Arguing about Infinity: The meaning (and use) of infinity and zero

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1 Introduction

The idea of infinity and zero are closely related, despite their inverse relationship. The symbol 0 intuitively refers to nothingness, whereas the symbol ∞ refers to "so much" that it cannot be quantified or captured. The notion of finititude rests somewhere between complete nothingness (0) and something having no end (∞).

My concern is that many of the philosophers arguing about the ontology (or possibility) of an actual infinite set are unaware or unfamiliar with the mathematical literature attempting to clearly and rigorously define these terms. I believe it is a mistake to leave mathematicians out of this conversation, as analysts in particular have defined (and used) infinity in a way that is relevant to the ongoing debate between philosophers.

For this paper, I have selected examples from Rudin's *Principles of Analysis* [11], which has become a standard text for Real Analysis classes taken by pure mathematicians and engineers. I will also restrict the scope of examples to Euclidean spaces for simplicity.

2 The Importance of Zero

The idea of a zero element, including the scalar zero and a zero vector, is an important element in topology [11], linear algebra [13], and algebraic structure more generally [2]. Letting $\vec{0}$ denote the zero vector (For Euclidian k-spaces with $k > 1$) and 0 denote the scalar zero, we have the following principles:

- For any field F, $x \cdot 0 = 0 \cdot x = 0$, $\forall x \in F$ [2] [1].
- For any field F, we have a zero element, denoted 0, such that $x + 0 = x$, $\forall x \in F$

[1] Also Proposition 1.16 in [11]

- Every vector space contains $\vec{0}$ (because one can choose 0 from the associated scalar field [2] [13]).
- $|x| = 0 \iff x = \vec{0}$ [11].

Zero serves as an additive identity (see the second bullet point) and can "nullifiy" every member of a Field (see the first bullet point). However we will soon see we need to be more precise about what zero actually means - is it a number that can produce a finite quantity when added enough (or an infinite) number of times, or is it a symbol that represents ontological "nothingness?" Here it is important to differentiate between the symbol $\vec{0}$, which represents the zero vector, with the zero vector itself, which is a vector that has zero length and thus does not "go" anywhere. Similarly, the symbol 0 refers to the number that is "nothing" (no length, no distance, etc). These symbols are elements that represent "nothingness," so we give them a name, despite the content being nothing. In a Fregarian sense, symbol 0 (for the reals, rationals, or integers) and the vector $\vec{0}$ (for Euclidan k-space) refers to the same thing, which is *nothing* in each case [4].

3 Defining Infinity Mathematically

Rudin (and other analysts) introduce the "extended real numbers," a number system that takes the field of real numbers, $\mathbb{R}$, and adds the *symbols* (they are careful to not call these *numbers*; as we will see later, field operations do not work on them) $+\infty$ and $-\infty$ [11]. This allows every subset of $\mathbb{R}$ to have an upper bound in the extended real numbers. The point here is these symbols are introduced to represent an upper bound rather; they do not refer to a quantity the way finite numbers do.

3.1 Infinity is not a number (in the traditional sense)

The error that Wes Morriston [8, 10, 9] and Alex Malpass [6] make is assuming infinity refers to a quantity that can be treated like any other finite number. Morriston's example in [9] involves two angels, Gabriel and Uriel, taking turns singing praises to God every minute. Note that Morriston's example is constructed so this is a potential infinite not an actual infinite, a distinction made by Aristotle and William Lane Craig [3] (who Morriston is specifically replying to), a worthwhile attempt to keep this example from begging the question by supposing an infinite set has already been completed. However Morriston's example still ends up treating infinity *like* a finite quantity in a question begging way. Morriston says,

> It's true, of course, that Gabriel and Uriel will never complete the series of praises. They will never arrive at a time at which they have

[2] 0 exists in all fields by the previous bullet point, also see the Field Axioms listed in 1.12 of [11]

> said all of them. Indeed, they will never arrive at a time at which they have said infinitely many praises. At every stage in the future series of events as I am imagining it, they will have said only finitely many. But that makes not a particle of difference to the point I am about to make. If you ask, "How many distinct praises will be said?" the only sensible answer is, infinitely many. [9]

Now it is clear that as time goes on and on towards infinity, the number of praises sung by Gabriel and Uriel tend towards infinity as well. However Morriston's question "how many distinct praises *will be* said" lacks a subject - namely *when* we are asking the count to end. For instance, if we ask "how many praises will be sung after four minutes," it is clear that the answer is four (Gabriel will have sang two and Uriel will have sung two as well). However the question "how many distinct praises will be said" without including a subject is like asking "if I start counting numbers now, how many numbers will I count if I do not stop?" The lack of a specific, definite subject makes the question ill-posed and under-determined.

If seems as if Morriston wants to ask the question "how many praises *will be sung* at infinity," however this would clearly beg the question if favor of an infinite quantity being reached by successive addition. Morriston could, at this point, argue there is a difference between *reaching* infinity and *completing* infinity, however I do not see how what distinction would be. Considering that Morriston creates this example to try and argue that a future series of possible events is an actual infinite (and that a regress of an infinite number of past events is possible), it is important that this argument does not rest on "counting to" an actual infinite in first place. I believe it does: asking "how many praises will be sung (implicitly at infinity, which is not a finite or determinate quantity)" is like asking "how many praises will be sung at the color blue." Morriston continues,

> As I have imagined the scenario, each of the praises is definite and discrete. What is their number? Since there is a first praise, the number of praises that have been said will always be finite. But that's not what I'm asking about. What I am asking is this: How many "definite and discrete" praises will be said after a given moment of time? (It's very important to keep our tenses straight here!) I do not see how the friends of the kalām argument can avoid the conclusion that the number of praises, each of which will be said, is (and always will be!) be greater than any natural number.

The point here is that "after a [or any] given moment of time" is also a "definite and discrete" number - the sum of a finite number of finite (integer, natural, rational, or real) numbers will *always* be a finite number - the field (of naturals, rationals, reals, etc) is "closed." In other words, the number of praises sung will never *exceed* a finite number - unless the number of terms we count itself *exceeds* a finite number. However numbers that exceed finite numbers are outside the field that constructs these numbers - it is "out of bounds" and ill-defined. My argument is that there is no "at infinity" at all - there is simply the tendency to get larger and larger (and thus closer, but still always "infinitely far") from the idea of infinitude.

Mathematicians from Gauss to Hilbert have all treated infinity as an idea and not a determinate quantity - indeed it is the treating of the symbol ∞ *as* a number algebraically that leads to a myriad of problems, some of which we will discuss in subsequent sections. My argument is the idea of "infinity" as a quantity that does not end is useful in an instrumentalist kind of way, but cannot be metaphysically (or logically) realized. Assuming infinity can be reached from a finite quantity violates the "closedness" of fields and arguably would be a category error.

3.2 What the heck are numbers?

To ask whether the symbol ∞ *could* refer to a number means we need to ask what numbers really are. Now it should be clear that by construction of the following, the symbol ∞ does not belong to the field of reals, rationals, integers, or natural numbers given the field axioms. However, would it be possible to construct a new "set," that perhaps is not a field at all, where ∞ refers to a number in the set?

Perhaps this is possible - one of the key features of numbers is that they are "ordered," and since fields are a subset of rings [1] (which themselves are a subset of groups), may be able to construct a consistent group where this is the case. I will not attempt to do so

$$\text{Fields} \subset \text{Rings} \subset \text{Groups}$$

However being able to construct a consistent system using a symbol to represent infinity does not necessarily imply infinity exists metaphysically. Indeed, there seems to be a problem in that a quantity of unending extant cannot be actualized in the "real world." The point here is that even if one can create a system of consistent mathematical operations involving the symbol ∞, there seems to be a logical issue of how this can be achieved in actuality.

4 Zeno's Paradox (Mostly) Resolved

4.1 Overview of Zeno's Paradox

Zeno's Dichotomy Paradox, often shortened to just "Zeno's Paradox" (Zeno had at least 9 paradoxes), involves a runner Atalanta trying to run a from point A to point B. Before Atalanta reaches B, she must first reach the halfway point, call this $B/2$. However before she reaches $B/2$, she must first reach $B/4$, and before this, $B/8$, and so on. As a result, despite the fact that the distance from A to B is finite, she would (apparently) need to complete an infinite number of distances to go from point A to point B. At least one philosopher has cited this as support for the metaphysical possibility of completing infinities, a point I will say more about in subsequent sections [5].

The sequence of points Atalanta must reach can be written as $a_n = \left\{\ldots, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1\right\}$. Thankfully, this sequence is absolutely convergent and thus by the Riemann Rearrangement Theorem we can reorder the terms while preserving the sum [11] [3]. We thus have a new sequence of partial sums

$$b_n = \sum_{i=1}^{n} \frac{1}{2^i}$$

In *Approaching Infinity*, Michael Huemer writes Zeno's paradox as a helpful syllogism I modify to fit this example:

1. To reach point B, Atalanta must arrive at each point in the sequence [4] $b_n = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right\}$
2. b_n has an infinite number of terms
3. It is impossible to complete a sequence with an infinite number of terms
4. Therefore, it is impossible for Atalanta to reach point B (In some formulations, the argument is that Atalanta cannot even begin to move).

On the surface, it appears we are stuck - Atalanta cannot move any finite distance, and thus cannot begin to move at all.

4.2 The Absurdity of Counting

To illustrate where Zeno's Paradox goes wrong, I will tweak the example a bit. Suppose I am counting real numbers and want to count from the number 1 to the number 3. Conventional wisdom tells me I can simply say "one, two, and three" and be done. But to get from 1 to 2, I must pass through the real number 1.5 and to get there, I must past through 1.25, and so on. It would be absurd to suggest I need to speak each possible real number in between 1 and 3 in order to "count" them.

My point here is that for any nontrivial counting to be done, one needs to pick a starting point (call this number a) and another number b, where $b \neq a$. However once a and b are picked, there will always exist a number between them: $\frac{a+b}{2}$. In fact, between any two real numbers, a and b with $a \neq b$, there are an *uncountably infinite* number of reals between them (one way to see this is to let $b_2 = \frac{a+\frac{a+b}{2}}{2}$). Regardless, the real numbers $1, 2$, and 3 are obviously finite (and so are the unaccountably infinite number of reals between them). Is

[3] Those unconvinced by the rearrangement theorem can consider the problem in reverse - another runner going from point B to point A

[4] Huemer describes this using the word "series" despite this being a *sequence* of partial sums. Huemer appears to use the word "series" to refer to sequences and series interchangeably - my guess is this is a regional difference in language. I will stick with the mathematical convention of a series being a summation of terms of a sequence.

it really the case that whenever I count the reals, I am completing an actual infinity?

Here it is important to distinguish the fact that while there are an infinite number of reals in any interval with nonzero (Lebesgue) measure, it is impossible to *enumerate* these in any practical way. In other words, regardless of whether infinite sets actually exist (in some Platonic Realm, in the mind of God, etc) we cannot hope write them down the way we can enumerate *finite* sets. One reason for this is there are a finite number of elementary particles in the universe, so any attempt to do so would run out of ink. While counting from one real number to another will pass an infinite number of reals, I am not "completing" or "enumerating" this infinite set [5].

4.3 Returning to Zeno

Returning to Zeno's paradox, it is clear that any step Atalanta "forward" (from A to B) will entail her crossing an infinite number of "points" or sub-intervals - at least, if these intervals or crossings form a continuum in the same way the real numbers do. My claim is that they do not - there are only a finite number of particles Atalanta will cross, and the length "A" to "B" is not a true continuum (at least not in the way we construct the real numbers). While we can imagine A and B as geometric points in a continuum of Euclidian space on the real numbers, this is an analogy based on mathematical axioms and abstract reasoning. I believe it is premature to argue this is the way "things really are."

To make the argument that Atalanta cannot move forward at all because space can be viewed as a continuum with an infinite number of points is like arguing one cannot count from 1 to 3 because there are an infinite number of reals between them. Of course, one cannot count from 1 to 3 by enumerating all the reals, but one *can* count a finite number of sub-intervals, each of which with finite Lebesgue measure, that construct said continuum.

What compounds the difficulty understanding this is that Arabic symbols 1 and 3 are overloaded in the sense that they can refer to real numbers, rational numbers, integers, and more. So when one counts $1, 2, 3$, they can be counting the integers (and thus are not skipping anything "in-between") or they can claim to be counting the reals (counting, not *enumerating*). Regardless, the jump from 1 to 2 means something different in the real number system than it does for the integers.

4.4 The problem with points and the Zero Dilemma

The paradox of geometric points is that shapes, lines and other geometric objects that take up an extended region of space are "made up of" points, which by construction have zero volume [12]. This creates problems because of the

[5] It is prudent to remember that the reals were constructed in such a way to make a continuum and fill in holes left by the rational numbers (such as $\sqrt{2}$) - thanks Dedekind and Cauchy!

intuitive notion that something with a value (or volume) of zero should stay zero, even if you add it an infinite number of times [6]. We face a choice between

1. Affirm an (infinite) sum of zeros can eventually give you a finite number, or

2. Reject the existence of zero-size parts of an object.

I agree with Huemer and take the second route - in fact the very word "zero-size part" seems to me an oxymoron: we can shrink our parts down very very small, but we cannot make these parts zero without changing what the symbol 0 represents. This illustrates what I call the "zero dilemma," and despite the fact that mathematicians have reached a general consensus about this, philosophers it seems have not.

To explain this dilemma, consider the line segment on the closed interval from $[1, 3] \subset \mathbb{R}$. The "length" (more formally, Lebesgue measure) of this interval is 2. However we can think of this length in many ways: the first is a continuum from 1 to 3 with measure 2. The second is a continuum from 1 to 2 (with measure 1) joined with another continuum from 2 to 3 (with measure 1). More generally, we can think of dividing up the interval up into k equally spaced parts, each with measure $\frac{1}{k}$. Conventional mathematics (field arithmetic) tells us that the overall length of this interval is $2k \cdot \frac{1}{k} = \frac{2k}{k} = 2$ so long as $k \neq 0$ (Recall that multiplicative inverses are specified to exist by the Field Axioms *except* for 0, see M5 in 1.12 of [11]).

Now, suppose that we broke convention and used field arithmetic on the extended real numbers [7]. By Field Axiom M5 we would have a multiplicative inverse for ∞, which would be $\frac{1}{\infty}$. Here comes the fun part: We know $\frac{2k}{k} = 2$, regardless of what k equals. So let us choose $k = \infty$, which gives us $\frac{2\infty}{\infty}$. However $2\infty = \infty$, so we have

$$2 = \frac{2k}{k} = \frac{2\infty}{\infty} = \frac{\infty}{\infty} = 1 \implies 2 = 1. \quad (1)$$

We can structure this absurd result in the following syllogism:

1. If infinity exists in a field (i.e. exists as a real or rational number), then $1 = 2$

2. But $1 \neq 2$

3. Therefore, infinity does not exist in a field.

Mathematicians do not like contradictions (unless it helps with their proofs), which is why there seems to be consensus that the symbol ∞ does not *refer to a number at all.* For many mathematicians, the symbol ∞ simply refers to the upper bound of every subset of the (extended) real number system.

[6] A sequence of all zeros is clearly absolutely convergent

[7] The extended reals include the real numbers plus the *symbols* (not numbers) $+\infty$ and $-\infty$; they do not form a field [11]

As we shrink our intervals down to zero measure (the "length" of a point): as $k \to \infty$, our intervals become smaller but we have more of them. We can make these intervals as small as we would like and still have a finite volume (hence the idea of limits), however as soon as these intervals reach *exactly* zero size we have problems like the one above. This forms what I call the "Zero Dilemma" [8]. One can choose from the following:

1. The symbol 0 does not refer to nothingness, but rather some quantity that, when added an infinite number of times, forms a nonzero number (i.e. $\sum_{i=1}^{\infty} 0 = 2$)

2. 0 refers to nothingness, but infinity is "powerful" enough to give you something from nothing (i.e. $\infty \cdot 0 = 2$)

3. ∞ is not a "number" and shouldn't be used as such - it is simply an idea, useful as an upper bound.

I hope the reader can understand why myself (and most mathematicians) take the third option of the zero dilemma: I personally would much rather ban infinity from field operations than either redefine zero (to not mean "nothing") or to allow what appears to be metaphysical absurdities like a violation of Leibniz's' principle of sufficient reason (PSR) [7]. Indeed we can understand why 0 was excluded from having a multiplicative inverse for fields - it causes more problems than it solves. Intuitively, I think this makes sense: we can take the "opposite" of any finite quantity and get another finite quantity. But the opposite of nothingness could be any*thing* [9]

5 Discrete Ontology

References

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[8] Not to be confused with Kotaro Uchikoshi's excellent game "Zero Time Dilemma"

[9] Recall the various absurdities that come from dividing by zero, and why mathematicians leave division by zero as "undefined" - attempting to define it as a fixed value in one situation causes more problems and contradictions in *other* situations.

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