Continuity and completeness of strongly independent preorders

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Abstract

A strongly independent preorder on a possibly infinite dimensional convex set that satisfies two of the following conditions must satisfy the third: (i) the Archimedean continuity condition; (ii) mixture continuity; and (iii) comparability under the preorder is an equivalence relation. In addition, if the preorder is nontrivial (has nonempty asymmetric part) and satisfies two of the following conditions, it must satisfy the third: (i′) a modest strengthening of the Archimedean condition; (ii) mixture continuity; and (iii′) completeness. Applications to decision making under conditions of risk and uncertainty are provided.

1 Introduction and main results

The completeness axiom of expected utility has long been regarded as dubious, while the usual continuity axioms are typically seen as innocuous. However, given a strongly independent preorder on a convex set, we show that the standard Archimedean and mixture continuity axioms together imply that the possibilities for incompleteness are highly restricted, in a sense made precise below. In particular, they rule out the most natural preference structures for agents who find they cannot exactly compare two alternatives. If the Archimedean axiom is slightly strengthened in a natural direction, the room for incompleteness vanishes entirely: the preorder must be complete. These claims generalize results of Aumann (1962) and Dubra (2011).

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In more detail, let X be a nonempty convex set, and $\succsim_X$ a preorder (a reflexive, transitive binary relation) on X. Consider the following axioms. The first is the standard strong independence axiom.

(SI) For $x, y, z \in X$ and $\alpha \in (0, 1)$,

$$x \succsim_X y \iff \alpha x + (1 - \alpha)z \succsim_X \alpha y + (1 - \alpha)z.$$

Thus $\succsim_X$ is an 'SI preorder'. We will be considering the following three Archimedean or continuity axioms.

(Ar) For x, y, $z \in X$, if $x \succ_X y \succ_X z$, then $(1 - \epsilon)x + \epsilon z \succ_X y$ for some $\epsilon \in (0, 1)$.

(Ar$^+$) For $x, y, z \in X$, if $x \succ_X y$, then $(1 - \epsilon)x + \epsilon z \succ_X y$ for some $\epsilon \in (0, 1)$.

(MC) For x, y, $z \in X$, if $\epsilon x + (1 - \epsilon)y \succ_X z$ for all $\epsilon \in (0, 1]$, then $y \succsim_X z$.

The axiom Ar is weaker than, but for SI preorders equivalent to, the standard Archimedean axiom introduced by Blackwell and Girshick (1954).[1] It is weaker than the axiom Ar$^+$, essentially introduced by Aumann (1962).[2] But both Ar and Ar$^+$ express a similar heuristic. Suppose x is strictly preferred to y, and z is some third alternative. Then Ar says that z cannot be so radically worse than y that a sufficiently small chance of z would disturb the original preference. The axiom Ar$^+$ extends this by replacing 'worse than' with 'worse than or incomparable with'. For SI preorders, MC is equivalent to the 'mixture-continuity' axiom of Herstein and Milnor (1953), that $\{\alpha \in [0, 1] \colon \alpha x + (1 - \alpha)y \succsim_X z\}$ is closed in $[0, 1]$. The displayed formulation is especially normatively natural, and suggests that MC should be seen as just as much an Archimedean condition as Ar and Ar$^+$.

The final two axioms restrict the possibilities for incomparability. Say that two members x and y of X are *comparable* if $x \succsim_X y$ or $y \succsim_X x$. They are *incomparable* if they are not comparable. They have a common upper bound if $z \succsim_X x$ and $z \succsim_X y$ for some $z \in X$, and similarly for common lower bound. The next axiom is nonstandard, while the last is the standard completeness axiom.

(Eq) Comparability is an equivalence relation.

[1] See Proposition 6 below.

[2] The axiom Aumann actually discusses is $\epsilon_0 x + (1 - \epsilon_0)z \succ_X y \Rightarrow \epsilon x + (1 - \epsilon)z \succ_X y$ for all ϵ close enough to ϵ_0, but for SI preorders, this is equivalent to Ar$^+$.

(C) All members of X are comparable.

Note that Eq is equivalent to the claim that pairs of incomparable elements have neither a common upper bound nor a common lower bound. This is a demanding requirement. In realistic cases where an agent finds it hard to compare two alternatives, she will find it easy to imagine an alternative that she finds superior to both, and also one she finds inferior.

To state our main result, say that $\succsim_X$ is *nontrivial* if it has a nonempty strict part; that is, for some x, $y \in X$, $x \succ_X y$.

Theorem 1. *For any SI preorder $\succsim_X$ on a convex set X:*
(1) Any two of the following imply the third: MC, Ar and Eq.
(2) For nontrivial $\succsim_X$, any two of the following imply the third: MC, Ar$^+$, and C.

The conditions in (1) do not suffice for C for trivial or nontrivial SI preorders, and nontriviality is essential to (2):

Example 2. (a) Let X contain at least two elements. Set $x \succsim_X y \Leftrightarrow x = y$. Then $\succsim_X$ is trivial, Ar, Ar$^+$ and MC hold, but C fails.
(b) Let $X = \mathbb{R}^2$. Set $x \succsim_X y \Leftrightarrow x_1 \geq y_1 \wedge x_2 = y_2$. Then $\succsim_X$ is nontrivial, Ar, MC and Eq hold, but Ar$^+$ and C fail.

Theorem 1 has several precedents. In a seemingly overlooked observation, Aumann (1962) claimed without proof that MC and Ar$^+$ imply Eq. Thus both parts of the theorem strengthen his claim. Aumann also claimed that when $X = V$ is a finite dimensional vector space, MC and Ar$^+$ imply that X may be written as the direct sum of two subspaces such that two elements are comparable if and only if their second coordinates are identical. The following strengthens this claim, by dropping finite dimensionality and weakening Ar$^+$ to Ar.

Corollary 3. *Suppose $\succsim_X$ is an SI preorder satisfying Ar and MC, and that $X = V$ is a vector space. Then $V = V_1 \oplus V_2$, and v, $w \in V$ are comparable if and only if $v_2 = w_2$.*

In the case where X is the set of probability functions on a given finite set, and thus can be identified with the standard simplex of a finite dimensional vector space, the second part of Theorem 1 was proved by Dubra (2011), building on Schmeidler (1971). Dubra's proof makes essential use of finite dimensionality. But placing no restrictions on the dimension of X allows for considerably broader applications,[3] including general sets of probability measures.

Schmeidler's result was that if a nontrivial preorder on a connected topological set has closed weak upper and lower contour sets, and open strict upper and lower contour sets, it must be complete. The axioms we discuss are purely algebraic, making them applicable to cases in which X is not naturally equipped with a topology. Our proof of Theorem 1 is purely algebraic. Indeed the main technical tool, stated in Theorem 7 below, states equivalences between the three continuity conditions and conditions involving algebraic openness or closedness in ambient vector spaces.

1.1 Discussion

The abstract structure of incomplete SI preorders on convex sets has been discussed, but the relevance of Theorem 1 perhaps has more to do with its compatibility with the typical concrete settings that are used to represent objective risk and subjective uncertainty. To illustrate, let Y be an arbitrary set, Y_c be a compact metric space, and Y_m an arbitrary measurable space; these are typical consequence spaces. Let $P(Y)$ be the set of finitely supported probability measures on Y, $P(Y_c)$ be the set of Borel probability measures on Y_c, and $P(Y_m)$ be an arbitrary convex set of probability measures on Y_m. These are obviously all convex sets, and cover typical cases involving objective risk. Let S_0 and S be a finite and arbitrary sets of states of nature respectively. Then $P(Y)^{S_0}$ is the set of Anscombe-Aumann 'horse lotteries'. Here, members of S_0 are bearers of subjective uncertainty, while the outcomes of horse lotteries are 'roulette lotteries' involving objective risk. For any x, $y \in P(Y)^{S_0}$ and $\alpha \in [0, 1]$, $\alpha x + (1 - \alpha)y \in P(Y)^{S_0}$ is defined by setting $(\alpha x + (1 - \alpha)y)(s) = \alpha x(s) + (1 - \alpha)y(s)$ for any $s \in S_0$, making $P(Y)^{S_0}$ into a convex set. Finally, the set Y^S is the set of Savage-acts associating states of nature with consequences; states of nature continue to be the bearers of subjective uncertainty, but no objective risk is modelled. The space Y^S is not naturally a convex set, but given a preorder on Y^S that satisfies reasonably modest axioms, Y^S can be endowed with convex structure; see for example Ghirardato, Maccheroni, Marinacci and Siniscalchi (2003). The importance of allowing X to be infinite dimensional can be seen from the fact that none of these typical domains can be identified with a finite dimensional X. There are many works discussing incomplete SI preorders in the settings just mentioned. Some focus only on SI strict partial orders,

[3] Recall that the dimension of a convex set X is the dimension of $\mathrm{Span}(X - X)$, or, equivalently, the dimension of the smallest affine space containing X.

but these are consistent with our model as they can be seen as studies of the class of SI preorders that are compatible with the partial orders.[4]

Given Theorem 7, it is natural to think of Ar and Ar^+ as 'open' conditions, and MC as a 'closed' condition. Both styles of condition have been used extensively in discussions of incomplete SI preorders on the kinds of convex sets just described. In almost every case we know of,[5] the open conditions are at least as strong as Ar in the given model, and the closed conditions are at least as strong as, and typically much stronger than, MC.[6] Thus Theorem 1 has considerable relevance.

The continuity conditions in the literature just mentioned are typically presented without discussion, and are sometimes said just to be 'technical assumptions'. But it is not so clear what this means. Each of our three continuity conditions expresses a normatively natural idea that might nevertheless be questioned, but so too are conditions like completeness and strong independence. Both Ar^+ and MC express different ways of ruling out infinitesimal value differences, and since this basic idea is so widely accepted in discussions of complete SI preorders (when all three continuity conditions are equivalent), it is rather remarkable that ruling out infinitesimal value differences across the board forces one to accept the heavily criticized completeness axiom. Thus we suggest that Theorem 1 deserves to be seen as an impossibility result. We end by canvassing some possible responses.

First, one one could try to argue that of the two styles of continuity condition, open and closed, one is more normatively or descriptively plausible than the other. But we side with Aumann (1962) and Manzini and Mariotti (2008) in thinking that Ar^+ and MC are comparably plausible in the abstract. Thus such an argument would have to pay attention to the specific interpretation of the preorder in question. Second, for applications, one could try to develop mirror theories for open and closed conditions, and analyze

[4]This class is always nonempty, as the reflexive closure of an SI strict partial order is an SI preorder whose asymmetric part is identical to the partial order.

[5]The exceptions are Aumann (1962), who imposes a continuity condition that is strictly weaker than both Ar and MC, and Seidenfeld, Schervish and Kadane (1995) who impose a similar condition in the Anscombe-Aumann setting.

[6]For open conditions, see Bewley (2002); Manzini and Mariotti (2008); Galaabaatar and Karni (2012, 2013); Evren (2014); McCarthy, Mikkola, and Thomas (2017b). For closed conditions, see Shapley and Baucells (1998); Ghirardato *et al* (2003); Dubra, Maccheroni and Ok (2004); Nau (2006); Baucells and Shapley (2008); Evren (2008); Kopylov (2009); Gilboa, Maccheroni, Marinacci and Schmeidler (2010); Danan, Guerdjikova and Zimper (2012); Ok, Ortoleva and Riella (2012); McCarthy, Mikkola, and Thomas (2017a). Without any continuity condition, one faces incomplete analogues of the situation first discussed by Hausner and Wendel (1952), discussed further in McCarthy, Mikkola, and Thomas (2017c).

the sensitivity of their implications to these conditions. Third, one could choose between the two styles of conditions on the basis of the convenience of the representation theorems they support (compare Evren, 2014). Fourth, to try to bypass the impossibility, one could adopt a nonstandard model of the relationship between strict partial orders and associated preorders (see further Karni, 2007; Galaabaatar and Karni, 2012). Fifth, one could argue that in some settings, the case for both styles of condition is strong enough that they provide a novel normative argument for completeness (in a different context, compare Broome, 1999), or even a new argument against strong independence.

2 Proofs

2.1 Preliminaries

When X is a nonempty convex set of a vector space V, the following provides a useful representation of the subspace $\mathrm{Span}(X - X)$.

Lemma 4. *Let X be a nonempty convex subset of a vector space V. Then*

$$\mathrm{Span}(X - X) = \{\lambda(x - x') \colon x, x' \in X, \lambda > 0\}.$$

Proof. The right-hand side is clearly included in the left. For the converse, let $v \in \mathrm{Span}(X - X)$. The case $v = 0$ is trivial, so let $v = \sum_{i=1}^{n} \lambda_i (x_i - x_i')$ with x_i, $x_i' \in X$, $\lambda_i \neq 0$ for all i, and $n \in \mathbb{N}$. Exchange x_i with x_i' if necessary to have each $\lambda_i > 0$. Set $\lambda = \sum_{i=1}^{n} \lambda_i$, $x = \frac{1}{\lambda} \sum_{i=1}^{n} \lambda_i x_i$, and $x' = \frac{1}{\lambda} \sum_{i=1}^{n} \lambda_i x_i'$. Then x, $x' \in X$ by convexity, and $v = \lambda(x - x')$ as needed. □

Recall that a vector preorder $\succsim_V$ is a preorder on a vector space V such that for any v, w, $u \in V$ and $\alpha > 0$, $v \succsim_V w$ implies $\alpha v + u \succsim_V \alpha w + u$. We define $\{\succsim_V 0\} \coloneqq \{v \in V \colon v \succsim_V 0\}$, and similarly $\{\succ_V 0\}$, $\{\sim_V 0\}$. We also define $\precsim_X$, $\precsim_V$ and sets such as $\{\precsim_V 0\}$ in the obvious way; for example, $x \precsim_X y \Leftrightarrow y \succsim_X x$.

Proposition 5. *Let $\succsim_X$ be a SI preorder on a nonempty convex subset X of a vector space V. For any $v, w \in V$, define $\succsim_V$ by*

$$v \succsim_V w \iff v - w = \lambda(x - y) \textit{ for some } x, y \in X, \lambda > 0 \textit{ with } x \succsim_X y.$$

Then $\succsim_V$ is a vector preorder on V, and $\succsim_X$ is its restriction to $X \times X$. Moreover, $\succsim_X$ is complete if and only if $\succsim_V$ is complete on $\mathrm{Span}(X - X) = \mathrm{Span}\{\succsim_V 0\}$.

Proof. Clearly $\succsim_V$ is reflexive. Suppose $u \succsim_V v$ and $v \succsim_V w$. Then for some λ, $\mu > 0$ and $x_1, x_2, y_1, y_2 \in X$, we have $u - v = \lambda(x_1 - x_2)$, $v - w = \mu(y_1 - y_2)$, $x_1 \succsim_X x_2$ and $y_1 \succsim_X y_2$. The former imply $u - w = (\lambda + \mu)\left(\frac{\lambda x_1 + \mu y_1}{\lambda + \mu} - \frac{\lambda x_2 + \mu y_2}{\lambda + \mu}\right)$; the latter and applications of SI imply $\frac{\lambda x_1 + \mu y_1}{\lambda + \mu} \succsim_X \frac{\lambda x_2 + \mu y_2}{\lambda + \mu}$. It follows that $u \succsim_V w$, establishing transitivity of $\succsim_V$, so $\succsim_V$ is a preorder on V.

Suppose $v \succsim_V w$, $u \in V$, $\alpha > 0$. For some $\lambda > 0$, x, $y \in X$, we have $v - w = \lambda(x - y)$ with $x \succsim_X y$. Then $(\alpha v + u) - (\alpha w + u) = \alpha\lambda(x - y)$. This implies $\alpha v + u \succsim_V \alpha w + u$, so $\succsim_V$ is a vector preorder.

Clearly $x \succsim_X y$ implies $x \succsim_V y$. Conversely, suppose $x \succsim_V y$ for some x, $y \in X$. Then for some x', $y' \in X$, $\lambda > 0$, $x - y = \lambda(x' - y')$ with $x' \succsim_X y'$. The former implies $\alpha x + (1-\alpha)y' = \alpha y + (1-\alpha)x'$ where $\alpha := \frac{1}{1+\lambda}$; the latter and SI imply $\alpha x + (1-\alpha)x' \succsim_X \alpha x + (1-\alpha)y'$. Substituting, then using SI again, yields $x \succsim_X y$, hence $\succsim_X$ is the restriction of $\succsim_V$.

The completeness claim follows from Lemma 4. □

A convex set can always be embedded in a vector space, so without loss of generality we henceforth assume that $(X, \succsim_X)$ and $(V, \succsim_V)$ are as in Proposition 5. It clearly follows that $\mathrm{Span}\{\succsim_V 0\} \subset \mathrm{Span}(X - X)$.

2.2 Algebraic conditions

The following assembles facts about the Archimedean conditions.

Proposition 6.
(a) Ar is equivalent to each of the following two conditions.

$$\text{For all } x, y, z \in X\colon x \succ_X y \text{ and } y \succ_X z \implies (1-\epsilon)x + \epsilon z \succ_X y \text{ and } y \succ_X \epsilon x + (1-\epsilon)z \text{ for some } \epsilon \in (0,1). \tag{1}$$

$$\text{For all } v, w \in V\colon\ v, w \succ_V 0 \implies v \succ_V \epsilon w \text{ for some } \epsilon > 0. \tag{2}$$

(b) If one of Ar, (1), (2) and Ar$^+$ holds for some ϵ_0, it holds for all $\epsilon \in [0, \epsilon_0]$.
(c) Ar$^+$ implies Ar (but not conversely).

Proof. (a) To show (2) ⇒ (1), assume (2) and suppose $x \succ_X y \succ_X z$. Set $x_\alpha := (1-\alpha)(x-y) + \alpha(z-y)\ \forall \alpha \in [0,1]$. Then $x_0 \succ_V 0 \succ_V x_1$, and from (2) one deduces that $(1-\epsilon)x_0 \succ_V \epsilon(-x_1)$ for small $\epsilon > 0$, i.e. $(1-\epsilon)(x-y) \succ_V \epsilon(y-z)$, implying $(1-\epsilon)x + \epsilon z \succ_X y$ for some $\epsilon \in (0,1)$. From (2) one also deduces $(1-\epsilon)(-x_1) \succ_V \epsilon x_0$ for small $\epsilon > 0$, hence $y \succ_X \epsilon x + (1-\epsilon)z$ for some $\epsilon \in (0,1)$. This establishes (1), and clearly (1) ⇒ Ar.

To show Ar ⇒ (2), assume Ar and suppose v, $w \succ_V 0$. By Proposition 5, $v = \lambda(x-y)$, $w = \mu(s-t)$ for some $x \succ_X y$, $s \succ_X t$, λ, $\mu > 0$. By SI, $\frac{1}{2}(x+s) \succ_X \frac{1}{2}(y+s) \succ_X \frac{1}{2}(y+t)$. By Ar, for some $\epsilon \in (0,1)$, $(1-\epsilon)\frac{1}{2}(x+$

$s) + \epsilon\frac{1}{2}(y+t) \succ_X \frac{1}{2}(y+s)$. This implies $(1-\epsilon)(x-y) \succ_V \epsilon(s-t)$, hence $v \succ_V \frac{\epsilon}{1-\epsilon} \cdot \frac{\lambda}{\mu} w$, establishing (2).

(b) Assume Ar^+, and suppose $x \succ_X y$ and $(1-\epsilon_0)x + \epsilon_0 z \succ_X y$ for some $\epsilon_0 > 0$. These and SI imply $(1-\epsilon)x + \epsilon z \succ_X y$ for all $\epsilon \in [0, \epsilon_0]$. The claims about Ar and (1) are proved similarly, and the claim about (2) is clear.

(c) That Ar^+ implies Ar is obvious. The failure of the converse is shown by Example 2(b). □

Recall that a subset S of a vector space W is *algebraically open in* W if for all $v \in S$, $w \in W$, $v + \epsilon w \in S$ for all sufficiently small $\epsilon > 0$. S is *algebraically closed* if for all v, $w \in W$: $(1-\alpha)v + \alpha w \in S$ for all $\alpha \in (0,1] \Rightarrow v \in S$. Given v, $w \in W$, we sometimes write $[v, w) \subset W$ for the line segment $\{(1-\alpha)v + \alpha w \colon \alpha \in [0,1)\}$. Then S is algebraically closed if $w \in S$ whenever $[v, w) \subset S$. The following connects these algebraic notions with our continuity axioms.

Theorem 7.
(a) Ar holds if and only if $\{\succ_V 0\}$ *is algebraically open in* $\mathrm{Span}\{\succ_V 0\}$.[7]
(b) Ar^+ *holds if and only if* $\{\succ_V 0\}$ *is algebraically open in* $\mathrm{Span}(X - X)$.
(c) MC holds if and only if $\{\succsim_V 0\}$ *is algebraically closed.*

Proof. (a) Suppose $\succsim_X$ satisfies Ar. Let $v \in \{\succ_V 0\}$, $w \in \mathrm{Span}\{\succ_V 0\}$. Clearly we can write $w = a - b$ where each of a and b is either 0 or in $\{\succ_V 0\}$. Since $\succsim_V$ is a vector preorder, $v + \epsilon_1 a \succ_V 0$ for all $\epsilon_1 > 0$. By Proposition 6(a)(2), we have $v \succ_V \epsilon_2 b$ for all sufficiently small $\epsilon_2 > 0$. These imply $v + \epsilon w \succ_V 0$ for all small enough $\epsilon > 0$. This shows that $\{\succ_V 0\}$ is algebraically open in $\mathrm{Span}\{\succ_V 0\}$.

Conversely, suppose $\{\succ_V 0\}$ is algebraically open in $\mathrm{Span}\{\succ_V 0\}$, and that v, $w \succ_V 0$. Since $-w \in \mathrm{Span}\{\succ_V 0\}$, $v + \epsilon(-w) \in \{\succ_V 0\}$ for all sufficiently small $\epsilon > 0$. By Proposition 6, $\succsim_X$ satisfies Ar.

(b) Assume Ar^+. Let $c \in \{\succ_V 0\}$, $v \in \mathrm{Span}(X - X)$. By Lemma 4 and Proposition 5, $c = \alpha(x-y)$, $v = \beta(p-q)$ for some $x, y, p, q \in X$ with $x \succ_X y$, α, $\beta > 0$. Then SI, Ar^+ and Proposition 6(b) imply $(1-\epsilon)\frac{1}{2}(x+q) + \epsilon\frac{1}{2}(x + p) \succ_X \frac{1}{2}(y+q)$, hence $(x-y) + \epsilon(p-q) \succ_V 0$, for all for all sufficiently small $\epsilon > 0$. Consequently $c + \epsilon\frac{\alpha}{\beta}v \succ_V 0$ for small enough $\epsilon > 0$.

Conversely, suppose $\{\succ_V 0\} \cap \mathrm{Span}(X - X)$ is algebraically open in $\mathrm{Span}(X - X)$. Suppose $x \succ_X y$ and $z \in X$. Then $x - y \in \{\succ_V 0\}$ and $z - y \in \mathrm{Span}(X - X)$, hence for some $\epsilon \in (0,1)$, $(1-\epsilon)(x-y) + \epsilon(z-y) \in \{\succ_V 0\}$, implying $(1-\epsilon)x + \epsilon z \succ_X y$.

(c) This is proved in McCarthy *et al* (2017a, Thm. 2.2). □

[7]The latter condition is often phrased as '$\{\succ_V 0\}$ is relatively algebraically open.'

Corollary 8. $\precsim_X$ *satisfies MC if and only if* $\succsim_X$ *satisfies MC.*

Proof. The set $\{\precsim_V 0\} = -\{\succsim_V 0\}$ is algebraically closed if and only if $\{\succsim_V 0\}$ is. □

2.3 Proof of Theorem 1 and Corollary 3.

Define $x \bowtie_X y \Leftrightarrow x \succsim_X y \vee y \succsim_X x$; that is, x and y are comparable.

Lemma 9. *Let* $x_i \bowtie_X y_i$, $\alpha_i \in \mathbb{R}$ $(i = 1, \ldots, n)$. *Then* $\sum_{i=1}^n \alpha_i(x_i - y_i) = \alpha(p - q)$, *where* $\alpha > 0$ *and* $p \bowtie_X z \bowtie_X q$ *for some* $z \in X$.

Proof. 1° *Case* $n = 2$. Without loss of generality, assume $\alpha_2, \alpha_1 > 0$; $\alpha_2 \geq \alpha_1$; and $\alpha_2 = 1$. Set $p_k = x_k$, $q_2 = y_2$, $q_1 = (1 - \alpha_1)x_1 + \alpha_1 y_1$ to have $\sum_{i=1}^2 \alpha_i(x_i - y_i) = p_1 - q_1 + p_2 - q_2$. Clearly $x_1 \bowtie_X q_1$. Set $p := \frac{1}{2}(p_1 + p_2)$, $q := \frac{1}{2}(q_1 + q_2)$, $z := \frac{1}{2}(q_1 + p_2)$, to have $\sum_{i=1}^2 \alpha_i(x_i - y_i) = 2(p - q)$ and $p \bowtie_X z \bowtie_X q$.

2° *General case.* Without loss of generality, assume $x_i \succsim_X y_i$ for all i. If $\alpha_1, \alpha_2 > 0$, then

$$\alpha_1(x_1 - y_1) + \alpha_2(x_2 - y_2) = \alpha'(x_1' - y_1')$$

where $\alpha' = \alpha_1 + \alpha_2 > 0$, $x_1' = (1 - \alpha'')x_1 + \alpha'' x_2$, $y_1' = (1 - \alpha'')y_1 + \alpha'' y_2$, $\alpha'' = \alpha_2/(\alpha_1 + \alpha_2)$, and hence $x_1' \succsim_X y_1'$, by SI. This way, by induction, we combine all terms having $\alpha_i > 0$. If all the α_i are strictly positive (or similarly, strictly negative), the result is immediate. Otherwise, similarly combine the terms with $\alpha_i < 0$, then apply 1° to the two. □

Lemma 10. *Let* W *be any vector space, and* $S \subset W$. *If* S *is algebraically open, then* $W \setminus S$ *is algebraically closed. The converse holds if* S *is convex.*

Proof. The claim is Ok (2007, Exercise G.1.5.30). The first claim is clear. For the converse, suppose S is not algebraically open. Then for some $v \in S, w \in W$, $\{(1 - \alpha)v + \alpha w \colon \alpha \in [0, \epsilon]\} \not\subset S$ for all $\epsilon > 0$. If S is convex, this implies $\{(1 - \alpha)v + \alpha w \colon \alpha \in (0, \epsilon_0]\} \subset W \setminus S$ for some $\epsilon_0 > 0$. If $W \setminus S$ is algebraically closed, we have $v \in W \setminus S$, a contradiction. □

Recall that $\succsim_X$ is nontrivial if $\succ_X \neq \varnothing$.

Proposition 11.
(a) *The following conditions are equivalent.*

(i) Both Ar and MC hold.

(ii) $\{\succsim_V 0\} = \{\sim_V 0\} + [0, \infty)c$ *for some* $c \succsim_V 0$.

(iii) $V = V_1 \oplus V_2$ *where* $V_1 = \{\sim_V 0\} + \mathbb{R}c$ *for some* $c \succsim_V 0$*, and* $v \bowtie_V w \Leftrightarrow v_2 = w_2$.

(iv) Ar holds and $\succsim_V$ *is complete on* $\mathrm{Span}\{\succsim_V 0\}$.

(b) $\succsim_V$ *is complete on* $\mathrm{Span}\{\succsim_V 0\}$ *if and only if* $\bowtie_X$ *is an equivalence relation.*
(c) *If* $\succsim_X$ *is nontrivial, then both Ar*$^+$ *and MC hold if and only if* $\succsim_X$ *is complete and Ar holds.*
(d) *If* $\succsim_X$ *is nontrivial, then* $\mathrm{Span}\{\succ_V 0\} = \mathrm{Span}\{\succsim_V 0\}$.

Proof. (a) We show (iii) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (i) $\Leftrightarrow$ (iv). It is clear that (iii) $\Leftrightarrow$ (ii). We obtain (ii) $\Rightarrow$ (i) by using Proposition 6(a)(2) for Ar and by Theorem 7(c) for MC. Conversely, assume (i). If $\succ_V = \varnothing$, (ii) is immediate, so pick $c \succ_V 0$. Suppose for a contradiction $\{\succsim_V 0\} \neq \{\sim_V 0\} + [0,\infty)c =: Q$. Clearly Q is contained in $\{\succsim_V 0\}$, so there exists $d \succ_V 0$ such that $d \notin Q$. By Ar and Theorem 7(a), since $\{\succ_V 0\}$ is algebraically open in $\mathrm{Span}\{\succ_V 0\}$, $\alpha(-c) + (1-\alpha)d \succ_V 0$ for sufficiently small $\alpha > 0$. Since $\succsim_V$ is a vector preorder, the set of such α is an interval and is bounded above by 1. Let α_0 be its supremum, and set $e := \alpha_0(-c) + (1-\alpha_0)d$. By MC and Theorem 7(c), $e \succsim_V 0$, and hence $\alpha_0 \in (0,1)$. By Ar and Theorem 7(a), $e \not\succ_V 0$. Hence $e \sim_V 0$, implying $(1-\alpha_0)d \sim_V e + \alpha_0 c \in Q$, a contradiction. Thus (ii) $\Leftrightarrow$ (i).

Assume (i). Since (i) implies (ii), we have $\{\succsim_V 0\} = \{\sim_V 0\} + [0,\infty)c$ and $\mathrm{Span}\{\succsim_V 0\} = \{\sim_V 0\} + \mathbb{R}c$ for some $c \succsim_V 0$, so $\succsim_V$ is complete on $\mathrm{Span}\{\succsim_V 0\}$, establishing (iv). Conversely, assume (iv), and for a contradiction suppose MC does not hold. By Theorem 7(c), there is some $[a,b) \subset \{\succsim_V 0\}$ with $b \not\succsim_V 0$. Clearly $b \in \mathrm{Span}\{\succsim_V 0\}$, so by completeness of $\succsim_V$ on $\mathrm{Span}\{\succsim_V 0\}$, $-b \succ_V 0$. This is a contradiction, since Ar and Theorem 7(a) imply $\{\succ_V 0\}$ is algebraically open in $\mathrm{Span}\{\succsim_V 0\}$, but $[-a,-b) \subset \{\precsim_V 0\}$. Hence (i) $\Leftrightarrow$ (iv).

(b) Suppose $\succsim_V$ is complete on $\mathrm{Span}\{\succsim_V 0\}$. Then $x \bowtie_X y$ and $y \bowtie_X z \Rightarrow x - y, y - z \in \pm\{\succsim_V 0\} \subset \mathrm{Span}\{\succsim_V 0\}$, implying $x - z \in \mathrm{Span}\{\succsim_V 0\}$, so $x \bowtie_X z$, implying $\bowtie_X$ is transitive, and hence an equivalence relation as it is clearly reflexive and symmetric.

Conversely, suppose $\bowtie_X$ is an equivalence relation. Let $v \in \mathrm{Span}\{\succsim_V 0\} = \mathrm{Span}\{x - y \colon x \succsim_X y\}$, by Proposition 5. Then $v = \sum_{i=1}^n \alpha_i(x_i - y_i)$ for some $\alpha_i \in \mathbb{R}$, $x_i \succsim_X y_i$. By Lemma 9, $v = \alpha(p - q)$, where $\alpha > 0$, $p, q \in X$ with $p \bowtie_X z$ and $z \bowtie_X q$ for some $z \in X$. Transitivity of $\bowtie_X$ implies $p \bowtie_X q$, hence $v \bowtie_V 0$, implying that $\succsim_V$ is complete on $\mathrm{Span}\{\succsim_V 0\}$.

(c) Assume $\succsim_X$ is nontrivial. Suppose $\succsim_X$ satisfies Ar$^+$ and MC. By Proposition 6(c), $\succsim_X$ satisfies Ar. By Theorem 7(b), $\{\succ_V 0\}$ is algebraically open in $\mathrm{Span}(X - X)$. Let $w \in X - X$, and by nontriviality, pick $v \in \{\succ_V 0\}$.

Then $v + \epsilon w \in \{\succ_V 0\}$ for sufficiently small $\epsilon > 0$, so $w \in \text{Span}\{\succ_V 0\}$; that is, $X - X \subset \text{Span}\{\succ_V 0\}$. By (a), $\succsim_V$ is complete on $X - X$, hence $\succsim_X$ is complete.

Conversely, when $\succsim_X$ is complete and satisfies Ar, it must satisfy Ar$^+$ by SI. By Proposition 5, $\succsim_V$ is complete on $\text{Span}\{\succsim_V 0\}$, hence by (a), $\succsim_X$ satisfies MC.

(d) The left-hand side is clearly contained in the right. But for nontrivial $\succsim_V$, $\{\sim_V 0\} \subset \text{Span}\{\succ_V 0\}$, hence $\text{Span}\{\succsim_V 0\} \subset \text{Span}\{\succ_V 0\}$. □

Proof of Theorem 1. (1) If $\succsim_X$ is trivial, clearly all three of Ar, MC, and Eq hold, so suppose $\succsim_X$ is nontrivial. Now if $\succsim_X$ satisfies Ar and MC, Eq must hold by Proposition 11(a, b).

Assume Eq. Then $\succsim_V$ is complete on $\text{Span}\{\succsim_V 0\}$ by Proposition 11(b). If Ar holds, then so does MC, by Proposition 11(a)(iv, i). Assume MC. Then $\{\precsim_V 0\}$ is algebraically closed, by Corollary 8 and Theorem 7(c), hence $\{\succ_V 0\}$ is algebraically open in $\text{Span}\{\succsim_V 0\}$, by Lemma 10 and the completeness on $\text{Span}\{\succsim_V 0\}$. But $\text{Span}\{\succsim_V 0\} = \text{Span}\{\succ_V 0\}$, by Proposition 11(d), hence Ar holds, by Theorem 7(a).

(2) By Proposition 11(c), if Ar$^+$ and MC hold, then so does C. By Proposition 6(c) and Proposition 11(c), if Ar$^+$ and C hold, then so does MC. Finally, MC and C imply Ar by Theorem 1(1), and C and Ar imply Ar$^+$ by SI. □

Proof of Corollary 3. This is immediate from Proposition 11(a). □

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