Arguing about Infinity: The meaning and use of infinity, infinitecimals, and zero.

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1 Introduction

The idea of infinity and zero are closely related, despite what many perceive as an intuitive inverse relationship. The symbol 0 generally refers to nothingness, whereas the symbol ∞ refers to "so much" that it cannot be quantified or captured. The notion of finititude rests somewhere between complete nothingness (0) and something having no end (∞).

My concern is that many of the philosophers arguing for or against the ontology (or possibility) of an actual infinite set are unaware or unfamiliar with the mathematical literature attempting to clearly and rigorously define these terms. I believe it is a mistake to leave mathematicians out of this conversation, as analysts in particular have defined (and used) infinity in a way that is relevant to the ongoing debate between philosophers.

For this paper, I have selected examples from *Principles of Analysis* by Rudin [16], which has become a standard text for Real Analysis classes taken by pure mathematicians and engineers. I will also restrict the scope of examples to Euclidean spaces for simplicity. Finally, I will be using addition, $+$ and multiplication $*$ in their usual way.

2 What are numbers?

2.1 Numbers and Numerals

To ask whether the symbol ∞ *could* refer to a number means we need to ask what numbers really are. Numbers are mathematical objects used to measure and count. Numerals, the symbols used to represent numbers (in English, we use Arabic numerals), are different from the "numbers" themselves (hence the Arabic numeral 2, the word "two", and two tally marks refer to the same number, despite having different names in each case.

Important sets of numbers include the natural numbers, which can either mean the positive integers $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$, or the "non-negative integers" $\mathbb{Z}^* = \{0, 1, 2, 3, \dots\}$. The integers, represented by $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots,\}$

are a set of positive and negative numbers. Next we have the rational numbers, $\mathbb{Q} = \left\{\frac{p}{q}\right\}$, where p and q are integers and $q \neq 0$. Finally, we have the real numbers, $\mathbb{R}$ which include all the rational numbers plus irrational numbers that form "gaps" in the rationals, such as $\sqrt{2}$ which cannot be written as a terminating or repeating decimal. Note that

$$\mathbb{Z}^+ \subset \mathbb{Z}^* \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

2.2 Sets of numbers

Two important sets of numbers are groups [20] and fields [19]. A group G is a set of elements along with an associated group operation that satisfies a specific set of properties [20]. One of these properties is that the group operation must have have an "identity element" along an "inverse operation" that allows one to get the identity element from any other element of the group. In the case of usual addition, the identity element is 0 and the inverse operation is $x + (-x) = 0$. As a result, the positive integers $\mathbb{Z}^+$ under addition is not a group (it does not contain the needed identity element 0), and neither are the non-negative integers $\mathbb{Z}^*$ (as they do not contain the negative numbers needed to "invert" addition to get the additive inverse, 0). On the other hand, the Integers ($\mathbb{Z}$), Rationals ($\mathbb{Q}$), and Reals ($\mathbb{R}$) do form a group under standard addition.

Fields contain two operators, $+$ and $*$, which must must satisfy the Field Axioms described in 1.12 of [16] [1]. One of these axioms is the existence of multiplicative inverses, so the integers (plus the subsets $\mathbb{Z}^+$ and $\mathbb{Z}^*$) do not form a field since the multiplicative inverses of integers are not integers. The rationals $\mathbb{Q}$ and reals $\mathbb{R}$ do form fields. Table 1 summarizes whether each set is a field and a group.

A mathematical structure, meaning a set with some associated operator(s), is said to be closed under a given operator if the operator applied to members of the set yields another element in the set [18]. The positive integers, non-negative integers, integers, rationals, and reals are all closed under standard addition since the sum of any two elements is an element in the set. In other words, $x, y \in S \implies (x + y) \in S$. This will be important when discussing infinity since the real numbers are closed before introducing the symbol ∞.

2.3 The Importance of Zero

A zero element, including the scalar zero and a zero vector, is an important element in topology [16], linear algebra [17], and algebraic structure more generally [22]. Letting $\vec{0}$ denote the zero vector (For Euclidian k-spaces with $k > 1$) and 0 denote the scalar zero, we have the following principles:

[1] There are also rings, [21], which include $+$ and $*$, but where multiplication need not be commutative. Rings are more general than fields because they do not need to follow all Field Axioms; thus all fields are rings but not all rings are fields.

Name	Symbol	Group?	Field?
Positive Integers	$\mathbb{Z}^+$	No	No
Non-Negative Integers	$\mathbb{Z}^*$	No	No
Integers	$\mathbb{Z}$	Yes	No
Rationals	$\mathbb{Q}$	Yes	Yes
Reals	$\mathbb{R}$	Yes	Yes

Table 1: Whether each set of numbers forms a group and a field under standard addition and multiplication.

- For any group G where the group operator is standard addition, 0 is the additive identity: $x + 0 = x$, $\forall x \in G$ [2].
- For any field F, $x \cdot 0 = 0 \cdot x = 0$, $\forall x \in F$ [22] [3].
- For any field F, we have a zero element, denoted 0, such that $x + 0 = x$, $\forall x \in F$
- Every vector space contains $\vec{0}$ (because one can choose 0 from the associated scalar field [4] [17]).
- $|x| = 0 \iff x = \vec{0}$ [16].

Zero serves as an additive identity (see the second bullet point) and can "nullifiy" every member of a field (see the first bullet point). However we will soon see we need to be more precise about what zero actually means - is it a number that can produce a finite quantity when added enough (or an infinite) number of times, or is it a symbol that represents ontological "nothingness?" Here it is important to differentiate between the symbol $\vec{0}$, which represents the zero vector, with the zero vector itself, which is a vector that has zero length and thus does not "go" anywhere. Similarly, the symbol 0 refers to the number that is "nothing" (no length, no distance, etc). These symbols are elements that represent "nothingness," so we give them a name, despite the content being nothing. In a Fregarian sense, symbol 0 (for the reals, rationals, or integers) and the vector $\vec{0}$ (for Euclidan k-space) refers to the same thing, which is *nothing* in each case [6].

To give an exampl of the importance of an additive, suppose I have five apples, and I pick three more. I now have $3 + 5 = 8$ apples. In this case, these numbers refer to the quantity (and presence) of apples. But suppose I have five apples and pick no apples (zero apples). The number of apples I have is $5 + 0 = 5$. Zero represents the "absence of" any object or unit, which is why if we were to change the analogy from apples to peaches, 0 in either case means "nothing."

[2] It should be clear this is also true for fields under standard addition as well

[3] Also Proposition 1.16 in [16]

[4] 0 exists in all fields by the previous bullet point, also see the Field Axioms listed in 1.12 of [16]

3 Infinity

3.1 Defining Infinity Mathematically

Rudin (and other analysts) introduce the "extended real numbers," a number system that takes the field of real numbers, $\mathbb{R}$, and adds the *symbols* (they are careful to not call these numbers; as we will see later, field operations do not work on them) $+\infty$ and $-\infty$ [16]. This allows every subset of $\mathbb{R}$ to have an upper and lower bound in the extended real numbers. It is worth pointing out that these symbols are introduced so any subset of the real numbers has an upper bound in the real numbers, and Rudin (among others) are clear to point out they do not refer to a definite quantity the way finite numbers do [5].

3.2 Infinite Praises - A cautionary Tale

The error that Wes Morriston [11, 13, 12] and Alex Malpass [9] make is assuming infinity refers to a quantity that can be treated like any other finite number. Morriston's example in [12] involves two angels, Gabriel and Uriel, taking turns singing praises to God every minute. Note that Morriston's example is constructed so this is a potential infinite not an actual infinite, a distinction made by Aristotle and William Lane Craig [3] (who Morriston is specifically replying to), a worthwhile attempt to keep this example from begging the question by supposing an infinite set has already been completed. However Morriston's example still ends up treating infinity *like* a finite quantity in a question begging way. Morriston says,

> It's true, of course, that Gabriel and Uriel will never complete the series of praises. They will never arrive at a time at which they have said all of them. Indeed, they will never arrive at a time at which they have said infinitely many praises. At every stage in the future series of events as I am imagining it, they will have said only finitely many. But that makes not a particle of difference to the point I am about to make. If you ask, "How many distinct praises will be said?" the only sensible answer is, infinitely many. [12]

Now it is clear that as time goes on and on towards infinity, the number of praises sung by Gabriel and Uriel tend towards infinity as well. However Morriston's question "how many distinct praises *will be* said" lacks a subject - namely *when* we are asking the count to end. For instance, if we ask "how many praises will be sung after four minutes," it is clear that the answer is four (Gabriel will have sang two and Uriel will have sung two as well). However the question "how many distinct praises will be said" without including a subject is like asking "if I start counting numbers now, how many numbers will I count if I do not stop?" The lack of a specific, definite subject makes the question ill-posed and under-determined.

[5] See Cauchy and Dedekind for the construction of the reals

If seems as if Morriston wants to ask the question "how many praises *will be sung* at infinity," however this would clearly beg the question if favor of an infinite quantity being reached by successive addition. Morriston could, at this point, argue there is a difference between *reaching* infinity and *completing* infinity, however I do not see how what distinction would be. Considering that Morriston creates this example to try and argue that a future series of possible events is an actual infinite (and that a regress of an infinite number of past events is possible), it is important that this argument does not rest on "counting to" an actual infinite in first place. The point that infinity is not a determinate quantity is also shared by Michael Huemer, Morriston's colleague at University of Colorado, Boulder [7].

Notice how Morriston's argument rests on the subject of this question being larger than a natural number. He continues,

> As I have imagined the scenario, each of the praises is definite and discrete. What is their number? Since there is a first praise, the number of praises that have been said will always be finite. But that's not what I'm asking about. What I am asking is this: How many "definite and discrete" praises will be said after a given moment of time? (It's very important to keep our tenses straight here!) I do not see how the friends of the kalām argument can avoid the conclusion that the number of praises, each of which will be said, is (and always will be!) be greater than any natural number.

The point here is that "after a [or any] given moment of time" is also a "definite and discrete" number - the sum of a finite number of finite (integer, natural, rational, or real) numbers will *always* be a finite number because the the integers, rationals, and reals, are closed under addition. In Morriston's example, the number of praises sung will never *exceed* a finite number due to the closure of the integers, rationals, and reals under addition [18]. In other words, there is no "at infinity" at all - there is simply the tendency to get larger and larger (and thus closer, but still always "unaccountably far") from the idea of infinitude. This is because we are only ever adding finite quantities.

3.3 An Endless Series of Future Events

Malpass and Morriston [9] argue that if a future endless series of events is possible, a beginningless series of events cannot be rejected on the grounds that it will form an actual infinite. Their claim is Craig and Sinclair [4] equivocates what *will be* and what *will have been* [6]. However, if we assume events started from a given moment in time, and that such events are ordered or labelled in a way analogous real or natural numbers, then there never "will be" an infinite number of events (i.e. we will never reach "event ∞"). Any future event that occurs will *always* be a finite distance away from whatever the "first moment"

[6] My focus is not on whether Craig's Kalam argument succeeds; rather it is on the particularities of infinity

was. Furthermore, there never "will have been" an infinite number of past events at a given moment in time either - both of these "will always be" and "will always have been" finite numbers due to the assumed closure of addition. Infinity is simply the upper bound on the total number of events that ever "will have been by x" where x is a number.

The closure of addition ensures we can never count *to* (or have counted *from*) infinity by adding or subtracting finite numbers. Just as it is absurd to think I could start counting $1, 2, 3, \ldots$ and eventually count $\ldots, \infty - 2, \infty - 1, \infty$ it is equally absurd to think I can start counting $-\infty, -\infty + 1, -\infty + 2, \ldots$ and eventually reach $\ldots, -2, -1, 0$. The only way out of this is to argue addition on infinities is not commutative in the backwards direction, providing a kind of "symmetry breaker" that allows backwards counting from $-\infty$ but not forwards counting to ∞, but this seems ad-hoc.

Morriston's reply to this seems to be "...although it will never be the case that infinitely many [events] have been counted, ...each of infinitely many numbers is such that it will be counted — and, indeed, is such that it will have been counted [13]." We need to be careful about our language here – the number of future events does not end, and is thus not finite. If we were to imagine these events as existing in a set, the cardinality of this set would be written as ∞ because there is no end to the number of events. However we never "will have counted" an infinite number of them at any point in time, nor will we "have ever counted" all of them: there always "will be" an infinite number of future events to count.

One can, of course, ask "will we count an infinite number of numbers given an infinite amount of time to count," but this question assumes the word "infinity" refers to a quantity that makes this question well-defined. My argument is such a question is ill-posed, and breaking such a question down in plain English will help understand why: we are asking "is it possible to finish counting an unending series of numbers if I never stop counting." The problem here is the assumption that one can "finish counting" an unending series of numbers in the first place - no matter how much time you have, there are always more numbers. Even if you never stop counting (i.e. count for an infinite time), you will never reach an "end" of numbers to count.

My concern is Morriston's reply begs the question in favor of completing infinity. He argues, "although it will never be the case that infinitely many have been counted, it is also true that each of infinitely many numbers is such that it will be counted—and, indeed, is such that it will have been counted." While the first part of this sentence is true, it is premature to conclude that infinitely many numbers "will be counted" or "will have been counted," *even after an infinite amount of time.* Of course, if I start counting now and do not stop, for any number x I choose, I can find a time when I count $\ldots, x-2, x-1, x, x+1$. But this does not imply I can *finish* the count, or count "all" such numbers. In other words, just because I can find a time that I will count *any* given number I can imagine does not mean I will finish counting *all* of them, because there is no end to the number of numbers in question.

Of course, Malpass and Morriston [9] can argue that the set of all events that already *have* happened is infinite (if there is no beginning to the universe), but this does not change the fact that any given moment in the future will always be a finite distance away from any other given moment, nor would it change the fact that there is no end to the number of future events that 'will" or "will have" happened (unless of course there *is* an end to the universe). In conclusion, due to the closure of numbers under addition, it appears there is no non-circular justification for completing infinity without first assuming one *can* complete infinity.

3.4 Infinity and Provability

One difficulty here is that the 9 axioms of Zermelo–Fraenkel set theory, which form the basics of modern mathematics, [2], already *presuppose* that infinite sets exist. Given Gödel's incompleteness theorems, [15], this means we cannot *prove* that infinite sets exist within this system - they are simply assumed to exist. If the introduction of infinity does cause the ZFC axioms to be consistent, it cannot be proven as such within ZFC.

3.5 Conceptualizing infinity

Mathematicians from Gauss to Hilbert have all treated infinity as an idea and not a determinate quantity - indeed it is the treating of the symbol ∞ *as* a number algebraically that leads to a myriad of problems, some of which we will discuss in subsequent sections. My argument is the idea of "infinity" as a quantity that does not end is useful in an instrumentalist kind of way, but cannot be metaphysically (or logically) realized. Assuming infinity can be reached from a finite quantity violates the assumed closure of numbers.

4 Zeno's Paradox (Mostly) Resolved

4.1 Overview of Zeno's Paradox

Zeno's Dichotomy Paradox, often shortened to just Zeno's Paradox (Zeno had at least 9 paradoxes), involves a runner Atalanta trying to run a from point A to point B. Before Atalanta reaches B, she must first reach the halfway point, call this $B/2$. However before she reaches $B/2$, she must first reach $B/4$, and before this, $B/8$, and so on. As a result, despite the fact that the distance from A to B is finite, she would (apparently) need to complete an infinite number of distances to go from point A to point B. At least one philosopher has cited this as support for the metaphysical possibility of completing infinities, a point I will say more about in subsequent sections [8].

The sequence of points Atalanta must reach can be written as $a_n = \left\{\ldots, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\right\}$. Thankfully, this sequence is absolutely convergent and thus by the Riemann Rearrangement Theorem we can reorder the terms while preserving the sum [16]

[7]. We thus have a new sequence of partial sums

$$b_n = \sum_{i=1}^{n} \frac{1}{2^i}$$

In *Approaching Infinity*, Michael Huemer writes Zeno's paradox as a helpful syllogism I modify to fit this example:

1. To reach point B, Atalanta must arrive at each point in the sequence [8] $b_n = \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \right\}$
2. b_n has an infinite number of terms
3. It is impossible to complete a sequence with an infinite number of terms
4. Therefore, it is impossible for Atalanta to reach point B (In some formulations, the argument is that Atalanta cannot even begin to move).

On the surface, it appears we are stuck - Atalanta cannot move any finite distance, and thus cannot begin to move at all. But obviously, Atalanta (or any runner for that matter) *can* move. This forms a common argument for the existence of actual infinities: namely that distances, which can be described by an infinite sequence of sub-intervals, can be crossed. However, what is often not mentioned in such an argument is this "infinite sequence of sub-intervals" themselves must have zero measure (in which case, this is a trivial point, as we will see in the next section) or the sub-intervals must converge to zero measure "fast enough." Note further that not all sub-intervals converging to zero measure will work, they must shrink to zero measure "fast enough". Additionally, in the latter case, you can get as close as you would need to get to the actual interval in a finite number of sub-intervals. So long as all the sequence of regions have nontrivial measure (meaning the sub-regions are non-zero nor do they converge to zero measure), there will only be a finite number needed to describe the interval. If the sub-intervals have smaller measure, we may need more, but this number will always remain finite.

As a result, I find it odd that this is used as an argument for the existence of an actual infinite. The reals are constructed in such a way where there are no "gaps" between any two numbers, so of course there are an infinite number of real numbers between these two endpoints. However this was done by construction to ensure upper and lower bounds exist, not necessarily for any analogy to reality . These upper and lower bounds are themselves finite numbers, and so is the distance between any given pair of them.

[7]Those unconvinced by the rearrangement theorem can consider the problem in reverse - another runner going from point B to point A

[8]Huemer describes this using the word "series" despite this being a *sequence* of partial sums. Huemer appears to use the word "series" to refer to sequences and series interchangeably - my guess is this is a regional difference in language. I will stick with the mathematical convention of a series being a summation of terms of a sequence.

4.2 Infinitely Large Quantities vs Continuums

Suppose for the sake of argument we grant that infinities can be completed; that moving any finite distance is "completing" an infinite number of (infinitely small) sub-intervals. Would this imply that the symbol ∞ now refers to a definite value? Would it also imply could "count to" infinity by enumerating reals, thus logically violating the assumed closure of the field of reals?

I do not think so. In particular, it is important to distinguish between finite continuums (the only thing unbounded about them is the cardinality of subintervals that *could* be used to describe them, and in such a case, the measure of these sub-intervals must shrink to zero "fast enough" that we can get arbitrarily close within a finite number of terms) and infinitely large numbers. In the sake of finite continuums, one can also always describe them in terms of a finite number of (finite in measure) sub-intervals. In the case of infinitely large quantities, one cannot. As a result, there is a difference between a finite continuum and an infinitely large number which cannot be constructed from a finite number of finite pieces.

4.3 The "Absurdity" of Counting?

To illustrate where Zeno's Paradox goes wrong, I will tweak the example a bit. Suppose I am counting real numbers and want to count from the number 1 to the number 3. Conventional wisdom tells me I can simply say "one, two, and three" and be done. But to get from 1 to 2, I must pass through the real number 1.5 and to get there, I must past through 1.25, and so on. It would be absurd to suggest I need to speak each possible real number in between 1 and 3 in order to "count" them.

My point here is that for any nontrivial counting to be done, one needs to pick a starting point (call this number a) and another number b, where $b \neq a$. However once a and b are picked, there will always exist a number between them: $\frac{a+b}{2}$. In fact, between any two real numbers, a and b with $a \neq b$, there are an *uncountably infinite* number of reals between them (one way to see this is to let $b_2 = \frac{a+\frac{a+b}{2}}{2}$). Regardless, the real numbers $1, 2,$ and 3 are obviously finite (and so are each of the unaccountably infinite number of reals between them). Is it really the case that whenever I count the reals, I am completing an actual infinity?

Here it is important to distinguish the fact that while there are an infinite number of reals in any interval with nonzero (Lebesgue) measure, it is impossible to *enumerate* these in any practical way. In other words, regardless of whether infinite sets actually exist (in some Platonic Realm, in the mind of God, etc) we cannot hope write them down the way we can with *finite* sets. One practical reason to illustrate this is to notice there are a finite number of elementary particles in the universe, so any attempt to do so would run out of ink. While counting from one real number to another will pass an infinite number of reals, I am not "completing" or "enumerating" this infinite set.

4.4 Returning to Zeno

Returning to Zeno's paradox, it is clear that any step Atalanta "forward" (from A to B) will entail her crossing an infinite number of "points" or sub-intervals - at least, if these intervals or crossings form a continuum in the same way the real numbers do. My claim is that they do not - there are only a finite number of particles Atalanta will cross, and the length "A" to "B" is not a true continuum (at least not in the way we construct the real numbers). While we can imagine A and B as geometric points in a continuum of Euclidian space on the real numbers, this is an analogy based on mathematical axioms and abstract reasoning. I believe it is premature to argue this is the way "things really are."

To make the argument that Atalanta cannot move forward at all because space can be viewed as a continuum with an infinite number of points is like arguing one cannot count from 1 to 3 because there are an infinite number of reals between them. Of course, one cannot count from 1 to 3 by enumerating all the reals, but one *can* count a finite number of sub-intervals, each of which with finite Lebesgue measure, that construct said continuum.

What compounds the difficulty understanding this is that Arabic numerals 1 and 3 are overloaded in the sense that they can refer to real numbers, rational numbers, integers, and more. So when one counts $1, 2, 3$, they can be counting the integers (and thus are not skipping anything "in-between") or they can claim to be counting the reals (counting, not *enumerating*). While the jump from 1 to 2 means something different in the real number system than it does for the integers, it is possible to do such counting in either case.

4.5 The problem with points and the Zero Dilemma

The paradox of geometric points is that shapes, lines and other geometric objects that take up an extended region of space are "made up of" points, which by construction have zero volume [23]. This creates problems because of the intuitive notion that something with a value (or volume) of zero should stay zero, even if you add it an infinite number of times [9]. We face a choice between

1. Affirm an (infinite) sum of zeros can eventually give you a finite number, or
2. Reject the existence of zero-size parts of an object.

I agree with Huemer and take the second route - in fact the very word "zero-size part" seems to me an oxymoron: we can shrink our parts down very very small, but we cannot make these parts zero without changing what the symbol 0 represents. This illustrates what I call the "zero dilemma," and despite the fact that mathematicians have reached a general consensus about this, it appears philosophers have not.

To explain this dilemma, consider the line segment on the closed interval from $[1, 3] \subset \mathbb{R}$. The "length" (more formally, Lebesgue measure) of this interval is 2.

[9] A sequence of all zeros is clearly absolutely convergent

However we can think of this length in many ways: the first is a continuum from 1 to 3 with measure 2. The second is a continuum from 1 to 2 (with measure 1) joined with another continuum from 2 to 3 (with measure 1). More generally, we can think of dividing up the interval up into k equally spaced parts, each with measure $\frac{1}{k}$. Conventional mathematics (field arithmetic) tells us that the overall length of this interval is $2k \cdot \frac{1}{k} = \frac{2k}{k} = 2$ so long as $k \neq 0$ (Recall that multiplicative inverses are specified to exist by the Field Axioms *except* for 0, see M5 in 1.12 of [16]).

Now, suppose that we broke convention and used field arithmetic on the extended real numbers [10]. By Field Axiom M5 we would have a multiplicative inverse for ∞, which would be $\frac{1}{\infty}$. Here comes the fun part: We know $\frac{2k}{k} = 2$, regardless of what k equals. So let us choose $k = \infty$, which gives us $\frac{2\infty}{\infty}$. However $2\infty = \infty$, so we have

$$2 = \frac{2k}{k} = \frac{2\infty}{\infty} = \frac{\infty}{\infty} = 1 \implies 2 = 1. \tag{1}$$

We can structure this absurd result in the following syllogism:

1. If infinity exists in a field (i.e. exists as a real or rational number), then $1 = 2$
2. But $1 \neq 2$
3. Therefore, infinity does not exist in a field.

Mathematicians do not like contradictions (unless it helps with their proofs), which is why there seems to be consensus that the symbol ∞ does not *refer to a number at all*. For many mathematicians, the symbol ∞ simply refers to the upper bound of every subset of the (extended) real number system. For the philosopher who still thinks infinity can be reached by successive addition of finite numbers, I argue they have the following options:

1. Reject that the field of rationals or reals are truly closed under addition
2. Reject that infinity can be reached by addition of reals or rationals

As we shrink our intervals down to zero measure (the "length" of a point): as $k \to \infty$, our intervals become smaller but we have more of them. We can make these intervals as small as we would like and still have a finite volume (hence the idea of limits), however as soon as these intervals reach *exactly* zero size we have problems like the one above. This forms what I call the "Zero Dilemma" [11]. One can choose from the following:

1. The symbol 0 does not refer to nothingness, but rather some quantity that, when added an infinite number of times, forms a nonzero number (i.e. $\sum_{i=1}^{\infty} 0 = 2$)

[10] The extended reals include the real numbers plus the *symbols* (not numbers) $+\infty$ and $-\infty$; they do not form a field [16]

[11] Not to be confused with Kotaro Uchikoshi's work "Zero Time Dilemma"

2. 0 refers to nothingness, but infinity is "powerful" enough to give you something from nothing (i.e. $\infty \cdot 0 = 2$)

3. ∞ is not a "number" and shouldn't be used as such - it is simply an idea, useful as an upper bound.

I hope the reader can understand why myself (and many mathematicians) take the third option of the zero dilemma: I personally would much rather ban infinity from field operations than either redefine zero (to not mean "nothing") or to allow what appears to be metaphysical absurdities like a violation of Leibniz's' principle of sufficient reason (PSR) [10]. Indeed we can understand why 0 was excluded from having a multiplicative inverse for fields - it causes more problems than it solves. Intuitively, I think this makes sense: we can take the "opposite" of any finite quantity and get another finite quantity. But the opposite of nothingness could be any*thing* [12]

4.6 Relationship to Hyperreals

Like the extended real numbers, hyperreals are a way of extending the real numbers, however hyperreals introduce infinitesimals in addition to infinite quantities. Here the reciprocal of infinite quantities are infinitesimals - numbers that are infinitely small but still non-zero.

5 Discrete Ontology

In this section, I propose a description of reality that avoids any of the contradictions or difficulties associated with infinity. In this description, the fundamental level of reality is constructed of discrete phenomena (hence the name "discrete ontology"), even though it can be approximated by continuous mathematics. In other words, discrete ontology is a description of reality where *everything* is fundamentally quantized – there are no infinities, infinitesimals, or actual continuums. I will first discuss how items themselves are made of discrete parts before moving into the more controversial claim about the discreteness of space and time.

5.1 Discrete Items

At the fundamental level, objects are made up elementary particles. The most popular estimation of the number of particles in the (observable) universe is Eddington's calculation of approximately 10^{79} [5]. which is a unfathomably large, yet still finite. Now consider a flat surface like a tabletop: despite being able to describe it as a continuous "line," we know it is made up of a finite number of such particles, arranged in such a way that it feels and appears continuous.

[12] Recall the various absurdities that come from dividing by zero, and why mathematicians leave division by zero as "undefined" - attempting to define it as a fixed value in one situation causes more problems and contradictions in *other* situations.

5.2 Discrete Space

Consider a computer simulation of a 3-dimensional space: objects in the simulation will have a position represented by numbers stored inside these computers. These numbers can be integers, or more commonly, floating point numbers that allow for very large (and very small) values. The point here is a computer is clearly a discrete object: it is made up of a finite number of transistors and can only represent finite numbers (finite in quantity, and numbers that are either finitely large or finitely close to zero). In other words, even a highly detailed simulation of a 3D environment that appears to be continuous is quantized to a certain degree in reality. The "value" of infinity in computers is often represented as a special number that forms an "upper bound" - it is a number achieved when results of mathematical operations overflow [1]. As per the IEEE 754 standard, the "value" of (positive) infinity is a floating-point number where the sign and fraction bits are all zeroes while the biased exponent bits are all ones. In other words, the *number* infinity is not stored; rather a certain bit sequence is reserved to "act like" infinity in an expected way.

Likewise, in space there is a fundamental limit to how precise we can measure space, given by Planck length ($\approx 3.5 \times 10^{-35} m$) [14]. As a result I believe we should at least remain agnostic about whether below this limit, space itself is made up of a quantized "grid" which is finite in nature.

5.3 Discrete Time

Returning to the computer analogy, computers are driven by a clock, often made of crystal, which allows for pulses or triggers at a regular interval. The clock's trigger advances the "state" of the computer, which is necessary for sequential logic. The appearance of continuous motion from a computer (in videos, computer games, simulations) is illusory since each frame is a (discrete) image generated by a discrete number of computations and displayed at a regular rate. Just like a film can appear continuous (despite being made of a discrete number of frames or images), I am proposing that time itself may be fundamentally discrete. Similar to space, there is a limit to how finely we can measure time, given by Planck time ($\approx 5.39 \times 10^{-44} s$), which can be seen as an effective "sampling rate." As of this writing, no current physical theory can describe timescales shorter than Planck time. It is at least possible that at this period (or perhaps below it) time itself is quantized.

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