

TRIAL AND ERROR MATHEMATICS: DIALECTICAL SYSTEMS AND COMPLETIONS OF THEORIES

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ABSTRACT. This paper is part of a project that is based on the notion of dialectical system, introduced by Magari as a way of capturing trial and error mathematics. In [2] and [3], we investigated the expressive and computational power of dialectical systems, and we compared them to a new class of systems, that of quasidialectical systems, that enrich Magari's systems with a natural mechanism of revision. In the present paper we consider a third class of systems, that of p -dialectical systems, that naturally combine features coming from the two other cases. We prove several results about p -dialectical systems and the sets that they represent. Then we focus on the completions of first-order theories. In doing so, we consider systems with connectives, i.e. systems that encode the rules of classical logic. We show that any consistent system with connectives represents the completion of a given theory. We prove that dialectical and q -dialectical systems coincide with respect to the completions that they can represent. Yet, p -dialectical systems are more powerful: we exhibit a p -dialectical system representing a completion of Peano Arithmetic which is neither dialectical nor q -dialectical.

1. INTRODUCTION AND BACKGROUND

Formal systems represent mathematical theories in a rather static way, in which axioms of the represented theory have to be defined from the beginning, and no further modification is permitted. It has often been argued that this representation is not comprehensive of all aspects of real mathematical theories: see, for instance, the seminal work of Lakatos [13] for arguments against any hastily correspondence between formal systems and the way in which mathematicians deal with real theories. Our goal is to model cases in which a mathematician, when defining a new theory, chooses axioms through some trial and error process, instead of fixing them, once for all, at the initial stage. A possible way of characterizing such cases is provided by the so-called experimental logics, firstly studied by Jeroslow in the 1970's [10] (for a nice discussion about these logics and their philosophical meaning the reader is referred to Kåsa [11]). Our approach is based on another notion, that of dialectical systems, introduced by Magari [14] in the same period. In doing so, we continue the investigations initiated in [2] and [3].

The basic ingredients of *dialectical systems* are a number c , encoding a contradiction; a deduction operator H , that tells us how to derive consequences from a finite set of statements D ; and a proposing function f , that proposes statements to be accepted or rejected as provisional theses of the system. In [2], we introduced a new class of systems, that of *q -dialectical systems* (there called

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“quasidialectical”), by enriching Magari’s systems with a natural mechanism of revision. This is obtained by means of two additional ingredients: a replacement function f^- , that provides for all axioms a substituting axiom, and a symbol c^- , that encodes any sort of problem, possibly weaker than mathematical contradiction, that can justify the replacement of a certain axiom. In [2] and [3], we drew an accurate comparison between the expressiveness of these two systems. In particular, we showed the following: dialectical sets and q -dialectical sets (i.e., the set of statements that are eventually accepted by, respectively, dialectical and q -dialectical systems) are always Δ_2^0 ; the two systems have the same computational power, in the sense that the class of Turing degrees that contains a dialectical set coincide with the class of Turing degrees that contains a q -dialectical set (and in fact they are equivalent to the class of computably enumerable Turing degrees); yet, q -dialectical sets form a class which is much larger than that of dialectical sets, since q -dialectical sets inhabit each level of Ershov hierarchy, while dialectical sets are all ω -computably enumerable.

In this paper we consider a third class of systems, called *p -dialectical systems*. Their introduction is motivated by two main reasons. First, p -dialectical systems naturally combines, in their behaviour, features characterizing the other two classes: they have a mechanism of revision, as in the case of q -dialectical systems, but they do not distinguish between c and c^- , having only c in their syntax, as in the case of dialectical systems. In fact, dialectical and q -dialectical systems can be defined as modifications of p -dialectical systems, as we will do in the next section. The second important reason for focusing on this new class is connected with the completions of first-order theories.

As is shown below, if we restrict to the case of *systems with connectives*, i.e. systems in which the deduction operator H has to satisfy the rules of classical logic, then we do obtain the following: if S is a system that does not derive the contradiction from the empty set of premises, then S is the *completion* of a given theory. We make use of this fact to compare the expressiveness of our systems regarded as machines to build, in the limit, completions. We show that dialectical and q -dialectical completions coincide, all lying in the class of ω -c.e. sets. On the contrary, p -dialectical systems are much more powerful: for every effectively indexed class of Δ_2^0 sets we exhibit a concrete example of a p -dialectical system representing a p -dialectical set which is a completion of Peano Arithmetic not lying in that class. Each such p -dialectical system can be also looked at as an example of how a p -dialectical system works in concrete, perhaps the first such examples even considering [2, 3], more concerned with laying down the theoretical bases rather than examples and applications.

We would like to remark at this point that although dialectical systems may be viewed as a possible approach to trial and error mathematics, the emphasis in this paper is of a rather abstract nature. More than on the adequacy of these systems to formalize trial and error mathematics, we are mainly interested in the computability theoretic properties of the dialectical sets (i.e. the sets represented by these systems), and in the use of dialectical systems and our suggested variations of the dialectical procedure as tools for producing more and more complicated Δ_2^0 completions of consistent formal theories having a strong enough expressive power. In Section 3 however we sketch a brief comparison between dialectical systems and other approaches based on knowledge or assumptions revision.

Although the exposition of this paper is rather self-contained, a certain familiarity with the definitions of dialectical and q -dialectical systems, as presented in [2], might help the reader that aims at fully understanding the behaviour of the p -dialectical systems we will introduce next. Our computable theoretic notions are standard and as in Soare [21].

1.1. The p -dialectical systems. A p -dialectical system shall be thought as a machine for constructing a theory in stages, by adjusting the set of axioms whenever a contradiction is derived. This is the same intuition that both dialectical and q -dialectical systems aim at modelling (see [2]). What distinguishes the three cases is how they respond to the emergence of a contradiction, and whether they are allowed to revise an axiom, when this is temporary rejected by system, instead of being forced to fully dismiss it. We shall begin with the formal definition of p -dialectical systems. In what follows, if f is the so-called proposing function, we will denote $f(i)$ with f_i .

Definition 1.1. A p -dialectical system is a quadruple $p = \langle H, f, f^-, c \rangle$, where

- (1) H is an enumeration operator such that $H(\emptyset) \neq \emptyset$, $H(\{c\}) = \omega$, and H is an algebraic closure operator, i.e., H satisfies, for every $X \subseteq \omega$,
 - $X \subseteq H(X)$;
 - $H(X) \supseteq H(H(X))$.
- (2) f is a computable permutation of ω ;
- (3) f^- is an *acyclic* computable function, i.e., for every x , the f^- -orbit of x , i.e. the set

$$\{x, f^-(x), f^-(f^-(x)), \dots, (f^-)^n(x), \dots\},$$

is infinite.

We call f the *proposing function*, f^- the *revising function*, c the *contradiction*.

The p -dialectical procedure. Given such a $p = \langle H, f, f^-, c \rangle$, and starting from a a computable approximation $\alpha = \{H_s\}_{s \in \omega}$ (i.e. a computable sequence of finite sets, given by their canonical indices, such that $H_s \subseteq H_{s+1}$ and $H = \bigcup_s H_s$), define by induction values for several computable parameters, which depend on α : A_s (a finite set), r_s (a function such that for every x , $r_s(x)$ is a finite string of numbers, viewed as a vertical string, or *stack*), $m(s)$ (the greatest number m such that $r_s(m) \neq \langle \rangle$, where the symbol $\langle \rangle$ denotes the empty string). In addition, there are the derived parameters: $\rho_s(x)$ is the top of the stack $r_s(x)$, $L_s(x) = \{\rho_s(y) : y < x \text{ and } r_s(y) \neq \langle \rangle\}$, and, for every i , $\chi_s(i) = H_s(L_s(i+1))$.

Stage 0. Define $m(0) = 0$,

$$r_0(x) = \begin{cases} \langle f_0 \rangle & x = 0 \\ \langle \rangle & x > 0, \end{cases}$$

and let $A_0 = \emptyset$.

Stage $s + 1$. Assume $m(s) = m$. We distinguish the following cases:

- (1) there exists no $k \leq m$ such that $\{c\} \cap \chi_s(k) \neq \emptyset$: in this case, let $m(s+1) = m+1$, and define

$$r_{s+1}(x) = \begin{cases} r_s(x) & \text{if } x \leq m \\ \langle f_{m+1} \rangle & \text{if } x = m+1 \\ \langle \rangle & \text{if } x > m+1; \end{cases}$$

- (2) there exists $k \leq m$ such that $c \in \chi_s(k)$: in this case, let z be the least such k , let $m(s+1) = z$, and define, where $\rho_s(z) = f_y$,

$$r_{s+1}(x) = \begin{cases} r_s(x) & x < z \\ r_s(x) \wedge \langle f^-(f_y) \rangle & x = z \\ \langle \rangle & x > z + 1. \end{cases}$$

Finally let

$$A_{s+1} = \bigcup_{i < m(s+1)} \chi_{s+1}(i).$$

Notice that $A_{s+1} = H_{s+1}(L_{s+1}(m(s+1)))$ if $m(s+1) > 0$; otherwise $A_{s+1} = \emptyset$.

Figures 1 and 2 illustrate how we go from stage s to stage $s+1$, according to (1) and (2), respectively, of the definition. The vertical strings above the various slots represent the various stacks $r(x)$ at the given stage. In each figure, only the relevant slots are depicted.

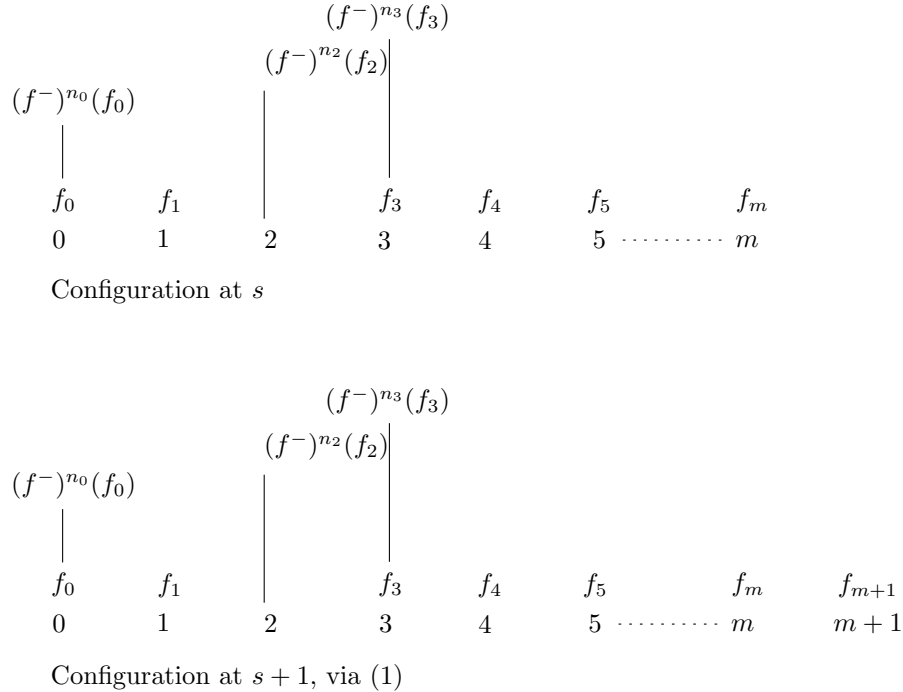


FIGURE 1. From stage s to $s+1$ using (1).

We say that a p -dialectical system with enumeration operator H is *consistent* if $c \notin H(\emptyset)$. We call A_s the set of *provisional theses* of p with respect to α at stage s . The set A_p defined as

$$A_p = \{f_x : (\exists t)(\forall s \geq t)[f_x \in A_s]\}$$

is called the set of *final theses* of p : notice that we write A_p and not A_p^α because we are going to show in next theorem that this set does not in fact depend on the approximation.

In the following theorem and its proof, relatively to any given approximation α we agree that $\lim_s r_s(u)$ *exists finite* if there exists t such that $r_s(u) = r_t(u)$ for all $s \geq t$; $\lim_s r_s(u)$ *exists infinite*

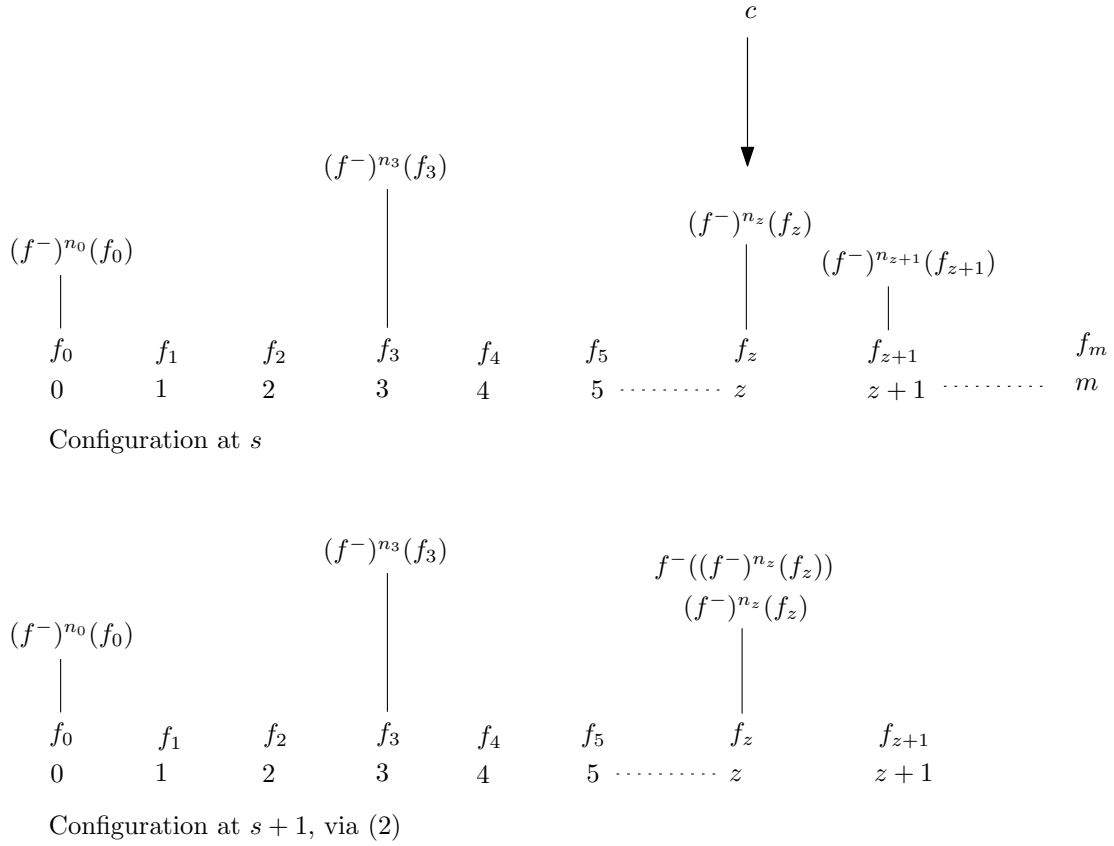


FIGURE 2. From stage s to $s+1$ using (2).

if there exists a stage t , such that for all $s \geq t$ $r_s(u)$ is an initial segment of $r_{s+1}(u)$ and $\bigcup_{s \geq t} r_s(u)$ is an infinite string; finally we say that $\lim_s r_s(u)$ does not exist if for infinitely many s we have $r_s(v) = \langle \rangle$.

Lemma 1.2. *The set of final theses of a p -dialectical system does not depend on the chosen approximation of the enumeration operator H , and, independently of the approximation, for every u , either $\lim_s r_s(u)$ exists finite, or $\lim_s r_s(u)$ exists infinite and in this case for every $v > u$ $\lim_s r_s(v)$ does not exist; or $\lim_s r_s(u)$ does not exist, and in this case also for every $v > u$ $\lim_s r_s(v)$ does not exist.*

Proof. Let $p = \langle H, f, f^-, c \rangle$ be a p -dialectical system and $\alpha = \{H_s\}$ an approximation to H .

First of all, if $\lim_s r_s(u)$ exists infinite, the every time we redefine $r_s(u)$ we also set $r_s(v) = \langle \rangle$ for every $v > u$; moreover it is easy to see that if $\lim_s r_s(v)$ does not exist then there is some $u < v$ such that $\lim_s r_s(u)$ exists infinite.

So the claim about $\lim_s r_s(u)$ amounts to show that either $\lim_s r_s(u)$ exists finite for every u or there is a least u such that $\lim_s r_s(u)$ exists infinite.

Now, $L(0) = \lim_s L_s(0) = \emptyset$ and clearly this value does not depend on the approximation. Suppose that $L(u)$ reaches limit and the limit does not depend on the approximation, and let us consider

$u + 1$: we show by induction on i that after $L(u)$ has reached limit, the i -th bit $r(u)_i$ is the same (whether defined or undefined) whatever approximation one considers. Now, $r(u)_0 = \langle f_u \rangle$ whatever the approximation; and clearly, assuming the claim for i , we have that $r(u)_{i+1} = f^-(r(u)_i)$ if and only if $c \in H(L(u) \cup \{r(u)_i\})$, which shows independence from the approximation. In particular $\lim_s r_s(u)$ exists either finite or infinite, independently of the approximation. \square

Theorem 1.3. *As granted by the previous lemma, let u be the greatest number $\leq \omega$ such that the limit value $L(u)$ exists finite, i.e. for every $v < u$ $\lim_s r_s(v)$ exists finite. If $u > 0$ then, independently of the approximation, $A_p = H(L(u))$ (where $L(\omega) = \bigcup_{v \in \omega} L(v)$ if $u = \omega$), and $A_p = L(\omega)$ if $u = \omega$. If $u = 0$ then, independently of the approximation, $A_p = \emptyset$.*

Proof. Let u be as in the statement of the theorem. Let us show first that $H(L(u)) \subseteq A_p$. If $u < \omega$ then cofinitely many times we have $L(u) \subseteq L_s(m(s))$, which implies $H_s(L(u)) \subseteq H_s(L_s(m(s)))$; hence cofinitely many times we have $H_s(L(u)) \subseteq A_s$, which implies that $H(L(u)) \subseteq A_p$, independently of the approximation. If $u = \omega$ then for every v an argument similar to the previous case shows that $L(v) \subseteq H_s(L(v)) \subseteq A_s$ for cofinitely many s , and thus $L(\omega) \subseteq A_p$, independently of the approximation.

We want now to show now that $A_p \subseteq H(L(u))$ and $A_p \subseteq L(\omega)$ if $u = \omega$: in the latter case by properties of H this implies also that $A_p \subseteq H(L(\omega))$. We distinguish again the two possible cases:

- $u < \omega$: for infinitely many s , we have that (whatever the approximation) $A_s = H_s(L(u))$ which shows that $A_p \subseteq H(L(u))$.
- $u = \omega$: suppose now that $f_x \notin L(\omega)$. Then, whatever the approximation, $c \in H(L(x) \cup \{f_x\})$, and thus $f_x \notin H_s(L_s(x))$ for every big enough stage. Let t be a stage such that starting from this stage $L(x)$ has reached limit already and $f_x \notin H_s(L_s(x))$ for every $s \geq t$. Let $t_0 \geq t$ be such that $f_x \in H_{t_0}(L_{t_0}(v))$ ($v = m(t_0)$): notice that $x < v$; then there is a stage $s_1 \geq t_0$ such that $c \in H_{s_1}(L_{t_0}(v))$; it follows that there is a stage $t_1 \geq t_0$ such that $L(v)$ changes value at t_1 , giving $m(t_1) < v$. If $f_x \notin A_{t_1}$ then we have found a stage $\geq t_0$ at which $f_x \notin A_p$ has changed; otherwise we repeat the same argument, but taking $v = m(t_1)$. By choice of t and properties of x , it is clear that proceeding in this way we end up with some $t' \geq t_0$ such that $f_x \notin A_{t'}$. We have shown that for every t_0 such that $f_x \in A_{t_0}$ there is a later stage t' such that $f_x \notin A_{t'}$. As this works for whatever approximation we use, this shows that $f_x \notin A_p$ whatever the approximation. We have thus shown that $A_p \subseteq L(\omega)$.

Finally we consider the case $u = 0$. In this case $m(s) = 0$ infinitely many times, then $A_p = \emptyset$, whatever the approximation. \square

Definition 1.4. A pair (p, α) where $p = \langle H, f, f^-, c \rangle$ is a p -dialectical system and α is an approximation to H is called *loopless* if for every u , the set $\{\rho_s(u) : s \in \omega\}$ is finite.

Remark 1.5. In view of the previous theorem if there is a loopless approximation then all approximations are loopless, and we will be justified in talking about a *loopless* p -dialectical system, and referring to the *final theses* A_p of p , without mentioning any special approximation α to the enumeration operator H of p .

Corollary 1.6. *If p is loopless then $A_p = L(\omega)$.*

Proof. See the proof of Theorem 1.3. \square

A set $A \subseteq \omega$ is called *p-dialectical* if $A = A_p$ for some *p-dialectical* system, and we say in this case that A is *represented* by p .

1.2. Dialectical systems and q-dialectical systems. Dialectical systems and *q-dialectical systems* have been extensively studied in [2] and [3]; the reader is referred to these papers for both full definitions of them and philosophical motivations for their study. For our present interests, let us show where the definition of a *p-dialectical* system is to be modified in order to obtain these others systems.

Definition 1.7. A *dialectical system* is a *p-dialectical* system with no revising function. That is to say, a dialectical system is a triple $d = \langle H, f, c \rangle$, in which H, f, c satisfy the same conditions formulated within Definition 1.1. All the others parameters we have introduced for *p-dialectical* systems ($A_s, r_s, m(s), \rho_s(x), r_s(x), L_s(x)$, and $\chi_s(i)$) hold the same meaning for dialectical systems.

Dialectical procedure. The dialectical procedure is equal verbatim to the *p-dialectical* procedure for stage 0, and for any application of Clause (1) of any given stage $s + 1$. Thus the only difference is with Clause (2), which in the case of dialectical systems has to be modified as follows:

(2) there exists $k \leq m$ such that $c \in \chi_s(k)$: in this case, let z be the least such k , and distinguish two cases:

(2.1) if $c \in H_s(\emptyset)$, then let $m(s + 1) = 0$, and define

$$r_{s+1}(x) = \begin{cases} \langle f_0 \rangle & \text{if } x = 0 \\ \langle \rangle & \text{if } x > 0; \end{cases}$$

(2.2) otherwise, let $m(s + 1) = z + 1$, and define

$$r_{s+1}(x) = \begin{cases} r_s(x) & \text{if } x < z \\ \langle f_{z+1} \rangle & \text{if } x = z + 1 \\ \langle \rangle & \text{if } x = z \text{ or } x > z + 1. \end{cases}$$

We say that a dialectical system with enumeration operator H is *consistent* if $c \notin H(\emptyset)$. The sets of final theses of dialectical systems and dialectical sets are defined in a similar way to *p-dialectical* systems.

Let us then move to *q-dialectical* system. A *q-dialectical* system, intuitively, incorporates both distinguishing features of dialectical and *p-dialectical* systems, in the sense that some axiom f_x can be either discarded, as in the case of a dialectical system, or revised by f^- , as in the case of a dialectical system. Since the formal defintion of a *q-dialectical* system (that can be found in [2]) for the most part is identical to that of a *p-dialectical* system, we limit ourselves to point the differences between the two.

Definition 1.8. A *q-dialectical* system is a quintuple $q = \langle H, f, f^-, c, c^- \rangle$, such that $\langle H, f, c \rangle$ is a dialectical system, f^- satisfies the condition expressed for a *p-dialectical* system, and finally $c^- \in \omega \setminus \text{range}(f^-)$.

We call c^- the *counterexample*.

q -dialectical procedure. Stage 0 of the q -dialectical procedure is identical to the same stage of both the p -dialectical and the dialectical procedure. Concerning stage $s + 1$, we have now three different clauses instead of two (the additional one being introduced since we deal with both c and c^-):

- (1) there exists no $k \leq m$ such that $\{c, c^-\} \cap \chi_s(k) \neq \emptyset$: in this case, let $m(s + 1) = m + 1$, and define

$$r_{s+1}(x) = \begin{cases} r_s(x) & \text{if } x \leq m \\ \langle f_{m+1} \rangle & \text{if } x = m + 1 \\ \langle \rangle & \text{if } x > m + 1; \end{cases}$$

- (2) there exists $k \leq m$ such that $c \in \chi_s(k)$, and for all $k' < k$, $c^- \notin \chi_s(k')$: in this case, let z be the least such k and distinguish two cases:

- (2.1) if $c \in H_s(\emptyset)$, then let $m(s + 1) = 0$, and define

$$r_{s+1}(x) = \begin{cases} \langle f_0 \rangle & \text{if } x = 0 \\ \langle \rangle & \text{if } x > 0; \end{cases}$$

- (2.2) otherwise, let $m(s + 1) = z + 1$, and define

$$r_{s+1}(x) = \begin{cases} r_s(x) & \text{if } x < z \\ \langle f_{z+1} \rangle & \text{if } x = z + 1 \\ \langle \rangle & \text{if } x = z \text{ or } x > z + 1. \end{cases}$$

- (3) there exists $k \leq m$ such that $c^- \in \chi_s(k)$, and for all $k' \leq k$, $c \notin \chi_s(k')$: in this case, let z be the least such k , let $m(s + 1) = z$, and define, where $\rho_s(z) = f_y$,

$$r_{s+1}(x) = \begin{cases} r_s(x) & x < z \\ r_s(x) \wedge \langle f^-(f_y) \rangle & x = z \\ \langle \rangle & x > z + 1. \end{cases}$$

Finally define

$$A_{s+1} = \bigcup_{i < m(s+1)} \chi_{s+1}(i).$$

Thus $A_{s+1} = \emptyset$ if $m(s + 1) = 0$, and $A_{s+1} = H_{s+1}(L_{s+1}(m(s + 1)))$ otherwise.

As is clear, Clause (1) is almost identical to the Clause (1) of the p -dialectical procedure; Clause (2) is essentially the same of Clause (2) of the dialectical procedure; Clause (3) is essentially the same of Clause (2) of the p -dialectical procedure.

We say that a q -dialectical system, with enumeration operator H , is *consistent* if $\{c, c^-\} \cap H(\emptyset) = \emptyset$. We call A_s the set of *provisional theses* of q with respect to α at stage s . The set A_q^α defined as

$$A_q^\alpha = \{f_x : (\exists t)(\forall s \geq t)[f_x \in A_s]\}$$

is called the set of *final theses* of q with respect to α . We often write $A_s = A_{q,s}^\alpha$ when we want to specify the q -dialectical system q and the chosen approximation to the enumeration operator. A pair (q, α) as above is called an *approximated q -dialectical system*. A set $A \subseteq \omega$ is called *q -dialectical* if $A = A_q^\alpha$ for some approximated q -dialectical system, and we say in this case that A is *represented* by the pair (q, α) .

We summarize some of the main properties of A_d and A_q^α . As in the case of p -dialectical systems, we say that an approximated q -dialectical system is *loopless* if the set $\{\rho^s(u) : s \in \omega\}$ is finite, for all u . For more information and properties about loopless approximated q -dialectical system, and in particular for a complete characterization of approximated q -dialectical system with loops, the reader is referred to [2].

Theorem 1.9 ([2, 14]). *If d and (q, α) are respectively a dialectical system or a loopless approximated q -dialectical system then the following hold:*

- (1) A_d and A_q^α are Δ_2^0 sets;
- (2) for every x , $\lim_s r_s(x) = r(x)$ and $\lim_s L_s(x) = L(x)$ exist finite (whether the functions $r_s(x)$, $L_s(x)$ refer to d , or (q, α)) and

$$\begin{aligned} A_d &= \{f_x : r(x) = \{f_x\}\} \\ A_q^\alpha &= \{f_x : r(x) = \langle f_x \rangle\}, \end{aligned}$$

and

$$\begin{aligned} f_x \in A_d &\Leftrightarrow c \notin H(L(x) \cup \{f_x\}) \\ f_x \in A_q^\alpha &\Leftrightarrow \{c, c^-\} \cap H(L(x) \cup \{f_x\}) = \emptyset. \end{aligned}$$

(For q -dialectical systems, the values of r and L depend in general on the chosen approximation α).

Proof. The claim that A_d is a Δ_2^0 set comes from [14]. The other claims come from [2, Lemma 3.8, Lemma 3.18] (to show that A_q^α is Δ_2^0 see also the proof of [3, Lemma 3.4] which amends a previous bug in [2]). \square

Notice that for a p -dialectical system, being loopless implies being consistent.

Most of the results proved for q -dialectical sets extend to p -dialectical sets. In particular,

Theorem 1.10. *If p is a loopless p -dialectical system then $\lim_s L_s(x)$ exists for every x and*

$$f_x \in A_p \Leftrightarrow c \notin H(L(x) \cup \{f_x\}).$$

Proof. The proof follows from Theorem 1.3, and an easy induction. Following the last stage at which $L(x)$ ceases to change, we propose $r_s(x) = \langle f_x \rangle$, and it is easy to see that

$$r(x) = \langle f_x \rangle \Leftrightarrow c \notin H(L(x) \cup \{f_x\}).$$

\square

Notwithstanding the independence of L from the chosen approximation to H established in Lemma 1.2 and Theorem 1.3 nothing guarantees that the sequence $\{A_s\}_{s \in \omega}$ of sets of provisional theses is independent of the approximation, or does even give a Δ_2^0 approximation to A_p . The following lemma shows however that from any given H one can find an approximation for which the sequence $\{A_s\}_{s \in \omega}$ is in fact a Δ_2^0 approximation to A_p .

Lemma 1.11. *If H is an algebraic closure operator then from any computable approximation to H we can effectively find an approximation $\{\hat{H}_s : s \in \omega\}$ to an enumeration operator \hat{H} such that for every s , the enumeration operator given by \hat{H}_s is an algebraic closure operator (more precisely it*

satisfies $X \subseteq \hat{H}_s(X)$ if $\max X \leq s$, and $\hat{H}_s(\hat{H}_s(X)) \subseteq \hat{H}_s(X)$ for all X , and H and \hat{H} coincide as enumeration operators, i.e. for every X , $\hat{H}(X) = H(X)$.

Proof. Given any enumeration operator G , we can effectively find a closure operator G^ω which extends G : the details of this construction can be found for instance in [2]. Moreover if $G \subseteq K$ then $G^\omega \subseteq K^\omega$; G is a closure operator if and only if (as enumeration operators, not as c.e. sets) $G = G^\omega$; if G is finite then G^ω is finite and the canonical index of G^ω can be effectively computed from that of G . Suppose now that $\{H_s : s \in \omega\}$ be a computable approximation to a closure operator H : we may assume that the approximation satisfies

- (1) if $\langle x, D \rangle \in H_s$ then $x, \max D < s$;
- (2) for every $i < s$, $\langle i, \{i\} \rangle \in H_s$.

For every s define $\hat{H}_s = (H_s)^\omega$. By the above remarks, this is a full-fledged computable approximation to H^ω , still satisfying (1) and (2). But (as enumeration operators, not as c.e. sets) $H = H^\omega$, as H is a closure operator. So $\{\hat{H}_s : s \in \omega\}$ is the desired approximation, effectively found from $\{H_s : s \in \omega\}$, to a suitable closure operator \hat{H} (namely $\hat{H} = H^\omega$) which coincides as an operator with H . \square

The next definition summarizes the properties of the approximation built in the proof of the previous theorem.

Definition 1.12. If H is an algebraic closure operator and $\{H_s\}$ is a computable approximation to it, we say that the approximation is *good* if for every s the following hold: $X \subseteq H_s(X)$ if $\max X \leq s$, and $H_s(H_s(X)) \subseteq H_s(X)$ for all X .

Corollary 1.13. If $p = \langle H, f, f^-, c \rangle$ is a p -dialectical system, and $\{H_s\}$ is a good approximation to H then the corresponding p -dialectical approximation $\{A_s : s \in \omega\}$, given by the p -dialectical procedure, is a Δ_2^0 approximation.

Proof. If p is not consistent then the claim follows from the fact that starting from the stage at which $c \in H(\emptyset)$ we have that $m_s(0) = 0$ and thus $A_s = \emptyset$.

If p is consistent then we can use Theorem 1.10. Let $f_u = x$, and assume that $x \notin A_p$. Let t_0 be a stage such that $L(u)$ has already reached limit $L(u)$. As $x \notin A_p$, we have that $c \in H(L(u) \cup \{x\})$: let $t_1 \geq t_0$ be such that $L(u) \subseteq L_s(m(s))$ for every $s \geq t_1$ and $c \in H_{t_1}(L(u) \cup \{x\})$, and suppose that $s > t_1$ is a stage such that $x \in A_s$, i.e. $x \in L_s(m(s))$ and $s > x, \max(L_u)$. It follows that $L(u) \subseteq H_s(L(u)) \subseteq H_s(L_s(m(s)))$ and $\{x\} \subseteq H_s(L_s(m(s)))$, hence $L(u) \cup \{x\} \subseteq H_s(L_s(m(s)))$, hence by goodness of the approximation, $H_s(L(u) \cup \{x\}) \subseteq H_s(L_s(m(s)))$, giving that $c \in H_s(L_s(m(s)))$, contradicting the definition of $m(s)$. \square

2. COMPARING DIALECTICAL SETS, p -DIALECTICAL SETS, AND q -DIALECTICAL SETS

In this section we compare under inclusion the notions of p -dialectical system, dialectical system, and q -dialectical system. Throughout the section we will use superscripts appended to the parameters L, r, ρ etc. (for instance L^p, L^d, L^q or r^p, r^d, r^q) to distinguish whether the parameters refer to the p -dialectical system, or the dialectical system, or the q -dialectical system we will happen to be talking about.

Theorem 2.1. *Given any dialectical system $d = \langle H, f, c \rangle$ such that $H(\emptyset)$ is infinite, we can build a p -dialectical system p such that $A_d = A_p$.*

Proof. Let $d = \langle H, f, c \rangle$, and being $H(\emptyset)$ an infinite c.e. set, let $Z = \{z_0 < z_1 < \dots < z_i < \dots\} \subseteq H(\emptyset)$ be a computable set. Then, let p be the p -dialectical system $p = \langle H, f, f^-, c \rangle$ where

$$f^-(x) = \begin{cases} z_0 & x \notin Z, \\ z_{i+1} & x = z_i \in Z. \end{cases}$$

This definitions obeys the requirement that the orbits of f^- be infinite. Now, we know that $\lim_s r_s^d(u)$ exists for every u . Notice that for every X and for every $z \in H(\emptyset)$, from H being a closure operator it follows that

$$c \in H(X \cup \{z\}) \Leftrightarrow c \in H(X).$$

Using this, it is easy to show by induction on u that

- if $r^d(u) = \langle f_u \rangle$ then $r^p(u) = \langle f_u \rangle$, and if $r^d(u) = \langle \rangle$ then $r^p(u) = \langle f_u, z_0 \rangle$.

It follows that

$$\bigcup_u L^p(u) = \bigcup_u L^d(u) \cup \{z_0\}.$$

$A_d = H(\bigcup_u L^p(u))$ (see [2], but the proof is similar to the proof of Theorem 1.3). On the other hand, by Theorem 1.3

$$A_p = H(\bigcup_u L^p(u)) = H(\bigcup_u L^d(u) \cup \{z_0\}) = H(\bigcup_u L^d(u))$$

because $z_0 \in H(\emptyset)$ and H is an algebraic closure operator. \square

Theorem 2.2. *Any p -dialectical set is a q -dialectical set. In fact, given a p -dialectical system p we can effectively build a q -dialectical system q such that $A_p = A_q^\alpha$ for any approximation α to the operator of q .*

Proof. Let $p = \langle H, f, f^-, c \rangle$. We first observe that the claim is trivial if $A_p = \omega \setminus \{c\}$, and if p has loops.

If not, let u_0 be the least number such that $z_0 = f_{u_0} \neq c$ and $z_0 \notin A_p$, and denote $\rho_p(u_0)$ with z_1 . Consider the q -dialectical system $q = \langle H^*, f^*, f^-, z_0, c^- \rangle$, where $c^- = c$, f^* is defined as follows

$$f^*(x) = \begin{cases} z_1 & \text{if } x = 0, \\ f(x-1) & \text{if } x > 0, \end{cases}$$

and

$$H^* = (H \setminus \{\langle z_0, D \rangle : z_0 \notin D\}) \cup \{\langle x, \{z_0\} \rangle : x \in \omega\}.$$

Notice that for every set X , if $z_0 \in H^*(X)$ then $z_0 \in X$.

We now show that H^* is an algebraic closure operator.

- We first show that $X \subseteq H^*(X)$ for every set X . Let X be given. If $z_0 \in X$, we have that $X \subseteq \omega \subseteq H^*(X)$. If $z_0 \notin X$ and $x \in X$ then (as H is an algebraic closure operator) there is an axiom $\langle x, D \rangle \in H$ with $D \subseteq X$, but then then $\langle x, D \rangle \in H^*$ as well and thus $x \in H^*(X)$.

- Next we show that $H^*(H^*(X)) \subseteq H^*(X)$. Let X be given, and assume that $x \in H^*(H^*(X))$. We may also assume that $z_0 \notin H^*(H^*(X)) \cup H^*(X) \cup X$, otherwise in any case $z_0 \in X$ by definition of H^* and thus $H^*(H^*(X)) \subseteq \omega \subseteq H^*(X)$.

So assume that $x \neq z_0$ and let $\langle x, D \rangle \in H^*$ be an axiom with $D \subseteq H^*(X)$: to this axiom by our assumptions (which imply $z_0 \notin D$) must correspond an axiom $\langle x, D \rangle \in H$. For every $y \in D$ there is an axiom $\langle y, E_y \rangle \in H^*$ with $E_e \subseteq X$ and by our assumptions again, each such axiom must correspond to an axiom $\langle y, E_y \rangle \in H$. We thus obtain $x \in H(H(X))$, and since H is an algebraic closure operator, this gives $x \in H(X)$ via an axiom, say, $\langle x, E \rangle \in H$: but this is also an axiom of H^* , thus $x \in H^*(X)$.

Let us now work with any approximation α to H^* . We want now to prove that $A_p = A_q^\alpha$. In particular, we show by induction on u that, for all u , we have that

$$r^q(u) = \begin{cases} \langle z_1 \rangle, & \text{if } u = 0, \\ r^p(u-1), & \text{if } u > 0 \text{ and } z_0 \notin \text{range}(r^p(u-1)), \\ \in \{ \langle \rangle, r^p(u-1) \}, & \text{otherwise,} \end{cases}$$

where the third clause means that $r^q(u) = \langle \rangle$ or $r^q(u) = r^p(u-1)$ depending on which one between z_0 and c appears first, enumerated in $H^*(L^q(u) \cup \{z_0\})$, at the relevant stage of the q -dialectical procedure. Moreover, we show by induction on $u > 0$ that

$$r^q(u) = \langle \rangle \Rightarrow \rho_p(u-1) = z_1,$$

so that $L^q(u) = L^p(u-1) \cup \{z_1\}$.

Since $f_0^* = z_1$, it is immediate to notice that that $r^q(0) = z_1$. Indeed, we can not have $z_0 \in H^*(\{z_1\})$ by definition of H^* , but we cannot have $c \in H^*(\{z_1\})$ either, otherwise $c \in H(\{z_1\})$ against the fact that $z_1 \in A_p$.

Then consider the case $u > 0$, and assume by induction that $L^q(u) = L^p(u-1)$. It is easy to see that if $z_0 \in \text{range}(r^p(u-1))$ then $\rho^p(u-1) = z_1$. Suppose that $r^p(u-1)$ has length n : we claim that for every $i < n$, $(r^q(u))_i = (r^p(u-1))_i$, and $\rho_q(u) = \rho_p(u-1)$. This is clearly true when $i = 0$ by definition of f^* . Assume the claim is true of $i < n-1$. If $(r^p(u-1))_i = (r^q(u))_i \neq z_0$, then (as $z_0 \notin L^q(u) \cup \{(r^q(u))_i\}$ by induction), we have that $z_0 \notin H^*(L^q(u) \cup \{(r^q(u))_i\})$; but (since $i < n-1$) $c \in H(L^p(u-1) \cup \{(r^p(u-1))_i\})$, thus $c \in H^*(L^q(u) \cup \{(r^q(u))_i\})$ (by the way H^* is defined), hence

$$r^q(u)_{i+1} = f^-((r^q(u))_i) = f^-((r^p(u-1))_i) = r^p(u-1)_{i+1}.$$

On the other hand, when we reach the top, $c \notin H(L^p(u-1) \cup \{\rho_p(u-1)\} \cup \{z_1\})$, and thus again $\{z_0, c\} \cap H^*(L^q(u) \cup \{\rho_p(u-1)\}) = \emptyset$, giving that $\rho_q(u) = \rho_p(u-1)$.

Let us consider now the case $(r^p(u-1))_i = (r^q(u))_i = z_0$. Now both $\{z_0, c\} \subseteq H^*(L^q(u) \cup \{(r^q(u))_i\})$. If at the relevant stage of the q -dialectical procedure, α shows z_0 derivable from $H^*(L^q(u) \cup \{(r^q(u))_i\})$ no later than c is so derivable, then $r^q(u) = \langle \rangle$ and $\rho_p(u-1) = z_1$; if α shows c derivable first, then by an argument similar to the one for the case when $(r^p(u-1))_i \neq z_0$, we conclude that $r^q(u)_{i+1} = r^p(u-1)_{i+1}$. Since f^- is not cyclic, we now have that $z_0 \neq (r^p(u-1))_j$ for all $i < j < n$, thus again as in the case seen above when $(r^p(u-1))_i \neq z_0$, we conclude that for all $i < j < n$, $r^q(u)_j = r^p(u-1)_j$, and eventually $\rho_q(u) = \rho_p(u-1)$.

It follows that $L^p(\omega) = L^q(\omega)$, and thus $A_p = A_q^\alpha$ for every approximation α to H^* . \square

The next problem is left open.

Problem 2.3. Are there q -dialectical sets that are not p -dialectical?

3. A BRIEF COMPARISON WITH OTHER APPROACHES TO TRIAL AND ERROR MATHEMATICS

Having set the formal definitions of the three systems (dialectical, q -dialectical, and p -dialectical systems) and laid down the theoretical bases, before moving to a detailed investigation of the computability theoretic properties of the sets they represent, including certain completions of formal theories, it is perhaps time to pause and briefly compare these systems with other popular models of trial and error mathematics.

3.1. Belief revision. The central problems facing the theory of belief revision are how to revise a knowledge system in the light of new information that turns out to be inconsistent with the old one. The AGM axiomatic theory [1] is the most famous theory of belief revision: in this model, beliefs are represented as sentences held by an agent. Such sentences form a deductive closed set: a *belief set*. To formalize how agents revise their beliefs, AGM describes various actions by which a belief set can be modified in response to new information. If this new information does not contradict the set of acquired knowledge, it is simply added to the belief set and we have the *expansion*. On the contrary, *revision* takes place when a new sentence turns out to be inconsistent with the belief set to which it is added. In order to maintain consistency, some of the old sentences are deleted by an action called *contraction*. What is kept of the old beliefs is the consequence of some guiding rules. Two *dogmas*, in particular, have been singled out (see [18] for more details): first, one's prior beliefs should be changed as little as possible; second, whenever there is a choice about which sentence should be deleted, the agent should abandon the least one with respect to some ordering of *epistemic entrenchment*, where “ q is more entrenched than p ” intuitively means that the sentence q has more epistemic value than the sentence p . So, the overall goal of these dogmas is to keep the loss of information minimal when a belief set is updated.

Dialectical systems, and the variations considered in this paper, aim at modeling similar actions, but they implement them in a rather different way. In this context, expansion is not limited to the addition of a new sentence (or axiom, in our terminology) but it consists also in increasing the deductive power of the deduction operator H (whereas in AGM each action leads to an already deductively closed set of beliefs).

More importantly, the dialectical model lacks an explicit entrenchment ordering: when a conflict emerges, i.e., c or c^- is derived, we reject/revise the last proposed axiom of the minimal inconsistent set, instead of evaluating the epistemic value of the axioms contained in it. Nevertheless, the behavior of the proposing function f and that of the revising function f^- to some extent surrogate the entrenchment: f encodes a certain priority to the axioms to be proposed, and f^- (in the case of p - and q - dialectical systems) can dynamically change this priority by swapping the ordering of two given axioms and thus modifying their mutual priority. One might go further and develop a dialectical model where to each axiom is assigned a certain weight: whenever a conflict arises, the system keeps as provisional theses the consistent subset X of the old knowledge that realizes the maximum weight. A similar line of research has been explored in [15], where the authors investigate generalized dialectical systems embodied with probability weights. Yet, also this approach differs from the AGM proposal, since entrenchment is more concerned with the *explanatory power* of the sentences. In Gärdenfors' and Makinson's words [7]:

Rather than being connected with probability, the epistemic entrenchment of a sentence is tied to its explanatory power and its overall informational value within the belief set. For example, lawlike sentences generally have greater epistemic entrenchment than accidental generalizations. This is not because lawlike sentences are better supported by the available evidence (normally they are not) but because giving up lawlike sentences means that the theory loses more of its explanatory power than giving up accidental generalizations.

Studying dialectical systems that incorporate some measures of explanatory power (as the ones discussed for instance in [19]) is a topic for future work.

3.2. Lakatos’ philosophy of mathematics. It would be incorrect to assert that dialectical systems attempt to formalize Lakatos’ philosophy of mathematics: the dialectical model is way too abstract to offer a convincing rendering of the dynamic of mathematical discovery characterized, e.g., in [13]. Yet, Lakatos’ intuition that mathematical knowledge is subject to constant refinement motivates Magari’s original proposal. Indeed, according to Magari [14], a dialectical system is best understood as modeling a mathematician (or even, a mathematical community) that in developing a mathematical theory proceeds by trial and errors, instead of merely accumulating more and more deductions (as classical formal systems prescribe). Moreover, the main conceptual reason for moving from dialectical to q -dialectical systems in [2] was precisely that of including in our systems a revision mechanism more adherent to that of mathematical practice, rather than just limiting ourselves to logical contradiction.

To sketch a more precise parallel between our systems and Lakatos’ approach, it is worth to briefly contrast the dialectical model with the way in which Lakatos’ theory has been computationally represented: in [16], the authors make use of abstract argumentation systems (in Dung-style, see [6]) to offer an automated realization of Lakatos’ view. In the field of structured argumentation (the interested reader is referred to [5]), an *abstract argumentation framework* is a directed graph, where the nodes are *arguments* and the arcs are *attacks*, and a set of arguments is conflict-free if no pair of argument belongs to the set of attacks. An *argument system* is then given by a logical language, a set of rules (that can be either *strict* or *defeasible*), and a partial function from rules to formulas. In a nutshell, Lakatos’ account is represented in [16] as a *formal dialogue game* between a Proponent and an Opponent (roles that are possibly embodied by many speakers) and proofs are carefully represented as arguments that correspond to the artifacts collaboratively created by the participants in a Lakatosian dialogue, such as the one famously exemplified by the classroom debate about Euler’s conjecture on polyhedra in [13]. This dialogue game is a rather complex game, in which players can perform different types of moves (such as raising counterexamples, piecemeal exclusion, monster barring, monster adjusting, etc.), corresponding to crucial ingredients of Lakatos’ informal logic.

The dialectical model is of course way less adherent to Lakatos’ perspective. A game-theoretic formulation of it can however be readily obtained: the Proponent makes a proposal via the function f and, and at each step of the computation, the Opponent tries to reject by either proving its inconsistency or its implausibility with acquired knowledge. So, the game can be roughly intended as a debate between the Proponent and the Opponent about whether any given sentence is to be accepted or not. However, such a game is much more rigid than the one formulated in [16]. For instance, unlike Lakatos’ game where the roles are interchangeable, in our models the Opponent always attacks and the Proponent always proposes new hypotheses. Another major difference is that Lakatos’ game does not contain strict rules (i.e, rules of the form “ B is always a consequence

of A ”), but only defeasible rules (i.e, rules of the form “typically B is a consequence of A ”). On the contrary, no defeasible reasoning is allowed in the dialectical game: in fact, an argument can be attacked only by showing some undesirable *deductive* consequences, and this depends only on the set of premises and the deduction operator.

Finally, the strategy of the Proponent and the Opponent are completely deterministic, being defined once for all at the beginning of the computation and eventually producing a unique set of final theses (modulo the approximation to H in the case of the q -dialectical systems). This is why our analysis is centered on the class of sets represented by the p - or q - dialectical systems, rather than focusing on the behavior of a particular system.

3.3. Algorithmic learning theory. Algorithmic learning theory (ALT) is a vast research program, initiated by Gold [8] and Putnam [17] in the 60s that comprises different models of learning in the limit. It deals with the question of how a *learner*, provided with more and more data about some *environment*, is eventually able to achieve systematic knowledge about it. For instance, a classic paradigm in ALT concerns the learning of total computable functions: the learner receives as input the stream of values of a function g to be learned and, at any stage, outputs a conjecture of a program that computes the function. The learning is successful if the learner eventually infer a correct program for g . Different formalizations of this and similar intuitions gave rise to a vast research area (for an introduction to the field see for instance [9]).

In analogy with the learning criteria explored in ALT, a dialectical system also embeds a stabilization process, by which we eventually converge to a set of final theses (and in fact, by Theorem 1.9 and Theorem 1.10 we have that, if a set is represented by our system, then it is computable in the limit, i.e., Δ_2^0). More importantly, the existence of a similar stabilization mechanism hints at a deeper similarity between the two models: they both display and manage information essentially *by stages*, in a way that is naturally apt to be analyzed by computable theoretic tools. The significance of this common trait is well described by the following remark of Van Benthem in [23]:

Perhaps the key activity tied up with theory change is learning, whether by individuals or whole communities. Modern learning theory (...) describes learning procedures over time, as an account of scientific methods in the face of steadily growing evidence, including surprises contradicting one’s current conjecture. In this perspective, update, revision, and contraction are single steps in a larger process, whose temporal structure needs to be brought out explicitly (...). Learning theory is itself a child of recursion theory, and hence it is one more illustration of a computational influence entering philosophy.

Dialectical systems, and our related models, are children of recursion theory as well. They do not offer a *logic* of trial and error mathematics, nor do they aim at spelling out a variety of principles by which we might want to change or preserve a given axiom. This can be seen as a limitation of dialectical systems. But note that no logic of learning (or of inductive inference) is provided in ALT, and no axiomatization of computability is contained in Turing’s 1936 paper [22]. This is because the emphasis of a computable theoretic investigation (such as the present one) is typically more process-oriented and focuses on exploring the computational costs of such processes. Dialectical, p -dialectical, and q -dialectical systems are attempts at characterizing the evolution of abstract mathematical theories by defining highly idealized agents that follow few mechanic rules – by which, nonetheless, a rich class of theories can be produced. One might insist that such an idealization is too extreme; in fact, in this section we offered enough evidence that other frameworks

might give a better understanding of, e.g., what belief change is. Yet, a measure of the fruitfulness of a given idealization also comes from whether it sheds new light on some well-established notion. The goal of the second half of this paper is to show that, for the dialectical model, this is exactly the case: our systems turn out to be a remarkably good machinery for dealing with a key-concept of classical logic, i.e., completions of first-order theories.

4. SYSTEMS WITH CONNECTIVES AND COMPLETIONS

By a *system* we will mean in general a p -dialectical system or a dialectical system, or a q -dialectical system. From now on we will restrict attention to systems in which, via identification of numbers with the sentences of some formal language, H is regarded as a logical deduction operator, i.e. $H(X)$ is the set of sentences which can be logically derived from the premises X . In this identification sentential connectives can be viewed as just computable functions.

The following definition is taken from [14].

Definition 4.1. A *system with connectives* is a system with an enumeration operator H , a contradiction c , and injective computable functions $\neg, \rightarrow, \wedge, \vee$ such that for every $X \subseteq \omega$ and $x, y \in \omega$,

- (1) $c \in H(\{x, \neg x\})$;
- (2) $H(\{\neg\neg x\}) = H(\{x\})$;
- (3) $x \vee \neg x \in H(\emptyset)$;
- (4) $H(X \cup \{x \vee y\}) = H(X \cup \{x\}) \cap H(X \cup \{y\})$;
- (5) if $c \in H(X \cup \{x\})$ then $\neg x \in H(X)$;
- (6) $x \in H(X \cup \{y\})$ if and only if $y \rightarrow x \in H(X)$.

Definition 4.2. Given a system with connectives and finale theses A , we say that the system is a *completion*, if for every x , $A \cap \{x, \neg x\}$ has exactly one element.

4.1. q -dialectical completions. It is known from [3] that there are (loopless) q -dialectical sets that are not dialectical. Unfortunately if we consider connectives, nothing is gained from passing from dialectical systems to q -dialectical systems.

We first show that if a loopless q -dialectical system with connective is consistent (i.e. $\{c, c^-\} \cap H(\emptyset) = \emptyset$, where H is the operator of q), then A_q is a completion.

Theorem 4.3. *If $q = \langle H, f, f^-, c, c^- \rangle$ is a consistent loopless q -dialectical system with connectives, α an approximation to H such that (q, α) is loopless, then A_q^α is a completion.*

Proof. Let q, α be as in the statement of the theorem; for simplicity, let us write $A_q = A_q^\alpha$

Assume now that x is the least number such that $f_x \notin A_q$, and $\neg f_x \notin A_q$: let $f_y = \neg f_x$, and assume without loss of generality that $y < x$, the other case $x < y$ being similar. By Theorem 1.9 this is the consequence of one of the following circumstances:

- (1) $c \in H(L(x) \cup \{f_x\})$, and $c \in H(L(y) \cup \{\neg f_x\})$: hence, $c \in H(L(x) \cup \{f_x\})$ and $c \in H(L(y) \cup \{\neg f_x\})$, and by (d) of Definition 4.1, we have that $c \in H(L(x) \cup \{f_x \vee \neg f_x\})$. But then, as $f_x \vee \neg f_x \in H(\emptyset)$, we have $c \in H(L(x))$, contrary to the fact that $L(x)$ is the limit set.
- (2) $c \in H(L(x) \cup \{f_x\})$, and $c^- \in H(L(y) \cup \{\neg f_x\})$: in this case, it is easy to see (under the assumption that $y < x$) that $c^- \in H(L(x) \cup \{\neg f_x\})$ and $c^- \in H(L(x) \cup \{f_x\})$, giving that

$c^- \in H(L(x) \cup \{f_x \vee \neg f_x\})$, and thus $c^- \in H(L(x))$, contrary to the fact that $L(x)$ is the limit set.

- (3) $c^- \in H(L(x) \cup \{f_x\})$, and $c \in H(L(y) \cup \{\neg f_x\})$: the argument is similar, having this time (under the assumption $y < x$) $c \in H(L(x) \cup \{\neg f_x\})$, and thus $c \in H(L(x))$.
- (4) $c^- \in H(L(x) \cup \{f_x\})$, and $c^- \in H(L(y) \cup \{\neg f_x\})$: Similar to (1), just replacing c with c^- .

It remains to show that exactly one of f_x and $\neg f_x$ lies in A_q , but this is obvious otherwise $c \in H(\emptyset)$ as H is with connectives. \square

Theorem 4.4. *If $p = \langle H, f, f^-, c \rangle$ is a loopless (hence consistent) p -dialectical system with connectives, then A_p is a completion.*

Proof. Let $p = \langle H, f, f^-, c \rangle$ a p -dialectical system with connectives where $c \notin H(\emptyset)$ (in such a way that something is not derivable). Let $f_u = x$ and $f_v = \neg x$ and without loss of generality assume $u < v$. Suppose that $x \notin A_p$; then $c \in H(L(u) \cup \{x\})$, and by property (6) of definition 3.1, we have $x \rightarrow c \in H(L(u))$, from which $\neg x \in H(L(u))$. Suppose now that also $c \in H(L(v) \cup \{\neg x\})$, and therefore by the same argument $x \in H(L(v))$. But there will be a stage s such that for all $t \geq s$ we will have $L(v) = L_t(v)$. Moreover, since $L(u) \subseteq L(v)$ and H is an algebraic closure operator we can assume that a t is big enough to have $L(v) \subseteq L_t(v) \subseteq H_t(L_t(v))$ from which $L(u) \subseteq H_t(L_t(v))$, giving that both $\neg x$ and x belong to $H_t(L_t(v))$ and therefore $c \in H_s(L_s(v))$ for some $s \geq t$, giving that $L_s(v)$ must change after t : contradiction. \square

4.2. Comparing dialectical, q -dialectical, and p -dialectical completions. We now consider the relationships under inclusion of the various systems with connectives.

An immediate consequence of Theorem 2.1 is the following:

Corollary 4.5. *Every dialectical completion is also a p -dialectical completion.*

Proof. The proof of Theorem 2.1 shows that starting from a dialectical system $d = \langle H, f, c \rangle$, with $H(\emptyset)$ infinite then one can build a p -dialectical system p with the same H , and the same c , p has connectives as H does. On the other hand, the condition that $H(\emptyset)$ be infinite is granted by the fact that H has connectives, and thus, for instance, if $x \in H(\emptyset)$ then $x \wedge x \in H(\emptyset)$ as well. \square

Theorem 4.6. *If (q, α) is a consistent loopless q -dialectical pair, with $q = \langle H, c, c^-, f, f^- \rangle$ a q -dialectical system with connectives, and α a good approximation to H , then A_q^α is a dialectical completion.*

Proof. Suppose that (q, α) is a loopless q -dialectical pair, $q = \langle H, c, c^-, f, f^- \rangle$ is a q -dialectical system with connectives, $c \notin H(\emptyset)$ and α is a good approximation to H . Then A_q^α is a completion by Theorem 4.3, and thus $\neg c^- \in A_q^\alpha$: let u be such that $f_u = \neg c^-$, hence $r(u) = \langle \neg c^- \rangle$, and let t_0 be the least stage such that $L(u+1)$ has reached limit already, $\neg c^- \leq t_0$ (thus each $s \geq t_0$ has an axiom $\langle \neg c^-, \{\neg c^-\} \rangle \in H_s$), and $c \in H_{t_0}(\{c^-, \neg c^-\})$. Suppose now that $s \geq t_0$ is a stage such that $c^- \in H_s(L_s(v))$ with $v > u$. But H_s is an algebraic closure operator, as α is good: therefore $\neg c^- \in H_s(L_s(v))$ since $\neg c^- \in L_s(v)$. This gives $\{c^-, \neg c^-\} \subseteq H_s(L_s(v))$, hence $c \in H_s(H_s(L_s(v))) \subseteq H_s(L_s(v))$. It is then clear that starting from t_0 , the q -dialectical procedure behaves as a dialectical procedure, since f^- no longer plays any role.

Let v be the greatest slot such that for every $s \geq t_0$, $L_s(v) = L_{t_0}(v)$ (clearly $v > u$; such a maximum exists since at t_0 almost all $r(v)$ are empty), and let $d = \langle H, g, c \rangle$ be the dialectical system where g

is defined as follows. First fix a strictly increasing computable sequence $z_0 < z_1 < \dots$ of elements of $H(\emptyset)$. Then

- if $v' < v$ then let

$$g_{v'} = \begin{cases} f_{v'}, & \text{if } r(v') = \langle \rangle, \\ \rho(v'), & \text{otherwise and } \rho(v') \notin \{g_{v''} : v'' < v'\}, \\ \min z_i \notin \{g_{v''} : v'' < v'\}, & \text{otherwise;} \end{cases}$$

- if $v' \geq v$ then let

$$g_{v'} = \begin{cases} f_{v'}, & \text{if } f_{v'} \notin \{g_{v''} : v'' < v\}, \\ \min z_i \notin \{g_{v''} : v'' < v'\}, & \text{otherwise.} \end{cases}$$

Then g is a computable permutation and by the above remarks, it is easy to see that $A_q^\alpha = A_d$. \square

Theorem 4.7. *If $p = \langle H, f, f^-, c \rangle$ is a loopless p -dialectical system with connectives, in which $f^- = \neg$, then A_p is both a dialectical completion, and a q -dialectical completion.*

Proof. Let $p = \langle H, f, f^-, c \rangle$ be a p -dialectical system with connectives in which $f^- = \neg$ and c is a contradiction. Let $d = \langle H, f, c \rangle$: we claim that $A_p = A_d$. Let us use the superscripts p and d , to distinguish the relevant parameters of p and d , respectively. We will prove by induction on u that

- if $r^d(u) = \langle f_u \rangle$ then $r^p(u) = \langle f_u \rangle$, and if $r^d(u) = \langle \rangle$ then $r^p(u) = \langle f_u, \neg f_u \rangle$;
- for every $v \leq u$, if $r^d(v) = \langle \rangle$ then $\neg f_v \in H(L_d(v))$.

Notice that from this and the fact that A_q and A_p are completions, it easily follows that $A_p = A_d$.

Case $u = 0$ (base of the induction). This case easily follows from the assumptions and the basic definitions.

Suppose that the claim is true of u . If $c \in H(L_p(u+1) \cup \{f_{u+1}\})$ then $c \in H(L_d(u+1) \cup \{\neg f_v : v \leq u \ \& \ r^d(v) = \langle \rangle\} \cup \{f_{u+1}\})$. By the inductive assumption, $\{\neg f_v : v \leq u \ \& \ r^d(v) = \langle \rangle\} \subseteq H(L_d(u+1))$, hence $c \in H(L_d(u+1) \cup \{f_{u+1}\})$. This shows that if $r^d(u+1) = \langle f_{u+1} \rangle$ then $r^p(u+1) = \langle f_{u+1} \rangle$. The claims that if $r^d(u+1) = \langle \rangle$ then $r^p(u+1) = \langle f_{u+1}, \neg f_{u+1} \rangle$, and if $r^d(u+1) = \langle \rangle$ then $\neg f_{u+1} \in H(L^d(u+1))$, come straight from the definitions.

The remaining claim (i.e. A_p is a q -dialectical completion) follows from the following lemma.

Lemma 4.8. *For every dialectical completion A_d there exists a loopless q -dialectical pair (q, α) such that $A_d = A_q^\alpha$.*

Proof. Let $d = \langle H, f, c \rangle$ be a consistent dialectical system with connectives. By Lemma 1.11 let α be a good approximation to H ; let c^- be $c \wedge c$ (thus $c \in H(\{c'\})$); finally let f^- be any proposing function. Notice that $q = \langle H, f, f^-, c, c^- \rangle$ is a (proper) q -dialectical system as $c \neq c^-$. We claim that $A_d = A_q^\alpha$. This follows from the fact that c^- does not play any role in the q -dialectical procedure, as if $c^- \in H_s(X)$, then by goodness of the approximation, we also have $c \in H_s(X)$ since $c \subseteq H_s(\{c^-\}) \subseteq H_s(H_s(X)) \subseteq H_s(X)$. \square

\square

5. p -DIALECTICAL SETS AND DEGREES

The characterizations of the Turing degrees of the dialectical sets and of the q -dialectical sets has been given in [3]:

Lemma 5.1. *The Turing degrees of the dialectical sets, and of the q -dialectical sets, are exactly the c.e. Turing degrees.*

Proof. See [3]. □

Let us now consider the case of p -completions. If T is a formal theory with set of theorems Thm_T , and d is a dialectical system with connectives such that $H(\emptyset) = \text{Thm}_T$, then we say that d is a *dialectical system for T* . If d is a consistent dialectical system for T , and T is consistent, then A_d is a completion of T . Let us consider a propositional calculus with propositional atoms $\{p_i : i \in \omega\}$: by codes, we assume that this set coincides ω . Given a set $A \subseteq \omega$, let T_A be the propositional calculus, obtained by adding to the classical propositional calculus the axioms $\{p_i : i \in A\}$. The following is due to [4].

Lemma 5.2. *For every c.e. A there exists a dialectical system $d = \langle H, f, c \rangle$ for the theory T_A , such that:*

- (1) $A \leq_m A_d$,
- (2) $A_d \leq_{tt} \text{Thm}_{T_A}$,
- (3) $\text{Thm}_{T_A} \leq_{tt} A$,

and therefore $A_d \equiv_{tt} A$.

Proof. See [4]. □

Corollary 5.3. *The c.e. Turing degrees coincide with the degrees of p -completions, and with the degrees of p -dialectical sets.*

Proof. If A is a c.e. set then by the above lemma there is a dialectical completion A_d with the same tt -degree as A . But every dialectical completion is a p -completion by Corollary 4.5, and thus every c.e. Turing degree contains a p -completion. On the other hand every p -dialectical set is also q -dialectical, thus by Lemma 5.1 we have that the degree of any p -dialectical set is c.e. □

6. A p -DIALECTICAL COMPLETION, WHICH IS NEITHER A DIALECTICAL COMPLETION, NOR A q -DIALECTICAL COMPLETION

In the following T is taken to be Peano Arithmetic (assumed to be sound).

The following lemma has been known to logicians for many years already, and a proof-theoretic proof can be found in, or at least worked out from, Smoryński [20, p. 362]. This proof uses a version of the fixed point theorem originally due to Kent [12]. Notably it is based on Rosser’s method of comparison of witnesses and includes a relativized proof predicate as in Kreisel-Levy Essential Unboundedness Theorem, asserting that a certain formula is derivable from a true formula of a certain fixed complexity ([20, p. 362])

We propose a purely computability-theoretic proof, which looks perhaps simpler than [20]. Being a Σ_n (Π_n) sentence means of course being provably equivalent in T to a sentence which is syntactically Σ_n (Π_n).

Lemma 6.1. *For every $n \geq 1$, there exists a sentence $\psi \in \Sigma_{n+1}$ such that, for every $\varphi \in \Delta_{n+1}$, if $\not\vdash_T \varphi$, then $\vdash_T \psi \rightarrow \varphi$ and $\vdash_T \neg\psi \rightarrow \varphi$.*

Proof. Suppose S is the set of all Δ_{n+1} -sentences. We need a ψ such that, for all $\varphi \in S$, if $T + \neg\varphi$ is consistent, then ψ is independent of $T + \neg\varphi$.

Recall that S is c.e., so let $\varphi_0, \varphi_1, \dots$ be a recursive enumeration of S . Let Dim_T denote the standard provability predicate, expressing, via codes, whether a given number is a proof of a given formula. For each j , we define the function f_j as follows: On input s , search for the least i such that either

- (a) $\text{Dim}_T(s, \ulcorner \Phi_j^{\emptyset(n)}(0) = 1 \rightarrow \varphi_i \urcorner) \wedge \neg\varphi_i$, or
- (b) $\text{Dim}_T(s, \ulcorner \neg\Phi_j^{\emptyset(n)}(0) = 1 \rightarrow \varphi_i \urcorner) \wedge \neg\varphi_i$,

and define

$$f_j(s) = \begin{cases} 1 & \text{if } i \text{ is found, and (a) holds,} \\ 0 & \text{if } i \text{ is found, and (b) holds,} \\ \uparrow & \text{if no such } i \text{ is found.} \end{cases}$$

By the Relativized Parameter Theorem, $f_j = \Phi_{h(j)}^{\emptyset(n)}$, for some computable function h ; and let g be a computable function so that

$$\Phi_{g(j)}^{\emptyset(n)}(x) = \begin{cases} \uparrow & \text{if } f_j \text{ has empty domain,} \\ 1 & \text{if the first value of } f_j \text{ is 1 (i.e. } f_j(m) = 1 \text{ where } m \text{ is the least} \\ & \text{number in the domain of } f_j), \\ 0 & \text{if the first value of } f_j \text{ is 0} \end{cases}$$

In the following, we often identify statements relative to f_j or $\Phi_j^{\emptyset(n)}$ with their formal arithmetical translations. Let e be a fixed point for g . That is:

$$\Phi_e^{\emptyset(n)} = \Phi_{g(e)}^{\emptyset(n)}.$$

Let ψ be the sentence which says that $\Phi_e^{\emptyset(n)}(0) = 1$.

Claim:

- (1) If s is a proof from ψ to φ_i for some i , then T proves φ_i (and thus φ_i is true);
- (2) If s is a proof from $\neg\psi$ to φ_i for some i , then T proves φ_i (and thus φ_i is true).

Proof. We induct on s , assuming the lemma for all $t < s$. Since the claim is true for all $t < s$, $f_e(t)$ diverges for all such t . Note that T can prove that $f_e(t)$ diverges for all $t < s$. For each $t < s$, T can determine if t is a proof of $\psi \rightarrow \varphi_i$ or $\neg\psi \rightarrow \varphi_i$ for some i . If not, then clearly $f_e(i)$ diverges. If it is a proof of that form, then by our inductive hypothesis, T also proves φ_i . Thus, T proves that $f_e(t)$ diverges, since $\neg\varphi_i$ is a condition for convergence of $f_e(t)$.

- (1) If s is a proof from ψ to φ_i , then s is a proof of φ_i from $\Phi_e^{\emptyset^{(n)}}(0) = 1$. T can argue: Either φ_i is true or φ_i is false. If φ_i is false, then $f_e(s)$ converges to 1. This means that $\Phi_{g(e)}^{\emptyset^{(n)}}(0) = 1$. But then this means that φ_i is true (from the proof s). Thus, T has proved that φ_i is true.
- (2) If s is a proof from $\neg\psi$ to φ_i , then s is a proof of φ_i from $\neg\Phi_e^{\emptyset^{(n)}}(0) = 1$. T can argue: Either φ_i is true or φ_i is false. If φ_i is false, then $f_e(s)$ converges to 0. This means that $\Phi_{g(e)}^{\emptyset^{(n)}}(0) = 0$. But then this means that φ_i is true (from the proof s). Thus, T has proved that φ_i is true.

Hence, for any i such that φ_i is not a theorem of T , there can be no proof in T of $\psi \rightarrow \varphi_i$, or $\neg\psi \rightarrow \varphi_i$. \square

\square

Remark 6.2. Notice that in the previous lemma, the sentence ψ associated with the set of all Δ_{n+1} -sentences is Σ_{n+1} .

A class \mathcal{A} of Δ_2^0 sets is called *computable* if there is a Δ_2^0 predicate $A(e, x)$ such that $\mathcal{A} = \{V_e : e \in \omega\}$, where

$$V_e = \{x : A(e, x)\}.$$

If $\{A(e, x, s) : s \in \omega\}$ is a computable approximation to $A(e, x)$, i.e. $\lim_s A(e, x, s) = A(e, x)$ for every x , then we let $V_{e,s}(x) = A(e, x, s)$.

Theorem 6.3. *If a class \mathcal{A} of Δ_2^0 sets is computable, then there is a p -dialectical system p with connectives such that A_p is a completion of Peano Arithmetic and $A_p \notin \mathcal{A}$.*

Proof. Suppose we are given a computable class of Δ_2^0 sets $\mathcal{A} = \{V_e : e \in \omega\}$. We want to build a p -dialectical system $p = \langle K, f, f^-, c \rangle$ with connectives, satisfying the requirements

$$N_e : A_p \neq V_e,$$

and such that A_p is a completion of Peano Arithmetic. Let again T denote Peano Arithmetic, and let H be the enumeration operator given by

$$H = \{\langle x, D \rangle : D \vdash_T x\}.$$

Via a suitable Gödel numbering, throughout the proof, numbers should be thought of as sentences of the language of T . We choose c to be the usual contradiction $0 = 1$.

The construction is by stages. At the end of stage s we will have defined a finite set Ax_s of axioms to be added to the axioms of T , and finite approximations f^s, f_s^- to computable functions f, f^- , respectively, so that $f = \bigcup_s f^s$, $f^- = \bigcup_s f_s^-$, and $Ax = \bigcup_s Ax_s$ is a c.e. set. In order to define a p -dialectical system, we will have also to specify a suitable enumeration operator K : since the construction is computable, the theory S_∞ obtained by adding all axioms $\bigcup_s Ax_s$ to those of T is a c.e. extension of T , and we will let

$$K = \{\langle x, D \rangle : D \vdash_{S_\infty} x\}.$$

Lemma 6.4. *K is a algebraic closure operator with connectives, and for every set X , $H(X) \subseteq K(X)$.*

Proof. Immediate. \square

By Lemma 6.1 let Γ be a computable function which with every finite set S of sentences associates a sentence x such that

$$(\forall S' \subseteq S)[c \notin H(S') \implies x \notin H(S') \ \& \ \neg x \notin H(S')].$$

We say in this case that x has been chosen to be *independent* of every such S' . In the rest of the proof, we will distinguish between K -consistency (i.e. consistency in S_∞ : a set X is *K-consistent* if $c \notin K(X)$) and H -consistency (i.e. consistency in T : a set X is *H-consistent* if $c \notin H(X)$).

The strategy to meet N_e . We outline the construction and the strategy to meet the requirement N_e , and we describe what our desired p -dialectical system should achieve. In addition to f^s, f_s^- , throughout the construction we use several computable parameters, which are modified stage by stage: $x_{e,s}, \hat{\rho}_s(u), \hat{r}_s(u), A^s$. In particular A^s stands for a finite set, such that, for every u , $A(u) = \lim_s A^s(u)$ exists; the parameters $x_{e,s}, \hat{\rho}_s(u), \hat{r}_s(u)$ will be such that $x_e = \lim_s x_{e,s}, \lim_s \hat{\rho}_s(u) = \hat{\rho}(u)$, and $\lim_s \hat{r}_s(u) = \hat{r}(u)$ exist, and $\rho_s(u)$ will coincide with $\rho(u)$ of the p -dialectical system we are aiming at; moreover $A(x_e) = A_p(x_e)$ for every e .

We reserve the two slots $3e, 3e + 1$ to attack and satisfy N_e . The action may take place at several different stages: at each stage s we denote by $C = C_s$ the set consisting of all (finitely many) Boolean combinations of the sentences corresponding to the numbers so far mentioned and used in the construction.

The first time at which we attack requirement N_e we let $f_{3e} = y_e, f_{3e+1} = x_e$, where $C := C_s$, and

$$\begin{aligned} x_e &= \Gamma(C) \\ y_e &= \Gamma(C \cup \{x_e\}). \end{aligned}$$

We then execute the following cycle, which starts with $k = 0, y_e(0) = y_e$:

- (1) wait until the least stage $t > s$ such that $x_e \in V_{e,t}$, then add the axiom $\neg(y_e(k) \wedge x_e)$ in Ax; extract x_e from A (i.e., define $A^t(x_e) = 0$); go to (2) with $s := t$;
- (2) wait until the least stage $t > s$ such that $x_e \notin V_{e,t}$ then add the axiom $\neg y_e(k)$ in Ax, define $y_e(k+1) = f^-(y_e(k)) = \Gamma(C)$; add x_e into A (i.e., define $A^t(y_e(k+1)) = 1$ and $A^t(x_e) = 1$); go to (1) with $s := t$ and $k := k + 1$.

Outcomes of the strategy. The cycle eventually stops since $V_e(x_e)$ may change only finitely many times, and eventually $A(x_e) \neq V_e(x_e)$. Having in mind the p -dialectical system which we want to build and its characterizing parameters ρ, r, L , this cycle must be viewed as our attempt to build a stack $r(3e)$ of which the number $y_e(k)$ becomes the top when it is appointed; similarly, when x_e is initially appointed we have $r(3e+1) = \langle x_e \rangle$. Our intended goal is that if $L(3e)$ is K -consistent (i.e. $c \notin K(L(3e))$) and we add the axiom $\neg y_e(k)$ in Ax then $y_e(k)$ will be discarded by the p -dialectical procedure (as $L(3e) \cup \{y_e(k)\}$ is not K -consistent) and it will be replaced by $y_e(k+1)$ so as to momentarily have $L(3e) \cup \{y_e(k+1)\}$ K -consistent; so the p -dialectical procedure will put back x_e as a thesis. If $L(3e) \cup \{y_e(k)\}$ is K -consistent, and we add the axiom $\neg(y_e(k) \wedge x_e)$ in Ax, then the p -dialectical procedure keeps $L(3e) \cup \{y_e(k)\}$ K -consistent and discards x_e as a thesis. This process is repeated as many times as are needed to diagonalize $A(x_e) \neq V_e(x_e)$. Use of the function Γ in choosing each $y_e(k)$ and x_e allows us to conclude that $y_e(k)$ does not clash with $L(3e)$ to derive c , and x_e does not clash with $L(3e+1)$ to derive c , for any reasons other than those due to which we add axioms in Ax, i.e. in order to diagonalize A against V_e . If our p -dialectical system is able to mirror faithfully the cycle for N_e as described, then $A_p(x_e) = A(x_e)$ and thus $A_p(x_e) \neq V_e(x_e)$.

Other issues in defining f and f^- . We must also come up with f being a permutation of ω , and with f^- being total.

At non-zero even stages we take care of surjectivity of f , by picking the least available slot u with $u = 3k + 2$ for some k , and the least x which has not as yet been proposed by f : we define $f_u = x$ and if $f^-(x)$ is as yet undefined then we define $f^-(x) = a_0$, where $a_0 \in H(\emptyset)$. Note that injectivity of f is immediate by construction.

At non-zero even stages we also pick the least z such that $f^-(z)$ has not as yet been defined, and we put $f^-(z) = \Gamma(C)$ with C evaluated at these stages.

The construction. The modifications imposed on a parameter during a stage of the construction will determine the final value of the parameter at the end of the stage. It is understood that a parameter which is not explicitly modified at a stage s , maintains, at the end of stage s , the same value as the one it possessed at the beginning of stage s . Throughout a stage $s + 1$, if a parameter is mentioned without specifying any stage of approximation, then it is understood to be evaluated with the value it possessed by the end of stage s . When we apply the function Γ at stage $s + 1$, without loss of generality, we may assume that Γ picks an element which is different from all numbers so far mentioned in the construction, in particular from every number already in the domain of f^- : otherwise, as in Craig's trick for computable axiomatizability of c.e. theories, take as many iterations $w \wedge w \wedge \dots$ of the conjunctive connective \wedge on the value w provided by Γ as are needed to achieve this goal.

Step 0. Choose a c.e. injective sequence $a_0 < a_1 < \dots$ in $H(\emptyset)$, and define $f_0^-(a_i) = a_{i+1}$, for every i . Define also $f^0 = A^0 = \emptyset$ and $Ax_0 = T$. All the other parameters are undefined.

Step $s + 1$, odd. We say that a requirement N_e requires attention at $s + 1$, if either

- (r1) x_e is not defined; or
- (r2) either $x_e \in V_e$ but no axiom $\neg(\hat{\rho}(3e) \wedge x_e)$ lies in Ax , or $x_e \notin V_e$ and $\neg\hat{\rho}(3e) \in Ax$.

Consider the least e such that N_e requires attention, and take action accordingly:

- (a1) if N_e requires attention through (r1), then define (where C is the current approximation to the set C as in the above description of the strategy for N_e)

$$\begin{aligned} x_{e,s+1} &= \Gamma(C) \\ y_{e,s+1} &= \Gamma(C \cup \{x_{e,s+1}\}). \end{aligned}$$

Define $f_{s+1}^-(x_{e,s+1}) = \Gamma(C \cup \{x_{e,s+1}, y_{e,s+1}\})$.

- (a2) if N_e requires attention through (r2) then we further distinguish the following two cases:

- (a21) if $x_e \in V_e$ then add the axiom $\neg(\hat{\rho}(3e) \wedge x_e)$ in Ax_{s+1} ; let

$$\hat{r}_{s+1}(3e + 1) = r(3e + 1) \frown \langle f_{s+1}^-(x_{e,s+1}) \rangle,$$

and $A^{s+1}(x_e) = 0$;

- (a22) if $x_e \notin V_e$, then add an axiom $\neg\hat{\rho}(3e)$ in Ax_{s+1} ; define $f_{s+1}^-(\hat{\rho}(3e)) = \Gamma(C)$ if $f^-(\hat{\rho}(3e))$ is undefined; let $\hat{\rho}_{s+1}(3e) = f_{s+1}^-(\hat{\rho}(3e))$; let

$$\hat{r}_{s+1}(3e) = \hat{r}(3e) \frown \langle \hat{\rho}_{s+1}(3e) \rangle.$$

Let also $A^{s+1}(x_e) = 1$.

Resetting. For all relevant $u > 3e + 1$ i.e. u of the form $3e'$ or $u = 3e' + 1$, let $\hat{r}_{s+1}(u) = \langle \rangle$, and consequently $\hat{\rho}_{s+1}(u)$ be undefined. Add the axioms $\neg\hat{\rho}(u)$ in Ax_{s+1} .

On all remaining u which are different from the x_i that are still defined at the end of this stage, define $A^{s+1}(u) = 0$.

Step $s + 1 > 0$, even. Let u be the least available slot of the form $u = 3e + 2$, and let x be the least number such that $x \notin \text{range}(f)$: define $f_u^{s+1} = x$ and $f_{s+1}^-(x) = a_0$ if $f^-(x)$ has not been already defined; otherwise, let i be the greatest number such that $(f^-)^i(x)$ is already defined, and define $f_{s+1}^-((f^-)^i(x)) = a_0$.

Pick the least z such that $f^-(z)$ is not as yet defined, define $f_{s+1}^-(z) = \Gamma(C)$.

The verification. The following lemma is an easy consequence of the construction.

Lemma 6.5. *The function f is bijective and the function f^- is acyclic. Hence $p = \langle K, f, f^-, c \rangle$ is a p -dialectical system.*

Proof. f^- is acyclic because we define it through Γ which picks at each stage numbers not in the domain of f^- and because of the way we have arranged things when we define or have defined at 0, $f^-(v) = a_i$ for some i . The rest of the claim is obvious by Lemma 6.4. \square

Let $S_\infty (= K(\emptyset))$ be the c.e. extension of Peano Arithmetic, having, as additional axioms, the axioms added during the construction. Define the *entry stage* of a number v which ever appears in a string $\hat{r}(u)$ with $u \in \{3e, 3e + 1 : e \in \omega\}$ to be the least s at which v enters the range of f or f^- . Next we define the entry stage of an axiom (one of the axioms added during the construction): the *entry stage* of an axiom $\neg v$ is the entry stage of v , and that of an axiom $\neg(v \wedge z)$ is the entry stage of v . In the verifications below, it will be useful to keep in mind that if X, Y are sets such that $X \subseteq \text{Ax}$, and Y is K -consistent, then $Y \cup X$ is H -consistent.

Lemma 6.6. *Each requirement acts finitely often. In particular, for every e $\lim_s x_{e,s} = x_e$ exists, for every $u \in \{3e, 3e + 1 : e \in \omega\}$, $\lim_s \hat{r}_s(u) = \hat{r}(u)$ exists, and thus $\lim_s \hat{\rho}_s(u) = \hat{\rho}(u)$ exists; finally, $\lim_s A_s(x_e) = A(x_e)$ exists and $A(x_e) \neq V_e(x_e)$.*

Proof. Assume inductively that each N_i , $i < e$, eventually stops acting. After every N_i , with $i < e$ has ceased to act, there is a least stage t_0 such that N_e defines the final value of x_e . At that point we keep modifying $\hat{r}(3e)$ and $\hat{r}(3e + 1)$, only in response to changes in $V_e(x_e)$, but as V_e is a Δ_2^0 set, this can happen only finitely many times. The claim about $A(x_e) \neq V_e(x_e)$ easily follows from the construction. \square

The following lemmata intend to explicitly relate the above construction to the p -dialectical system $p = \langle K, f, f^-, c \rangle$. In the lemma and its proof, ρ, r, L are the parameters associated with the p -dialectical system $p = \langle K, f, f^-, c \rangle$.

Lemma 6.7. *Suppose that $u = 3e$ is such that $L(u)$ exists in the limit. Then $x_e = \lim_s x_{e,s}$ exists. Moreover assume that $L(u) \cup \{\neg x_e\}$ is K -consistent; then for every s following the last stage at which $L(u)$ changes,*

- (1) $c \in K(L(u) \cup \{\hat{\rho}_s(u)\})$ if and only if $\neg\hat{\rho}_s(u)$ is among the final axioms of S_∞ ;
- (2) if $L(u) \cup \{\hat{\rho}_s(u)\}$ is K -consistent then $c \in K(L(u) \cup \{\hat{\rho}_s(u)\} \cup \{x_e\})$ if and only if $\neg(\hat{\rho}_s(u) \wedge x_e)$ is among the final axioms of S_∞ .

Proof. Suppose that $u = 3e$ satisfies the assumptions. In particular from the existence of $L(u)$ in the limit it is clear that $x_e = \lim_s x_{e,s}$ exists: in fact, once appointed after $L(u)$ ceases to change, we have that x_e does not change any more. Notice that in both claims (1) and (2) the right-to-left implication is trivial. So we need only prove the left-to-right implication of each equivalence. Let us first consider the first item.

- (1) Suppose that $c \in K(L(u) \cup \{\hat{\rho}_s(u)\})$. This means that there is a finite subset $X \subseteq \text{Ax}$ such that $c \in H(L(u) \cup \{\hat{\rho}_s(u)\} \cup X)$: assume that X is \subseteq -minimal with this property. It can not be $X = \emptyset$ as $L(u)$ is H -consistent and thus $\hat{\rho}_s(u)$ would be chosen independently of $L(u)$: notice that the value $\hat{\rho}_s(u)$ has not been chosen before $L(u)$ stops changing because of the resetting procedure at the end of odd stages. Let $a \in X$ be of greatest entry stage. Notice that, by minimality, $c \notin H(L(u) \cup \{\hat{\rho}_s(u)\} \cup (X \setminus \{a\}))$, i.e. the set $L(u) \cup \{\hat{\rho}_s(u)\} \cup (X \setminus \{a\})$ is H -consistent.

We distinguish the two possible cases due to which a can occur as a new axiom:

- (a) Case $a = \neg v$, for some v . In this case, if $v \notin L(u) \cup \{\hat{\rho}_s(u)\}$ then (as v has greatest entry stage) we have chosen v to be independent of $L(u) \cup \{\hat{\rho}_s(u)\} \cup (X \setminus \{a\})$, contradicting that $c \in H(L(u) \cup \{\hat{\rho}_s(u)\} \cup X)$ which implies, by logic, that $v \in H(L(u) \cup \{\hat{\rho}_s(u)\} \cup (X \setminus \{a\}))$. Hence $v \in L(u) \cup \{\hat{\rho}_s(u)\}$. On the other hand, it can not be $v \in L(u)$ since $L(u)$ is K -consistent. Therefore, $v = \hat{\rho}_s(u)$ as desired.
- (b) Case $a = \neg(v \wedge z)$, for some v, z . (Recall that in this case the entry stage of a is by definition that of v .) Then $v \wedge z \in H(L(u) \cup \{\hat{\rho}_s(u)\} \cup (X \setminus \{a\}))$, hence $v \in H(L(u) \cup \{\hat{\rho}_s(u)\} \cup (X \setminus \{a\}))$: if $v \notin L(u) \cup \{\hat{\rho}_s(u)\}$, then $v \notin L(u) \cup \{\hat{\rho}_s(u)\} \cup (X \setminus \{a\})$ contradicting that v (being of greatest entry stage) has been chosen to be independent of $L(u) \cup \{\hat{\rho}_s(u)\} \cup (X \setminus \{a\})$. Thus $v \in L(u) \cup \{\hat{\rho}_s(u)\}$. If $v \in L(u)$ then $v = \hat{\rho}(u')$ and $z = \hat{\rho}(v' + 1)$ with $v' + 1 < u$, i.e. the pair v', z' refer to a requirement $N_{e'}$ with $z = x_{e'}$ for some $e' < e$, contradicting that $L(u)$ is K -consistent. Thus we conclude that $v = \hat{\rho}_s(u)$ and $z = x_e$, and thus $v \rightarrow x_e \in H(L(u) \cup (X \setminus \{a\}))$. By logic we have that $\neg v \in H(\{\neg x_e\} \cup L(u) \cup (X \setminus \{a\}))$, contradicting the fact that v is independent of $L(u) \cup \{\neg x_e\} \cup (X \setminus \{a\})$, as by assumption $L(u) \cup \{\neg x_e\}$ is K -consistent, and v has greatest entry stage, and thus by construction v has entry stage greater than or equal to that of x_e , but, if equal, the claim holds as well by the way we choose $v = y_e$ in (a1) of the construction.
- (2) Suppose that $c \in K(L(u) \cup \{\hat{\rho}_s(u)\} \cup \{x_e\})$. As before, let $X \subseteq \text{Ax}$ be a finite set such that $c \in H(L(u) \cup \{\hat{\rho}_s(u)\} \cup \{x_e\} \cup X)$, and X is minimal with this property. As in the previous case, we may assume $X \neq \emptyset$ as $L(u) \cup \{x_e\}$ is H -consistent (being $x - e$ appointed after $L(u)$ has reached limit and being $L(u)$ H -consistent) and thus by resetting $\hat{\rho}_s(u)$ is chosen independently of this set (the entry stage of $\hat{\rho}_s(u)$ is greater than or equal to that of x_e : if equal the claim follows by the way we choose $v = y_e$ in (a1) of the construction). Let again $a \in X$ be of greatest entry stage.
- (a) Case $a = \neg v$, for some v . As in (1a), we are forced to conclude that $v = \hat{\rho}_s(u)$, which yields a contradiction the fact that $L(u) \cup \{\hat{\rho}_s(u)\}$ is K -consistent.
- (b) Case $a = \neg(v \wedge z)$, for some v, z . As in (1b) we can argue that $v = \hat{\rho}_s(u)$, and thus $z = x_e$.

□

Lemma 6.8. *For every u , $L(u)$ exists in the limit; moreover if $u = 3e$ then $L(u) \cup \{x_e\}$, $L(u) \cup \{\neg x_e\}$ are K -consistent.*

Proof. The proof is by induction on u .

Cases $u = 0, 1, 2$.

Assume first that $u = 0$. Then $L(0) = \emptyset$. It is enough to show that $\{\neg x_0\}$ is K -consistent. We show only that $\{\neg x_0\}$ is K -consistent: the other case is similar. Notice that $x_0 = x_{0,s}$ for ever $s > 0$, thus x_0 has least possible entry stage. So, suppose that $c \in K(\{\neg x_0\})$ and let $X \subseteq \text{Ax}$ be a minimal set such that $c \in H(\{\neg x_0\} \cup X)$. It can not be $X = \emptyset$ because x_0 is chosen independently of \emptyset as $c \notin H(\emptyset)$. Let $a \in X$ be of greatest entry stage.

We distinguish the two possible cases due to which a can occur as a new axiom:

- (1) Case $a = \neg v$, for some v . It follows that $v \in H(\{\neg x_0\} \cup (X \setminus \{a\}))$, contradicting that v has greatest entry stage and thus it is chosen independent of $\{\neg x_0\} \cup (X \setminus \{a\})$ which is H -consistent.
- (2) Case $a = \neg(v \wedge z)$, for some v, z . In this case we have that $v \wedge z \in H(\{\neg x_0\} \cup (X \setminus \{a\}))$, and thus $v \in H(\{\neg x_0\} \cup (X \setminus \{a\}))$, contradicting that v has greatest entry stage (at most equal to that of x_0 , but this, as in (1b) of the proof of the previous lemma does not make things different) and thus it is chosen independent of $\{\neg x_0\} \cup (X \setminus \{a\})$.

By Lemma 6.7 this implies also the claim for $u = 1, 2$, as for these slots the p -dialectical procedure perfectly mirrors the construction. Notice that consistency of $L(1)$ follows from Lemma 6.7 and the fact that, for the final values of ρ , we never add the axiom $\neg\rho(0)$, and consistency of $L(2)$ follows from the fact that we never add the axiom $\neg(\rho(0) \wedge x_0)$ in which case $\rho(1) = x_e$, or we do add this axiom and thus $\rho(1) = a_0 \in H(\emptyset)$.

Cases $u = 3e, 3e + 1, 3e + 2$, with $e > 0$.

Assume that $u = 3e$, with $e > 0$. We first observe that $L(3e)$ exists in the limit, and is K -consistent. After $L(3e - 1)$ remains unchanged, at subsequent stages for $r(3e - 1)$ we observe the following: if there is $u < 3e - 1$ such that there are strings σ, τ with $r(u) = \sigma \widehat{\langle f_{3e-1} \rangle} \tau$ (this can happen for at most one u) then $r(3e - 1) = \langle f_{3e-1} \rangle \widehat{\tau}$; on the other hand by the way we define f^- , we see that f_{3e-1} never enters any of the stacks $r(u)$ for $u > 3e - 1$; the only remaining possibilities are that either $r(3e - 1)$ becomes $\langle f_{3e-1} \rangle$ if $L(3e - 1) \cup \{f_{3e-1}\}$ is K -consistent, or $\langle f_{3e-1}, a_0 \rangle$ otherwise. It follows also that $L(3e)$ is K -consistent.

It remains to see that $L(3e) \cup \{\neg x_e\}$ is K -consistent. Assume that it is not K -consistent. Then there is a finite set $X \subseteq \text{Ax}$ such that $c \in H(L(3e) \cup \{\neg x_e\} \cup X)$, and X is minimal with this property. As in the case $u = 0$ we can exclude the possibility $X = \emptyset$. Let $a \in X$ be of greatest entry stage. By an argument similar to that for the case $u = 0$, we conclude that either possible case, i.e. a is of the form $a = \neg v$ or $a = \neg(v \wedge z)$, leads to a contradiction.

By Lemma 6.7, the claim extends to $3e + 1$ and $3e + 2$ as well. □

Lemma 6.9. *If $u \in \{3e, 3e + 1 : e \in \omega\}$ then $\hat{\rho}(u) = \rho(u)$.*

Proof. For these u the p -dialectical procedure faithfully mirrors the construction. The only exception is that, by resetting, $\hat{r}(u)$ may not coincide with $r(u)$ but is in any case a final segment of $r(u)$, as the string $r(u)$ keeps records of all proposals made by f^- including those made even before $L(u)$ has stopped changing, whereas $\hat{r}(u)$ is reset every time $L(u)$ changes. But part of the resetting procedure is adding the axiom $\neg\hat{\rho}(u)$ every time there is a change in L_u . Therefore each such $\hat{\rho}(u)$

is discarded by the p -dialectical procedure, and after $L(u)$ has reached limit the stack $r(u)$, after a few consecutive discarding moves, starts to copy $\hat{r}(u)$. \square

Lemma 6.10. *For every e , the p -dialectical set A_p is a completion of Peano Arithmetic which satisfies $A_p \neq V_e$.*

Proof. The system p is loopless and so by Theorem 4.4 A_p is a completion. By Lemma 6.9, it follows that $\rho(3e + 1) = \hat{\rho}(3e + 1)$, and thus by Lemma 6.6, $A(x_e) \neq V_e(x_e)$. On the other hand, $A_p(x_e) = A(x_e)$. For this we use also that by Theorem 1.3 A_p consists exactly of the numbers that eventually occupy $L = \bigcup_u L(u)$, for the final values $L(u)$, and $x_e \in L$ if and only if $x_e \notin V_e$. \square

This ends the proof of Theorem 6.3. \square

Corollary 6.11. *There exists a p -dialectical system p with connectives such that A_p is a completion of Peano Arithmetic, and A_p is not dialectical.*

Proof. Apply the previous theorem, taking \mathcal{A} to be the class of ω -c.e. sets, which by a result in [3] contains all dialectical sets, and is known to be a computable class of Δ_2^0 sets. \square

Remark 6.12. Notice that the q -dialectical system defined in the proof of Theorem 2.2 need not preserve connectives, even if the original H does. This is fairly clear from the way H^* is defined: on the other hand, if the construction of q from p preserved connectives, then as the result is independent of the approximation to H^* , it would be that $A_d = A_q^\alpha$ where α is a good approximation to H^* . But then, by Theorem 4.6 A_q^α and thus A_p would be dialectical. It would follow that every p -completion is a d -completion, contrary to Theorem 6.3.

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