make little sense if the points were not parts of the continuum[12]. This is confirmed by a closer inspection of the theory. For in (13) Leibniz refers to these indivisible points as parts of space, albeit parts smaller than any given part. Indeed they must be, in order for a moving body to be said to occupy a greater part of space at the point of contact than it would if at rest (this is crucial to his theory of cohesion):

> "(13) *One point of a moving body in the time of its endeavour*, i.e. in a time smaller than can be given, *is in several* places or *points of space*, that is, it will fill a part of space greater than itself, or greater than it fills when it is at rest, or moving more slowly, or endeavouring in only one direction; yet this part of space is still unassignable, or consists in a point, although the ratio of a point of a body (or of the point it fills when at rest) to the point of space it fills when moving, is as an angle of contact to a rectilinear angle, or as a point to a line" (A VI, II, 265; *LLC*, 340–41).

It seems very probable that Leibniz was inspired to construct this theory by Hobbes's attempt to provide a sound philosophical foundation for Cavalieri's Method of Indivisibles[13]. For not only does Leibniz interpret Cavalieri's indivisibles similarly to Hobbes, but two other features of Hobbes's analysis are also to be found in his own: (i) a proposed redefinition of 'point' intended to replace Euclid's, which is considered defective; (ii) a justification of these arbitrarily small points in terms of "horn angles" (a horn angle is the angle of contact between a straight line and a curve, usually the arc of a circle).

Leibniz's theory is by no means just a version of Hobbes's, however. In the first place, he rejects Hobbes's definition of a point as a line "whose length is not considered" (more precisely, a body whose length, breadth and depth are not considered)[14], opting instead for an interpretation of points as actually infinitely small, in opposition to Hobbes's finitism. He interprets the horn angles as support for this position, in that one horn angle may be bigger than another while both are less than any rectilinear angle that can be assigned[15]. Interesting too in this connection is Leibniz's passing mention of the Scholastic Theory of Signs in *fundamentum* 18. This appears to have emboldened him in his idea that points, though unextended, may nevertheless have a structure or situation of (unextended) parts. That is, the parts will have a situation even though they are "indistant" or "lack distance" from one another.

The importance of this property of points is that it enables Leibniz to evade some of the traditional objections to composing the continuum from points. In his *Parmenides* (138a) Plato had argued that a thing without parts cannot have a situation, and Aristotle had built on this argument in his *Physics* (231b), where he argued that indivisibles, being partless, cannot be joined. Similarly, Sextus Empiricus argued that if a line were composed of points one would not be able to divide it, since a point has no parts[16]. Again, if a line were composed of partless points or minima, there would be as many points in the diagonal as in the side of a given square, since they can be put into a 1-1 correspondence; but then there will be none in the line that is their difference, contrary to the initial supposition that every line is composed of points.

Leibniz addresses both of these objections by acknowledging that they apply to true minima, or partless points, in contradistinction to the points he has defined:

> "(3) *There is no minimum in space or body*, that is, there is nothing which has no magnitude or part. For such a thing has no situation, since whatever is situated somewhere can be touched by several things simultaneously that are not touching each other, and would thus have several faces; nor can a minimum be supposed without it following that the whole has as many minima as the part, which implies a contradiction" (A VI, II, 264; *LLC*, 339).

The first objection does not apply to his own points because these are asserted to have parts, albeit unextended ones, and thus a situation to one another, even though the parts are indistant. Moreover, since magnitude of a quantity is defined as "the multiplicity [*multitudo*] of its parts", Leibniz's points (unlike Galileo's *parti non quànte*) may have a magnitude. Because of this, he assumes, they avoid Sextus's objection too.

The theory of magnitude of these points is further clarified by (6) and (10);

> "(6) The ratio of rest to motion is not that of a point to space, but that of nothing to one.
> (10) Endeavour is to motion as a point is to space, i.e. as one to infinity, for it is the beginning and end of motion" (A VI, II, 265; *LLC*, 340–41).

That is, the ratio of a point to a line is 1 to ∞, not 0 to 1. Points are not "nothings", as Wallis termed them[17], but are proportional to the motions generating them. Take, for instance, a line segment of finite magnitude F. This is composed of an infinity of parts, each smaller than any assignable, whose magnitude is therefore F/∞. Points of different magnitudes are generated by motions at different uniform speeds:

> "(18) *One point is greater than another, one endeavour is greater than another, but one instant is equal to another*, whence time is expounded by a uniform motion in the same line, although its parts do not cease in an instant, but are indistant. In this they are like the angles at a point, which the Scholastics (whether following Euclid's example, I do not know) called *signs*, as

12 "But what do I anticipate being clarified by this [theory of points]? I believe the Labyrinth of the Continuum can scarcely be escaped in any other way" (to Henry Oldenburg, 11 March 1671; A II, I 90); "[the TMA] examines the reasons for abstract motions, and unfolds the wonderful nature of the continuum... so that as one endeavour is greater than another, so is one point greater than another, in which way I not only escaped from that whole labyrinth of the continuum, but also saved the Cavalierian geometry of indivisibles" (to Lambert van Velthuysen, May 1671; A II, I, 97).

13 See Jesseph, *Foundations*, for a detailed treatment of Leibniz's debt to Hobbes. For a succinct account of Cavalieri's method, see K. Andersen: "Cavalieri's Method of Indivisibles", in *Archive for History of Exact Sciences* 31, 4, 1985, pp. 291–367.

14 All Hobbes's mathematical objects are bodies: a surface is a body whose depth is not considered, a line a surface whose breadth is not considered. See T. Hobbes: *De Corpore*, VIII, 12; excerpts in *LLC*, 559.

15 That Leibniz was not mistaken in taking horn angles for actual infinitesimals is shown by an interesting article by S. K. Thomason: "Euclidean Infinitesimals", in: *Pacific Philosophical Quarterly* 63, 168–185. Thomason shows that one could construct a consistent theory of horn angles within Euclidean geometry, in which they would indeed count as non-Archimedean infinitesimals.

16 Sextus Empiricus, *Against the Physicists* I, 288.

17 Again, see Jesseph, *Foundations*, for an illuminating treatment of the relationship of the views of Wallis and Leibniz on the infinitely small.

there appear in them things that are simultaneous in time, but not simultaneous by nature, since one is the cause of the other" (A VI, II, 266; *LLC*, 341).

Thus if we take two points *p* and *q* that are the beginnings of two different lines described in time T by the unequal uniform motions (whose speeds are) M and N, they will be proportional to the endeavours that are the beginnings of these motions, M/∞ and N/∞, resp. Therefore even though they are infinitely small they will be in the ratio M:N, i.e. in the same ratio as their generating motions. An infinity of points of length MT/∞ will compose a line of length MT, just as an infinity of endeavours M/∞ will compose the motion M.

In this last respect, the composition of a continuous motion M from an infinity of endeavours M/∞, the theory contrasts with Leibniz's earlier theory of metaphysically discontinuous motion, as he implicitly observes:

"(7) Motion is continuous, i.e. not interrupted by any little intervals of rest. For
(8) once a thing comes to rest, it will always be at rest, unless a new cause of motion occurs" (A VI, II, 265; *LLC*, 340–41).

Finally, Leibniz justifies the existence of these endeavours or beginnings of motions with the following ingenious inversion of Zeno's dichotomy argument[18]:

"(4) *There are indivisibles or unextended things*, otherwise neither the beginning nor the end of a motion or body is intelligible. This is the demonstration: any space, body, motion and time has a beginning and an end. Let that whose beginning is sought be represented by the line *ab*, whose midpoint is *c*, and let the midpoint of *ac* be *d*, that of *ad* be *e*, and so on. Let the beginning be sought to the left, on *a*'s side. I say that *ac* is not the beginning, since *dc* can be taken away from it without destroying the beginning; nor is *ad*, since *ed* can be taken away, and so on. Therefore nothing is a beginning from which something on the right can be taken away. But that from which nothing having extension can be taken away is unextended. Therefore the beginning of a body, space, motion, or time (namely, a point, an endeavour, or an instant) is either nothing, which is absurd, or is unextended, which was to be demonstrated" (A VI, II, 264; *LLC*, 339).

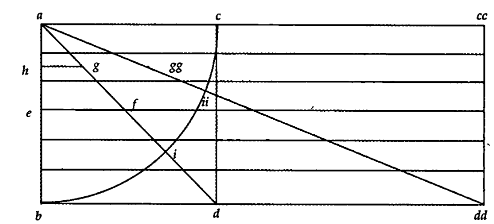
In calling this an inversion of Zeno's dichotomy argument I mean that, while Zeno argued for the unreality of motion on the grounds that the motion could never begin, Leibniz takes the reality of motion for granted and uses the dichotomy argument to argue that the beginning must be unextended. Indeed, since this argument is applicable to any subinterval of the motion, it entails the stronger conclusion that any subinterval whatever must contain an unextended beginning. Given the proportionality of points to endeavours, this argument therefore provides a powerful justification for Leibniz's notion of extensionless points.

There is, however, a problem of consistency with this theory that has been pointed out by other commentators. For the assumption that a line is composed of points – even points like Leibniz's that have parts and magnitude, but no extension – is just as susceptible to Sextus Empiricus's objection as the assumption of *true minima* which Leibniz had rejected in the *TMA*. He appears to have realized this

18 For a detailed analysis of the this inversion of Zeno's dichotomy and its place in Leibniz's thought, see R. T. W. Arthur: "Leibniz's Inversion of Zeno's Dichotomy", forthcoming in: *Corporeal Substances and the Labyrinth of the Continuum in Leibniz*, eds. M. Mugnai and E. Pasini. [=*Studia Leibnitiana, Supplementa*].

late in 1671[19], but the argument for it is given explicitly in a paper written in the winter of 1672/3 (*De minimo et maximo*)[20], where he now identifies indivisibles with minima and rejects both. His version there of Sextus' argument (which I have elsewhere dubbed "Leibniz's Diagonal Argument") runs as follows:

"*There is no minimum, or indivisible, in space and body.*
For if there is an indivisible in space or body, there will also be one in the line *ab*. If there is one in the line *ab*, there will be indivisibles in it everywhere. Moreover, every indivisible point can be understood as the indivisible boundary of a line. So let us understand infinitely many lines, parallel to each other and perpendicular to *ab*,



to be drawn from *ab* to *cd*. Now no point can be assigned in the transverse line or diagonal *ad* which does not fall on one of the infinitely many parallel lines extending perpendicularly from *ab*. For, if this is possible, let there exist some such point *g*: then a straight line *gh* may certainly be understood to be drawn from it perpendicular to *ab*. But this line *gh* must necessarily be one of all the parallels extending perpendicularly from *ab*. Therefore the point *g* falls – i.e. any assignable point will fall – on one of these lines. Moreover, the same point cannot fall on several parallel lines, nor can one parallel fall on several points. Therefore the line *ad* will have as many indivisible points as there are parallel lines extending from *ab*, i.e. as many as there are indivisible points in the line *ab*. Therefore there are as many indivisible points in *ad* as in *ab*. Let us assume in *ad* a line *ai* equal to *ab*. Now since there are as many points in *ai* as in *ab* (since they are equal), and as many in *ab* as in *ad*, as has been shown, there will be as many indivisible points in *ai* as in *ad*. Therefore there will be no points in the difference between *ai* and *ad*, namely in *id*, which is absurd" (A VI, III, 97; *LLC*, 8–11).

From a modern perspective this argument is apt to seem fallacious: it looks as though Leibniz has conflated the measure of the set of points in a line with the number of points contained in it. Just because there is the same number of indivisible points in *ai* as in *ad*, it does not follow that their difference *id* has zero measure. But this presupposes a rather anachronistic point of view for appreciating this argument, for Leibniz's whole theory precisely depends on a notion of point as possess-

19 In a letter to Arnauld dated November, 1671, Leibniz wrote: "there are no indivisibles, but there are unextended things" (A II, I, 172). P. Beeley takes this to have been Leibniz's position all along: see his *Kontinuität und Mechanismus*, esp. pp. 258–9. For criticisms, see O. B. Bassler: "The Leibnizian Continuum in 1671", in: *Studia Leibnitiana* 30 (1998), no. 1, pp. 1–23; p. 19, and R. T. W. Arthur: "The Enigma of Leibniz's Atomism", in: *Oxford Studies in Early Modern Philosophy*, I (2003), D. Garber and S. Nadler ed.s, pp. 183–227; 196.

20 *De minimo et maximo. De corporibus et mentibus* (*On Minimum and Maximum; on Bodies and Minds*): A VI, III, N5; *LLC*, 8–19.

ing a non-zero magnitude: this is what enables Leibniz to claim that one point may have a ratio to another. Also, prior to modern measure theory there was no way to compose a magnitude from points which lack magnitude[21]. Adopting a perspective that is more historically sensitive, one can treat Leibniz's argument on its own terms as follows. It can be seen to depend on four assumptions: (i) that there are points everywhere in a given line, each of which can be considered to be the beginning of any other line, and; (ii) that the given line can be regarded as composed of these points as parts; (iii) that all the points of any given line are of equal magnitude; and (iv) that the whole is greater than the part. Assumption (i) allows the establishment of a 1–1 correspondence between the points of any two lines, by connecting them with parallel straight lines. The trouble is that by (iii) each of the points on any one of the parallels connecting the lines *ab* and *ad* will be of equal magnitude, so that by (ii) the magnitudes of *ab* and *ad* will be equal. By a similar argument the magnitudes of *ab* and *ai* will be equal. Thus the magnitude of *ad*, the whole, will equal the magnitude of *ai*, the part, contradicting (iv).

Leibniz's solution is to give up his identification of the actually infinitely small with unextended points or indivisibles. That is, if the infinitely small "beginnings" in a line are taken to be indivisible in the sense of having zero extension, then there is nothing to prevent such points being taken as the endpoints of other lines, as in assumption (i). But this enables the Diagonal Paradox, as explained above. Consequently the idea of indivisibles or points of zero extension composing an extended line must be dropped. Leibniz's attempt to distinguish minima (having zero magnitude) from indivisibles (having zero extension) does not succeed.

Another way of expressing this point is in terms of dimensional homogeneity[22]. In characterizing his points as indivisible beginnings, Leibniz was trying to justify the idea of a point as a rudiment or beginning from which the line could be considered as generated. But the diagonal paradox throws into question the whole idea of the composition of the line from unextended points, and thus the composition of any continuum of dimension d from elements of dimension $d - 1$. The saving of Cavalieri requires the "points" to have an infinitely small extension, rather than to be unextended indivisibles. If points are considered as truly dimensionless or unextended, then the Diagonal Paradox shows that they cannot compose a line: their ratio to a finite line would be 0 to 1, not 1 to ∞, as intended. This realization leads Leibniz to modify his theory accordingly.

21 Cf. Spinoza, from his Letter on the Infinite: "For it is the same thing for a duration to be composed out of moments as for a number to arise solely by the addition of noughts (*Idem enim est durationem ex momentis, quam numerum ex sola nullitatum additione oriri*"; quoted from Leibniz's version, A VI, III, 280; *LLC*, 110–11).

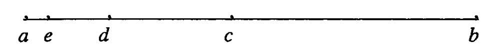
22 This point about dimensional homogeneity has been lucidly explained by Bassler in "The Leibnizian Continuum in 1671".

Phase 3: Infinitely small lines proportional to endeavours

(iii) a continuous line is composed of infinitely many infinitesimal lines, each of which is divisible and proportional to a generating motion at an instant (*conatus*) (1672–75).

In *De minimo et maximo*, as we have seen, Leibniz uses the Diagonal Argument to reject indivisibles. But immediately afterwards he reaffirms the existence of infinitely small actuals or beginnings of motion with a reiteration of the Inverted Zeno argument[23]:

> *"There are some things in the continuum that are infinitely small, that is, infinitely smaller than any given sensible thing.*



> First I show this for the case of space as follows. Let there be a line *ab*, to be traversed by some motion. Since some beginning of motion is intelligible in that line, so also will be a beginning of the line traversed by this beginning of motion. Let this beginning of the line be *ac*. But it is evident that *dc* can be cut off from it without cutting off the beginning. And if *ad* is believed to be the beginning, from it again *ed* can be cut off without cutting off the beginning, and so on to infinity. For even if my hand is unable and my soul unwilling to pursue the division to infinity, it can nevertheless in general be understood at once that everything that can be cut off without cutting off the beginning does not involve the beginning. And since parts can be cut off to infinity (for the continuum, as others have demonstrated, is divisible to infinity), it follows that the beginning of the line, i.e. that which is traversed in the beginning of the motion, is infinitely small" (A VI, III, 98–99; *LLC*, 12–13).

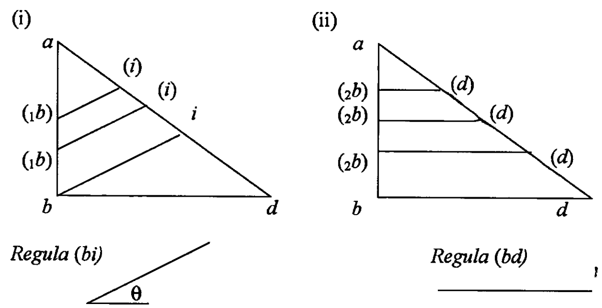
This argument, as before, depends on an assumption (in contradiction to Zeno) that the phenomenon of motion is real, and (in agreement with Zeno) that, in order for there to be a real motion, it must have an intelligible beginning. From this, however, a contradiction is derivable, if infinitesimals are thought of as preexisting parts of space and body:

> "I shall show that if there is some space in the nature of things distinct from body, and if there is some body distinct from motion, then indivisibles must be admitted. But this is absurd, and contrary to what has been demonstrated. Suppose we understand a point as an infinitely small line, there being one such line greater than others, and this line is thought of as designated in a space or body; and suppose we seek the beginning of some body or of a certain space, i.e. its first part; and suppose also that anything from which we may cut off something without cutting off the beginning cannot be regarded as the beginning: with all this supposed, we shall necessarily arrive at indivisibles in space and body. For that line, however infinitely small it is, will not be the true beginning of body, since something can still be cut off from it, namely the difference between it and another infinitely small line that is still smaller; nor will this cease until it reaches a thing lacking a part, or one smaller than which cannot be imagined, which kind of thing has been shown to be impossible" (A VI, III, 99–100; *LLC*, 14–17).

23 Although Leibniz appears to have already distinguished his points from indivisibles in his letter to Arnauld of 1671 (see note 19 above), here he goes further, characterizing the infinitely small not as unextended points, but as infinitely small lines.

This is a curious line of reasoning. Leibniz argues that if the infinitely small elements of a line are unextended or indivisible, as he had concluded in the *TMA*, then they would be susceptible to Sextus's refutation. If, on the other hand, they are regarded as infinitely small lines, then, so long as they are extended, they will not be true beginnings as required by the Inverted Zeno argument. Here he finds a third option. This is to regard them as infinitely small lines *modulo* a particular generating motion: infinitely small lines are contingent on, and must be defined in terms of, the infinitely small beginnings of motion or *conatûs* (or endeavour).

> "But if a body is understood as that which moves, then its beginning will be defined as an infinitely small line. For even if there exists another line smaller than it, the beginning of its motion can nonetheless be taken to be simply something that is greater than the beginning of some other slower motion. But the beginning of a body we define as the beginning of motion itself, i.e. endeavour, since otherwise the beginning of the body would turn out to be an indivisible" (A VI, III,100; *LLC*, 16–17).



We can make sense of this as follows. Infinitely small lines are intelligible only in terms of their proportionality to the endeavours of corresponding generating motions. Thus if the infinitesimals ($_1b$) and (*i*) in the lines *ab* and *ai* are generated by the motion of a *regula*[24] parallel to *bi* moving from *a* to *bi*, the infinitesimals in *ab* will be equal to those in *ai*; but they will be of a different magnitude than the infinitesimals ($_2b$) in *ab* generated by the motion of a *regula* parallel to *bd* moving from *a* to *bd*. In fact, if the *regula* (*bi*) moves with velocity *v* for a time *t* to reach *bi*, and the second *regula* reaches *bd* in the same time, the infinitesimals ($_1b$) and (*i*) will be of magnitude (vt/∞) and the infinitesimals ($_2b$) and (*d*) of magnitude ($vt\sec\theta/\infty$), since the latter will be generated by a motion whose effective velocity is $v\sec\theta$. But what this means is that infinitesimals exist as elements or actual parts of a line only relative to a given generating motion. But the same real line cannot really be composed of infinitesimals corresponding to different motions, as *ab* is in

24 For the importance of the *regula* to Cavalieri's method, see Andersen: "Cavalieri's Method", pp. 299ff.

figures (i) and (ii). Yet the infinitesimals of lines from the two distinct motions can be compared.

Now the interpretation of Cavalierian points as indefinitely small lines is also in keeping with the interpretation Pascal gives them in his *Lettres de A. Dettonville, contenant quelques-unes de ses inventions de géometrie* (1659)[25]. Thus when Leibniz reads the *Lettres* on Huygens' suggestion in the first half of 1673, he is already in a state of total receptivity to Pascal's reading. Actually, however, as Enrico Pasini has perceptively observed, Pascal does not interpret Cavalieri's indivisibles directly as infinitely small lines. Rather, he interprets indivisible points as marking the divisions of a line into indefinitely many such infinitesimal lines, and indivisible lines as dividing a plane into indefinitely small rectangles or parts. On this reading, the parts are in each case homogeneous with the continuum they compose, rather than being indivisible elements of one fewer dimensions. Pascal had written

> "Let there be understood to be an indefinite multiplicity of planes between them, parallel and equally distant (this means that the distance from the first to the second is equal to the distance from the second to the third, and to that from the third to the fourth, and so on), which planes cut all the proposed magnitudes into an indefinite multiplicity of parts, each one comprised between any two of these neighbouring planes" (Pascal, *Lettres de A. Dettonville*, 7–8).

Pasini comments: "Such parts are, in distinction from the usual version of the method of indivisibles, comprised between the lines that individuate them, and not identical with them. They are therefore extended, and for this reason dimensionally homogeneous with everything of which they are a part"[26]. This contrasts, for instance, with John Wallis (whom Leibniz had also just read at Huygens' suggestion), who had regarded Cavalieri's planes as directly composed from lines, which he allowed might be equated with parallelograms[27]. Thus Wallis's method fudges over a dimensional difference, and cannot be said to be either clear or rigorous. On Pascal's interpretation, on the other hand, as Pasini explains, whenever a surface is covered with lines that divide the area, they are understood to be distributed over the infinitely small parts of the straight line taken as the base of the figure, each of which functions as a unity, so as to generate equal rectangles of indefinitely small size:

> "When one speaks of the *sum of an indefinite multiplicity of lines* one always has in view a certain line by the equal and indefinite parts of which they are multiplied. But when this line (by the equal portions of which they are understood to be multiplied) is not expressed, it is necessary to understand that it is that by whose division they originate [or by which they are multiplied]." (Pascal, *Lettres de A. Dettonville*, 11; Pasini, p. 53)

25 This is argued in detail by E. Pasini: *Il reale e l'immaginario*, esp. pp. 50–59.

26 "*Tali parti sono, a differenza che nella abituale versione del metodo degli indivisibili, comprese tra linee che le individuano e non identiche con esse. Sono dunque estese e percìo omogenee per dimensione con il tutto di cui sono parte...*" Pasini, pp. 51–52, my translation.

27 In his *Treatise on Conic Sections*, 1655, Prop. 1, Wallis had written of Cavalieri's planes as "composed of infinite parallel lines, or rather (as you may prefer) of infinite parallelograms of equal height, the height of each of which is therefore 1/∞ of the height of the whole". Cf. Pasini, pp. 45ff.

This neatly resolves the difficulty of dimensional homogeneity. Each line (or ordinate) is multiplied by an infinitesimal segment of the line which functions as a unity (since the ratio of such successive equal parts is one), so that the area of the figure is composed from an indefinite multiplicity of indefinitely small areas. On this Leibniz follows Pascal:

> "[I]n the Geometry of Indivisibles, when it is said that the sum of lines equals a certain surface or that the sum of surfaces equals a given solid, it is necessary for there to be given a unity, that is, for there to be a certain line to which they are understood to be applied, or into one of whose infinitely many equal parts, which represents the unity, they are multiplied, so that from them arise infinitely many surfaces, each of which is, however, smaller than any given surface" (Leibniz, Lh 35 15 1, f. 20; Pasini, p. 53).

Leibniz notes: "the *indivisibles* [of Cavalieri's *Geometria*] must be defined as infinitely small, or that whose ratio to a sensible quantity is infinity"[28]. Similarly, in *On Minimum and Maximum* he had defined the "infinitely small things" in the continuum as "things infinitely smaller than any given sensible thing"[29].

A full account of this stage of Leibniz's thinking on infinitesimals would include a detailed description of his method of sums and differences. As is well known, he generalized results obtained with difference series involving discrete finite differences to the case of continuous geometrical lines, which were regarded as composed of an infinity of infinitely small differences, or *differentia*. Thus given a series A, such as that of the reciprocal natural numbers $^1/_1 + {}^1/_2 + {}^1/_3 + {}^1/_4 + {}^1/_5 + {}^1/_6 \ldots$, and a second series B whose terms are the differences of consecutive terms of the original series, here $^1/_2 + {}^1/_6 + {}^1/_{12} + {}^1/_{20} + {}^1/_{30} + \ldots$, the sum of the B series of differences is the difference between the first and last terms of the original A series. Generalizing to infinitely small differences, the area under a curve $B(x)$, consisting in the sum ≡ of the infinitesimal elements $B(x)dx$ from $x = a$ to $x = b$, could be obtained analogously by taking the difference between the values of a second curve $C(x)$, $C(b) - C(a)$, where the curve $C(x)$ is constructed so that value of B at x is the slope of C at x, $\{C(x + dx) - C(x)\}/dx$.

Different "progressions of the variables" would correspond to which infinitesimal was regarded as being held constant, acting as the "unit" multiplied into the ordinates to preserve their dimensional homogeneity. Thus, in contrast to Wallis's "arithmetic of the infinite", an area would not be composed of an infinity of lines, but of an infinity of infinitesimal rectangles, the ordinates $B(x)$ of the derivative curve "applied to" (i.e. multiplied into) dx. But the elements $B(x)dx$ are not elements in an absolute sense, since one could equally have taken the dy's as units.

28 *De admirandis arithmeticae infinitorum paradoxa* (On the Wonderful Paradox of the Arithmetic of the Infinite); Lh 35 15 1, f. 20v; first half of 1673; translated from the passage quoted in Pasini, p. 54.

29 "*Infinite parva, seu infinities minora, quovis sensibili dato*"; A VI, III, 98; *LLC*, 12–13. The talk of "a more profound contemplation" also evokes Leibniz's boast in *De minimo* that "This wonderful method of demonstration, unnoticed by anyone else, became clear to me from a more intimate knowledge of indivisibles (*Mira et a nemine observata hâec demonstrandi ratio mihi patuit, ex interiore indivisibilium cognitione*)"; A VI, III, 99; *LLC*, 14–15.

Leibniz had thus come a long way from Cavalieri. But he retained the connection at the foundation of Cavalieri's method between the infinitesimals and the motions generating the figures. As he wrote to Malebranche in 1675, "it is necessary to maintain that the parts of the continuum exist only insofar as they are effectively determined by matter or motion"[30]. The relativity of the composition of the continuum from infinitesimal parts to the progression of the variables selected is still understood in terms of infinitesimals being defined by the endeavours of the corresponding generating motion.

By Spring of 1676, however, this situation has changed dramatically. In a paper written in early April, he refers to a "very recent demonstration" that endeavours are not, after all, infinitely small motions:

> "But on the other hand there is the great difficulty that endeavours are along tangents, so that motions will be too. For I have demonstrated elsewhere very recently that endeavours are true motions, not infinitely small ones" (A VI, III, 492; *LLC*, 75)[31].

The significance of this change of view cannot be understated. For it spells the demise of the actualist interpretation of infinitesimals of Leibniz's third theory. In a series of papers he strives to work out the significance of this for understanding the continuity of motion. But regarding infinitesimals themselves, from now on he regards them as useful fictions, without status as actual parts of the continuum.

His mathematical investigations, it is true, had already been pulling him in this direction. The regarding of the infinite and the infinitely small as fictions would seem to be concomitant on his rejection of infinite number already in 1672/73. For since the infinitely small quantities of *De minima* are inverses of infinitely large ones, as explained above, a rejection of infinite number and infinite wholes would seem to require some such interpretation of the infinitely small. And in October 1674, Leibniz explicitly describes infinite wholes as fictions (A VII, III, 468).

Pulling against this, on the other hand, were a variety of considerations concerning matter and substance. One of these was his belief that the unequal flow of fluid matter around a solid in a plenum would divide matter into a "multiplicity of infinitely many points [*infinitorum punctorum*] or bodies smaller than any that can be assigned" (A VI, III, 473; *LLC*, 47), that is, actually infinitely small parts of matter. To prevent such a dissolution of matter into a "powder of points", as he later called it, there must be atoms, "indissectible bodies", infinite multiplicities of points "held together by motion or a mind of some sort" (*ibid.*)[32]. Mind here is conceived as an organizing principle analogous to the "substantial forms" of the Scholastics[33]. Thus as late as February 1676 Leibniz was still vacillating over whether "there is something infinitely small, though not indivisible" ("On the Secrets of the Sublime", A VI, III, 474; *LLC*, 49): "Since we see that the hypothesis of infinites and

30 Letter to Malebranche, March-April 1675 (?): GP I, 322; Malebranche, *Oeuvres*, 97

31 Exactly what demonstration Leibniz is referring to here is unclear.

32 Leibniz's motivations, biological and theological, for believing in atoms, are explored in R. T. W. Arthur: "The Enigma of Leibniz's Atomism", in: *Oxford Studies in Early Modern Philosophy*, Volume 1 (2003), 183–227.

33 See also R. T. W. Arthur: "Animal Generation and Substance in Sennert and Leibniz", in: *The Problem of Animal Generation in Modern Philosophy*, ed. J. Smith (2005), pp. 304–359.

the infinitely small [*hypothesin infinitorum et infinite parvorum*] is splendidly consistent and successful", he reasons, "this increases the likelihood that they really exist" (A VI, III, 475; *LLC*, 51).

In April 1676, however, Leibniz comes to see that the infinite division of matter can be interpreted syncategorematically: "Being divided without end is different from being divided into minima, in that [in such an unending division] there will be no last part, just as in an unbounded line there is no last point" (A VI, III, 513; *LLC*, 119). In "Infinite Numbers" of April 10th any entity such as a line smaller than any assignable, or the angle between two such lines, is firmly characterized as "fictitious" (A VI, III, 498–99; *LLC*, 89). There are no such things *in rerum natura*, even though they express "real truths": "these fictitious entities are excellent abbreviations of propositions, and are for this reason extremely useful" (A VI, III, 499; *LLC*, 89–91). But if the unassignable is a fiction, then perfectly fluid matter consisting in unassignable points must be impossible, and so must atoms composed of such points. This is the view we find Leibniz adopting in the dialogue *Pacidius Philalethi* of November (NS) of the same year, and the arguments given there against the reality of atoms will be repeated for the rest of his intellectual career.

Conclusion

In this paper I have tried to document the changes in Leibniz's understanding of the infinitely small in his early work. What we find there is surprisingly rich and varied. Leibniz appears to have entertained in succession several significantly different theories of the infinitely small, from the one implicit in his original conception of continuous creation and motion in 1670 through to the interpretation of infinitesimals as fictions in 1676. In between he had developed a continuist and non-Archimedean theory, based on Hobbes's endeavours and Cavalieri's indivisibles, involving points lacking extension, and then a second interpretation of Cavalieri that made the infinitely small extended and homogeneous to the continuum they compose, but made their existence relative to a given motion. Even had he not developed the differential calculus, these theories of the infinitely small would hold great interest, and not only for their impact on the development of Leibniz's thought on natural philosophy and its metaphysical foundations. It is remarkable that the same thinker who provided one of the most subtle and convincing interpretations of infinitesimals as fictions should have first articulated three accounts of infinitesimals as actuals that closely anticipate features of several modern theories[34].

34 It is a pleasure to acknowledge Mark Kulstad for his diligence in arranging the Young Leibniz conference at which this paper was presented, and I thank those in attendance for their helpful comments. I am indebted also to my former institution, Middlebury College, for the sabbatical leave during which an earlier draft of this paper was written.

APPROACHING INFINITY: PHILOSOPHICAL CONSEQUENCES OF LEIBNIZ'S MATHEMATICAL INVESTIGATIONS IN PARIS AND THEREAFTER

Philip Beeley (Oxford)

I. Introduction

It is a commonplace in Leibniz studies that the young doctor of law and revision counsel at the Higher Court of Appeal in Mainz only became a mathematician during his stay in Paris between March 1672 and early October 1676. Although there is clear evidence that despite his lack of any formal mathematical training he had consulted works by authors such as Harsdörffer and Cardano as well as numerous books employing the mathematical method early in his career[1], it was only in the French capital and first and foremost under the tutelage of Christiaan Huygens[2] that he began to occupy himself seriously with questions in the exact sciences. As is also well known, this preoccupation was ultimately decisive for the length of his stay. Writing to Duke Johann Friedrich of Braunschweig-Lüneburg in 1679 he points out that the pursuit of mathematical knowledge was the true reason for his having remained as long in Paris as he did[3].

Nowhere is his earlier deficit in mathematics more apparent than in his bold claim in *Theoria motus abstracti* (1671) that by means of the innovative concept of point which he had presented in that work he had been able to place both Cavalieri's method of indivisibles and Wallis's analysis of infinites on a solid foundation[4].

1 See J. E. Hofmann: *Leibniz in Paris 1672–1676. His Growth to Mathematical Maturity*, Cambridge 1974, pp. 3–4. All dates in the following paper are given new style, i.e. according to the Gregorian calendar.

2 Leibniz acknowledged this publicly in "De solutionibus problematis catenarii vel funicularis in Actis Junii A. 1691, aliisque a Dn. J. Bernoullio propositis", in: *Acta eruditorum*, September 1691, pp. 435–439, p. 438.

3 Leibniz to Herzog Johann Friedrich, Autumn 1679 (?), A II, 1, 490: "je pretendois pour desabuser le monde la dessus, de me tirer un peu hors du pair en mathematiques, ou je croy avoir des découvertes, qui sont déja dans l'approbation generale des plus grands hommes de ce temps, et qui paroistront avec éclat quand je voudray. Ce fut là la veritable raison qui m'a fait rester si long temps en France, pour me perfectionner la dessus, et pour m'y mettre en quelque estime, car alors que j'y allois je n'estois pas encor assez geometre, ce qui m'estoit pourtant necessaire pour me rendre capable de proposer mes demonstrations avec rigueur".

4 *Theoria motus abstracti*, praef., A VI, 2, 262, fund. praedem. §5, A VI, 2, 265. See also Leibniz to Oldenburg, 11 March 1671, A II, 1, 90; Leibniz to van Velthuysen, beginning of May 1671, A II, 1, 97; Leibniz to Carcavy, 22 June (?) 1671, A II, 1, 126; Leibniz to Carcavy, 17 August 1671, A II, 1, 143. In all of these letters Leibniz only makes the claim with regard to Cavalieri.

While there had indeed been philosophical criticisms leveled against both these methods, most notably by Paul Guldin[5] and Thomas Hobbes[6] respectively, Leibniz failed to recognize the genuinely mathematical weaknesses they contained. Prominent among these is the apparent lack of rigor in Wallis's inductive method, to which Hobbes, who was certainly one of Leibniz's sources[7], explicitly refers. In effect, Leibniz's efforts at "saving" Cavalieri and Wallis make clear that before Paris he was able to mistake philosophical polemics for truly mathematical arguments.

Although Leibniz years later would write that mathematicians have just as much need to be philosophers as philosophers have to be mathematicians[8], one of the conclusions which he drew from criticisms of his own endeavors in analysis was that questions over the existence of infinitely small or infinitely large quantities, the "metaphysics of the geometers"[9], are largely misplaced in a mathematical context. Decisive considerations are rather those of suitability of concepts and efficacity of procedures. As he writes in *De quadratura arithmetica circuli* (1676?), whether or not the nature of things allows such quantities as infinitesimals to exist is for metaphysicians to dispute. For the geometer it is sufficient to demonstrate the correctness of that which follows, when these quantities are posited[10]. The demonstration of mathematical rigor alone suffices for such concepts, since this ultimately guarantees their usefulness and thus their benefit for human life, or as Leibniz occasionally puts it: they serve *ad usum vitae*[11].

5 See E. Festa: "Quelques aspects de la controverse sur les indivisibles", in: *Geometria e atomismo nella scuola galileiana*, ed. M. Bucciantini and M. Torrini, Florence 1992, pp. 193–207; P. Mancosu: *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*, New York and Oxford 1996, pp. 50–56.

6 See D. Jesseph: *Squaring the Circle: The War between Hobbes and Wallis*, Chicago and London 1999, pp. 177–81. On the background to the dispute between Hobbes and Wallis see also S. Probst: *Die mathematische Kontroverse zwischen Thomas Hobbes und John Wallis*, Hanover 1997 (Dissertation Univ. of Regensburg).

7 See Hobbes: *Six Lessons to the Professors of the Mathematiques* §5, in: *The English Works of Thomas Hobbes*, ed. William Molesworth, 11 vols., London 1839–45, VII, 308; Jesseph: *Squaring the Circle*, pp. 176–8. There is no evidence that Leibniz had even seen John Wallis's *Arithmetica infinitorum* (1656) when he had first set about placing the English mathematician's method on a firm foundation in Mainz.

8 Leibniz to Malebranche, 23 March 1699, GP I, 356: "Les Mathematiciens ont autant besoin d'estre philosophes que les philosophes d'estre Mathematiciens".

9 *Elementa rationis*, A VI, 4, 721: "[...] ut quod dixi Lumen Matheseos post tanti temporis Eclipsin rursus effulgeret, detectis atque auctis Archimedeis per indivisibilia et infinita inveniendi artificiis, quae Metaphysicam Geometrarum appellare possis".

10 G. W. Leibniz: *De quadratura arithmetica circuli ellipseos et hyperbolae cujus corollarium est trigonometria sine tabulis*, ed. E. Knobloch, Göttingen 1993, p. 133: "[...] An autem hujusmodi quantitates ferat natura rerum Metaphysici est disquirere; Geometrae sufficit, quid ex ipsis positis sequatur, demonstrare". See also *Reponse aux reflexions contenues dans la seconde edition du Dictionnaire Critique de M. Bayle*; GP IV, 569: "Les Mathématiciens cependant n'ont point besoin du tout des discussions métaphysiques, ni de s'embarrasser de l'existence réelle des points, des indivisibles, des infiniment petits, et des infinis à la rigueur"; *Quelques remarques sur le livre de Mons. Lock intitulé Essay of Understanding*, A VI, 6, 7.

11 See *De arte characteristica inventoriaque analytica combinatoriave in mathesi universali*, A

But philosophically-motivated foundational disputes such as those initiated by certain Cartesians[12] against concepts employed by Leibniz in his infinitesimal calculus are not the topic of this paper. Rather, we seek to address the question how, and to what extent, his prodigious investigations on algebra, number theory, geometry, and analysis from the beginning of his Paris sojourn up to his move to Hanover and thereafter had immediate consequences for the philosophical views he chose to adopt.

We shall begin by discussing the direct impetus which these investigations had on his ideas about infinity and then proceed to consider what implications these had in turn for his conception of the nature of mind. In this context we hope to show that there are close ties between Leibniz's general program of seeking a numerical approach to mathematical problems and his understanding of the source of our knowledge of eternal truths. An important part of the development of this in his early thought, it will be argued, was the rejection from 1672 onwards of his geometrical model of the mind and his focussing instead on the nature of truth and concepts. By taking a look, finally, at the role of universal character and its intimate relation to the nature of human thought, we shall endeavor to demonstrate that Leibniz was perfectly consistent in subscribing to a restricted form of Plato's doctrine of anamnesis, as far as mathematical knowledge is concerned and that this is part and parcel of his strong rationalist view that things are essentially expressed in number.

II. Early Views on the Infinite

As we know from Leibniz's mathematical letters and papers which have already appeared in the Academy Edition or which are about to do so, one of his central points of interest during his first nine months in Paris was arithmetical and geometrical progressions and series. Probably in September 1672[13], he informed Huygens at a meeting that he was in possession of a method which allowed the summation of certain progressions, whose sum was not yet known. Huygens thereupon set Leibniz the task of finding the sum of reciprocal triangular numbers, the result of which was already known to him but which he had not yet published. The young philosopher and mathematical apprentice eventually succeeded in summing triangular numbers as a series of differences of the harmonic series as well as finding the sums of other reciprocal figured numbers. These results were partly incorporated in

VI, 4, 329; Leibniz for Huygens, October 1674, A III, 1, 168; Leibniz for Prince Ferdinand of Tuscany, 28 May 1692, A I, 8, 260.

12 The main adversaries were a group of mathematicians at the Académie des Sciences, including Rolle, Philippe de la Hire, and Gallois. See Leibniz to Varignon, 2 February 1702, GM IV, 91–2; P. Costabel: "Pierre Varignon (1654–1722) et la diffusion en France du calcul différentiel et intégral", in: *Conférences du Palais de la Découverte*, series D, No. 108 (4 December 1965), p. 21; Mancosu: *Philosophy of Mathematics*, pp. 165–77. Comparing Rolle's criticism with that of Berkeley, Mancosu writes, "For Rolle, this finitism was embedded in the Cartesian refusal to admit infinitary mathematics as a rigorous discipline; for Berkeley, more explicit epistemological considerations accounted for the finitist commitment" (p. 177).

13 See *Historia et origo calculi differentialis*, GM V, 404.

the tract *Accessio ad arithmeticam infinitorum*, which he prepared for Jean Gallois at the end of 1672, probably with the intention that it be published in the *Journal des Sçavans* of which the French mathematician was editor at that time[14].

The *Accessio* bears testimony to the enormous strides which Leibniz made in mathematics in the space of less than a year. At the same time, it displays a remarkable growth in his understanding of the nature of infinity compared to the views put forward earlier in the *Theoria motus abstracti*. While there he had adopted an ontological approach to the continuum, seeking to reconcile infinite divisibility with the actual existence of parts by postulating points in such a sense that they could be conceived as constitutive entities[15], he now appeals to the argumentative force provided by mathematical proofs, such as those concerning infinite progression within finite limits:

> "He namely who is led by the senses will persuade himself that there cannot be a line of such shortness, that it contains not only an infinite number of points but also an infinite number of lines (as an infinite number of actually separated parts) having a finite relation to what is given, unless demonstrations compel this"[16].

Part of the goal of the *Accessio*, as Leibniz indeed makes clear in the first paragraph, is to demonstrate the impossibility of an infinite number. Here, as in numerous other contemporary pieces, he develops his position in contradistinction to arguments put forward by Galileo in the *Discorsi e dimostrazioni matematiche* (1638), in which the infinite number, understood as the number of all numbers, is purportedly compared to the unity. As Galileo argued, every number into infinity has its own square, its own cube, and so on, with the result that there are as many squares and cubes as there are roots or integers, which however is impossible[17]. The Pisan mathematician famously concludes from this that quantitative relations such as those of equality or greater than or less than do not apply to the infinite.

Effectively, Galileo negated the validity of the axiom *Totum esse majus parte* in infinite number. For Leibniz, who alongside the *Discorsi* had also carefully studied

14 It is probable that the *Accessio* was never published simply because the *Journal des Sçavans* temporarily ceased publication on 12 December 1672. When publication of the *Journal* resumed on 1 January 1674, the article would no longer have been considered topical.

15 For a detailed account of Leibniz's early model of the continuum see the author's *Kontinuität und Mechanismus. Zur Philosophie des jungen Leibniz in ihrem ideengeschichtlichen Kontext*, Suttgart 1996. This interpretation has recently been criticized by O. B. Bassler in "The Leibnizian Continuum in 1671", in: *Studia Leibnitiana* XXX/1 (1998), pp. 1–23. Bassler's view, that Leibniz around 1671 "attempts to develop a position in which the continuum is both composed out of unextended indivisibles and subject to the analysis of quantitative variation" (p. 21) is difficult to reconcile with the concept of the actual division of points which Leibniz postulated at that time.

16 *Accessio ad arithmeticam infinitorum*, A III, 1, 3: "Constat scientiam minimi et maximi, seu indivisibilis et infiniti inter maxima documenta esse, quibus mens humana sibi vendicat incorporalitatem. Quis enim sensu duce, persuaderet sibi, nullam dari posse lineam tantae brevitatis, quin in ea sint non tantum infinita puncta, sed et infinitae lineae (ac proinde partes a se invicem separatae actu infinitae) rationem habentes finitam ad datam, nisi demonstrationes cogerent".

17 *Accessio ad arithmeticam infinitorum*, A III, 1, 10–11. See also *Aus und zu Galileis Discorsi*, A VI, 3, 168; *Mathematica*, A VII, 1, 656–6.

the *Opus geometricum* (1647) of Grégoire à Saint Vincent, this was comparable to the Jesuit mathematician's negation of the validity of the axiom in horn angles (anguli contingentiae). In both cases the fundamental mistake in Leibniz's view was not so much the making of exceptions as the concept of infinity which motivated these: "that this axiom should fail is impossible, or, to say the same in other words, this axiom never fails except in the case of null or nothing". From the unrestricted validity of this axiom he draws the conclusion that such an infinite number is impossible, "it is not one, not a whole, but nothing"[18]. Then, as he proceeds to explain in the *Accessio*, employing an argument which is also to be found in contemporary algebraic studies, not only is $0 + 0 = 0$, but also $0 - 0 = 0$. An infinity which is produced from all units or which is the sum of all is namely on his opinion nothing, "about which nothing can be known or demonstrated and which has no attributes"[19].

In negating the possibility of truly infinite magnitudes, Leibniz rejects the concept of there being parts to the infinite or of one infinite quantity being larger, smaller or equal in relation to another. But, as he makes clear in one of numerous contemporary studies, mathematical practice is not affected by this. Wallis's arithmetic of the infinites and Cavalieri's geometry of indivisibles no more fail on account of the absence of a genuine metaphysical infinite than do surd roots or imaginary dimensions[20]. When it comes to reconciling his philosophy with the conceptual demands of mathematics and the physical sciences, Leibniz is decidedly pragmatic[21]. It is later one of his favorite topoi in discussions on the infinite that the infinitely small quantities employed in his calculus are simply useful fictions[22] or are tolerably true (toleranter verae)[23] concepts, allowing proofs to a degree of error which is smaller

18 *Accessio ad arithmeticam infinitorum*, A III, 1, 11: "At axioma illud fallere impossibile est seu, quod idem est, axioma istud nunquam, id est non nisi in nullo seu nihilo fallit. Ergo numerus iste infinitus est impossibilis, non unum, non totum, sed nihil". See also *De minimo et maximo, de corporibus et mentibus*, A VI, 3, 98; *Aus und zu Galileis Discorsi*, A VI, 3, 168; *Mathematica*, A VII, 1, 656; E. Knobloch: "Galileo and Leibniz: Different Approaches to Infinity", in: *Archive för History of Exact Sciences* 54 (1999), pp. 87–99, p. 94.

19 *Mathematica*, A VII, 1, 657: "Nam: 0+0=0. Et 0–0=0. Infinitum ergo ex omnibus unitatibus conflatum, seu summa omnium est nihil, de quo scilicet nihil potest cogitari aut demonstrari, et nulla sunt attributa". See also *De bipartitionibus numerorum eorumque geometricis interpretationibus*, A VII, 1, 227.

20 *De progressionibus et de arithmetica infinitorum*, A VII, 3 69: "Infinitum ergo nihil est, nec totum habens nec partes et infinitum unum altero nec est maius nec minus nec aequale, quia nulla est infiniti magnitudo. Sed arithmetica infinitorum et geometria indivisibilium, non magis fallunt quam radices surdae et dimensiones imaginariae et numeri nihilo minores".

21 Leibniz's pragmatism with respect to contemporary results in the physical sciences is discussed in the author's "A Question of Approach: Material Bodies, Ideal Entities, and the Continuum in Leibniz" (forthcoming).

22 See for example Leibniz to Masson, after 12 October 1716, GP VI, 629.

23 See for example *Epistola G. G. L. ad V. Cl. Christianum Wolfium, professorem matheseos Halensem, circa scientiam infiniti*, GM V, 385; *Observatio quod rationes sive proportiones non habeant locum circa quantitates nihilo minores, et de vero sensu methodi infinitesimalis*, GM V, 388; C 581.

than any error which can be given. This, he asserts, is sufficient in order to demonstrate certainty as well as usage[24].

On a practical level Leibniz allows that the infinitely small in a true sense of magnitudes diminishing towards zero be substituted by quantities smaller than any quantities which can be given, thus enabling the mathematician to avoid metaphysical disputes right from the outset. From this point of view, he regards it as being sufficient that we distinguish three grades of the infinite. These range, as he makes clear in remarks he wrote on Spinoza's *Ethics* (1677), from the lowest or mathematical infinite, understood in the pragmatic, non-metaphysical sense, to the highest or absolute which is anterior to every form of composition and which is to be identified with the deity. The requirements of mathematics are thus met outside the sphere of metaphysics:

> "I generally say that there are three grades of infinity; the lowest is for example the asymptote of a hyperbola and this is alone that which I generally call infinite. It is larger than any particular magnitude that can be given. About the others this can be said. One is namely the maximum of its sort, as the maximum of extension is the whole of space, the maximum of all succeeding things is eternity. The third grade of infinity is itself the highest, everything, just as the infinite in God, then this is one totality, in this namely the requisites for the existence of everything else are contained"[25].

The development of Leibniz's views on the infinite from 1672 onwards evidently went hand in hand with his own mathematical investigations. In numerous manuscripts on number theory, algebra, and series we can find precursors of more overtly philosophical writings on the topic. The recognition of the need to develop the concept of the infinitely small outside the constraints of metaphysics is just a part of this development. However, precisely through his work on mathematical topics he was able to acquire a deeper understanding of the nature of the infinite. It is thus a reflection of his own experience when he emphasizes the importance of geometry in increasing the perfection of judgement and invention in that which is most powerful in us, namely the mind[26]. Similarly, he describes the mathematical sciences in

24 *De organo sive arte magna cogitandi*, A VI, 4, 159: "Quid autem de tribus his continuis sentiendum sit videtur pendere ex consideratione perfectionis divinae. Sed Geometria ad haec assurgere necesse non habet. Nam etiamsi non darentur in natura nec dari possent rectae ac circuli, sufficiet tamen dari posse figuras, quae a rectis et circularibus tam parum absint, ut error sie minor quolibet dato." See also Leibniz to Des Bosses, 11 March 1706, GP II, 305; *Reponse aux reflexions contenues dans la seconde edition du Dictionnaire Critique de M. Bayle*, GP IV, 569; *Théodicée* § 70, GP VI, 90; Leibniz to Masson, after 12 October 1716, GP VI, 629; *Nouveaux Essais* II, 18, §3, A VI, 6, 158; Leibniz to the Electress Sophie, 31 October 1705, GP VII, 561–2.

25 *On Spinoza's Ethics*, A VI, 3, 385: "Ego soleo dicere: tres esse infiniti gradus, infimum v.g. ut exempli causa asymptoti hyperbolae; et hoc soleo tantum vocare infinitum. Id est majus quolibet assignabili; quod et de caeteris omnibus dici potest; alterum est maximum in suo scilicet genere, ut maximum omnium extensorum est totum spatium, maximum omnium successivorum est aeternitas. Tertius infiniti, isque summus gradus est ipsum, omnia, quale infinitum est in Deo, is enim est unus omnia; in eo enim caeterorum omnium ad existendum requisita continentur". See also *Communicata ex literis Domini Schulleri*, A VI, 3, 281–2; Leibniz to Des Bosses, 11 March 1706, GP II, 305; *Nouveaux Essais* II, 17, § 1, A VI, 6, 157.

26 *De usu geometriae*, A VI, 3, 449: "[...] perfectio autem nostra sit inprimis perfectio ejus quod in nobis potissimum est, id est mentis, Mentis autem vim ac judicandi atque inveniendi potestatem egregie augeat geometria".

a letter to the Duchess Sophie in 1691, as being "of marvellous assistance" for the very reason that through them we are able to have "accurate and solid knowledge of the infinite itself"[27].

III. Ancient Learning and the Reduction of Things to Numbers

But the position which Leibniz holds from Paris onwards goes much deeper than simple recognition of the insights which mathematics is able to provide into the infinite. In this regard another remark which he makes in the *Accessio* is of decisive importance. He writes namely that he considers it to be an established fact that knowledge of the maximum and the minimum, or of the indivisible and the infinite is "among the most important proofs through which the human mind ascribes itself incorporality"[28]. While mind is able to grasp infinity by means of mathematics, the body is essentially determined by limits. Although already in *De quadratura arithmetica circuli* he appears to diverge from this position when he remarks that the nature of the mind itself and its operations, particularly reflection, suffice in order to distinguish it from body[29], this is a view he develops primarily in response to Gaston Pardies. Fundamental considerations on infinity continue to play an important role in Leibniz's philosophy. Thus in *Rationale fidei catholicae* he presents first of all the uncontroversial view that the infinite in the sense of whole does not pertain to things formed by the composition from parts. Conceived absolutely, the infinite is naturally and conceptually (natura sive conceptu) prior to the finite[30]. The important part of Leibniz's argument proceeds from this. Nothing, he asserts, prevents the mind from being conceived as infinite, such that it does not accept its thoughts from elsewhere. This is evidently a reference to the mind of God, understood as that which truly and absolutely encompasses all. However, he goes on to place knowledge of eternal truths such as those of mathematics, into which human minds gain insight, in a remarkable relation to divine knowledge, drawing thereby on Plato's doctrine of anamnesis:

> "However, in us eternal truths are not learnt from sense and experience, but flow from the nature of the mind itself, which is what Plato intended with his concept of a certain reminiscence. And every single mind is more perfect the less it has need to be taught by experience. The most perfect mind therefore conceives all a priori out of itself in the form of eternal truths"[31].

27 Leibniz to the Duchess Sophie, 2 November 1691, A I, 7, 48: "Les sciences Mathematiques sont d'un secours merveilleux pour nous faire avoir des connoissances justes et solides de l'infini même".

28 *Accessio ad arithemticam infinitorum*, A III, 1, 3: "Constat scientiam minimi et maximi, seu indivisibilis et infiniti inter maxima documenta esse, quibus mens humana sibi vendicat incorporalitatem".

29 *De quadratura arithmetica circuli*, p. 132. Cf. *Théodicée* § 69, GP VI, 89.

30 *Rationale fidei catholicae*, A VI, 4, 2308. See also *Aus und zu Malebranche, De la recherche de la verité*, A VI, 4, 1859: "L'idée de l'infini est avant celle du fini".

31 *Rationale fidei catholicae*, A VI, 4, 2309: "Mentem autem nihil prohibet infinitam concipi eam

As we shall see, this concept is of central importance to understanding the role of mathematics in Leibniz's philosophy. It finds its expression in his life-long interest for deciphering[32]–an art which he unfortunately did not master–, in his fascination with the tradition of the Jewish cabbala[33], and his approval of the Pythagorean[34] doctrine according to which everything is subject to or can be expressed by number. And, finally, it lies at the root of and is thus a prerequisite to understanding what he sets out to achieve by means of his concept of universal character.

From a mathematical point of view an important part of Leibniz's program consists as far as possible in the reduction of geometrical problems to expressions employing numbers. He states this quite explicitly in a letter to Gallois from the end of 1675, in which he notes at the same time the shortcomings of Descartes, who in his *Géométrie* (1637) had spoken as if all problems could be reduced to equations[35]. Not only is Cartesius's opinion in this respect untenable, Leibniz remarks, but also the closely associated one that most curvilinear problems are insolvable. While he concedes that Descartes, were he still alive at the time, would have recognized these views to have been mistaken, he nevertheless makes clear the considerable distance between him and the French philosopher, as being precisely represented by developments in analysis. Specifically, he is able to point out that where equations fail, nature has provided us with another means of reducing problems to numbers, namely the employment of numerical progressions. Then here the question of giving the magnitude of a certain curvilinear figure is reduced to finding the sum of arithmetical series. He refers thereby, perhaps not surprisingly, to none other than his own recent achievements:

> "Archimedes was the first to use this for the quadrature of the parabola: in our times Cavalieri and both Mr Fermat and Mr Wallis have prosecuted the matter further. But no one has up to now been able to find a row of numbers which expresses the ordinates of a circle which are always irrational"[36].

scilicet quae cogitationes suas aliunde non accipit, nam et in nobis aeternae veritates non a sensu et experientia discuntur, sed fluunt ex ipsa natura mentis, quod sua quoque reminiscentia voluit Plato; et unaquaeque Mens eo est perfectior, quo minus per experientiam discere opus habet. Mens perfectissima concipit omnia a priori et ex se ipsa ad modum aeternarum veritatum".

32 See the author's "Un de mes amis. On Leibniz's Relation to the English Mathematician and Theologian John Wallis", in: *Leibniz and the English-Speaking World*, ed. P. Phemister and S. Brown, Dordrecht 2007, pp. 63–81.

33 See *De arte characteristica ad perfeiciendas scientias ratione nitentes*, A VI, 4, 911; *Guilielmi Pacidii plus ultra*, A VI, 4, 675 and the author's "Leibniz on Wachter's Elucidarius cabalisticus: A Critical Edition of the so-called Réfutation de Spinoza", in: *Leibniz Review* 12 (2002), pp. i-viii, 1–14.

34 See *De numeris characteristicis*, A VI, 4, 264.

35 Leibniz to Gallois, end of 1675, A III, 1, 358: "[...] et les trop grandes promesses de Mons. des Cartes, qui parle dans sa geometrie, comme si tous les problemes se reduisoient aux equations".

36 Leibniz to Gallois, end of 1675, A III, 1, 358: "Archimede s'en servit le premier pour la quadrature de la Parabole: de nostre temps Cavalieri, et Messieurs Fermat et Wallis ont poussé la chose plus avant. Mais personne a pû encor trouver un rang de nombres qui exprimât les ordonnées du cercle, qui sont tousjours irrationelles".

The reduction of a relation to numbers is on Leibniz's view not only satisfying to the mind, since "we can understand numbers more perfectly than any other magnitudes"[37], but also corresponds to the simplicity and economy of nature. In this sense he calls his quadrature of conic sections arithmetical, since the magnitude of the circle, the ellipse, the parabola and so on is discovered by means of convergent infinite series, or as he writes, in terms of "the most simple series of numbers"[38]. On other occasions he refers to the drawbacks resulting from reliance on the imagination and indeed sees this as the reason why progress in the ancient tradition of analysis had long been impeded[39]. Thus in a draft review of L'Hospital's *Analyse des infiniment petits* (1696) he notes that the ancients had had an *Ars inveniendi* or analysis of their own "but did not pursue it so far, since they were obliged to employ the aid of imagination, which confounds itself in entangled figures"[40].

Along the same lines, too, he criticizes the approach adopted by François Viète in the modern tradition of algebra, describing the French mathematician as having "taken it into his head to consider numbers as lines and figures"[41]. He rejects this all the more, because in what appears to him to be the ancient origin of algebra, namely in Diophantus, this art is applied principally to numbers.

Not infrequently, Leibniz contrasts the limited scope of the problems dealt with in Descartes's geometry with the ability of his own method to calculate the magnitudes of most kinds of curves as representing the distinction between the traditions of Apollonios and Archimedes. While Descartes had been forced to exclude curves such as the cycloid and the quadratrix from his calculus, calling them mechanical, Leibniz emphasizes that he is able to find their properties by calculation without his having to employ the imagination (*sans me gêner l'imagination*)[42].

37 Leibniz to Colbert, December 1679, A III, 2, 918: "Tum enim sibi satisfacit mens nostra, cum rem ad numeros reduxit, quas prae caeteris magnitudinibus perfecte intelligimus".

38 Leibniz to Colbert, December 1679, A III, 2, 918: "Quadraturam ejusmodi Arithemticam voco, quoniam circuli magnitudinem ad Quadratum relatam simplicissima numerum serie exprimit, tametsi nondum eam ideo lineis exhibeat". See also Leibniz to Oldenburg, 30 March 1675, A III, 1, 203; Leibniz to Tschirnhaus, end of December 1679, A III, 2, 934; Leibniz to Gallois, end of 1675, A III, 1, 356.

39 Leibniz for Etienne Chauvin for the *Nouveau Journal des Sçavans*, A I, 13, 358. See also E. Knobloch: "Im freiesten Streifzug des Geistes (liberrimo mentis discursu): zu den Zielen und Methoden Leibnizscher Mathematik", in: *Wissenschaft und Weltgestaltung. Internationales Symposion zum 350. Geburtstag von Gottfried Wilhelm Leibniz vom 9. Bis 11. April 1996 in Leipzig*, ed. K. Nowak and H. Poser, Hildesheim, Zurich, and New York 1999, pp. 211–229, p. 222: "nach Leibniz' eigenem Verständnis war die Etablierung des Differentialkalküls ein dreifacher Akt der Befreiung: der Geometrie, des Geistes und der Vorstellung".

40 Leibniz for Etienne Chauvin for the *Nouveau Journal des Sçavans*, A I, 13, 358: "Les Mathematiciens ont coustume d'entendre l'art d'inventer sous le nom de l'Analyse. On sçait assez que les anciens en avoient une à leur mode mais elle ne les menoit pas fort loin, par ce qu'ils estoient obligés de s'aider de l'imagination, qui se confond dans les figures embrouillées".

41 Leibniz for Etienne Chauvin for the *Nouveau Journal des Sçavans*, A I, 13, 358: "Ils [sc. les anciens] avoient quelque chose de l'Algebre comme il paroist par Diophante, mais ils l'appliquoient principalement aux nombres; jusqu'à François Viete s'avisa de considerer les lignes et figures comme des nombres".

42 Leibniz to La Loubère, 27 October 1692, A I, 8, 485. See also Leibniz to Molanus (for Eck-

It suffices here to point out that Leibniz gives Descartes's mechanical curves the name transcendents, since instead of being second, third or any other degree the indeterminates enter into the degree itself[43]. All quadratures, centers of gravity, and a large part of the problems of mechanics, once these are reduced to pure geometry, are of this kind[44]. They are, he suggests, aptly dealt with by means of infinite series, not being reducible to equations[45]. But precisely in this respect the intimate relation between the infinite in mathematics and the infinite in nature becomes apparent, something to which Leibniz himself explicitly refers in a letter to Simon de la Loubère:

> "For this method serves principally to deal analytically with physico-geometrical problems, since my analysis is truly the analysis of the infinites (completely different from the geometry of indivisibles of Cavalieri and the Arithmetic of infinites of Wallis) and nature acts always by an infinity of changes"[46].

There are abundant remarks of this kind, reflecting Leibniz's conviction that nature can be shown to approach certain norms readily found in mathematics[47]. The architectonics of his metaphysical model in effect provides the foundation for the successful application of mathematics in modern scientific explanation which he conversely interprets as expressing not only divine benevolence but also a deep-rooted economy of the world system. It is precisely this combination of metaphysical and mathematical considerations which enables Leibniz to utter the expectation that by means of his calculus natural events will be more adequately grasped than has previously been the case:

> "For this reason it is now not surprising that certain problems on receipt of my calculus have found solutions which earlier could scarcely have been hoped for and which especially concern the transition from geometry to nature. Then traditional geometry is of little use as soon as the question of the infinite is involved, which is suitably involved in many operations in nature and whereby the Creator finds better expression"[48].

hard), beginning of April (?) 1677, A II, 1, 308–9; Leibniz to Gallois, end of 1675, A III, 1, 358; Leibniz to Kochánski, 20 August 1694, A I, 10, 513; Leibniz to Molanus, beginning of April (?) 1677, A II, 1, 307–8.

43 Leibniz to Tschirnhaus, end of May/beginning of April 1678, A III, 2, 427. See also Leibniz to Reyher, 20 August 1680, A III, 3, 253; Leibniz to Clüver, 10 September 1680, A III, 3, 263; G. W. Leibniz: *Ein Dialog zur Einführung in die Arithmetik und Algebra*, ed. E. Knobloch, Stuttgart-Bad Cannstatt 1976 (cited hereafter as *Dialogue on Arithmetic and Algebra*), pp. 86/7; Leibniz to La Loubère, 15 October 1691, A I, 7, 399; Leibniz to Bignon, 5 February 1694, A I, 10, 244–5. See also H. Breger: "Leibniz's Einführung des Transzendenten", in: *300 Jahre Nova methodus von G. W. Leibniz (1684–1984)*, ed. A. Heinekamp, Stuttgart 1986 (= *Studia Leibnitiana, Sonderheft* 14), pp. 119–132, p. 120.

44 Leibniz to La Chaise, end of April/beginning of May 1680, A III, 3, 191.

45 Leibniz to Clüver, 10 September 1680, A III, 3, 263: "Caeterum Transcendentia commode tractantur per series infinitas".

46 Leibniz to La Loubère, 15 October, 1691, A I, 7, 400: "Car cette Methode sert principalement à traiter analytiquement les problemes physico-geometriques parce que mon Analyse est proprement l'Analyse des Infinis (infiniment differente de la Geometrie des indivisibles de Cavalieri et de l'Arithmetique des infinis de Wallis) et la nature va tous jours par une infinité de changemens"; Leibniz to Papebroch, 20 August 1694, A I, 10, 517.

47 See for example *Response aus reflexions contenues dans la seconde Edition du Dictionnaire Critique de M. Bayle, article Rorarius, sur le systeme de l'Harmonie preétablie*, GP IV, 568.

Apart from the ability to deal with curves which were previously considered recalcitrant, an essential part of Leibniz's program consists in the employment of rigorous methods–one compares here his criticism of Wallis's use of induction[49]–and the overcoming of reliance on the imagination. "My arithmetic of the infinite is pure"[50], he writes in *De progressionibus et de arithmetica infinitorum*, "that of Wallis is figurate". Similarly, when Malebranche suggests that one of the advantages of geometry is that lines can represent to the imagination more things than the spirit can recognize, Leibniz makes the remark that this advantage is of no weight whatsoever in the search for truth "since these sensible expressions of incommensurable magnitudes reveal nothing to the spirit"[51].

It is exactly for this reason that Charinus, alias Leibniz, in the *Dialogue on Arithmetic and Algebra*[52] expresses his astonishment at Hobbes's attack on the use of algebra in geometry in *De corpore* (1655) and even more forcefully in the context of his dispute with Wallis in *Six Lessons to the Professors of the Mathematiques* (1656). While Leibniz always had a strong admiration for the English philosopher's reduction of thought to calculation, he could only put it down to age that Hobbes had such little insight into the importance of algebra, noting that he had made these attacks at the same time as he denied the fundamental proposition of Pythagoras.

For Leibniz, on the other hand, algebra, which he describes as "the science of magnitude or of the equal and the unequal", is only part of the higher science of combinatorics, which treats of forms of the similar and dissimilar[53]. As he writes to

48 Leibniz to Wallis, 28 May 1697, GM IV, 26: "Unde jam mirum est, Problemata quaedam post receptum calculum meum soluta haberi, quae antea vix sperabantur: et praesertim quae ad transitum pertinet a Geometria ad Naturam, quoniam scilicet vulgaris Geometria minus sufficit, quoties infiniti involvitur consideratio, quam plerisque naturae operationibus inesse consentaneum est, quo melius referat Autorem suum". See also Leibniz to Grimaldi, January/February 1697, A I, 13, 523; Leibniz to Chauvin, 7 May 1697, A I, 14, 155: "Feu Mons. Hugens, un des premiers Geometres du monde declara publiquement, que cette Methode [sc. de l'Analyse] donnoit des découvertes, aux quelles l'entrée paroissoit fermée auparavant. Et on reconnoit sur tout qu'elle sert pour faciliter le passage de la Geometrie à la Physique, par ce que la consideration des effects de la nature enveloppe ordinairement l'infini pour exprimer le charactere de son auteur"; Leibniz to the Duchess Sophie, 2 November 1691, A I, 7, 52; Leibniz to Masson, after 12 October 1716, GP VI, 629.

49 See Leibniz to Gallois, end of 1675, A III, 3, 359, 361; Leibniz to Tschirnhaus, end of June 1682, A III, 3, 655.

50 *De progressionibus et de arithmetica infinitorum*, A VII, 3, 102: "Arithmetica infinitorum mea est pura, Wallisii figurata".

51 *Aus und zu Malebranche, De la Recherche de la verité*, A VI, 4, 1892: "Mais cet avantage n'est pas fort considerable pour la recherche de la verité, puisque ces expressions sensibles des grandeurs incommensurables, ne découvrent rien à l'esprit".

52 *Dialogue on Arithmetic and Algebra*, pp. 76/77, 134/135, 196.

53 Leibniz to Tschirnhaus, end of May/beginning of April 1678, A III, 2, 449: "Verum mihi aliud longe est ars Combinatoria scilicet: scientia de formis seu de simili et dissimili, quemadmodum algebra est scientia de magnitudine seu de aequali, et inaequali"; *Remarques sur les Elemens de mathematiques de M. Prestet*, A VII, 2, 806; Leibniz to Jean Gallois, 19 December 1678, A

Tschirnhaus in 1678, combinatorics appears to differ little from general characteristic science, with the help of which suitable characters for algebra, music, as well as logic have been and can be thought out[54]. Elsewhere, he points to the importance of characters for the advancement of human thought[55]. It is then perhaps not surprising that one of the arts he groups under combinatorics is that of deciphering, "although in this it is not so much a question of composing as of breaking down composita and so to speak of investigating roots"[56]. Indeed, he finds a direct similarity between roots in algebra and the key which needs to be discovered in order to resolve a particular cryptogram[57]. And this already suggests a further comparison which Leibniz often draws, namely between *Ars deciphrandi* and the art of creating hypotheses[58]. Here he has in mind in particular the search for a rule or regularity when faced with a set of empirical or experimental data, corresponding broadly to the analytical side of the analysis-synthesis dichotomy. Similarly, he envisages an important role for the art of deciphering in mathematics, noting for example the occasional need to reduce irregular series to a rule[59].

At first glance, Leibniz's interest in deciphering appears to be explainable wholly through the question of the need to formulate hypothetical rules on the basis of a certain set of empirical or numerical data. But the deeper significance is that for Leibniz, when putting across his strong rationalist point of view, there is nothing in the world which is not capable of being grasped numerically, so that just as number is a kind of metaphysical figure, arithmetic itself represents a kind of statics of the universe[60]. In this context, too, he regards the ancient Jewish cabbala and the teachings of the Pythagoreans as expressing, each in their own way, the arcana rerum, the secrets of nature. However, Leibniz's position again goes further than this. Then, as he writes in *De numeris characteristicis*, it is as if God in giving the human race the sciences of arithmetic and algebra wanted to teach us that in our understanding a much greater secret lies hidden, of which these are only shadows[61]. It is precisely this concept which lies at the root of his famous remark that God in calculating and exercising thought brings about the world–"cum Deus calculat et cogitationem exercet, fit mundus"[62]–, against which in more exoteric writings he presents God instead as the most perfect geometer[63], or where he suggests that the base of divine reason in none other than the geometrical continuum[64]. In a very profound sense there is a numerical base to Leibniz's strong rationalist philosophy.

IV. Infinity and Conceptual Analysis

One aspect in which the infinite in mathematics is intimately tied up with Leibniz's philosophical thought is in the broader theory of concepts. On one of the rare occasions where he talks about the motives for the development of his ideas, he points out in *De natura veritatis, contingentiae et indifferentiae* that it was precisely knowledge of geometrical matters and of the analysis of the infinite which provided him with the insight that concepts are resolvable into infinity[65]. This knowledge showed him that while essential propositions such as those of mathematics could be demonstrated through resolution into terms which are necessarily or virtually identical, existential propositions are such that their truth can only be understood a priori by the one infinite mind and cannot be proven by any degree of resolution[66].

In view of this connection, it comes perhaps as no surprise that Leibniz provides a certain metaphysical explanation for the nature of contingent propositions, namely in the actual division of all natural bodies into infinity. Then this very correspondence between resolution of parts and resolution of concepts already formed the basis of the approach he adopted in the *Dissertatio de arte combinatoria* (1666)

III, 2, 566; Leibniz to Jakob Bernoulli, 4 October 1690, A III, 4, 582; *De synthesi et analysi seu arte inveniendi et judicandi*, A VI, 4, 545,

54 Leibniz to Tschirnhaus, end of May/beginning of April 1678, A III, 2, 449: "Imo Combinatoria parum differe videtur, a Scientia Characteristica generali, cujus ope characteres apti ad Algebram ad Musicam, imo et ad Logicam excogitati sund aut excogitari possunt".

55 Leibniz to Mariotte, Juli 1676, A II, 1, 271. See J. Maat: *Philosophical Languages in the Seventeenth Century: Dalgarno, Wilkins, Leibniz*, Dordrecht, Boston, London 2004, p. 296.

56 Leibniz to Tschirnhaus, end of May/beginning of April 1678, A III, 2, 449: "Hujus scientiae [sc. combinatoria] etiam portio est Cryptographia, quamquam in ea non tam componere quam resolvere composita et ut ita dicam radices investigare difficile sit".

57 Leibniz to Tschirnhaus, end of May/beginning of April 1678, A III, 2, 449–50: "Nam quod radix in Algebra, id clavis in Cryptographia Divinatoria." See also Leibniz to Schmidt, 20 March 1699, A I, 16, 639; *De synthesi et analysi universali seu arte inveniendi et judicandi*, A VI, 4, 545; *Remarques sur les Elemens de mathematiques de M. Prestet*, A VII, 2, 806.

58 See for example *De methodi quadraturarum usu in seriebus*, A VII, 3, 253; *De serierum summis et de quadraturis pars tertia*, A VII, 3, 406; *Praefatio ad libellum elementorum physicae*, A VI, 4,1999.

59 *De methodi quadraturarum in seriebus*, A VII, 3, 252–3; *De serierum summis et de quadraturis pars tertia*, A VII, 3, 406.

60 *De numeris characteristicis ad linguam universalem constituendam*, A VI, 4, 263–4: "Vetus verbum est, Deum omnia pondere, mensura, numero fecisse. Sunt autem quae ponderari non possunt, scilicet quae vim ac potentiam nullam habent; sunt etiam quae carent partibus ac proinde mensuram non recipiunt. Sed nihil est quod numerum non patiatur, Itaque numerus quasi figura quaedam metaphysica est, et Arithmetica est quaedam Statica Universi, qua rerum gradus explorantur.[...] nemo tamen aggressus est linguam aut Characteristicen, in qua simul ars inveniendi et judicandi contineretur: id est cujus notae sive characteres praestarent idem, quod notae Arithmeticae in numeris et Algebraicae in magnitudinibus abstracte sumtis. Et tamen videtur Deus cum has duas scientias generi humano largitus est, admonere nos voluisse, latere, in nostro intellectu arcanum longe majus, cujus hae tantum umbrae essent". See also *Principes de la nature et de la grace* § 14, GP VI, 604. Cf. Leibniz to Masson, after 12 October 1716, GP VI, 629.

61 *De numeris characteristicis ad linguam universalem constituendam*, A VI, 4, 264.

62 *Dialogus*, A VI, 4, 22.

63 *Definitiones cogitationesque metaphysicae*, A VI, 4, 1395: "Denique operationes Die sunt tanquam excellentissimi Geometrae qui optimas problematum constructiones exhibere novit"; *Specimen inventorum de admirandis naturae generalis arcanis*, A VI, 4, 1616–17.

64 Leibniz to the Electress Sophie, 31 October 1705, GP VII, 564.

65 *De natura veritatis, contingentiae et indifferentiae atque de libertate et praedeterminationae*, A VI, 4, 1516: "Sed cognitio rerum Geometricarum ataque analysis infinitorum hanc mihi lucem accendere, ut intelligerem, etiam notiones in infinitum resolubiles esse".

66 *De natura veritatis, contingentiae et indifferentiae atque de libertate et praedeterminationae*, A VI, 4, 1517. Cf. *Communicata es literis Domini Schulleri*, A VI, 3, 276.

draw thoughts together on the restricted level of mathematics, so too combinatorics or characteristica universalis is able to carry this out on a more general level. But this means, too, that the advantages of the mathematical approach are equally valid here. Thus truths can be demonstrated by handling characters "without any work of imagination or effort of the mind, just as occurs in arithmetic and algebra"[74].

Leibniz of course considered his characteristica universalis to have a much more profound significance than that of other universal language schemes[75] and indeed maintained that, for example, the improvement of communication, on which much weight had been placed within the Comenian framework in which these schemes had been largely developed was its least important aspect. Rather, it would serve to provide a readily graspable cognitive thread (filum meditandi), a method "coarse and perceptible", through which truths could be discovered and questions resolved[76]. But more than this, Leibniz saw it as being a means to improving the perfection of the human mind[77], precisely because he assumed that it would agree perfectly with our thoughts. Then, as he writes in *De modis combinandi characteres*, all our reasoning is nothing else but the connection and substitution of characters, whether these characters be words, marks or in some way likenesses of the things they represent[78].

On the basis of this assumption of a direct correlation between characteristica universalis and the nature of the human mind, Leibniz is moved on a number of occasions to draw a parallel to the optical instruments which played such a decisive role in the development of modern science. Surmising on the benefits his age might have accrued from the characteristica universalis, had work on it begun a hundred years earlier, he writes in a draft letter to Oldenburg 1675/6 that "no telescopic tube

riam, nunc characteristicam appellare soleo, longe diversam ab illis, quae auditis his vocibus statim alicui in mentem venire possent".

74 *De alphabeto cogitationum humanarum*, A VI, 4, 272: "Ajo atque: omnes veritates quae de rebus hac lingua exprimibilibus demonstrari possunt, sine adhibitione novarum notionum hac lingua nondum expressarum; eas omnes posse demonstrari solo calculo, sive sola tractatione characterum secundum certam quandam formam, sine ullo imaginationis labore aut mentis nisu, prorsus quemadmodum fit in Arithmetica et Algebra". See also Leibniz to Gallois, 19 December 1678, A III, 2, 570; Leibniz to Huygens, 20 October 1679, A III, 2, 875; Leibniz to Rödeken, 1708, GP VII, 32.

75 See *De numeris characteristicis ad linguam universalem constituendam*, A VI, 4, 264; Leibniz to Haak, February 1680, A III, 3, 83–4; Leibniz to Kochánski, July 1692, A I, 8, 350; Leibniz to Verjus, 12 December 1697, A I, 14, 840–1; O. Pombo: *Leibniz and the Problem of a Universal Language*, Münster 1987, pp. 79–81, 84–6; Maat: *Philosophical Languages*, pp. 301–2.

76 Leibniz to Bertet, September 1677, A III, 2, 237: "Mais le principal [sc. avantage] seroit qu'elle nous donneroit filum meditandi, c'est à dire une methode grossiere et sensible, mais asseurée de découvrir des veritez, et resoudre des questions ex datis; comme les operations et formules qu'on apprend aux apprentifs d'arithmetique conduisent en même temps pour ainsi dire leur main et leur esprit". See also Leibniz to Oldenburg, 1675/6, A II, 1, 241.

77 See Leibniz to Oldenburg, 1675/6, A II, 1, 241–2; Leibniz to Kochánski, July 1692, A I, 8, 350; Leibniz to Gallois, 19 December, A III, 2, 570.

78 *De modis combinandi characteres*, A VI, 4, 922: "Omnis Ratiocinatio nostra nihil aliud est quam characterum connexio, et substitutio. Sive illi characteres sint verba, sive notae, sive denique imagines".

or microscope would have added so much to vision as that instrument would have given to the capacity for reasoning"[79]. Even more effusive is his description in *De numeris characteristicis*, where after drawing the same comparison he goes on to allude to the image employed by Bacon in his *Novum organon*: "This constellation will bring us more use to those who traverse the oceans of research than the magnet ever gave the seafarers"[80].

It is, however, important to recognize that Leibniz's characteristica universalis has an even more profound aspect for the young philosopher, namely that it will bring our minds closer to us, in the sense of apperception, and at the same time disclose the inward form of things[81]. Since our thought on his view takes the form of a calculus, proceeding by the rules of combinatorics, a characteristic based on such a model will not only aid rational processes but also essentially mirror them as well. There is, however, another part to this conception which is equally important, and which refers back to the idea of there being a mathematical core to nature, something which Leibniz sets out in his writings already in the early 1670s. The fundamental consequence of this is that in employing the characteristica universalis we are in effect using a language in which things are essentially written. The role he ascribes to universal character goes hand in hand with his philosophical programm for the mathematization of nature.

V. Minds, Nature and Mathematics

Here, finally, appears the key to Leibniz's approval of a restricted form of Plato's doctrine of anamnesis. The intellectual world, of which the ancients spoke so strongly, he writes in the *Reponse aux reflexions de M. Bayle*, "is in God, and to a certain extent in us too"[82]. Since the human mind is conceived as an image of the divine mind, it contains not only – to varying degrees of distinction – knowledge of everything, but is also able to gain access to the highest form of knowledge by re-

79 Leibniz to Oldenburg, 1675/6, A II, 1, 241: "Non tubi, non microscopia tantum oculis adjecere, quantum istud cogitandi instrumentum menti capacitatis dedisset". See also *De numeris characteristicis ad linguam universalem constituendam*, A VI, 4, 268.

80 *De numeris characteristicis ad linguam universalem constituendam*, A VI, 4, 268: "Nec unquam acus magnetica plus commodi navigantibus attulit quam haec cynosura experimentorum mare tranantibus, feret".

81 Leibniz to Oldenburg, 1675/6, A II, 1, 241: "[...] nam post inventa pro visu proque auditu organa, menti ipsi age novum Telescopium construamus, quod non sideribus tantum, sed et ipsis intelligentiis nos propiores reddet; non tantum corporum superficies repraesentabit sed et interiores rerum formas deteget".

82 *Response aus reflexions contenues dans la seconde Edition du Dictionnaire Critique de M. Bayle, article Rorarius, sur le systeme de l'Harmonie preétablie*, GP IV, 571: "Ce Monde intellectuel, dont les Anciens ont fort parlé, est en Dieu, et en quelque façon en nous aussi". See also Leibniz to Weigel, mid-September 1679, A III, 2, 839, and the author's "Leibniz et la tradition platonicienne: les mathématiques comme paradigme de la connaissance innée", in: *Leibniz selon les Nouveaux Essais sur l'entendement humain*, ed. F. Duchesneau and J. Griard, Montréal and Paris 2006, pp. 35–47, p. 45.

flection. In this sense eternal truths in us "flow from the very nature of the mind"[83], they are "discovered by reasoning"[84]. Likewise in a letter to his former teacher Erhard Weigel, written in 1679, he asserts that individual minds result from a special mode of thinking of the divine mind, and adds: "in every individual mind there is a sort of omniscience, but confused, and the power to extend itself over the whole universe"[85].

It is well known that this is a concept which plays a pivotal role right through to the mature position of the doctrine of monads. Thus for example he writes in later years that the truth is that "we see everything in us and in our souls"[86] or that "it is through reflection on our thoughts that we know extension and bodies"[87]. But, as we have sought to show, one of the results of Leibniz's investigations into algebra and arithmetic was his recognition that truth in mathematics lies deeper than the symbolism in which it is expressed and that a suitably chosen system of symbols can not only approach but also actually mirror mathematical thought. The partial omniscience which Leibniz ascribes to us is therefore reflected first and foremost in the fact that the most fundamental knowledge, namely mathematical knowledge, is something to which we are not merely amenable, but which can with suitable tuition be drawn out of us.

Thus Charinus, alias Leibniz, in the *Dialogue on Arithmetic and Algebra* shows in a fashion redolent of Plato's *Meno* that a boy who is seemingly ignorant of mathematics can be directed to formulate certain basic algebraic propositions, and thereupon exclaims: "You see, that you are already to a certain extent an algebraist!"[88]. And similarly in the dialogue *Phoranomus* he alludes to his own experience in tutoring his friend Bodenhausen as a further example that Plato's doctrine is fundamentally correct[89]. In the dialogue *Pacidius philalethi*, written at a time when he began to have a marked interest both for the writings of Plato and for the topic of universal character[90], he sets out to do precisely the same in the context of the theory of motion. And finally Theophilus alias Leibniz in the *Nouveaux Essais* proclaims that the whole of arithmetic and the whole of geometry must be innate to our minds, so that we can discover mathematical knowledge as soon as we contemplate and order what our minds contain[91].

Now, then, it becomes clear why Leibniz in the *Praefatio ad libellum elementorum physicae* asserts that the most perfect method of investigating the innermost constitution of things is discovered a priori through contemplation of the divine author[92]. Everything is contained conceptually in God and everything is at root expressed in the most perfect way numerically. But our own minds are images of the divine. Correspondingly, the somewhat less perfect method which is open to us is to reflect on our own mental powers. And these powers are increased through applying ourselves to mathematics. The precise reason for this is that for Leibniz, according to his strong rationalist model, human thought is fundamentally reducible to a form of combinatorics or calculus. In this respect he is entirely in agreement with Hobbes.

From here it becomes clear, too, why Leibniz is able to claim that despite the ideality of mathematical considerations this does not diminish their usefulness. Then, as he explains in the *Reponse aux reflexions de M. Bayle*, existing things could not stray from their rules, indeed "one can say in effect that it is precisely in this that the reality of the phenomena consists"[93]. While on the one side the metaphysical foundation of mathematics is in the divine mind, so that mathematical knowledge is also discoverable in us a priori, on the other side everything in nature is essentially written in the language of mathematics and can be grasped by it. But the physical world cannot be completely grasped mathematically. Only by denying the reducibility of nature to a mathematical model is it possible for Leibniz to avoid that other consequence which for him ranks in its pernicious character alongside the materialism of Hobbes: the absolute necessity of the pantheistic monism of Spinoza.

83 *Rationale fidei catholicae*, A VI, 4, 2309, 2316–7: "nam et in nobis aeternae veritates non sensu et experientia discuntur, sed fluunt ex ipsa natura mentis, et conceptuum seu idearum"; *Sur ce qui passe les sens et la matiere*, GP VI, 490–1.

84 *De alphabeto cogitationum humanarum*, A VI, 4, 272: "Veritates enim (exceptis experimentis) non possunt inveniri aut dijudicari nisi per rationes"; *Animadversiones in partem generalem Principiorum Cartesianorum*, GP IV, 355; Leibniz to Bierling, 12 August 1711, GP VII, 500.

85 Leibniz to Weigel, mid-September 1679, A III, 2, 839: "Arbitror enim non tam mentem nostram in res agere quam Deum ad ejus voluntatem; et mentes oriri ex speciali quodam modo cogitandi divinae mentis; quin imo addo in omni mente esse quandam omniscientiam sed confusam et potentiam in totum universum sese extendentem sed refractam".

86 *Remarques sur l'ecrit de Locke: Examination of Malebranche's Opinion of Seeing all Things in God*, A VI, 6, 557: "La verité est, que nous voyons tout en nous et dans nos ames".

87 Ibid, "c'est par la reflexion sur nos pensées que nous connoissons l'etendue et les corps". See also *Principes de la nature et de la grace, fondés en raison*, GP VI, 601.

88 *Dialogue on Arithmetic and Algebra*, p. 29: "Hic autem modus calculandi dicitur Algebraicus, vides igitur te jam quodammodo Algebraicum esse".

89 *G. W. Leibniz: Phoranomus seu de potentia et legibus naturae*, ed. A. Robinet, in: *Physis* 28 (1991), dial. I, pp. 429–541, dial. II, pp. 797–885, p. 455. See also Leibniz to Oldenburg, 28 December 1675, A III, 1, 331.

90 As Leibniz reports in his letter to Gallois of September 1677, he used part of the time waiting on board boat on the Thames for suitable weather conditions for sailing to the Low Countries, to thinking about universal character. "En ce temps là ne sçachant que faire et n'ayant personne dans le vaisseau que des mariniers je meditois ces choses là, et sur tout je songeois à mon vieux dessein d'une langue ou ecriture rationelle, dont le moindre effect seroit l'universalité et la communication de differentes nations" (A III, 2, 228–9). It was also during this time on boat that he wrote the dialogue *Pacidius Philalethi*.

91 *Nouveaux Essais*, I, 1, §5, A VI, 6, 77: [Theoph.] "Dans ce sens on doit dire que toute l'Arithmetique et toute la Geometrie sont innées, et sont en nous d'une maniere virtuelle, en sorte qu'on les y peut trouver en considerant attentivement et rangeant ce qu'on a déja dans l'esprit sans se servir d'aucune verité apprise par l'experience, ou par la tradition d'autruy, comme Platon l'a montré dans un Dialogue, où il introduit Socrate menant un enfant à des verités abstruses, par les seules interrogations sans luy rien apprendre". See also the preface, A VI, 6, 52.

92 *Praefatio ad libellum elementorum physicae*, A VI, 4, 1998: "Methodus perfectissima est interiorem constitutionem corporum invenire a priori, ex contemplatione autoris rerum Dei".

93 *Reponse aux reflexions contenues dans la seconde edition du Dictionnaire Critique de M. Bayle*, GP IV, 569: "Ainsi quoyque les méditations Mathematiques soient idéales, cela ne diminue rien de leur utilité, parce que les choses actuelles ne sauroient s'écarter de leurs règles; et on peut dire en effect, que c'est en cela que consiste la réalité des phénomènes, qui les distingue des songes".