

Modal Set Theory*

Christopher Menzel

1 Modal Set Theory and Traditional Modal Metaphysics

Set theory is the study of sets using the tools of contemporary mathematical logic. *Modal* set theory draws in particular upon contemporary modal logic, the logic of necessity and possibility. One simple and obvious motivation for modal set theory is the fact that, from a realist perspective that takes the existence of sets seriously, sets have philosophically interesting modal properties. For instance, perhaps the most notable and distinctive property of sets is their *extensionality*: sets a and b are identical if they have exactly the same members; formally, where we take variables from the lower end of the alphabet to range over sets:

Ext $\forall a\forall b(\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b)$.

Intuitively, however, extensionality is not a contingent matter, a mere matter of happenstance. Rather, there simply *couldn't* have been distinct sets that shared all their members; there is no such possible world. That is, at a minimum, we want to be able to express that extensionality is a *necessary truth*:

$\Box\text{Ext}$ $\Box\forall a\forall b(\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b)$.

Clearly, however, there is much more than this to the modal connection between sets and their members. For note that, for all $\Box\text{Ext}$ tells us, one and the same set a could have vastly different members from one world to the next, so long as it remains the case that, in each world, no other set has exactly the same members as a in that world. Intuitively, however, the intimate connection between a set and its members is maintained *across* worlds; if a set has Angela Merkel, say, as a member, it could not possibly have failed to have her as a member. Sets, that is to say, have their members *essentially*; if x is a member of a , then it is a member of a in every world in which a exists, i.e., in every world in which something is identical to a ; formally, letting $E!t$ abbreviate $\exists yy = t$, for terms t :

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$$\mathbf{EE} \quad \Box \forall a \forall x (x \in a \rightarrow \Box (E!a \rightarrow x \in a)),$$

Likewise, *non-membership*; a set cannot “add” new members in one world that it lacks in another:

$$\mathbf{E\neg 11} \quad \Box \forall a \forall x (x \notin a \rightarrow \Box (E!a \rightarrow x \notin a)).^1$$

A related philosophical issue is not settled by the preceding principles. Suppose x is a member of a set a here in the actual world and that a exists in some other possible world w . Then by \mathbf{EE} , x is a member of a in w . But nothing follows about x 's *existence* in w . For all we know from \mathbf{EE} , all sets might exist necessarily, even those that have contingent members, members that might not themselves have existed. Hence, if that is so, the singleton set {Merkel}, for example, would have existed even if Merkel hadn't. However, on most conceptions of set, sets are *ontologically dependent* upon their members and, hence, could not themselves exist without their members existing; there could be no singleton set {Merkel} without Merkel. On such a conception, then, we are in need of a further modal principle, namely, that, necessarily, sets exist only if their members do:

$$\mathbf{Ex} \quad \Box \forall a \Box \forall x (x \in a \rightarrow \Box (E!a \rightarrow E!x)).^2$$

Given \mathbf{Ex} and the assumption that Angela Merkel is a contingent being, $\Diamond \neg E!m$, it now follows that the set {Merkel} too is contingent, as expected; it fails to exist at any Merkel-free possible world.

Surprisingly, at first blush anyway, the assumption that there are contingent beings is neither philosophically nor mathematically trivial. Regarding the latter, it is in fact a well-known theorem of the simplest and most straightforward system of modal predicate logic that there neither are nor could have been any contingent beings, i.e., that, necessarily, everything there is exists necessarily:

$$\mathbf{Nec} \quad \Box \forall x \Box E!x.^3$$

¹ $\mathbf{E\neg 11}$ follows from \mathbf{EE} given the principle \mathbf{Ex} below if the modal propositional base of the system is S5.

² The necessity operator between the two quantifiers ensures that a set a can't have a member that is “impossible” with it, that is, a member that fails to exist in every world in which a exists. Without it, \mathbf{Ex} would only say that if $x \in a$ is a member of a in *some* world, $x \in a$ in *all* worlds.

³ See (Menzel, 2014a), §2 for a formal proof and a discussion of the surrounding philosophical issues.

Avoiding this consequence requires choosing between restrictions (of varying severity) on one's logical system, each with its own virtues and liabilities.⁴ However, some philosophers — so-called *necessitists* — embrace **Nec** (from which, of course, **Ex** trivially follows) and choose instead to offer sophisticated philosophical explanations of the allegedly mistaken naïve intuition that some things might not have existed.⁵ The choice of a logic and the adoption of a philosophical standpoint about the metaphysics of sets are therefore interestingly interdependent.

It is not our purpose here to adjudicate these issues. Rather, the point of this initial section has been to illustrate one powerful motivation for modal set theory, namely, its usefulness as a tool for exploring quite traditional lines of inquiry in modal metaphysics concerning contingency, essentiality, ontological dependence, and the like that surface naturally in connection with the existence of sets.⁶ The remainder of this article, however, will be devoted to the development of modal set theory with regard to a rather more directed inquiry into both the nature and *structure* of sets that is motivated in particular by the attractive prospect of a satisfying explanation of Russell's Paradox.

2 ZF and Russell's Paradox⁷

The years 1897-1903 saw the emergence of a string of related paradoxes concerning the notions of number, set, class, property, proposition, and truth.⁸ Among those concerning sets, Russell's Paradox is undoubtedly the best known and arguably the one most directly responsible for subsequent developments in the foundations of mathematics. The argument is well-known. Its heart, of course, is the principle of *naïve comprehension*, i.e., the principle that, for any property of things, there is the set of things that have the property. More formally (and somewhat anachronistically) expressed in the language of first-order logic, it is the principle that, for any predicate $\varphi(x)$, there is the set of things it is true of:

$$\text{NC} \quad \exists a \forall x (x \in a \leftrightarrow \varphi(x)), \text{ where 'a' is not free in } \varphi(x)$$

⁴ The system Q of (Prior, 1968) is among the most severely restricted of well-known systems, abandoning in particular the interdefinability of \Box and \Diamond and most familiar principles of propositional modal logic. For examples of less severely restricted systems, see (Kripke, 1963), (Menzel, 1991), and (Fine, 1978).

⁵ The term "necessitism" and its cognates was coined in (Williamson, 2010), although the view was in large measure anticipated and developed in detail in (Linsky & Zalta, 1994). See also (Salmon, 1987) for an influential precursor.

⁶ The reader is encouraged to consult (Fine, 1981) for a deep and comprehensive study along these lines.

⁷ The exposition in this section and portions of the following is similar to that found in Sections 3-6 of (Menzel, n.d.), which was written largely in parallel with the current entry.

⁸ See (Cantini, 2014) for an excellent overview.

Intuitively, at first sight anyway, the principle seems airtight. For a well-defined predicate unambiguously picks out some existing things (or perhaps no things at all), and what more could you need for the existence of a set than the existence of its purported members, the things that constitute it? For all its intuitive appeal, of course, **NC** is inconsistent: given the well-defined predicate ‘ $x \notin x$ ’, by **NC** we have the “Russell set” r containing exactly those things x that are not members of themselves, i.e.,

$$\mathbf{R} \quad \forall x(x \in r \leftrightarrow x \notin x).$$

Instantiating for r , the contradiction that $r \in r$ if and only if $r \notin r$ follows immediately.

The best known and most influential response to Russell’s Paradox is of course that of Ernst Zermelo. To express Zermelo’s ideas, it is useful to speak of mere *pluralities* of things, where such talk is to be thought of as “ontologically innocent”. That is, talk of a plurality of things is not to be understood to refer to some *additional* thing over and above the things we are talking about — a set or class or mereological sum that they constitute — but, rather, simply as a convenient way of talking about those things jointly, or collectively, as we seem freely to do when we use plural noun phrases in sentences like “It took *three men* to lift the piano” and “*The fans* went wild”. The lesson of Russell’s Paradox, then, in these terms, is that not all pluralities of things can *safely* be assumed to constitute a further thing, viz., a set that contains them; in particular, to assume without qualification, as **NC** would have it, that the things an arbitrary predicate is true of constitute a set can be logically catastrophic. At the same time, some pluralities seem clearly safe. Zermelo’s brilliantly executed idea (Zermelo, 1908)⁹ — implemented in his axiomatic set theory Z — was to *stipulate* the existence of some initial sets to get things going and then introduce a variety of sound “set-building” operations that lead safely from given objects or sets to new sets. We will describe Z in some detail.

Zermelo begins with the extensionality axiom **Ext**. His next axiom, the axiom of *elementary sets*, is actually a combination of an existence axiom and a set-building axiom. Specifically, he postulates the existence of the empty set \emptyset ,

$$\mathbf{ES} \quad \exists a \forall x x \notin a,$$

⁹ Translated as (Zermelo, 1967); see also the informative introductory note by Felgner that accompanies the translation of this paper in the polyglot edition (Zermelo, 2010) of Zermelo’s collected works, pp. 160-89. Zermelo’s theory included the important but controversial axiom of *Choice*, though it will play no part here.

and introduces the axiom of *Pairing*, which says, in effect, that any pair of (not necessarily distinct) objects x and y are jointly safe and hence constitute a set $\{x, y\}$.¹⁰

$$\mathbf{Pr} \quad \exists a \forall z (z \in a \leftrightarrow (z = x \vee z = y)).$$

Assuming extensionality, these two axioms alone already give us the power to prove the existence of the infinite series of *Zermelo numbers* $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \dots$, so-called because they served as Zermelo's surrogates for the natural numbers. (For convenience, abbreviate them, respectively, as $\emptyset_0, \emptyset_1, \emptyset_2, \emptyset_3, \dots$.) However, in addition to so-called “pure” sets like these that are “built up” solely from the empty set, Zermelo also made room in his theory for the existence of arbitrarily many *urelements*, that is, things that are not themselves sets — persons, planets, natural numbers, etc. — and, hence, by **Pr**, for the existence of “impure” sets built up from them. And although he didn't explicitly assume it as an axiom in 1908, it is in the spirit of his theory to take the urelements to constitute a set U of their own.¹¹

$$\mathbf{Ur} \quad \exists a \forall x (x \in a \leftrightarrow (x \neq \emptyset \wedge \forall y y \notin x)).$$

Let ZU be Zermelo's theory Z with the additional axiom **Ur**.

NC is of course absent from Zermelo's theory but a significant remnant of it remains in the form of a set-building principle, *Separation*. Given a predicate $\varphi(x)$, **NC** called sets into being *ex nihilo* from the things of which $\varphi(x)$ is true. Separation, by contrast, only vouches for the things $\varphi(x)$ is true of that are already members of some previously given set:

$$\mathbf{Sep} \quad \forall b \exists a \forall x (x \in a \leftrightarrow (x \in b \wedge \varphi(x))), \text{ where 'a' is not free in } \varphi(x).^{12}$$

Zermelo's next two set-building axioms are *Union*,

$$\mathbf{Un} \quad \exists a \forall x (x \in a \leftrightarrow \exists c (c \in b \wedge x \in c)),$$

which says that the members of the members of a given set b constitute a set, and *Powerset*,

¹⁰ We will make free use of the common $\{x_1, \dots, x_n\}$ notation for finite sets without defining it formally.

¹¹ The existence of U is not entirely unproblematic, as it could turn out to be inconsistent with the other axioms if there are “too many” urelements. See, e.g., (Nolan, 1996) and (Menzel, 2014b). Zermelo himself wasn't sure how to work urelements into his theory until over two decades later; see (Zermelo, 1930) and Kanamori's informative introductory note to its translation in (Zermelo, 2010), pp. 390-430.

¹² **Sep** renders **ES** otiose, since it is a truth of (classical first-order) logic that something x exists, $\exists x E!x$, from which **ES** follows directly from **Pr** and the instance of **Sep** where $\varphi(x)$ is $x \neq x$.

Pow $\exists a \forall x (x \in a \leftrightarrow x \subseteq b)$,

which tell us that the subsets of a set safely constitute a set. Although formulated decades before the mature conception of the set theoretic universe in (Zermelo, 1930), these two axioms clearly anticipated it. For, given the initial set U of urelements — call it U_0 — by **Pow** the set $\wp(U_0)$ of all of their subsets exists. By **Un** there is the set $U_1 = U_0 \cup \wp(U_0)$ consisting of all the members of U_0 and $\wp(U_0)$. Applying **Pow** again we have the set $\wp(U_1)$ of all the subsets of U_1 which we can then join with U_1 itself to yield the set $U_2 = U_1 \cup \wp(U_1)$. In general:

$$\mathbf{D1} \quad \begin{array}{l} U_0 = U \\ U_{n+1} = U_n \cup \wp(U_n) \end{array}$$

Even in Zermelo's early work, then, the sets are naturally taken to have a structure that is *cumulative* and *hierarchical*, advancing “upwards” via iterations of the powerset and union operations, from an initial stock of urelements, in an ever-expanding series of *stages*, or *levels*, each successive level U_{n+1} consisting of everything in the preceding level together with all the sets that can be formed from them, as indicated in Figure 1.

Say that one level U_n is *higher* than another U_m just in case $n > m$ (equivalently, in light of their cumulative nature, just in case $U_m \subset U_n$) and that the *level* $\lambda(x)$ of an object x is the first level of the hierarchy at which it occurs. Since (a) we begin with a base set U_0 of urelements, (b) the hierarchy grows discretely from one level to the next, and (c) a set of level U_{n+1} is always constituted by objects in level U_n , it should also be clear that the sets on this conception are all *well-founded*: no set can be a member of itself and, more generally, there are no infinitely descending membership chains $\dots \in a_{n+1} \in a_n \in \dots \in a_1 \in a_0$. Since the axioms above do not explicitly rule out such structural impossibilities, it must be done independently by means of a separate principle; in Z , this is the axiom of *Foundation*, which requires every nonempty set a to have a member with which it shares no members:

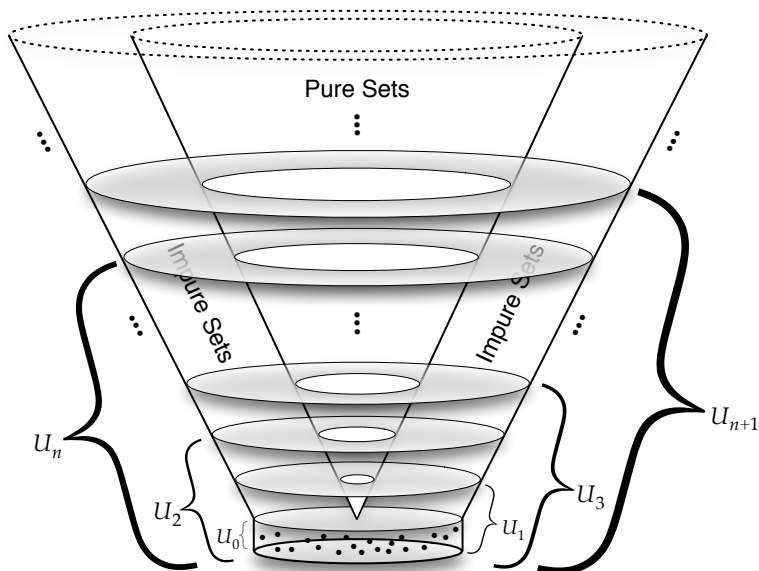


Figure 1: The finite levels of the cumulative hierarchy

Fnd $a \neq \emptyset \rightarrow \exists y(y \in a \wedge \forall z(z \in y \rightarrow z \notin a))$.¹³

We turn now to the critical Zermelian axiom of *Infinity*. What is particularly important about this axiom, especially for purposes here, is not merely that it asserts the safety of an infinite plurality but, rather, the safety of a plurality that is *unbounded* in our hierarchy of finite levels. Consider, in particular, the Zermelo numbers $\emptyset_0, \emptyset_1, \emptyset_2, \dots$. As $\lambda(\emptyset_i) = U_{i+1}$, for all natural numbers i , it follows that, for every level U_n , no matter how high, there is a Zermelo number (\emptyset_n , for example) that only first occurs in a higher level still; the Zermelo numbers are thus unbounded in the hierarchy of levels U_0, U_1, U_2, \dots and, hence, never constitute a set in any finite level.

Such unbounded pluralities, then, are of a rather different sort structurally than any we've encountered hitherto. But, ultimately, from the realist's standpoint, at least, there doesn't seem to be any more reason to question their safety than there is to question the safety of the urelements that we sanction in **Ur** that get the hierarchy going in the first place or the plurality of subsets of a given level that we sanction in **Pow** that enable us, at any given level, to extend the hierarchy to the next level. For, just as in those cases, all of the things in question are *there* and hence

¹³ Zermelo did not include Foundation in his 1908 axiomatization but, as non-well-founded sets were not defined and studied in any systematic way until (Mirimanoff, 1917) and the iterative conception was at most only beginning to take shape in Zermelo's mind, it seems likely that he did not at the time recognize any pressing need for the axiom.

available for collection into a set. Moreover, as the Zermelo numbers are unbounded in the hierarchy of finite levels and hence could not constitute a set at any such level, the set they would constitute would definitively not be a member of itself. So there also seems to be no hint of a Russell-style paradox in the offing.

Accordingly, Zermelo's Infinity axiom posits the collective safety of the Zermelo numbers the way that **Ur** does for the urelements and **Pow** does for the subsets of a given set. More exactly, it declares that there is a set that *contains* the Zermelo numbers:

$$\mathbf{Inf} \quad \exists a(\emptyset \in a \wedge \forall x(x \in a \rightarrow \{x\} \in a)).$$

The set of Zermelo numbers proper — call it ζ (zeta) — can then be derived straightaway by **Sep**.¹⁴

So the axiom of Infinity implies, not just that there are infinite sets, but that there are infinite sets whose members occur at arbitrarily high finite levels of the hierarchy. Hence, intuitively, the hierarchy U_0, U_1, U_2, \dots , of cumulative levels must continue *beyond* the finite. For this to be the case, however, they must first themselves form a set $\{U_0, U_1, U_2, \dots\}$. Given their structural similarity to the Zermelo numbers $\emptyset_0, \emptyset_1, \emptyset_2, \dots$, it certainly seems that they *should*. But **Inf** and the other axioms are not enough to guarantee this.¹⁵ Thus, a further principle is needed, one typically attributed to Abraham Fraenkel,¹⁶ the axiom schema of *Replacement*, the addition of which to Zermelo's theory $Z(U)$ gives us $ZF(U)$. Replacement captures the structural intuition that if the members of a set can be correlated systematically with a given plurality — each $\emptyset_n \in \zeta$ with U_n , for example, — then that plurality also constitutes a set. More formally, where $\exists!x\varphi$ as usual means that something is uniquely φ :

$$\mathbf{Rep} \quad \forall x\exists!y\psi(x, y) \rightarrow \forall b\exists a\forall y(y \in a \leftrightarrow \exists x(x \in b \wedge \psi(x, y))).$$

¹⁴ Specifically, by letting b be the set given by **Inf** and letting $\varphi(x)$ be $\forall y[(\emptyset \in y \wedge \forall z(z \in y \rightarrow \{z\} \in y)) \rightarrow x \in y]$, i.e., the predicate “ x is in every set y that contains \emptyset and the singleton of any of its members”.

¹⁵ To see this, very briefly: where V_ω is the set of hereditarily finite pure sets, let $W_0 = V_\omega \cup U$ and $W_{n+1} = W_n \cup \wp(W_n)$, for $n \in \mathbb{N}$, and let $W = \bigcup_{n \in \mathbb{N}} W_n$. It is easy to see that W is a model of Z and that $U_i \in W$ for all $i \in \mathbb{N}$ but that $\{U_0, U_1, U_2, \dots\} \notin W$. The author thanks Noah Schweber and Joel David Hamkins for this construction.

¹⁶ Skolem independently identified the need for Replacement, and his explicitly first-order formulation of the principle is essentially the one that is mostly used today. See (Fraenkel, 1922) and (Skolem, 1922); an English translation of the latter can be found in (van Heijenoort, 1967), pp. 290-301.

Let $L(y)$ mean that y is one of our finite levels.¹⁷ Letting $\psi(x, y)$ be ‘ $x \in \zeta \wedge L(y) \wedge x \in y \wedge \forall z((L(z) \wedge x \in z) \rightarrow y \subseteq z)$ ’, i.e., “ x is a Zermelo number and y is its level”, we have a 1-1 correlation between the members of ζ and the levels U_1, U_2, \dots and, hence, by **Rep**, they constitute a set to which (by **Pr** and **Un**) we can add the initial level U_0 of urelements; so all the levels jointly constitute our desired set $\{U_0, U_1, U_2, \dots\}$. By **Un** their union is a set, so all the members of all the finite levels do indeed form a set U_ω of their own, the first *transfinite* level of the hierarchy¹⁸ — the result, as it were, of putting a “disk” atop the hierarchy of finite levels depicted in Figure 1 indicating its “completion” in a further level. That of course is not the end of the hierarchy but simply a new starting point for iterating the powerset and union operations to generate yet further levels $U_{\omega+1}, U_{\omega+2}, \dots$, which (by **Rep**) jointly form a set and (by **Un**) hence constitute a new limit level $U_{\omega+\omega}$, and thus once again further levels $U_{\omega+\omega+1}, U_{\omega+\omega+2}, \dots$, and so on, as depicted in Figure 2. In general, then, by including a limit clause representing the continual “completion” of these unbounded series of levels, we can define the entire transfinite cumulative hierarchy for all ordinal numbers, finite and transfinite alike:

$$\begin{aligned} U_0 &= U \\ \mathbf{D2} \quad U_{\alpha+1} &= U_\alpha \cup \wp(U_\alpha) \\ U_\gamma &= \bigcup_{\alpha < \gamma} U_\alpha, \text{ for limit ordinals } \gamma \end{aligned}$$

This intuitive and deeply satisfying conception of the structure of the set theoretic universe yields a compelling explanation of Russell’s Paradox: a plurality safely constitutes a set if and only if it is *bounded* in the full cumulative hierarchy, that is, if and only if there is some level of the hierarchy at or before which the plurality “runs out”; more formally: a level U_α such that, for everything x in that plurality, $\lambda(x)$ is no higher than U_α , in which case those things are “available” to be collected into a set at the next level $U_{\alpha+1}$. However, some predicates — notably, ‘ $x \notin x$ ’ — pick out *absolutely unbounded* pluralities, pluralities that never “run out” by

¹⁷ $L(y)$ is definable without any mention of finitude as: $\forall a[(U \in a \wedge \forall b(b \in a \rightarrow b \cup \wp(b) \in a)) \rightarrow y \in a]$, i.e., “ y is in every set that contains the set U of urelements and also contains $b \cup \wp(b)$ whenever it contains b , for any set b ”.

¹⁸ ω is the first transfinite ordinal number, the first “counting number” after the natural numbers. It is also the first *limit* ordinal, i.e., the first ordinal $\alpha > 0$ such that, if $\beta < \alpha$, then $\beta + 1 < \alpha$. See (Devlin, 1991), Ch. 3 for a good introduction to transfinite ordinals and cardinals and their arithmetic.

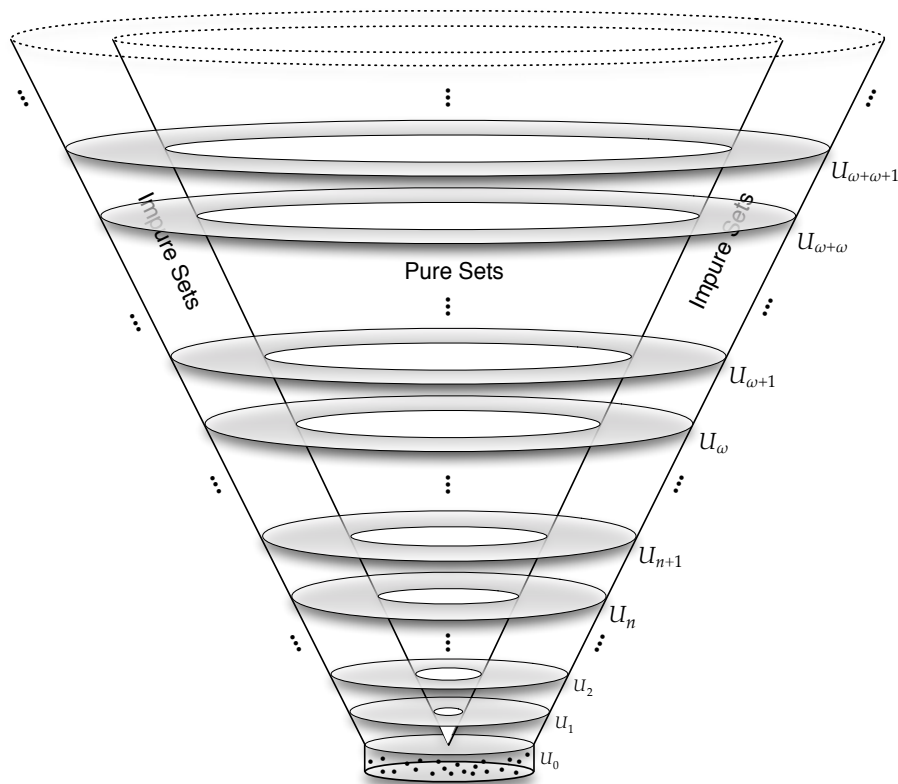


Figure 2: The transfinite cumulative hierarchy

any level.¹⁹ Accordingly, as in Russell’s Paradox, the assumption that they do — i.e., the assumption that they constitute sets — leads to contradiction.

3 Modal Set Theory and the Completion Problem

Unfortunately, as satisfying and illuminating as this explanation might be, a serious puzzle remains for the realist who wants to take the existence of sets seriously: the full cumulative hierarchy is itself a well-defined plurality; why is it not safe? That is, why does the hierarchy itself fail to be “completed”? Note the question is not: Why is there no universal set, i.e., no set containing all the urelements and all the sets? As we’ve just seen, the iterative conception of set provides a cogent answer to that question: only those pluralities that “run out” by some level of the cumulative hierarchy constitute sets at the next level and, obviously, the entire hierarchy is not such a plurality; there is no level at which the members of all the levels form a set. Rather,

¹⁹ In the cumulative hierarchy, where all sets are well-founded, no set is a member of itself so ‘ $x \notin x$ ’ is in fact true of everything.

the question is: Why is the hierarchy only as “high” as it is? Why do all the urelements and sets that there actually are fail to constitute a *further* level that kicks off yet another series of iterations? For the same justification we had for “completing” the hierarchy of finite levels appears to apply no less to the full hierarchy: all of the urelements and sets in all the levels are *there*, as robustly as the members of the finite levels; and, moreover, since the set they would constitute — call it U_Ω — would be of a higher level than its members and, hence, would not be a member of itself, just as in the case of U_ω , no obvious threat of paradox looms. Call this the *completion problem*.

Granted, the completion problem does not appear to be as grave and immediate a threat to the coherence of set theory as Russell’s Paradox. But it does raise disturbing questions for the realist: if there is no answer to the completion problem, then there is an essential element of randomness to set existence. For once we acknowledge that there are pluralities that inexplicably fail to constitute sets, it is hard to see what grounds there are for picking and choosing between those that do and those that don’t: in particular, if the same reasons for accepting that the finite levels U_0, U_1, U_2, \dots constitute a set seem to hold for all the levels of the hierarchy without their constituting a set, then what reason do we have for accepting that the finite levels U_0, U_1, U_2, \dots do? Or that *any* plurality does for that matter? Without a solution to the completion problem, then, the *actual* structure of the hierarchy appears to be unknowable; any claims to knowledge of it would appear to be groundless, as the objects of the purported knowledge might well concern entities that simply do not exist.

Putnam (1967) was the first to argue explicitly that such questions are answered by taking the principles underlying the iterative conception of set to be essentially *modal* and, more specifically, by suggesting that a set is not to be understood in terms of the actual existence of a finished thing but as the *possibility* of its formation (p. 12):²⁰

[T]here is not, from a mathematical point of view, any significant difference between the assertion that *there exists a set of integers* satisfying an arithmetical condition and the assertion that *it is possible to select* integers so

²⁰ See (Hellman, 1989) for a detailed development of Putnam’s ideas.

as to satisfy the condition. Sets, [to parody] John Stuart Mill, are permanent possibilities of selection.²¹

Parsons (1977) spells the idea out a little less metaphorically in a thesis — call it *Parsons' Principle* — that addresses the completion problem directly: any given plurality of things “*can* constitute a set, but it is not necessary that they do.” Thus, necessarily, no matter how many cumulative levels there might be, the *absolutely unbounded* pluralities that don't in fact constitute sets in any level nonetheless *could have* constituted sets. The answer to the completion problem on this *potentialist* conception of sets, then, is simply that there neither is nor could be a “completed” cumulative hierarchy. Rather, instead of the completed stages of the cumulative hierarchy, we have a *potential* hierarchy, i.e., roughly speaking, an infinite hierarchy of *possibilities* where, given any possible completion of the hierarchy up to a given level U_α , there is always a more expansive possibility in which some of the mere pluralities of U_α constitute sets — in the “maximal” case, a possibility comprising the entire next level $U_{\alpha+1}$. The completion problem only arises on the assumption that all the levels — hence all the sets — that there could be (relative to an initial set U_0 of urelements) are already *actual* and, hence, that the hierarchy of sets is complete, that there is no more “collecting” of pluralities into sets that can be done. For only under that assumption — call it *actualism* — is it mysterious why the hierarchy is only as high as it is, why it (or indeed any absolutely unbounded plurality) fails to constitute a further set. The potentialist rejects the actualist assumption: the unbounded pluralities of one possible world always constitute sets in further, more comprehensive worlds.

What becomes of ZF on the potentialist conception? Thought of semantically, the potentialist conception suggests (roughly put) that an assertion to the effect that a certain set exists — and hence occurs at some level of the cumulative hierarchy — should be understood as an assertion that it is *possible* that such a set exist; likewise, assertions about all sets should be understood, not simply as assertions about the sets that *in fact* exist but, roughly speaking, about all the sets there could be, all the sets in any possible world.²² Formalized, this insight yields what Linnebo (2013) calls the *potentialist translation* φ^\diamond of a sentence φ of ordinary set theory, viz., the result

²¹ There are intimations of Putnam's idea in Cantor, notably in an 1897 letter to Hilbert: “I say of a set that it can be thought of as *finished* ... if it is possible without contradiction (as can be done with finite sets) to think of *all its elements as existing together*, and so to think of the set itself as *a compounded thing for itself*; or (in other words) if it is *possible* to imagine the set as *actually existing* with the totality of its elements.” (Ewald, 1996), p. 927.

²² My exposition from this point draws heavily upon (Linnebo, 2010; 2013). This work in turn was strongly influenced by (Parsons, 1983).

of replacing every existential quantifier occurrence \exists in φ with its modalized counterpart $\diamond\exists$ and every universal quantifier occurrence \forall with $\Box\forall$. The idea, then, is that, if a statement φ of ordinary ZF set theory is purportedly true in the cumulative hierarchy, its modalized counterpart φ^\diamond will be true in the potential hierarchy. This in turn might suggest that modal set theory will simply consist in formulating the theory ZF^\diamond that results from replacing the axioms of ZF with their potentialist translations. But that would not in and of itself be terribly interesting. Instead, modal set theorists like Parsons and Linnebo opt for the far more illuminating tack of taking the potentialist conception itself as primary and, hence, by axiomatizing its fundamental principles, *deriving* the axioms of ZF^\diamond .

The first task toward that end is to identify the right propositional modal logic for the potentialist conception. Expressed in terms of possible worlds, the basic underlying intuition is that the universe of sets in a given world w can always be increased — for any world w there is an accessible world w' that includes, not only everything already in w , but new sets whose members are mere pluralities in w .²³ This can be captured more formally by means of several intuitive constraints on world domains $D(w)$ and the accessibility relation R . Specifically, growth is reflected in the constraint that, if Rww' , then $D(w) \subseteq D(w')$.²⁴ R itself should be a partial order (i.e., reflexive, transitive, and anti-symmetric) — each world w' accessible from a world w represents a way in which some of the mere pluralities of w constitute fully-fledged sets in w' , but different pluralities of w might constitute sets in different accessible worlds. Moreover, that the formation of new sets proceeds discretely is reflected in the requirement that R be weakly well-founded.²⁵ Finally, if w_1 and w_2 represent distinct expansions of the set theoretic universe of a world w — different “choices” of which mere pluralities of w to take to constitute sets — it should still be possible in each world to form the sets constituted in the other; given the constraint on domains, the pluralities that constitute sets in w_1 will still be available in w_2 and vice versa. Hence, a further natural condition is that accessibility be *directed*, that is, that,

²³ For fairly obvious reasons, possible world semantics can only be considered a useful heuristic, as Kripke models are themselves by definition *sets*. Hence, no Kripke model could purport to represent (in a one-to-one fashion) *all possible* sets without opening the door once again to Russell’s Paradox.

²⁴ This will of course have the consequence that, for any object x , including urelements, it will be a logical truth that x exists necessarily, $\Box E!x$. That is the right result, as the constraints in question are only meant intuitively to capture possible ways of *extending* the set theoretic universe of a given world w ; nothing in w “goes away” as new sets are added in accessible worlds.

²⁵ R is weakly well-founded if every nonempty set S of worlds contains an R -minimal world, i.e., a world $w \in S$ such that, for all other $u \in S$, not- Ruw . (R is well-founded if it is a strict partial order — irreflexive and transitive — that satisfies this definition.)

for any two worlds w_1, w_2 accessible from a given world w there is a third world accessible from both and, hence, one whose sets include all those formed in either world. The propositional modal logic determined by these conditions on accessibility is S4.2, the normal modal logic that includes the familiar axioms of the logic S4, viz.,

$$\mathbf{T} \quad \Box\varphi \rightarrow \varphi$$

$$\mathbf{4} \quad \Box\varphi \rightarrow \Box\Box\varphi$$

corresponding to reflexivity and transitivity, respectively, and the axiom

$$\mathbf{G} \quad \Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$$

corresponding to directedness.²⁶ And to S4.2 is added classical quantification theory with identity and the axiom ‘ $x \neq y \rightarrow \Box x \neq y$ ’ expressing the necessity of difference.²⁷

Now, as seen above, it is useful to describe the axioms of ZF informally in terms of pluralities. To characterize the potentialist conception properly with the tools of modal logic, it is essential to quantify over them explicitly in order to identify the logical principles that govern their behavior. Accordingly, we introduce plural variables ‘ xx ’, ‘ yy ’, etc. and a new type of atomic formula ‘ $y < xx$ ’ to indicate that y is one of the things xx . The inference rules for plural quantifiers parallel those for first-order quantifiers exactly.²⁸

Several principles capture the existence and nature of pluralities. First, given the ontological innocence of plural quantification, the plural counterpart to **NC** is harmless:

$$\mathbf{PC} \quad \exists xx \forall y (y < xx \leftrightarrow \varphi(y)), \text{ where ‘} xx \text{’ does not occur free in } \varphi(y),$$

that is, simply put, for any predicate $\varphi(y)$, there are the things it is true of.²⁹ Next, the modal properties of pluralities are captured in two axioms expressing the *stability* of the $<$ relation, that

²⁶ There is no axiom corresponding to antisymmetry because it is not definable in propositional modal logic. See, e.g., (Blackburn, de Rijke, & Venema, 2001), §4.5. Well-foundedness can be axiomatized in a bimodal extension of propositional modal logic with both “forward-looking” and “backward-looking” necessity operators. See (Studd, 2013).

²⁷ The necessity of identity is a well-known theorem of this logic.

²⁸ See (Linnebo, 2014) for a comprehensive overview of the logic of plural quantification.

²⁹ As Linnebo points out (2013, p. 210), **PC** entails the existence of an “empty” plurality. This can be avoided at the cost of some inconvenience. As to **PC**’s harmlessness, note that a Russell-style argument to a contradiction from **PC** with $\varphi(y) = ‘y \nless y’$ (assuming that construction is even permitted syntactically) breaks down, as it is in general invalid to infer from the fact that all things (distributively) have a property F , that any things (collectively) have F . Hence, from $\forall y \varphi(y)$ we cannot legitimately infer $\varphi(xx)$; in particular, we can’t infer $xx < xx \leftrightarrow xx \nless xx$ from

is, that, for any plurality xx and object y in a given world w , y will be among the things xx in an arbitrary accessible world w' if and only if it is among them in w :

$$\mathbf{Stb}_{\prec}^+ \quad y < xx \rightarrow \Box(y < xx)$$

$$\mathbf{Stb}_{\prec}^- \quad y \nprec xx \rightarrow \Box(y \nprec xx).$$

However, these axioms don't rule out the possibility that xx *grows* in a further world w' , that xx includes a new object z that only first comes to exist in w' . This possibility is ruled out by means of a schema that ensures that pluralities are *inextensible*:

$$\mathbf{InEx}_{\prec} \quad \forall y(y < xx \rightarrow \Box\varphi) \rightarrow \Box\forall y(y < xx \rightarrow \varphi).^{30}$$

Let *MPFO* be this system of plural first-order modal logic.

The next task is to extend MPFO to a basic modal set theory BMST. Like ZF, BMST axiomatizes the two fundamental structural properties of sets, viz., extensionality and foundation,³¹ which are captured simply by adopting the axioms **Ext** and **Fnd** (hence also their necessitations). Recall from Section 1 above, however, that sets also have their members essentially, as expressed in the principles **E \in** and **E \notin** .³² Both of these principles, as well as the inextensibility of membership, are entailed by the following:

$$\mathbf{ED}_{\in} \quad \exists xx\Box\forall y(y < xx \leftrightarrow y \in a).$$

Together with the stability and inextensibility principles for pluralities above, **ED \in** says that, for any set a , one and the same plurality of things constitute a in every (accessible) possible world.

$\forall y(y < xx \leftrightarrow y \nprec y)$. For more on distributive vs. collective quantification, see, e.g., (Scha, 1984) and (McKay, 2006).

³⁰ To see how this might work to ensure inextensibility, suppose xx is a plurality in w and that a exists in some accessible world w' but is not in w (and hence that, at w , $a \nprec xx$) and suppose for reductio that $a < xx$ at w' . Let φ be the formula ' $y \neq z$ '. Then, where ' z ' takes a as value, ' $\forall y(y < xx \rightarrow \Box y \neq z)$ ' is true at w but ' $\forall y(y < xx \rightarrow y \neq z)$ ' is false at w' , contradiction. Note that for this axiom to work, it is critical in general that, in the evaluation of φ at a world w , free variables are not taken to be implicitly universally quantified but, rather, that the free variables of φ be allowed to take values outside of $D(w)$. For in general there is no guarantee that there will be some predicate involving at most the things in w that is necessarily true of each of the things xx but of nothing else in any accessible world.

³¹ Foundation follows semantically from the weak well-foundedness of the accessibility relation R and the fundamental potentialist assumption that the members of any set that first comes to be in any world w is constituted by objects that exist in some world u such that Ruw . See (Linnebo, 2013), pp. 216-17.

³² Since $\Box E!a$ is a simple theorem of MPFO, the antecedents $E!a$ of the conditionals in the consequents of these two principles are superfluous in the context of BMST.

As noted above, the potentialist correlate to any statement asserting the existence of a given set is a statement asserting that it is *possible* that such a set exist. BMST has but a single such principle. Recall that the general intuitive motivation for the naïve comprehension principle **NC** was that the existence of the members of a purported set should suffice for the existence of the set, which we can now express generally, and formally, in terms of plural quantification:

$$\mathbf{GNC} \quad \forall xx \exists a \forall y (y \in a \leftrightarrow y < xx).$$

Given the innocent principle **PC**, **NC** (and catastrophe) follow immediately from **GNC**. As we've seen, both actualist and potentialist accounts have explanations of where **GNC** goes wrong based intuitively on the iterative conception of set. On both accounts, in one sense or another, sets are “constructed” level-by-level without end, the new sets of one level constituted by the mere pluralities of previous levels. Hence, necessarily, the plurality of *all* the sets in all the levels, and unbounded pluralities generally, fail to constitute sets. Both accounts thus agree on where **GNC** gets it wrong: necessarily, there are pluralities that don't constitute sets. Additionally, however, the potentialist conception avoids the completion problem by explaining where **GNC** *almost* gets it right: not every plurality constitutes a set, but any plurality *could*. From the potentialist standpoint, then, **GNC** simply missed the implicit modality in claims of set existence; what we need is just its potentialist translation:

$$\mathbf{C} \quad \Box \forall xx \Diamond \exists a \Box \forall y (y \in a \leftrightarrow y < xx).$$

That is, necessarily, any plurality of things *could* constitute a set consisting of exactly those things. This, of course, is a precise formalization of Parsons' Principle. Let BMST be the result of adding the axioms **Ext**, **Fnd**, **ED_ε**, and **C** to MPFO.

It is a simple matter to show that BMST proves the potentialist translations of all of the ZF axioms except **Pow**, **Inf**, and **Rep**.³³ As these are by far the most powerful axioms of ZF, this shows that our basic intuitions about pluralities and sets as expressed in BMST — in particular, Parsons' Principle **C** — only get us so far. To see this with regard to **Pow**, suppose we have a plurality xx in some world w . By principle **C** there is a world u accessible from w in which they constitute a set b . Obviously, all the “subpluralities” of xx — all the things yy such that each of

³³ See (Linnebo 2013, p. 221) and (Parsons 1983), Appendix 2. The addition of the urelement axiom **Ur**, which neither Linnebo nor Parsons includes in their discussions, would involve no significant complications to these results.

them is one of the things xx — also exist in w . But neither **C** nor the structure of worlds and their domains in the underlying semantics provides any guarantee that any of them (other than xx) constitute a set in u , let alone all of them. Indeed, for all we know on the potentialist conception to this point, the subpluralities of xx might be *inexhaustible* in the sense that, for any world u accessible from w , there is always a further world v accessible from u at which some subplurality of xx only first constitutes a set (hence, a subset of b). If so, there is no world where all possible subsets of b exist, in which case the power set of b is impossible, contrary to **Pow**[◇], i.e., $\diamond\exists a\Box\forall x(x \in a \leftrightarrow x \subseteq b)$.

To derive **Pow**[◇], then, an additional principle is required that rules out this sort of inexhaustibility. Intuitively, this is accomplished most naturally by assuming that worlds more directly reflect the levels of the cumulative hierarchy; that is, by assuming, not only that the newly-formed sets of a given world are mere pluralities of some preceding world, but that set formation is always *maximal*: that the newly-formed sets of a world are *all* those that can be constituted from the mere pluralities of a preceding world.³⁴ This assumption can be expressed elegantly in a single axiom to the effect that, much like its members, the subsets of a given set a are constant across possible worlds:

$$\mathbf{ED}_{\subseteq} \quad \exists xx\Box\forall y(y < xx \leftrightarrow y \subseteq a).$$

Pow[◇] now follows straightaway from **BMST** + **ED**_⊆.

Recall that **ES** and **Pr** alone suffice to generate the infinite plurality of Zermelo numbers \emptyset_0 , \emptyset_1 , \emptyset_2 , ... and it was left to **Inf** simply to sanction a set containing them. If we could prove the mere possibility that all the Zermelo numbers exist on the potentialist conception, we could immediately invoke principle **C** to prove the possible existence of a set containing them. However, the potentialist is in a slightly more fraught situation. For the potentialist principles to this point — the derived principles **ES**[◇] and **Pr**[◇] in particular — only yield a series of possible *initial segments* of the Zermelo numbers: by **ES**[◇] it is possible that \emptyset_1 exists; and by iterated applications of **Pr**[◇], it is possible that \emptyset_0 and \emptyset_1 exist and hence also that the numbers \emptyset_0 , \emptyset_1 ,

³⁴ Linnebo (2013, p. 209, fn 7) suggests that maximality entails that the accessibility relation is linear but that would follow only if worlds are individuated extensionally by their domains (which is plausible if, as in (Linnebo 2013), the existence of urelements is not assumed). Linearity can be forced axiomatically by any of a variety of axioms; see (Chellas, 1980). The system resulting from adding a linearity axiom to S4 in place of **G** is S4.3 though, as Parsons notes (1983, pp. 319-20), S4.3 seems to add nothing over S4.2 for the purpose of deriving set-theoretic consequences.

and \emptyset_2 exist, and so on, but without the entire series of Zermelo numbers ever being “completed” in a single possibility.

To derive the possible existence of all the Zermelo numbers thus requires a further principle asserting, roughly, that whatever is true of the potential hierarchy as a whole, as expressed in the potentialist translation φ^\diamond of some proposition φ of set theory (hence a proposition containing no plural quantifiers³⁵), is possible *simpliciter*:

Ref $\varphi^\diamond \rightarrow \diamond\varphi$.

Thus, in particular, **ES**[◇] and **Pr**[◇] yield a proposition φ^\diamond expressing the infinite series of possibilities involving larger and larger initial finite segments of the Zermelo numbers.³⁶ By **Ref** that series is reflected in a single possibility containing all — hence, by **PC**, the entire infinite plurality of — the Zermelo numbers, and so by Parsons’ Principle **C** it is possible that they constitute a single infinite set. **Inf**[◇] follows immediately by some simple modal logic. The potentialist translation **Rep**[◇] of the Replacement schema can be similarly proved by strengthening **Ref** to

Ref⁺ $\diamond\forall x(\varphi^\diamond(x) \rightarrow \varphi(x))$.³⁷

Importantly, it can be shown that the modalized quantifiers $\diamond\exists$ and $\square\forall$ “behave proof-theoretically very much like ordinary quantifiers” (Linnebo, 2013, p. 213), thus explaining why they are not found in ordinary set-theoretic practice — mathematicians can, in effect, talk about the potentialist hierarchy *as if* it were actual.

Concluding Philosophical Postscript. This article has focused chiefly on the technical development of modal set theory and its intuitive motivations without any close critical attention paid to surrounding philosophical questions. In closing we note briefly that perhaps the most significant philosophical challenge to the potentialist conception is to explain the nature of the modality in question. For, from an “absolute” metaphysical perspective outside of any given

³⁵ Allowing plural quantifiers in φ without restriction would in fact lead to Russell’s Paradox.

³⁶ Specifically, φ^\diamond is $\diamond\exists a\forall xx \notin a \wedge \square\forall x\diamond\exists y\square\forall z(z \in y \leftrightarrow z = x)$ and hence $\diamond\varphi$ is $\diamond(\exists a\forall xx \notin a \wedge \forall x\exists y\forall z(z \in y \leftrightarrow z = x))$.

³⁷ Again, where φ contains no plural quantifiers. Parsons (1983, p. 323) relies on a rather different principle to derive **Inf**[◇] and **Rep**[◇]. See Linnebo (2013, p. 223, fn 28) for further discussion.

possible world, even pure sets are metaphysically contingent beings — in particular, at any given possible world w , those sets in any accessible world w' whose members are mere pluralities in w . If this is in fact the sober truth about the metaphysics of sets, then set existence is modally capricious — two possible worlds can be in all respects identical but for the fact that there are pure sets in one that simply and inexplicably fail to exist in the other. Though not identical to the completion problem, the apparently inexplicable contingency of set existence on the potentialist conception, taken literally, seems to raise questions parallel to those arising from the apparently inexplicably nonexistence of certain sets, as noted in the completion problem.

Perhaps in response to this difficulty — though neither explicitly says so — both Parsons (1977, §IV) and Linnebo (2013, pp. 207-8) suggest that the modality of the potential hierarchy is more semantic than metaphysical: at any given time, one's conception of the “height” of the set theoretic universe, hence the range of one's quantifiers, is determined by one's strongest large cardinal assumptions.³⁸ Once convinced of the existence of a larger cardinal still, pluralities that had been absolutely unbounded constitute sets under the stronger assumptions and the range of one's quantifiers broadens accordingly. Thus Linnebo (p. 208):

A claim is possible, in this sense, if it can be made to hold by a permissible extension of the mathematical ontology; and it is necessary if it holds under any permissible such extension. Metaphysical modality would be unsuitable for our present purposes because pure sets are taken to exist of metaphysical necessity if at all.

However, if after all (pure) sets exist as a matter of metaphysical necessity (so that, in particular, any pure sets that *could have* existed *actually* exist), as Linnebo appears to suggest here, then the completion problem threatens once again to rear its head with all its original force: why are there only the pure sets there actually are? If, necessarily, all pure sets exist of metaphysical necessity, what explains the fact that there *couldn't* have been more, the fact that there are pluralities of things such that it is not even metaphysically possible that they constitute sets?³⁹

These and related metaphysical questions prompted by the potentialist conception of set point to a fertile area for exploration in the philosophical foundations modal set theory.

³⁸ See also (Fine, 2006).

³⁹ (Hewitt, 2015, §§2.4-2.6) raises related problems.

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