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Author(s): José M. Mendez and Francisco Salto

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JOSÉ M. MENDEZ and FRANCISCO SALTO

A NATURAL NEGATION COMPLETION OF URQUHART'S MANY-VALUED LOGIC C[1]

A. Urquhart introduced in [7] a positive propositional logic called C as a previous step in defining a relational semantics for Lukasiewicz's infinite-valued logic Lw. The logic C can intuitively be described as the positive fragment of Dummett's well-known system LC (see [1]) minus the contraction axiom. There are (essentially) two possibilities for extending C with a negation connective without collapsing it into classical logic or Dummett's LC. The first is a kind of semiclassical negation: the result is Lw; the second – the alternative that we consider in this paper – is a semi-intuitionistic negation.

Urquhart's C plus this semi-intuitionistic negation results in a system [let us use CI to refer to it] that can intuitively be described as Dummett's LC without the contraction and reductio axioms $[(A \to (A \to B)) \to (A \to B), (A \to \neg A) \to \neg A$, respectively]. And the system CI is, we think, interesting from two different points of view:

(a) As suggested by Urquhart, multivalent logics can be understood "as the logics of inference from multisets" ([7], p. 106; see [3] and references there). According to this suggestion, C (as remarked by Urquhart himself) and, so, CI seem more adequate than Lw to this "multiset interpretation".

(b) In the "concluding remarks" of their reference work on contractionless logic [4], Ono and Komori recommend the study of superintuitionistic logics without the contraction axiom. Now, CI is one of the most interesting items in this class.

In what follows we provide Routley–Meyer type relational semantics (see [5]) for CI with negation defined either as a primitive connective or by means of a falsity constant. In this sense, we note that the reader can find in the development of these semantics for CI some technical "detours" not required in the case of the standard semantics: unlike the

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logics Routley–Meyer type semantics were, in principle, designed for, CI only generates consistent theories.

1. URQUHART'S C

Urquhart's C is axiomatized with:

Axioms A1. $(B \to C) \to [(A \to B) \to (A \to C)]$.
A2. $[A \to (B \to C)] \to [B \to (A \to C)]$.
A3. $(A \wedge B) \to A \quad (A \wedge B) \to B$.
A4. $A \to [B \to (A \wedge B)]$.
A5. $A \to (A \vee B) \quad B \to (A \vee B)$.
A6. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$.
A7. $(A \to B) \vee (B \to A)$.

Rule: modus ponens: if $\vdash A$ and $\vdash A \to B$, then $\vdash B$.

2. ROUTLEY–MEYER TYPE SEMANTICS FOR URQUHART'S C

A C-model structure [C-m.s.] is a pair $\langle K, R\rangle$ where K is a set and R is a ternary relation on K subject to the following definitions and postulates for all $a, b, c, d \in K$ with quantifiers ranging over K:

d1. $a \leqslant b =_{\text{def}} \exists x Rxab$.
d2. $R^2abcd =_{\text{def}} \exists x[Rabx \text{ and } Rxcd]$.
P1. $a \leqslant a$.
P2. $a \leqslant b$ and $Rbcd \Rightarrow Racd$.
P3. $R^2abcd \Rightarrow \exists x[Rbcx \text{ and } Raxd]$.
P4. $Rabc \Rightarrow Rbac$.
P5. $Rabc$ and $Rade \Rightarrow b \leqslant e$ or $d \leqslant c$.

A C-model is a triple $\langle K, R, \vDash \rangle$ where $\langle K, R\rangle$ is a C-m.s. and $\vDash$ is a valuation relation from K to the sentences of C satisfying the following conditions for all $a \in K$:

(i) For each propositional variable p and $a, b \in K$, $a \vDash p$ and $a \leqslant b \Rightarrow b \vDash p$.
(ii) $a \vDash A \wedge B$ iff $a \vDash A$ and $a \vDash B$.
(iii) $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$.
(iv) $a \vDash A \to B$ iff for all $b, c \in K$, R abc and $b \vDash A \Rightarrow c \vDash B$.

A formula A is valid iff $a \vDash A$ for all $a \in K$ in all models. In [2] it was shown that A is valid iff A is a theorem of C.

3. THE SYSTEM CI

We add to the sentential language of C the propositional falsity constant F and the axiom:

A8. $F \to A$.

Now, we stipulate,

DEFINITION. $\neg A =_{def} A \to F$.

Note that, for example,

$$(A \to B) \to (\neg B \to \neg A).$$
$$(A \to \neg B) \to (B \to \neg A).$$
$$A \to \neg\neg A.$$
$$\neg A \to (A \to B)$$

are provable.

4. SEMANTICS FOR CI

A CI-model is just as a C-model but with the clause:

(v) For every $a \in K$, $a \nvDash F$

added to the conditions in §2. A formula $\underline{A}$ is CI-valid iff $a \vDash A$ for all $a \in K$ in all models. Semantic consistency is easy [using the results of [2], only A8 has to be proved valid, which is trivial with clause (v)]. As for completeness, we begin with some definitions and then we prove some previous lemmas.

Let us define a *theory* as a set of formulas of CI closed under adjunction and provable entailment [that is, $\underline{a}$ is a theory if whenever $A, B \in a$, then $A \wedge B \in a$, and whenever $A \to B$ is a theorem of CI, if $A \in a$, then $B \in a$]. A theory is *prime* if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$; and *regular* if it contains all theorems of CI. Finally $\underline{a}$ is *consistent* iff the negation of a theorem does not belong to $\underline{a}$. We now define the CI-canonical structure as the pair $\langle K^c, R^c \rangle$ where K^c is the set of all non-null prime consistent theories and R^c is defined on K^c as follows: for all formulas A, B and $a, b, c \in K^c$, $Rabc$ iff $A \to B \in a$ and $A \in b$, then $B \in c$. Now, we prove

LEMMA 1. *If* $\underline{a}$ *is a non-null theory, then* $\underline{a}$ *is regular.*

Proof. Suppose A is a theorem and let $B \in a$. By the theorem $A \to (B \to A)$, $B \to A$ is a theorem. Then, $B \in a$.

LEMMA 2. *If* A *is not a theorem of* CI, *then there is a non-null prime consistent theory* T *which does not contain* A.

Proof. Given that CI is a non-null consistent theory which does not contain A, by Zorn's Lemma there is a maximal non-null consistent theory T without A. If T is not prime, then $B \vee C \in T$, $B \notin T$, $C \notin T$ for some wffs B, C. Define the theories $[T, B] = \{E \mid \exists D(D \in T$ and $(B \wedge D) \to E \in CI\}$, $[T, C] = \{E \mid \exists D(D \in T$ and $(C \wedge D) \to E \in CI)\}$. It is easy to show that $[T, B]$ and $[T, C]$ are non-null theories strictly including T. By the maximality of T, there are three possible situations:

(a) $[T, B]$ and $[T, C]$ are inconsistent.

By definitions, $(B \wedge D) \to \neg E$, $(C \wedge D') \to \neg E' \in CI$ with D, $D' \in T$ and E, E' theorems of CI. Then $[(B \wedge D) \vee (C \wedge D')] \to (\neg E \vee \neg E') \in CI$, and, by distributive properties, $[(B \vee C) \wedge (D \wedge D')] \to (\neg E \vee \neg E') \in CI$. Thus, $\neg E \vee \neg E' \in T$ [since $(B \vee C) \wedge (D \wedge D') \in T$]. But $E \wedge E'$ is a theorem. So, $\neg(\neg E \vee \neg E')$ also is a theorem by the (weak) De Morgan laws. Therefore, T is inconsistent, since $\neg\neg(\neg E \vee \neg E') \in T$ by weak double negation, which is impossible.

(b) $A \in [T, B]$ and $A \in [T, C]$.

By definition, $(B \wedge D) \to A$, $(C \wedge D') \to A \in CI$ for some D, $D' \in T$. Then $[(B \vee C) \wedge (D \wedge D')] \to A \in CI$ and thus $A \in T$ [cfr. the argument in (a) above], which is impossible.

(c) $[T, B]$ is inconsistent and $A \in [T, C]$ or $[T, C]$ is inconsistent and $A \in [T, B]$.

Suppose $[T, B]$ inconsistent and $A \in [T, C]$. By definitions, $(B \wedge D) \to \neg E$, $(C \wedge D') \to A \in CI$ with D, $D' \in T$ and E a theorem of CI. As $\neg E \to G$ [G is any wff] is a theorem of CI, we have $(B \wedge D) \to A \in CI$. Then, a similar argument to that in (a) and (b) above shows that $A \in T$, which is impossible.

The proof that second alternative also leads to contradictions is similar.

Each one (a), (b) and (c) is untenable. Therefore, T is prime, a result which ends the proof of Lemma 2.

LEMMA 3. *Let* $\langle K^c, R^c \rangle$ *be the canonical structure. For all* $a, b \in K^c$, $a \leqslant b$ *iff* $a \subseteq b$.

Proof. Suppose $a \leqslant b$. Then, for some non-null prime consistent theory x, $Rxab$. Now, $A \rightarrow A \in x$. Hence, whenever $A \in a$, $A \in b$, i.e., $a \subseteq b$. Suppose now $a \subseteq b$. It is clear that RCIab [since RCIaa and $a \subseteq b$]. So, $\exists x Rxab$, i.e., $a \leqslant b$. It remains to be proved that x can be extended to a non-null prime consistent theory x' such that $Rx'ab$. Consider the set of all non-null consistent theories y such that $x \subseteq y$ and $Ryab$. By Zorn's Lemma, there is a maximal element x' in this set such that $x \subseteq x'$ and $Rx'ab$. Suppose x' is not prime. Then, $A \vee B \in x'$, $A \notin x'$, $B \notin x'$ for some wffs A, B. As in the proof of Lemma 2, define the non-null thoeries $[x', A]$ and $[x', B]$ strictly including x' [cf. Lemma 2].

By the maximality of x', there are three possible situations:

(a) $[x', A]$ and $[x', B]$ are inconsistent.

Then, x' is inconsistent [cf. Lemma 2].

(b) not-R$[x', A]ab$ and not-R$[x', B]ab$.

By definitions, $(A \wedge E) \rightarrow (C \rightarrow D) \in CI$, $C \in a$, $E \in x'$, $D \notin b$ and $(B \wedge E') \rightarrow (C' \rightarrow D') \in CI$, $C' \in a$, $E' \in x'$, $D' \notin b$ for some wffs C, D, E, C', D', E'. By elementary properties of conjunction and disjunction, $[(A \vee B) \wedge (E \wedge E')] \rightarrow [(C \rightarrow D) \vee (C' \rightarrow D')] \in CI$. Since $(A \vee B) \wedge (E \wedge E') \in x'$, $(C \rightarrow D) \vee (C' \rightarrow D') \in x'$. By the theorem $[(C \rightarrow D) \vee (C' \rightarrow D')] \rightarrow [(C \wedge C') \rightarrow (D \vee D')]$, $(C \wedge C') \rightarrow (D \vee D') \in x'$ whence by $Rx'ab$ and $C \wedge C' \in a$ we have $D \vee D' \in b$. But b is prime, so $D \in b$ or $D' \in b'$ contradicting our hypothesis.

(c) Not-R$[x', A]ab$ and $[x', B]$ is inconsistent or not-R$[x', B]ab$ and $[x', A]$ is inconsistent.

Suppose not-R$[x', A]ab$. By definitions, $(A \wedge E) \rightarrow (C \rightarrow D) \in CI$, $E \in x'$, $C \in a$, $D \notin b$ for some wffs E, C, D. Suppose now $[x', B]$ inconsistent. By definitions, $(B \wedge E') \rightarrow \neg G \in CI$ with $E \in x'$ and G a theorem. But for any wff H, $\neg G \rightarrow H \in CI$. So, $(B \wedge E') \rightarrow (C \rightarrow D) \in CI$. Thus, $[(A \vee B) \wedge (E \wedge E')] \rightarrow (C \rightarrow D) \in CI$ and, hence, $C \rightarrow D \in x'$. By $Rx'ab$, $D \in b$ contradicting our hypothesis.

The proof that the second alternative in untenable in similar.

Therefore, x' is prime, which ends the proof of Lemma 3.

LEMMA 4. *The canonical structure is indeed a model structure.*

Proof. We have to prove that the postulates P1–P5 hold in the canonical structure. Now, P1 and P2 are trivial by Lemma 3; P4 is simple using the theorem A $\rightarrow$ [(A $\rightarrow$ B) $\rightarrow$ B] and P5 can easily be proved with A7 and Lemma 3. So, it remains for us to be prove P3. $R^2abcd \Rightarrow \exists x[Rbcx$ and $Raxd]$. Given Raby and Rycd, we have to show that there is a prime non-null consistent theory x' such that Rbcx' and R$ax'd$. Then, define $x = \{B \mid \exists A[A \in c \text{ and } A \rightarrow B \in b]\}$. It is easy to verify that x is a non-null theory. Rbcx is trivial and Raxd follows easily using the hypothesis and A1. We now prove that x is consistent. Suppose it is not. Then, for some theorem A, $\neg A \in x$. Now, $\neg A \rightarrow (A \rightarrow \neg B)$ [with B a theorem] is a theorem. So, $\neg A \rightarrow (A \rightarrow \neg B) \in a$. By R$axd$, $A \rightarrow \neg B \in d$. But R$dCId$ [RCIdd and P4]. Therefore, $\neg B \in d$ which is impossible d being consistent.

Consider now the set of all non-null consistent theories y such that $x \subseteq y$ and Rayd. By Zorn's Lemma y has a maximal element x'. By definition of R, Rbcx' and by construction, R$ax'd$. Suppose x' is not prime. Then, for some wffs A, B, $A \vee B \in x'$, $A \notin x'$, $B \notin x'$. Define the non-null theories $[x', A]$ and $[x', B]$ that strictly include x' similarly as in previous lemmas. By the maximality of x', there are three situations:

(a) $[x', A]$ and $[x', B]$ are inconsistent.

Then x' is inconsistent [cf. Lemmas 2, 3].

(b) Not-R$a[x', A]d$ and not-R$a[x', B]d$.

By definitions, $C \rightarrow D \in a$, $(A \wedge H) \rightarrow C \in CI$, $H \in x'$, $D \notin d$ and $C' \rightarrow D' \in a$, $(B \wedge H) \rightarrow C' \in CI$, $H' \in x'$, $D' \notin d$ for some wffs C, D, H, C′, D′, H′. Using transitivity [A1] we have $(A \wedge H) \rightarrow D \in a$, $(B \wedge H' \rightarrow D' \in a)$ whence $[(A \vee B) \wedge (H \wedge H') \rightarrow (D \vee D') \in a$. By R$ax'd$, $D \vee D' \in d$ contradicting our hypothesis given the primeness of d.

(c) Not-R$a[x', A]d$ and $[x', B]$ is inconsistent or not-R$a[x', B]d$ and $[x', A]$ is inconsistent.

We consider the first alternative. By definitions, $C \rightarrow D \in a$, $(A \wedge H) \rightarrow C \in CI$, $H \in x'$, $D \notin d$ and $(B \wedge H') \rightarrow \neg G \in CI$, $H' \in x'$ for some wffs C, D, H, H′ with G a theorem. By transivity, $(A \wedge H) \rightarrow D \in a$; by the theorem $\neg G \rightarrow J$ [for any wff J], $(B \wedge H') \rightarrow D \in CI$ whence $(B \wedge H') \rightarrow D \in a$. Thus, $D \in d$ [cf. (b) above] contradicting our hypothesis.

Each one of the three possibilities leading to contradiction, we conclude that x' is prime, ending the proof of Lemma 4.

LEMMA 5. *Let* $\langle K^c R^c, \vDash^c \rangle$ *be the* CI-*canonical model where* $\langle K^c R^c \rangle$ *is the* CI-*canonical structure and* $\vDash^c$ *is a valuation relation from* K^c *to the* *sentences of* CI *such that for each wff* A *and* $a \in K^c$, $a \models^c$ A *iff* A $\in a$. *Then, the canonical model is indeed a model.*

Proof. We have to prove that the canonical $\models^c$ satisfies the conditions (i)–(v) of the valuation relation. Clauses (i)–(iii) are trivial.

Clause (iv)

subcase (a). If $a \models^c$ A $\rightarrow$ B, then for all b, $c \in K^c$, if $R^c abc$ and $b \models^c$ A, then $c \models^c$ B.

Proof simple.

subcase (b). If $a \not\models^c$ A $\rightarrow$ B, there are b', $c' \in K^c$ such that $R^c ab'c'$, $b' \models^c$ A and $c' \not\models^c$ B.

Proof: define the non-null theories $b = \{C \mid A \rightarrow C \in CI\}$, $c = \{C \mid \exists D(D \in b \text{ and } D \rightarrow C \in a)\}$.

Clearly, $R^c abc$. We now prove that b and c are consistent.

(i) b is consistent. Suppose it is not. Then, $\neg C \in b$ [C a theorem]. By definition, A $\rightarrow \neg$C $\in$ CI. By contraposition, C $\rightarrow \neg$A $\in$ CI. Thus, $\neg$A $\in$ CI and, by the theorem $\neg A \rightarrow (A \rightarrow B)$, A $\rightarrow$ B $\in$ CI. Then, A $\rightarrow$ B $\in a$ and, so, $a \models^c$ A $\rightarrow$ B, which contradicts our hypothesis.

(ii) c is consistent. Suppose c inconsistent. Then, $\neg C \in c$ [C is a theorem]. By definitions, D $\rightarrow \neg$C $\in a$, A $\rightarrow$ D $\in$ CI. Then, A $\rightarrow \neg$C $\in a$; by contraposition, C $\rightarrow \neg$A $\in a$. Now, given P1 and P4, $Raxa$ for some $x \in K^c$. Thus, C $\in x$. Then, $\neg$A $\in a$ and A $\rightarrow$ B $\in a$, i.e., $a \models^c$ A $\rightarrow$ B, contradicting the hypothesis.

Let X be the set of all non-null consistent theories x such that C $\subseteq x$ and B $\notin x$. A similar argument to that in the proof of Lemma 2 shows that there is a prime non-null consistent theory c' such that $c \subseteq c'$ and B $\notin c'$. By definition and $Rabc, Rabc'$. Next, define X' as the set of all consistent theories x such that $b \subseteq x$ and $Raxc'$. Reasoning as in the proof of Lemma 3, it is easy to show that there is a prime consistent theory b' such that $b \subseteq b'$ and $Rab'c'$. Clearly A $\in b$; so A $\in b'$. Thus, we have prime consistent theories b', c' such that $Rab'c'$, A $\in b'$ and B $\notin c'$. By definition of $\models^c$, $b' \models^c$ A and $c' \not\models^c$ B, which ends the proof of subcase (b).

Clause (v). We have to prove: for every $a \in K^c$, $a \not\models^c$ F [i.e., F $\in a$ iff a is inconsistent].

Suppose $F \in a$ and let B be a theorem. By the theorem $F \rightarrow (B \rightarrow F)$, $B \rightarrow F \in a$. Thus, a is inconsistent. Suppose now $A \rightarrow F \in a$, A being a theorem. As a is a theory, $R^c a CI a [R^c CI aa$ and P4]. So, $F \in a$ ending the proof of Lemma 5.
Finally, we prove

THEOREM (completeness). *If* A *is valid, then* A *is a theorem of* CI.

Proof. Suppose A is not a theorem. By Lemma 2, there is a non-null prime consistent theory T such that $A \notin T$. Therefore, A is invalid by Lemma 5.

5. C WITH SEMI-INTUITIONISTIC NEGATION ADDED AS A PRIMITIVE CONNECTIVE: THE SYSTEM CI′

To formulate the system CI′ we add to the sentential language of C the unary connective ¬ [negation] and the axioms:

A9. $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$.
A10. $\neg A \rightarrow (A \rightarrow B)$.

6. SEMANTICS FOR CI′

A CI′-model is just as a C-model but with the clause:

(vi) $a \vDash \neg A$ iff for all $b, c \in K$, not-$Rabc$ or $b \nvDash A$

added to the conditions in §2. A formula A is CI′-valid iff $a \vDash A$ for all $a \in K$ in all models. We note that, given the interpretation of F in §4, we have

$$a \vDash A \rightarrow F \text{ iff } a \vDash \neg A$$

as required.

Semantic consistency is easy. As for completeness, we define the canonical model similarly as in §4. In order to show the completeness of CI′ we only have to prove that the canonical $\vDash$ satisfies clause (vi).

Subcase (a): if $a \vDash^c \neg A$, then there are $b, c \in K^c$ such that not-$R^c abc$ or $b \nvDash^c A$.

Proof. Suppose $a \models^c \neg$A and (for reductio) that there are some $b, c \in$ K such that R$^c abc$ and $b \models^c$ A. By the theorem $\neg$A $\rightarrow$ (A $\rightarrow$ $\neg$(B $\rightarrow$ B)), A $\rightarrow$ $\neg$(B $\rightarrow$ B) $\in a$. Thus, $\neg$(B $\rightarrow$ B) $\in c$. So, c is inconsistent contradicting the hypothesis

Subcase (b): if $a \models^c \neg$A, then there are $b', c' \in$ K^c such that R$^c ab'c'$ and $b' \models^c$ A.

Proof. Suppose $a \models \neg$A. Define $b = \{$B $|$ A $\rightarrow$ B $\in$ CI$'\}$, $c = \{$C $|$ $\exists$B[B $\in b$ and B $\rightarrow$ C $\in a$]$\}$. As in the proof of Lemma 5, it is easy to show that b and c are non-null consistent theories such that R$^c abc$ and A $\in b$. It remains to be proved that b and c can be extended to prime theories b' and c' such that R$^c ab'c'$ and $b' \models^c$ A. So, define X as the set of all non-null consistent theories x such that $c \subseteq x$ and Rabx. By Zorn's lemma, X has a maximal element c'. By definition of R, Rabc'. But suppose c' is not prime. Then, for some wffs B, C, B $\vee$ C $\in c'$, B $\notin c'$, C $\notin c'$. As in previous lemmas, define the non-null theories $[c', \text{B}], [c', \text{C}]$. It is clear that c' is strictly included in $[c', \text{B}]$ and $[c', \text{C}]$. Thus, R$ab[c', \text{B}]$ and R$ab[c', \text{C}]$ by definition of R and Rabc'. So, $[c', \text{B}]$ and $[c', \text{C}]$ are inconsistent by the maximality of c'. Thus is, we have (B $\wedge$ D) $\rightarrow$ $\neg$E, (C $\wedge$ D$'$) $\rightarrow$ $\neg$E$'$ $\in$ CI$'$. Then, [(B $\vee$ C) $\wedge$ (D $\wedge$ D$'$)] $\rightarrow$ ($\neg$E $\vee$ $\neg$E$'$) $\in$ CI$'$ and, by contraposition, $\neg$[$\neg$E $\vee$ E$'$] $\rightarrow$ $\neg$[(B $\vee$ C) $\wedge$ (D $\wedge$ D$'$)] $\in$ CI$'$. By the (weak) De Morgan laws, (E $\wedge$ E$'$) $\rightarrow$ $\neg$[(B $\vee$ C) $\wedge$ (D $\wedge$ D$'$)] $\in$ CI$'$. Thus, $\neg$[(B $\vee$ C) $\wedge$ (D $\wedge$ D$'$)] $\in$ CI$'$, which is impossible given the consistency of c' and the fact that (B $\vee$ C) $\wedge$ (D $\wedge$ D$'$) $\in c'$ [whence, by weak double negation, $\neg\neg$[(B$\vee$C)$\wedge$(D$\wedge$D$'$)] $\in c'$], i.e., c' contains the negation of a theorem].

Therefore, c' is a prime consistent theory such that Rabc'.
A similar argument shows that there is a prime consistent theory b' such that R$ab'c'$ [cf. Lemma 4]. Now, A $\in b'$ because A $\in b$ and b is included in b' by construction. Thus, $b' \models$ A by definition of $\models$, which ends the proof of subcase (b) and the completeness of CI$'$.

NOTES

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Universidad de Salamanca,
Departamento de Filosofia,
Campus Unamuno, Edificio F.E.S.,
37007 Salamanca, Spain