfying this law, this can be seen as evidence that we have pushed Shehtman's method to its limits and that new ideas might be needed in order to construct, if at all possible, topologically incomplete logics.

7. Conclusion

We conclude by outlining some areas for further research.

First of all, we have only presented preliminary results regarding a correspondence theory between lattice equations and b-frame properties. While we have been able to isolate first-order conditions on b-frames that are equivalent to various properties of complete lattices, we are still lacking a systematic procedure for translating lattice equations into b-frame conditions, akin to Sahlqvist correspondence in modal logic.

Moreover, although we focused in our applications on certain classes of complete Heyting algebras, the adjunction we presented holds for all complete lattices. This means in particular that one could use bosets in the study of some categories of enriched lattices, including for example ortholattices, residuated lattices, or lattices expanded with various modal operators. In that respect, the connection with polarity-based semantics for non-classical logics developed in [12,13,21,28,30] should be explored further.

Finally, the dualities developed here are all discrete dualities between complete lattices and relational structures. This means that we decided to trade off the ability to deal with incomplete lattices for a greater simplicity of the geometric structures we work with. A natural next step would therefore be to topologize the duality presented here and to connect such a generalization both to the Dunn-Hartonas duality for bounded lattices [31,32] and to the choice-free duality recently developed in [9].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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