THE 1900 TURN IN BERTRAND RUSSELL'S LOGIC, THE EMERGENCE OF HIS PARADOX, AND THE WAY OUT

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Abstract

Russell's initial project in philosophy (1898) was to make mathematics rigorous reducing it to logic. Before August 1900, however, Russell's logic was nothing but mereology. First, his acquaintance with Peano's ideas in August 1900 led him to discard the part-whole logic and accept a kind of intensional predicate logic instead. Among other things, the predicate logic helped Russell embrace a technique of treating the paradox of infinite numbers with the help of a singular concept, which he called 'denoting phrase'. Unfortunately, a new paradox emerged soon: that of classes. The main contention of this paper is that Russell's new conception only transferred the paradox of infinity from the realm of infinite numbers to that of class-inclusion.

Russell's long-elaborated solution to his paradox developed between 1905 and 1908 was nothing but to set aside of some of the ideas he adopted with his turn of August 1900: (i) With the Theory of Descriptions, he reintroduced the complexes we are acquainted with in logic. In this way, he partly restored the pre-August 1900 mereology of complexes and simples. (ii) The elimination of classes, with the help of the 'substitutional theory',[1] and of propositions, by means of the Multiple Relation Theory of Judgment,[2] completed this process.

1. Russell as a Mereologist

In 1898, Russell abandoned his short period of adherence to the Neo-Hegelian position in the philosophy of mathematics and replaced it with what can be called the 'analytic philosophy of mathematics', substantiated by the logic of relations. To be more exact, Russell took his first step in this direction after reading A N Whitehead's *A Treatise on Universal Algebra*

[1] Cf. Landini (1998).

[2] Cf. Milkov (2013).

in March 1898. In contrast to his philosophy of mathematics, Russell's logic became predominantly analytical only after he read Cantor's *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* at the beginning of July 1899. This change is documented in the manuscript of 'Fundamental Ideas and Axioms of Mathematics' (1899), in which Russell adopted a full-blooded mereology for the first time. Now, he claimed that the relation of 'logical priority', understood as the relation between the whole and the part, is central in logic.

In Russell's part-whole logic, the logical consequence holds between both terms and propositions. Russell maintained that 'it is possible for simple concepts [i.e., not only propositions] to imply others' (1899, p. 293). In other words, at that point in time, what was later called the relationship of 'ontological dependence' played the central place in Russell's logic. He further argued that:

> '*A* implies *B*' cannot *mean* '*A*'s truth implies *B*'s truth'; for here a simpler case of implication is explained by one which is more complex. '*A* implies *B*' *implies* '*A*'s truth implies *B*'s truth' and also *implies* '*B*'s falsehood implies *A*'s falsehood'. But '*A* implies *B*' applies to *A* and *B* simply as propositions, and quite independently of their truth or falsehood.' (Ibid., p. 292)

Russell's mereological logic had three sources:

(i) His basic categories of whole and part were connected 'with Boole's logical system' (Moore, 1993a, p. xxiii). Indeed, in dropping 'any use of magnitude and studying objects defined by their laws of combination alone', Russell followed the spirit of Boole (see Bornet 1995, p. 238).

(ii) The 1899–1900 logic of Russell was inspired by G. E. Moore's Relational Theory of Judgment, as stated in Moore's paper 'The Nature of Judgment' (1899). It maintained that 'all concepts of a proposition are to be regarded at the same logical and ontological level, together with the "external" relations joining them, which must be seen as terms as real as the rest' (Rodríguez-Consuegra 1991, p. 32). In fact, this single-level logic was the archetype of Russell's logical explorations at the time.[3]

(iii) Another much older impulse to adopt mereology came directly from absolute idealism. Indeed, absolute idealism, both German and British, was essentially mereological. [4]

[3] It so happened, however, that Russell was urged two times to substantially revise it (cf. § 6).

[4] See Milkov (1997), i, pp. 82–3.

2. The Turn

In the first year of the new millennium (1900), Russell gradually adopted two novelties in his logic, developed in full first in the *Principles of Mathematics* (1903): the 'material implication' and the predicate logic in the form of the theory of denoting. Both, in particular, changed Russell's methods of constructing units.

2.1 Material Implication

Besides the relationship between the part and the whole, under the continuing influence of Moore's work in philosophical logic (especially of his paper 'Necessity', 1900) in the first few months of 1900, Russell embraced the view that there is another relationship, that of implication, which holds only between propositions, not between terms. Russell soon found out that this relation is logically more fundamental.

In fact, the idea of what was termed 'material implication' a year later, was already articulated in the manuscript '*The Principles of Mathematics*, Draft of 1899–1900'. The material implication here is set out in Moorean terms: 'Whenever *A* implies *B*, we have also the following propositions: *A*'s truth implies *B*'s truth, and *B*'s falsehood implies *A*'s falsehood' (1900, p. 36). However, Russell still believed that a proposition can imply a term.

The radical and consequent turn against the part-whole logic was taken only after Russell became acquainted with the works of Peano. In October 1900, two months after the First International Congress of Philosophy held in Paris where Russell met Peano, he noted: 'I have been wrong in regarding Logical Calculus as having specially to do with whole and part. *Whole* is distinct from *Class*, and occurs nowhere in the Logical Calculus' (1993b, plate II). A few weeks later, he wrote in an article on Peano:

> 'It has been one of the bad effects of the analogy with ordinary Algebra that most formal logicians (with the exception of Frege and MacColl) have shown more interest in logical equations than in implication' (1901*a*, p. 353).

Russell's argument for adopting material implication as a basic relationship in logic was that the part-whole relation, or the 'logical priority of *A* to *B* [,] requires not only "*B* implies *A*", but also "*A* does not imply *B*"' (1903, § 134). In contrast, material implication is transitive. Consequently, this relation is rather simple than part-whole, and thus, more fundamental and more appropriate as a logical constant.

Be as it may, some remnants of the old mereology remained in Russell's logic. Thus, he continued to speak about logical summing of terms, and not only of propositions. By Peter Geach's account, '[t]o a contemporary logician the idea of a disjunction of proper names may well seem alien' (Geach 1962, p. 66). This case shows that in the second half of 1900, Russell did not embrace his new logical conception without reservation.

The acceptance of the material implication as the simplest relation in logic, with the help of which all pure mathematics can be deduced (see 1903, § 1), made Russell's logic *intensional*. Apparently, this turn towards intensional logic was also connected with his treatment of infinity, which we will discuss in the next subsection.

2.2 Theory of Denoting and Predicate Logic

In *The Principles of Mathematics,* Russell also formulated his Theory of Denoting (see 1903, Ch. V) which, in fact, can be seen as nothing but his interpretation of Frege's technique of quantification. The central point of the Theory of Denoting is that there are two kinds of denoting: intuitive (immediate) or proper, and symbolic or improper.[5] *Things* have proper meaning, and *concepts,* improper. This is the case because while things occur in a proposition as terms, concepts give rise to 'classes as one' (or class-concepts), which are combinations of terms. They differ from 'classes as many', which are merely aggregates of terms. In other words, in a class-concept, 'one predicate occurs otherwise than as a term' (§ 57). More particularly, it gives rise to *denoting*. 'A concept *denotes* when, if it occurs in a proposition, the proposition is not *about* the concept, but about the term connected in a certain peculiar way with the concept' (§ 56).[6] It does not refer to the term directly, but in its interrelationship with other terms. In other words, Russell's Theory of Denoting employs what Michael Dummett later called 'Frege's context principle'. Typical examples of denoting concepts are phrases that are quantified, and in particular, phrases that contain propositional functions.

To use Frege's words, denoting phrases put stress on the 'organic connection' of the propositional function (or concept) with the arguments (or objects) that fall under it. Functions are not just aggregates (heaps) of arguments (or objects, or individuals). This explains why a propositional function is valid for *any* argument that falls under it, and a concept defines *eve-*

[5] Incidentally, Husserl had already accepted this in 1894 (see Coffa 1991, pp. 101–2).

[6] This conception gave rise to the fruitful ontology of ways, further explored by Wittgenstein (cf. Milkov 2017).

ry object (individual) that falls under it. In contrast, both mereologists and Boolean logicians completely disregard the distinction between function, or concept, and argument, or individual.

Moreover, predicate logic is the logic of many-orders (types). Its ontology embraces: (i) the order of individuals, (ii) that of functions (class-concepts), (iii) that of classes of classes, etc. It is true that the first-order predicate logic did not do obvious harm. The trouble became visible only when Frege introduced (in 'Function and Object', 1891) the notion of 'extension of concepts' that are objects themselves. Incidentally, in this way, Frege disparaged his own principle that logic has radically different orders—a mistake Russell tried to eliminate with his Theory of Types (more about this in § 7).

2.3 Theory of Denoting and the Treatment of Infinity

Russell was especially enthusiastic about his Theory of Denoting since it introduced a new technique for (intensively) treating infinite collections, including infinite numbers, with the help of a singular concept. Here is this story in short.

In his idealistic period, Russell, the Neo-Hegelian, was very sensitive to philosophical and logical paradoxes (cf. § 5). To be more exact, up to 1898, he believed that there are three paradoxes: of infinitesimals, of continuity, and of infinity. The latter has two forms: of actual infinity, and of infinite numbers.

After his turn of 1898, he changed his mind. This change happened in three steps:

(i) In 1898, Weierstrass convinced him to banish the *infinitesimals*: There is no such thing as the 'next'. There are no infinitesimal moments, places, etc.[7] There are only elements of finite size that are ordered in different ways.

(ii) Russell eliminated the antinomy of *continuity* after embracing Cantor's set theory in 1899.

(iii) Based on Cantor's diagonal method, he also resolved the problem of the actual infinity.

[7] Russell thought of logic ontologically (he would say 'realistically'). He was strongly convinced that 'logic is concerned with the real world just as truly as zoology, though with its more abstract and general features' (1919, p. 169) (see § 5).

Russell, however, did not adopt Cantor's treatment of infinite ordinal and cardinal numbers. More precisely, he did not accept their existence. As a consequence, he still believed that:

> 'Mathematical ideas are almost all infected with one great contradiction. This is the contradiction of infinity. All antinomies, I believe, so far as they are valid at all, will be found reducible to the antinomy of infinite number' (1900, p. 70).

Exactly at this point, Peano's logic helped Russell. Or rather, he tried to solve his 'great contradiction' via Peano's predicate logic.

A typical example of a denoting concept (phrase) is the infinite collection denoted by the concept 'all numbers' (the importance of this example will be discussed in § 3).[8] The point in question is that, having no direct connection with the referent, one denoting phrase can refer—in the most precise way—to many, including *infinitely* many, terms. In fact, one of Russell's reasons to introduce the denoting phrases thus understood was his endeavour to treat with their help infinite numbers without paradoxes.

To be more exact, according to Russell's Theory of Denoting (1903), there are five possible forms of denoting, which are nothing but five different ways of referring to terms of constructed unities, or totalities. These are characterised by five words 'of constant occurrence in daily life': *all*, *every*, *any*, *a*, *some*. (i) *All* means a *numerical conjunction* ('Brown and Jones are two of Miss Smith's suitors'). (ii) *Every* means a *propositional conjunction* ('Brown and Jones are paying court to Miss Smith'). (iii) *Any* is a *variable conjunction*, which is something between conjunction and disjunction ('if it was Brown or Jones you met, it was a very ardent lover'). (iv) *A* gives rise to the *variable disjunction* ('if it was one of Miss Smith's suitors, it must have been Brown or Jones'). (v) *Some* gives rise to the *constant disjunction* ('Miss Smith will marry Brown or Jones').[9]

[8] It deserves notice that already before the Paris Congress, Russell was conscious that the problem of totality, where *all* and *any* describe various forms of the permutations in a set, is indeed 'intimately connected' but nevertheless different from that of the whole and part. For example, he maintained that '*all* cannot be defined numerically', but it nevertheless means a perfectly specified notion. (1900, p. 41).

[9] This analysis of compositionality is a good example of Russell using ordinary language as a compass in philosophy. Ironically enough, in the 1950s, he was strongly against this approach.

Of special interest are the two ways of referring to infinite unities: points 'i' and 'iv'. As already mentioned, at the centre of Russell's argument was the claim that there are two kinds of constructing wholes: aggregates and units. The aggregate 'is definite as soon as [all] its constituents are known' (1903, § 135). In contrast, the unit is intensional. In the aggregate, we have a 'numerical conjunction of terms', while in units, we have a 'variable disjunction'. The first is a simple summative class and the second is a class-concept, or predicate, or propositional function. Russell further claimed that the unit is logically more fundamental than the aggregate. Indeed, it is the unit that helps resolve the paradox of infinity number.

Two final remarks:

(i) Besides aggregates and units, in *The Principles of Mathematics*, Russell also held that there is a relation between subordinate aggregates (not between an aggregate and a term) that can be called a relation between the whole and the part proper. This means that in 1903, Russell did not completely disregard mereology.

(ii) Towards the end of his study of the forms of denoting, Russell found a sixth form of denoting, indicated by the definite article '*the*'. Unfortunately, he had no time to make a precise analysis in the *Principles*. He did it two years later, in his famous Theory of Descriptions, in the paper 'On Denoting' (1905). This development, however, lies beyond the scope of this paper.

To sum up the results we achieved in this section: Russell was fascinated with the technique of the quantification of Frege and Peano because it allegedly resolved something of greatest importance to him—the paradox of infinite numbers. As a consequence, he started to consider unities (totalities) as theoretically more important than aggregates.

3. First Symptom that the Turn Produced Problems: the Paradoxes

So far, we have found that in an attempt to escape the paradox of infinity, in *The Principles of Mathematics*, Russell assumed that class-concepts (propositional functions) and objects, or individuals (arguments) are radically opposite things. In this, he followed the new many-ordered logic of Peano, and ultimately of Frege, which embraced an opposition between class-concepts on the one hand, and individuals and terms on the other.

Unfortunately, as a by-product of this conception, another paradox emerged—that of classes. All this suggests, and this point shall be examined in a while, that Russell's Peano-Fregean turn[10] did not eliminate the paradox of infinity, but merely *removed* it from one realm into another—that is, from the realm of infinite classes to that of class-inclusion.

This point did not go unnoticed by commentators. According to one of them, Gregory Moore, all the three paradoxes Russell tried to resolve have the same structure. These paradoxes are: (i) that of an infinite ordinal number (discovered in July 1899), (ii) that of the largest cardinal number (discovered in November 1900), and (iii) Russell's paradox proper (his paradox of classes, discovered in May 1901). Apparently, 'this structure was presented in the back of his mind as a kind of template that could be unconsciously applied to Cantor's work on infinite number' (Moore 1995, p. 236). The rest of this section shall try to specify this template in more concrete terms.

Russell was convinced that in each contradiction, 'there is a common characteristic, which we may describe as self-reference or reflexiveness' (1908, p. 61). Again, in 1959, he wrote: 'in all the logical paradoxes there is a kind of reflexive self-reference which is to be condemned on the same ground: viz. that it includes, as a matter of totality, something referring to that totality which can only have a definite meaning if the totality is already fixed' (p. 63).

Contrary to this conception of the unity of logical paradoxes, after the discovery of Russell's paradox proper, the paradox of classes, philosophers were inclined to multiply paradoxes *ad libitum*. Frank Ramsey (1978, p. 171) made the decisive step in this direction, splitting the paradoxes up into semantic and syntactical paradox. However, as it was conclusively shown recently, these have one and the same structure (Priest 1994).

The self-reference is also characteristic of Russell's initial paradox of infinite numbers. Indeed, he conceived the paradox of infinite numbers in exactly the same form: 'There are

[10] This section shall discuss 'Russell's Peano-Fregean turn' only figuratively. In fact, until June 1902, Russell had practically no knowledge of Frege. There are two reasons for using this figure: (i) The gist of Russell's turn from August 1900 was the assimilation of the philosophical consequences of Peano's theory of quantification. Today, however, it is widely accepted that Frege's theory of quantification decisively influenced Peano's theory of quantification (see Gillies 1982). This partly explains why, after assimilating Peano's conception, Russell so easily embraced the ideas of Frege. (ii) Russell was also impressed by Peano's elegant symbolism, which was developed 'partly under Frege's influence' (Moore 1998, p. 732a).

many numbers; therefore, there is a number of numbers. [But] If this be *N*, *N*+1 is also a number; therefore, there is no number of numbers' (1899, p. 265). In truth, this was a proto-variant of Russell's paradox proper but formulated two years earlier. Apparently, the two paradoxes, of infinite numbers and of classes, were only two sides of this proto-paradox.

All this explains why, when the problem of infinite numbers was 'resolved' after August 1900 by way of (Peano's) treating the number of numbers as a single class-concept, immediately after Russell began to work on his newly moulded pure mathematics in detail in October 1900, he found, in November 1900, a 'mistake in Cantor' exactly on this point (the class of all classes). It later turned out to be the 'new contradiction': Whenever a greatest cardinal number is accepted, the number of classes is the largest number.

Apparently, in November 1900, Russell merely transformed the paradox of infinite ordinal numbers into a paradox of cardinal numbers (of the largest cardinal number). To be sure, 'Russell's antinomy of infinite number ... has precisely the same formal structure as the paradox of the largest cardinal' (Moore 1995, p. 226). Thus, 'it was not a *new* discovery, but a *shift* in how he perceived an argument which he already possessed' (ibid., p. 231).

Some months later, in May 1901, Russell reformulated his antinomy in terms of predicates not predicable of themselves, thus articulating his paradox proper. Only in his letters to Frege from June 16, 1902, however, did he formulate it in terms of classes. Now his problem was: Is the class of the classes that are not members of themselves a member of itself? Russell's official answer was that, in fact, it is of a type different from that of the other objects. To eliminate the possibility of type-confusion, he introduced the 'vicious circle principle', according to which '[w]hatever involves *all* of a collection must not be one of the collection' (1910, i, p. 37). As it shall be shown below, his real answer was rather different.

4. Theoretical Source of Russell's Paradox

The gist of this discussion is that in treating ordered collections of any kind, paradoxes are unavoidable. Apparently, the problems in such cases pertain to the 'limits to thought whose very notion is dialetheic' (Priest 1991, p. 369):[11] the problem of truth (*aletheia*) is not relevant to them. What can be done in such cases is to treat them not in an orderly fashion but analytically, and structurally—for example, via quantifiers. In other words, the irreversible

[11] Kant called them 'antinomies of pure reason'.

order (and with this the infinity) in them can be put 'in brackets'; we can 'seal it off' and to proceed the calculation further.

Incidentally, Russell himself was conscious of the dialetheic limits of human thought. This is shown by the fact that the strategy he followed to resolve the problems of continuity and infinity was to go beyond conventional intuitions which presuppose an aletheic understanding of composing unities (totalities) with individuals (objects). To be more exact, following Cantor, he maintained that the problem of infinity can be only overcome if we banish a maxim of common sense:[12] the intuition that 'if one collection is part of another, the one which is a part has fewer terms than the one of which it is a part' (1901*b*, p. 373).

Ironically, the trouble with Russell's paradox was that he, just like Frege, did not banish the conventional intuitions that led to *his* paradox: that we can quantify objects of any kind, without restriction. To be more exact, Russell failed to structuralise the notion of class-membership. Instead, he followed the common-sense belief that the including class can comprise everything that falls under it. From the beginning, he rejected the idea that there can be subclasses that are not susceptible to class-inclusion (see §5).

To be more specific, as it has already been mentioned (in §2.1), in '*The Principles of Mathematics*, Draft of 1899–1900' (1900), Russell adopted Cantor's treatment of continuity (which developed some of Weierstrass's points), according to which there are no infinitesimals. The moments and places in his logical ontology were absolutely determined and finite (cf. n. 7). There are no intervals between them, no *next moments* or *places*. Between two moments (places), there are always other moments (places). Thus, 'next' was the first common-sense intuition that Russell, following Cantor, banished.[13]

This conception sees the world as an assemblage of structures. Indeed, in structures, there is no problem of neighbourhood; nor is there a topology of structures. This is because it is irrelevant where they are: they are mutually substitutive. With his turn of 1900—to be more exact, with the new, Peano-inspired treating of infinite numbers—Russell introduced another structuralist conception based on the one-many relationship. What he achieved by adopting

[12] Wittgenstein put this matter in similar, yet clearly different terms. The problem of infinity is a product of certain (grammatical) misunderstandings, which have to be removed from the calculation (Wittgenstein 1956, IV, §6).

[13] 'The banishment of the infinitesimal has all sorts of odd consequences, to which one has to become gradually accustomed. For example, there is no such thing as the next moment' (1901*b*, p. 371).

the technique of quantification was a new technique of 'putting' an infinite number into—rather, *including* it *in*—one concept or class-concept.

Despite his strong belief to the contrary, however, the problem of infinity was not 're-solved'. It appeared on the face of the new concept of class-*inclusion*. Indeed, Russell 're-solved' the antinomy of infinite numbers but only by transforming it into the antinomy of class-inclusion.

5. Motives for Asserting the Paradox of Classes

Apparently, the reason why Russell stuck to the paradoxes in the philosophy of logic and mathematics was his 'debt to German learning'.[14] Because of it, Russell was inclined to become infatuated with insolubilia,[15] so that when confronted with problems of this kind, he lost his ability to analyse—in this case, to analyse the composition of unities (totalities). Historians of analytic philosophy have already noticed that while Russell's official theory was that mathematics is free of paradoxes, deep in his mind, he continued to believe that mathematics is paradoxical (see Garciadiego 1992, p. 152). This explains why he was so sensitive to any sign of paradoxes in logic and mathematics.

Especially illuminating in this respect is the fact that it was Frege who made Russell's paradox a *paradox*. Indeed, as recent historical investigations reveal, '[t]he fact that Frege, whose logical work Russell admired intensely, found Russell's paradox devastating ... played a major role in convincing him of its fundamental importance' (Moore 1995, p. 235). Until Frege's reaction to Russell's problem in his historic letter of 22 June 1902, Russell's friends Couturat and Peano, as well as Whitehead, who shared most of his logical ideas, were not impressed with his trouble at all. This explains why, before communicating the 'contradiction' to Frege, Russell was uncertain 'as to how important his paradox actually was' (Moore 2013, p. xxxiii). But, why was that so?[16]

[14] Cf. with the title of Russell's paper (1955). It is important that Russell himself insisted on the nationality of different schools of mathematics and logic (see, for example, 1901*a*).

[15] Some authors have justly noted that 'in a neo-Hegelian vein—he [Russell] collected as many paradoxes as he could' (Grattan-Guinness 1986, p. 108).

[16] Griffin (2004) tried to answer this question without referring to Frege. He was convinced that 'Russell was not the first person to find a paradox in set theory, but he was the first to make a really big fuss about it' (p. 349). As just seen, however, the first to 'make a big fuss about it' was Frege, not

First of all, both Frege and Russell were more philosophically oriented than the aforementioned logicians. The trouble with Russell, in particular, was that, as we have already mentioned (in n. 7), he stuck to his philosophical realism. The ontology that underlined his logic consisted of realistic categories such like 'points' and 'moments'. Frege's problem was even greater. He was convinced that 'axioms should express truths and definitions should give the meanings. … If the terms in the proposed axioms do not have meaning beforehand, then the statements cannot be true (or false), and thus they cannot be axioms' (Shapiro 1996, p. 161). And this is even more important since logic is the science of truth.

The philosophical realism both Frege and Russell stuck to explains why they used 'quantifiers as varying over everything in the universe' (Moore 2013, p. xxiv). Today, we know that the most important lesson from the spectacular failure of Frege's logicist programme was that 'we cannot uncritically assume the existence of universal domain of quantification' (Simons 2007, p. 238).[17] Peano, for one, did not share this belief. He also did not make use of propositional functions (cf. 1903, § 22). That also explains why he was not impressed with Russell's 'contradiction'.

The main point, which supports our thesis that Russell himself created his paradox, or, to be more accurate, that he remoulded an old paradox of his into his paradox of classes, is that, as already mentioned, many other logicians did not see a paradox here. Among them were also Stanislaw Leśniewski and Kurt Gödel. Their approach is especially telling from our perspective since they directed their attention to analysing Russell's concept of unity or totality. In particular, both criticised Russell's understanding of classes as class-concepts.

Gödel was especially clear on this point: '[O]ne may, on good grounds, deny that reference to a totality necessarily implies reference to all single elements of it' (Gödel 1944, p. 135). *All* is not necessarily an infinite logical conjunction. Indeed, we have already seen (in § 2.3) that Russell himself enumerated five different ways of constructing unities. Oddly enough, though he knew these alternatives, he stuck to one—point 'iv' in our notation.

Russell. This point reveals an important difference between Griffin's (and that of some other mainstream historians of analytic philosophy) approach and the approach we follow in this paper. The former is fine-grained, going deep into formal details. Our approach, in contrast, tries to draw a more general logico-philosophical picture of the developments under review in order to suggest the best explanation.

[17] Simons refers here to Dummett (1973, p. 455 ff).

Much before Gödel, however, Leśniewski had already suggested an extensive solution to Russell's paradox in 1927, pointing out that it rests on the same one-sided use of the concept of class. To eliminate it, Leśniewski discriminated between a collective and distributive conception of class. Something is a member of a distributive class if and only if it is *ipso facto* this class.[18] In contrast, a member of a collective class need not be *ipso facto* that class (see Leśniewski 1927–31, p. 17). Russell's paradox would not have emerged had he assumed that all classes were distributive; in this case, there is no such object as the class of classes that are not members of themselves.

Of course, this was not the single possible way to face Russell's paradox. For example, David Hilbert and his acolytes in Göttingen followed an alternative direction. They (Ernst Zermelo, in particular) tried to 'reform logic' through axiomatizing set theory (see Peckhaus 2004, p. 510).

6. Russell's Main Trouble Inflicted by the Turn of 1900

George Santayana had once said that Russell 'was a failure'. Russell's task was to renew Frances Bacon's project for a great *instauratio magna* of all sciences. Instead, he was involved in exploring subjectivist epistemological problems. Hao Wang surmised that this failure was due to the harmful influence of Wittgenstein. This section will show that what occasioned Russell's 'failure' was, above all, Frege's influence on his logic, and not the influence of Wittgenstein alone. The latter simply transported Frege's influence to Russell.[19]

Russell himself felt dissatisfied with *The Principles of Mathematics* upon finishing it. On 2 August 1902, he wrote in a letter to Miss G. L. Dickinson: '…the proofs come occasionally, and seem to me very worthless; I have a poor opinion of the stuff when I think of what it ought to be' (Moore 1993b, p. xxxviii). Two and a half months before that, on 16 May, Russell wrote to his wife, Alys: 'This is not the true artistic conscience, but that is a luxury I can no longer afford for the present' (Moore 2013, p. xxx). And three days later, he wrote: '…the final product is not a work of art, as I had hoped it would be' (1992, p. 234).

So, what caused this disappointment?

[18] Leśniewski borrowed the very terminology of *collective* and *distributive* classes from Tadeusz Kotarbiński much later.

[19] Cf. Milkov (2013).

Apparently, to Russell, the 'beauty' of his initial project (1898–August 1900) came with the ease with which it treated different problems by way of one and the same concept—the concept of *order*, elaborated with the help of the logic of relations.[20] The chapters of this project were: logical *order* (based on implication), the *order* of whole and part (based on the extensive class-inclusion), the *order* of numbers, and spatial and temporal *order*.[21] Organized this way, the work really did promise to be 'as clear as a crystal'.[22]

This project was rooted in the one-order logic developed by Moore in 1899, which decisively influenced Russell's project to reform logic (cf. § 1 [ii], above). It was also closely connected with the method of reductive analysis, which can be especially well embedded in a kind of part-whole logic.

In contrast, Frege's philosophical logic assumed a *deep*, many-ordered, intentional logic that is *fruitful*, and in which the function is *organically* connected to all individuals who fall under it and so determines them.[23] It persuaded Russell to assume that logical terms are heteromorphic, divided into strata: terms, propositions, functions, propositional functions, and classes (terms and denoting phrases). This was the first lesson in logic Russell learned from Frege (cf. § 2.3).

This lesson had grave consequences for both Russell's logic and philosophy. As it can be seen in the next section (§ 7), between 1903 and 1910, Russell tried to mitigate the damages it induced in his programme of analysis. It also deserves notice at this point that, as it has been shown elsewhere (Milkov 2013), a 'second lesson' in the logic of Frege that Russell learned through Wittgenstein in 1913 urged him to further revise his philosophy and logic.

In Russell's defence, it can be mentioned that he took this turn with hesitation, making many efforts to evade those elements in it that were alien to his authentic, extensional logical intuitions. As a result, his 'logic remained of a quite different character from [Frege's]' (Grattan-Guinness 1988, p. 77b).[24]

[20] Hager (1994) partly supports this view.

[21] In accepting the concept of order as central to his logic, Russell followed—directly or indirectly—Hermann Lotze (cf. Milkov 2008).

[22] This was Wittgenstein's judgment on his *Tractatus*.

[23] Milkov (2015) demonstrated that these ideas of Frege's logic were influenced by ideas of German Idealism.

[24] A point often discussed in the newer literature (Mayer 1996, p. 135; Peregrin 2001; Beaney 2008).

From the point of view of the mathematical logic, Russell's doubts can be explained by the fact that with his turn of August 1900, Russell tried to synthesize two traditions in it: First, the algebraic line launched by De Morgan and developed further by Boole, Peirce, Schröder, and Whitehead. Second, the programme to improve the rigour of mathematical analysis laid down by Cauchy and developed further by some continental mathematicians and logicians like Weierstrass, Peano and Frege. '[T]he algebraists (like Grassman) used part-whole theory, whereas mathematical logicians used Cantorian set theory' (Grattan-Guinness 1996, p. 212).

From the point of view of philosophical logic, there were at least two symptoms to show how adopting the elements of Frege's logic alienated Russell from his authentic direction: (i) '[T]he recognition of denoting concepts was inconsistent with [Russell's metaphysical] monism' (Coffa 1991, p. 106) that was proclaimed with the acceptance of realism in 1898. Indeed, it introduced different layers of being. (ii) The adoption of propositional function in 1903

> forced to some extent the readmission of the subject-predicate pattern (by means of propositional functions) and, therefore, the abandonment of the relational ontology that was implicit in Moore's theory of judgment (Rodríguez-Consuegra 1993, p. 80).

In contrast to Russell, Edmund Husserl, who was very well informed about the new developments in logic and mathematics, was convinced that the most important achievement in the logic of the *fin the siècle* period was the new mereology (not Frege's technique of quantification, with which Husserl was well acquainted), or the logic of terms. Philosophers had to follow it as well.

7. Russell's Way out—Theory of Descriptions and other Emendations

After years of heroic efforts, Russell found some tentative solutions to his paradox. First, his Ramified Theory of Types set aside some philosophical assumptions in logic he and Frege had made which clearly contradicted their many-ordered logic: that there is a universal (one-levelled) domain of quantification. It deserves notice, however, that Russell, even in (1919), was not so confident about how sound this solution was. He clearly felt that 'the theory of types emphatically does not belong to the finished and certain part of our subject: much of this theory is still inchoate, confused, and obscure' (p. 135). Russell simply believed in 'the need of some doctrine of types' (ibid.).

Much more successful in solving his paradox was the Theory of Descriptions (1905). In fact, Russell felt from the very beginning that his paradox can be resolved through a correct theory of descriptions.[25] Later, he remembered that after years of abortive efforts to solve the paradox, the first success came with his Theory of Descriptions. ‘This was, apparently, not connected with the contradictions, but in time an unsuspected connection emerged’ (1959, p. 79).

But, what exactly was the connection between the solution of the paradoxes of self-reference and the Theory of Descriptions? First of all, with the introduction of objects of acquaintance as a legitimate part of logic, Russell also (re)introduced complexes we are acquainted with into his logic. In this way, Theory of Descriptions limited the competence of the propositional functions, and thus partly restored the realistic mereology of complexes and simples, embraced in 1898, but rejected in 1900.

Secondly, with the introduction of the concept of ‘incomplete symbols’, Russell also made two other emendations to his logic, both of which were directed at eliminating the splitting of logic into different levels:

(i) He first eliminated classes, and also relations, as entities. After 1905, he maintained that classes are only ‘incomplete symbols’ and hence, are not objects. To be more precise, Russell introduced the ‘no class’ theory in (1907, p. 45). There are no classes but also no propositions and no propositional functions (cf. § 4). ‘A propositional function standing alone may be taken to be a mere schema, a mere shell, an empty receptacle for meaning, not something already significant’ (1919, p. 157).

It is true that after 1907, with the introduction of the Ramified Theory of Types, Russell adopted a stratified language with ordered variables. But, he never spoke about orders of entities (objects) again. Only the constituents of judgements are stratified into individuals (particulars) and universals. In other words, now he maintained that there are types of classes (attributes), but these can be presented by only one type of variable. In other words, there is a hierarchy of classes but not of objects.

[25] In a letter to Jourdain on 14 March 1906, Russell wrote: ‘In April 1904 I began working at the Contradiction again, and continued at it, with few intermissions, till January 1905. I was throughout much occupied by the question of Denoting, which I thought was probably relevant, as it proved to be’ (quoted according to Grattan-Guinness 1977, p. 79).

(ii) Also in 1907, Russell discovered that propositions and propositional functions produce paradoxes of their own. (Of course they do: they are unities.) In consequence, he came to maintain that there are no propositions. To be more exact, these were eliminated with the help of the Multiple Relation Theory of Judgment. According to it, propositions only receive meaning (i.e. unity) through the judging mind: 'Propositions are incomplete symbols that require the context of judging mind in order to achieve a meaning.'[26] It follows that the truth-bearers are judgements, not propositions. The ontology of *Principia Mathematica* is also based on this conception.

As a result, in *Principia Mathematica*, Russell adopted a full rehabilitation of the ontology of the complex and simple. Both classes and propositions were declared incomplete symbols. With this step, the many-order logic left ontology; now it was restricted to the logical language exclusively (see Chihara 1972, pp. 262–3).

Incidentally, despite all the difference between their logics, at the beginning of the 1920s, towards the end of his days, Frege reached a similar conclusion: the only way to avoid paradoxes in logic is to refuse to use the concept of (intensive) class as totality (or class-concept). The adoption of this position was supported by the acceptance of explicit geometrism in logic. Frege realised that the assumption of classes cannot guarantee a convincing treatment of infinite unities (including infinite numbers); this can only be done by the spatial and temporal intuition (see Dummett 1981, p. 663). As it has been shown elsewhere (see Milkov 1999), Frege's change of heart was also helped by the fact that his logic was crypto-intuitive from the very beginning. In fact, his foremost preconception was that logical objects can be grasped and operated only via geometrical intuition.

8. Epilogue

Russell was a revolutionary philosopher who aimed at suggesting a clear programme to radically reform the philosophy of his time. In particular, he was convinced that the new techniques introduced in mathematics and mathematical logic can finally solve many obscure philosophical paradoxes. This was the gist of his 'scientific method in philosophy' which was to replace the dialectical method of the British Neo-Hegelians.[27]

[26] Stevens (2005, p. 79).

[27] It was also to replace the dialectical method of the British Neo-Lotzeans (see Milkov 2008).

The literature on Russell has so far mainly discussed the ways in which he abandoned the *method* of the Neo-Hegelians after he embraced his scientific method of analysis. The present paper follows another direction. Since Russell formed his philosophical intuitions in the context of German Learning, he cannot escape being influenced with Hegelian topics as well. This is true, in particular, about the problem how units (totalities) relate to their constituents, which was the central topic of Hegel's logic. Exactly this subject-matter gave also rise to Russell's paradox.

The conclusion we reached in this paper is that the elimination of classes, relations, propositional functions, and propositions from Russell's ontology was more efficient in solving the paradox of classes than the official device introduced for this purpose—his Ramified Theory of Types. In particular, Russell's eliminative programme banished the many-levelled ontology he was prone to follow after its 1900 turn, which, in many ways, hampered his method of analysis.

Unfortunately, Russell did not clearly realise that his paradox emerged in connection with the problem of constructing unities (totalities) from individuals (objects, terms). That is why he never considered the solution of his paradox in the wake later followed by Leśniewski and Gödel.

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