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On the Definition of the Centre of Gravity of an Arbitrary Closed System in the Theory of Relativity

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ON THE DEFINITION OF THE CENTRE OF GRAVITY OF AN ARBITRARY CLOSED SYSTEM IN THE THEORY OF RELATIVITY.*

By C. MØLLER.

I.—CLASSICAL THEORY.

I.—Centre of Gravity. Relative Angular Momentum Tensor.

One of the simplest and most fundamental conceptions in Newtonian mechanics is the notion of the centre of gravity of a system of masses. As is well known, this notion loses its simple significance already in the special theory of relativity on account of the changes in the conception of mass brought about by this theory. For an arbitrary closed system it is, however, as we shall see, comparatively easy to define a point which has very similar properties to the centre of gravity in Newtonian mechanics.

In Newtonian mechanics, the centre of gravity[1] of a system of mass points is defined in the following way. If $m_1, m_2, \ldots m_i, \ldots$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_i, \ldots$ denote the masses and coordinate vectors of the different mass points respectively, the coordinate vector $\mathbf{X}$ of the centre of gravity is defined by the equation

$$\mathbf{X} = \frac{1}{M} \sum_i m_i \mathbf{x}_i$$

where (1)

$$M = \sum_i m_i$$

is the total mass of the system. Or, if the mass is continuously distributed with a mass density $\mu(\mathbf{x}, t)$, we have

$$\begin{aligned} \mathbf{X} &= \frac{1}{M} \int \mu(\mathbf{x}, t) . \mathbf{x} \, dv \\ M &= \int \mu(\mathbf{x}, t) \, dv \end{aligned} \qquad (2)$$

where the integrals are extended over that part of the physical space in which μ is different from zero.

*Lectures delivered at the Dublin Institute for Advanced Studies, July. 1947.

[1] Since we are, in the following, nowhere concerned with gravitational fields it would be more adequate to speak of the centre of inertia instead of centre of gravity, but we shall follow the usual terminology.

According to (2), the centre of gravity is a *centre of mass* corresponding to a kind of average position of the mass in the system. For a finite closed system, i.e. a system contained in a finite region in space with no interaction with systems outside, the centre of gravity moves in a straight line with constant velocity. This follows at once from the theorems of conservation of mass and momentum. The first theorem is expressed by the continuity equation

$$\frac{\partial \mu}{\partial t} + \operatorname{div}(\mu \mathbf{u}) = 0 \tag{3}$$

where $\mathbf{u}(\mathbf{x}, t)$ is the velocity of mass at the point $\mathbf{x}$ and at time t. In the first place, we get from (3) by integration over the whole system

$$\frac{dM}{dt} = \int \frac{\partial \mu}{\partial t}\, dv = -\int \operatorname{div}(\mu \mathbf{u})\, dv = 0, \tag{4}$$

$$i.e. \quad M = \text{constant}.$$

Further, from (2) we obtain by means of (4) and (3)

$$\frac{d\mathbf{X}}{dt} = \frac{1}{M}\int \frac{\partial \mu}{\partial t} \cdot \mathbf{x}\, dv = -\frac{1}{M}\int \operatorname{div}(\mu \mathbf{u}) \cdot \mathbf{x}\, dv = \frac{1}{M}\int \mu \mathbf{u}\, dv = \frac{\mathbf{P}}{M} \tag{5}$$

where $\mathbf{P}$ is the total linear momentum of the system. Now, for a closed system $\mathbf{P}$ is constant in time and (5) shows that the centre of gravity is moving with the constant velocity

$$\mathbf{U} = \frac{\mathbf{P}}{M}. \tag{6}$$

The definitions (1) and (2) of the centre of gravity in Newtonian mechanics cannot be immediately used in the theory of relativity on account of the variation of mass with velocity. We could, of course, for a system of mass points use the definition (1) with m_i replaced by the proper mass $m_i{}^0$, but the centre defined in this way would *not* perform a uniform motion for a closed system of mutually interacting particles with no external forces. For

$$\frac{d\mathbf{X}}{dt} = \frac{1}{M^0} \sum_i m_i{}^0\, \mathbf{u}_i$$

would not be proportional to the constant total momentum of the system. A similar difficulty arises if m_i in (1) is to mean the relativistic mass

$$m_i = \frac{m_i{}^0}{\sqrt{1 - \frac{\mathbf{u}_i{}^2}{c^2}}};$$

in this case, even $M = \sum_i m_i$ will not in general be constant for a closed system. Only in the case of a system of free particles with no interactions at all, the centre of mass defined in this way will perform a uniform motion. But even in this trivial case, the centre of gravity can, as remarked by Fokker,[2] in general be defined by (1) in one definite frame of reference only, for instance, in the frame in which the total momentum of the system is zero. In another system of inertia the centre of gravity will then not in general be a mass centre. This is a characteristic property of the relativistic centre of gravity, which we shall find also in the case of an arbitrary closed system.

To get a useful definition of the centre of gravity of an arbitrary closed system, we must take into account that also the potential energies of the forces acting between the different parts of a system contribute to the total mass of the system. According to Einstein's theorem, an energy density $h(\mathbf{x}, t)$ corresponds to a mass density

$$\mu(\mathbf{x}, t) = \frac{h(\mathbf{x}, t)}{c^2} \tag{7}$$

and, if $h(\mathbf{x}, t)$ includes all types of energies in the system, the energy and, thus, also the mass will be conserved as in Newtonian mechanics. This obviously opens a possibility of a useful definition of the centre of gravity, also in the relativistic region.

In the special theory of relativity, an arbitrary physical system, say, an elastic body, an arbitrary field or a system of particles interacting by means of an intermediary field, is described by a symmetric energy-momentum tensor

$$T_{ik} = T_{ki} = T_{ik}(x_l)$$

whose components are functions of the space-time coordinates

$$x_i = (\mathbf{x}, \ddot{\imath}ct) = (x, y, z, \ddot{\imath}ct) \tag{8}$$

Throughout this paper, Latin indices $i, k \ldots$ are running from 1 to 4, while Greek indices $\iota, \kappa, \ldots$ are running from 1 to 3; $\ddot{\imath}$ denotes a quantity whose square is -1. (It is convenient to distinguish between the $\ddot{\imath}$ introduced in the theory of relativity and the i occurring in the commutation relations in quantum mechanics.) For a finite system, the components $T_{ik}(\mathbf{x}, t)$ are, for fixed t, zero outside a finite region in $\mathbf{x}$–space. This means that the T_{ik} are different from zero only inside a tube in the fourdimensional space-time continuum and the normals to the walls of this tube are everywhere represented by space-like vectors. The physical meaning of the components T_{i4} are given by

$$\frac{1}{\ddot{\imath}c} T_{i4} = g_i = \left(\mathbf{g}, \frac{\ddot{\imath}}{c} h\right) \tag{9}$$

where g and h are the momentum density and energy density, respectively.

[2] A. D. Fokker. Relativiteitstheorie, Noordhoff Groningen (1929), p. 190.

For a closed system, the theorems of conservation of momentum and energy are expressed by the equations

$$\frac{\partial T_{ik}}{\partial x_k} = 0. \tag{10}$$

(Dummy Latin indices like k in (10) are to be summed over from 1 to 4, while dummy Greek indices are to be summed from 1 to 3.) For $i = 4$ (10) may be written

$$\frac{\partial g_i}{\partial x_i} = 0 \tag{11}$$

an equation which on account of (7) and (9) is equivalent with the continuity equation (3) for mass conservation. If we multiply (10) by dx_1, dx_2, dx_3, and integrate over the whole **x**-space, we find at once that the four quantities

$$P_i = \int g_i(\mathbf{x}, t)\, dv = \left(\mathbf{P}, \frac{i}{c} H\right) \tag{12}$$

are constant in time for a finite closed system. From the physical meaning of $\mathbf{g}$ and h in (9) we see that $\mathbf{P}$ and H represent the total linear momentum and the total energy of the system, respectively.

Further, from (10) and by use of the Gauss' theorem applied to a suitable region in fourdimensional space-time, it follows in a well-known way that the four quantities P_i transform like the components of a four vector by Lorentz transformations. P_i are the components of the momentum-energy vector. Therefore, $P_i P_i$ will be an invariant, and for all real physical systems the value of this invariant must be negative or zero. We may, thus, define the proper mass M_0 of the system by the equation

$$P_i P_i = -M_0^2 c^2. \tag{13}$$

In what follows, we shall assume $M_0 > 0$, thus excluding the case of a plane electromagnetic wave. We can then always find a system of inertia S^0 in which the total linear momentum $\mathbf{P}^0$ is zero. Thus, the components of the momentum-energy vector in S^0 are by (13)

$$P_i^0 = (0, 0, 0, i M_0 c). \tag{14}$$

The system S^0 is usually called the centre of gravity system without any closer specification of the position of the centre of gravity in S^0. In view of (7) (9), (12), it is now natural to define the coordinate vector $\mathbf{X}^0$ of the centre of gravity in S^0 by the equations

$$\mathbf{X}^0 = \frac{c^2}{H^0} \int \frac{h^0(\mathbf{x}^0, t^0)}{c^2} \cdot \mathbf{x}^0\, dv^0 = \frac{1}{M_0} \int \frac{h^0(\mathbf{x}^0, t^0)}{c^2} \cdot \mathbf{x}^0\, dv^0 \tag{15}$$

in analogy with (2).

Although h^0 is time dependent, it is easily seen that $\mathbf{X}^0$ in (15) is constant in time. For, by means of (11), we get in any system of reference

$$\frac{\partial g_i x_k}{\partial x_i} = g_i \delta_{ik} = g_k$$

or by integration over **x**-space

$$\frac{d}{dx_4}\int g_4 x_k \, dv = P_k. \tag{16}$$

Now, in the rest system S^0, we have $\mathbf{P}^0 = 0$ and (16) becomes for $k = \iota = 1, 2, 3$

$$\frac{d}{dx_4^0}\int h^0 x_\iota^0 \, dv^0 = 0.$$

Thus, the centre of gravity defined by (15) is a point which has a fixed position in the rest system and it is a centre of mass in this system. In an arbitrary system of inertia S the centre of gravity will then move with a constant velocity

$$\mathbf{U} = \frac{c^2 \mathbf{P}}{H},$$

i.e. the relative velocity of S^0 with respect to S. The time track (world line) of the centre of gravity will, thus, be a straight line in space-time, the space-time coordinates $X_i = (\mathbf{X}, X_4)$ being linear functions, $X_i = X_i(\tau)$, of the proper time τ of the centre of gravity. If

$$U_i = \frac{dX_i}{d\tau}$$

denotes the components of the corresponding four velocity, we have

$$U_i = \left(\frac{\mathbf{U}}{\sqrt{1 - \frac{U^2}{c^2}}}, \frac{ic}{\sqrt{1 - \frac{U^2}{c^2}}}\right) \tag{17}$$

and

$$P_i = M_0 U_i. \tag{18}$$

The last relation is obviously true in the rest system S^0 and, on account of the four vector character of P_i and U_i, it holds in any system of reference. Thus, the system as a whole has similar properties as a particle of proper mass M_0 placed in the centre of gravity of the system.

Besides the four constants P_i any closed finite system has six other integrals: the components of the four angular momentum tensor M_{ik}. From (10), we get

$$\frac{\partial}{\partial x_l}(x_i T_{kl} - x_k T_{il}) = \delta_{il} T_{kl} - \delta_{kl} T_{il} = T_{ki} - T_{ik} = 0. \tag{19}$$

on account of the symmetry of the tensor T_{ik}. By integration of (19) over the whole **x**-space we get

$$\frac{d}{dx_4}\int (x_i T_{k4} - x_k T_{i4})\, dv = 0$$

which shows that the quantities

$$M_{ik} = -M_{ki} = \int (x_i g_k - x_k g_i)\, dv \qquad (20)$$

are constant in time. By Lorentz transformations, the M_{ik} transform like the components of an antisymmetrical tensor. This may be shown by a similar method as that used for the proof of the four vector character of P_i.

Let us introduce two space vectors **M** and **N** by

$$\begin{aligned} \mathbf{M} &= (M_x, M_y, M_z) = (M_{23}, M_{31}, M_{12}) \\ i\mathbf{N} &= i(N_x, N_y, N_z) = (M_{14}, M_{24}, M_{34}) \end{aligned} \qquad (21)$$

the vector

$$\mathbf{M} = \int (\mathbf{x} \times \mathbf{g})\, dv$$

being the total angular momentum vector of the system with respect to the arbitrary origin of our coordinate system.

We may now define a new antisymmetrical tensor m_{ik}, the relative angular momentum four tensor with respect to the centre of gravity, by replacing x_i in the definitions (20) by $x_i - X_i$. We then get

$$m_{ik} = M_{ik} - (X_i P_k - X_k P_i)\,. \qquad (22)$$

Although X_i is varying with the proper time τ, m_{ik} is independent of τ, for, on account of (18), we have

$$\frac{dm_{ik}}{dT} = -(U_i P_k - U_k P_i) = 0\,. \qquad (23)$$

m_{ik} is also independent of the choice of origin in our coordinate system.

We may now introduce two space vectors **m** and **n** related to m_{ik} in the same way as **M** and **N** in (21) are related to M_{ik}, i.e.

$$\begin{aligned} \mathbf{m} &= (m_{23}, m_{31}, m_{12}) \\ i\mathbf{n} &= (m_{14}, m_{24}, m_{34}) \end{aligned} \qquad (24)$$

m is the relative angular momentum vector with respect to the centre of gravity, i.e. the inner angular momentum. For $i = \iota = 1, 2, 3$ and $k = 4$ (23) becomes

$$i\mathbf{n} = i\mathbf{N} - (\mathbf{X} P_4 - X_4 \mathbf{P})$$

or

$$\mathbf{X} = \frac{\mathbf{P}}{P_4} \cdot X_4 + i\mathbf{N} P_4^{-1} - i\mathbf{n} P_4^{-1}. \qquad (25)$$

Now, we get from (21), (20), (9) and (12)

$$i\mathbf{N} = \int (\mathbf{x} \cdot g_4 - \mathbf{g} \cdot x_4)\, dv = \frac{i}{c}\int h\,\mathbf{x}\, dv - x_4 \cdot \mathbf{P}. \qquad (26)$$

Although each term in the right-hand side of this equation depends on x_4 we know that $\mathbf{N}$ is constant in time. We may, therefore, for instance, choose $x_4 = X_4(\tau)$ in (26). Introducing (26) into (25) we then get

$$\mathbf{X} = \frac{1}{H}\int h \cdot \mathbf{x}\, dv \Big|_{x_4 = X_4} - \frac{c\mathbf{n}}{H} \qquad (27)$$

where the time variable x_4 in h has to be put equal to $X_4(\tau)$. On account of the last term in (27) the centre of gravity is a centre of mass in every system of reference only if the relative angular momentum tensor m_{ik} is zero; generally, the two centres are differing from each other, which means that the centre of mass has no relativistically invariant meaning.[3]

A comparison of (27) and (15) shows, however, that the vector $\mathbf{n}$ must be zero in the rest system S^0 of the centre of gravity, i.e.

$$\mathbf{n}^0 = 0, \quad m_{\iota 4}{}^0 = 0. \qquad (28)$$

Hence, we get the covariant relation

$$m_{ik} P_k = 0 \qquad (29)$$

or by (18)

$$m_{ik} U_k = 0. \qquad (30)$$

The validity of (29) in S^0 follows at once from (14) and (28) and, on account of its covariant form, it must hold in any system of inertia. (29) contains a relation between the vectors $\mathbf{m}$ and $\mathbf{n}$; in fact, we get by means of (24) from the three equations (29) corresponding to $i = 1, 2, 3$

$$\mathbf{n} = -\frac{c}{H}(\mathbf{m} \times \mathbf{P}). \qquad (31)$$

[3] This result, as well as some of the other results about mass centres obtained in this paper, has already been obtained by A. Papapetrou in a paper in Prahliha de l'Académie d'Athènes, 14, 1939, p. 540, which has come to my eyes after completion of the present investigation.

Multiplication of (22) with $\dot{P}_k$ gives, on account of (29) and (13),

$$0 = M_{ik} P_k + X_i (M_0 c)^2 + (X_k P_k) P_i$$

or

$$X_i = -\frac{X_l P_l}{(M_0 c)^2} P_i - \frac{M_{il} P_l}{(M_0 c)^2}.$$

If this expression for X_i is introduced into (22) we get

$$m_{ik} = M_{ik} + \frac{1}{(M_0 c)^2} (M_{il} P_l P_k - M_{kl} P_l P_i). \qquad (32)$$

Thus, m_{ik} is that part of M_{ik} which is orthogonal to the direction of the fourvector P_i in accordance with (29). The physical meaning of the vector $\mathbf{n}$ is seen from (27), which gives

$$\frac{\mathbf{n}}{c} = \int \frac{h}{c^2} (\mathbf{x} - \mathbf{X})\, dv = \int \mu \,.\, (\mathbf{x} - \mathbf{X})\, dv$$

Thus $\frac{\mathbf{n}}{c}$ is the moment of mass of the system with respect to the centre of gravity.

2. Centres of Mass.

In an arbitrary Lorentz system S the centre of mass has by definition coordinates $\mathbf{X}(S)$ given by

$$\mathbf{X}(S) = \frac{1}{H} \int h\, \mathbf{x}\, dv. \qquad (34)$$

According to (16) or (26), the centre of mass is moving with the same constant velocity

$$\mathbf{U} = \frac{c^2 P}{H}$$

as the centre of gravity. Each system of reference has its own centre of mass, the centre of gravity[4] being the centre of mass corresponding to the rest system S^0, i.e.

$$\mathbf{X} = \mathbf{X}(S^0).$$

All these mass centres will coincide only if the relative angular momentum tensor m_{ik} is zero.

If we choose the same orientation of the spatial axes in S as in S^0 the Lorentz system S is uniquely defined by the relative velocity vector

$$\mathbf{v} = \mathbf{U} = \frac{c^2 \mathbf{P}}{H} \qquad (35)$$

[4] The most adequate name for the centre of gravity would therefore be *proper centre of mass*.

of S^0 with respect to S. From the transformation properties of the antisymmetrical tensor m_{ik} we then get on account of (28)

$$\mathbf{n} = \frac{\mathbf{v} \times \mathbf{m}^0}{c\sqrt{1 - \frac{v^2}{c^2}}} \tag{36}$$

where $\mathbf{m}^0$ is the relative angular momentum vector in S^0.

The difference between simultaneous positions of the centre of gravity and the centre of mass in the system S is, according to (27), given by the time-independent space vector

$$\mathbf{a}(S) = \mathbf{X}(S) - \mathbf{X} = \frac{c\,\mathbf{n}}{H}$$

or

$$\mathbf{a}(S) = \frac{\mathbf{v} \times \mathbf{m}^0}{M_0 c^2} \tag{37}$$

on account of (36) and (35). Since the transformation from S to S^0 is given by a Lorentz transformation without rotation of the spatial axes, and since $\mathbf{a}$ is perpendicular to the relative velocity $\mathbf{v}$, the distance between simultaneous positions of the two mentioned centres in the rest system S^0 is also given by (37).

In the rest system S^0 all mass centres $C(S)$ obtained by varying S or $\mathbf{v}$ in (37) form a two dimensional circular disc perpendicular to the angular momentum vector $\mathbf{m}^0$ with centre in the centre of gravity C and with radius

$$\rho = \frac{|\mathbf{m}^0|}{M_0 c}. \tag{38}$$

All these mass centres are at rest in the system S^0.

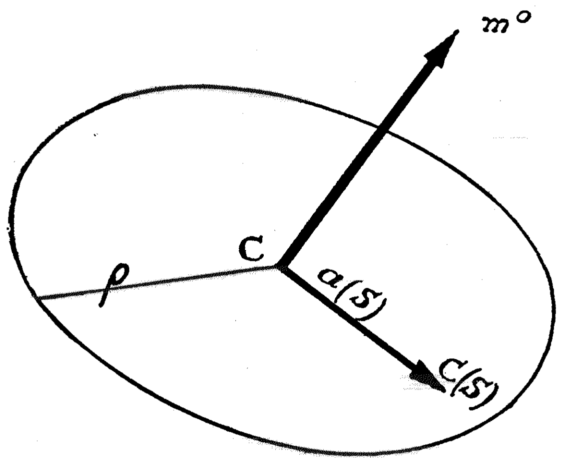
If $\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_{||}$ is written as a sum of two vectors $\mathbf{v}_\perp$ and $\mathbf{v}_{||}$ perpendicular and parallel to $\mathbf{m}^0$, respectively, we see that $\mathbf{a}$ in (37) depends on the perpendicular component $\mathbf{v}_\perp$ only. Each point on the disc is, thus, a mass centre in an infinite number of systems S corresponding to an arbitrary variation of $\mathbf{v}_{||}$ in the interval.

$$-\sqrt{c^2 - |\mathbf{v}_\perp|^2} \leqslant \mathbf{v}_{||} \leqslant \sqrt{c^2 - |\mathbf{v}_\perp|^2}\,.$$

Let us now consider a system which in S^0 lies entirely inside a sphere with centre in C and radius r, i.e. a system for which all components of the energy-momentum tensor are zero outside this sphere. If we further assume that the energy density h of the system is positive in any system of reference it is clear that the whole of the disc of mass centres must lie inside the sphere; for if we consider an arbitrary point, $C(S)$ say, on the disc, this point will in the system S be a centre of mass, and since h is positive it must then lie inside the system. We thus get

$$r \geqslant \frac{|\mathbf{m}^0|}{M_0 c} \qquad (39)$$

i.e. *a classical system with a positive energy density and with a given inner angular momentum* $|m^0|$ *and a given rest mass* M^0 *must always have a finite extension, given by (39) in the centre of gravity system.* If the system is smaller, h cannot be everywhere positive in all systems of reference.



3. Mathisson's Equations of Motions for a Spinning Body. Pseudo-Centres of Gravity.

In a paper in the Acta Polonica from 1937, M. Mathisson[5] has treated the motion of a classical spinning particle according to the general theory of relativity. The equations of motion derived by Mathisson have many strange consequences. In the first place, the motion of the particle should not be uniquely determined by the initial position and velocity of the particle, the equations of motion having an infinite number of solutions for fixed initial values of these quantities. In the case of a free particle without any gravitational fields, these solutions correspond

[5] M. Mathisson, Acta Phys. Pol. VI, 163 (1937); *ibid.*, 218 (1937).

to circular motions around a centre which itself moves with constant velocity. On account of the similarity between these motions and Schrödinger's "Zitterbewegung" of a Dirac electron, MATHISSON and WEYSSENHOF[6] have considered the Mathisson particle as a classical picture of the Dirac electron.

However, since a particle may be regarded as a limiting case of the general system considered here, MATHISSON's result is in contradiction with our result, if the coordinates of the particles are identified with the centre of gravity of the system.

Besides the centre of gravity defined in section 1 there exists, however, in every physical system an infinite number of points which have very similar properties as the centre of gravity, and a closer consideration shows that Mathisson's equations really are the equations of motion of these false centres of gravity and the multitude of solutions of these equations then merely shows that there are many of these pseudo centres of gravity in any physical system.

Among all the mass centres in the disc in Fig. 1, the centre of gravity only has the property of being a centre of mass in its rest system S^0, an arbitrary other point $C(S)$ with radius vector

$$\mathbf{a}(S) = \frac{\mathbf{v} \times \mathbf{m}^0}{M_0 c^2}$$

being a centre of mass in the system S moving with the velocity $-\mathbf{v}$ relative to S^0. For a given $\mathbf{a}$ and $\mathbf{m}^0$ we can always choose the direction of $\mathbf{v}$ in the plane of the disc, i.e. $\mathbf{v}_{||} = 0$, and $\mathbf{v}$ and the system S are then uniquely determined. A point p with radius vector

$$\underline{\xi}^0(p) = \mathbf{x}^0(p) - \mathbf{X}^0 = \frac{\mathbf{v} \times \mathbf{m}^0}{M_0 c^2} \tag{40}$$

and with velocity

$$\frac{d\,\xi^0(p)}{dt} = -\mathbf{v} \tag{41}$$

in S^0 is then a centre of mass in its momentary rest system S. From (41) and (40) we get

$$M^0 c^2 \xi^0 + \left(\frac{d\xi^0}{dt^0} \times \mathbf{m}^0\right) = 0. \tag{42}$$

[6] J. W. Weyssenhof, *Nature*, 141, 328 (1938); Acta Phys. Pol. IX, 1–62 (1947).

which shows that $P_i u_i$ is a constant of the motion. The physical meaning of this constant is found by introducing the momentary rest system S'

$$P_i u_i = i c P'_4 = -E' = -M' c^2$$

where M' is the (constant) value of the total mass in the rest systems S'.

If we introduce this value of $P_k u_k$ into the preceding equations, we get

$$M' c^2 x_i + M_{ik} u_k + P_i (x_k u_k) = 0,$$

$$M_{ik} \dot{u}_k + u_i M' c^2 - P_i c^2 + P_i (x_k \dot{u}_k) = 0.$$

If we work in the centre of gravity system S^0, the first equation is easily seen to be identical with the equations of motion (42) for the point p. The second equation becomes, by means of the equation

$$M_{ik} \dot{u}_k = \Omega_{ik} \dot{u}_k - P_i (x_k \dot{u}_k)$$

following from (22′),

$$P_i = M' u_i + \frac{1}{c^2} \Omega_{ik} \dot{u}_k .$$

This equation, together with (23′) and the equation

$$\dot{\Omega}_{ik} \dot{u}_k = 0$$

following from (23′), leads to the set of equations

$$M' \dot{u}_i + \frac{1}{c^2} \Omega_{ik} \ddot{u}_k = 0,$$

$$\dot{\Omega}_{ik} + \frac{1}{c^2} (u_i \Omega_{kl} \dot{u}_l - \Omega_{il} \dot{u}_l u_k) = 0, \qquad \text{(M)}$$

which are just the equations of motion at Mathisson (loc. cit. equation).

The solution

$$\Omega_{ik} = \text{constant}, \qquad u_i = \text{constant}$$

of these equations represents the motion of the centre of gravity; all other solutions describing the motions of the pseudo-centres of gravity defined above do not seem to have any simple physical meaning.

A closer investigation shows that the equations (42) governing the motion of the p-points are equivalent with the equations of motion of MATHISSON, so that these really are the equations of motion of the above mentioned pseudo-centres of gravity.

Let x_i for the moment denote the space time coordinates of the pseudo-centre of gravity p considered in an arbitrary system of inertia, and let τ be its proper time. The angular momentum four tensor Ω_{ik} with respect to this point is then

$$\Omega_{ik} = M_{ik} - (x_i P_k - x_k P_i) = m_{ik} - [(x_i - X_i)P_k - (x_k - X_k)P_i]. \qquad (22')$$

From this equation we get

$$\dot{\Omega}_{ik} = \frac{d\Omega_{ik}}{d\tau} = -(u_i P_k - u_k P_i) \qquad (23')$$

where

$$u_i = \frac{dx_i}{d\tau}, \quad u_i u_i = -c^2$$

is the four velocity of this point p.

If S' denotes the system of inertia, which is the rest system of p at a definite proper time τ, p is by definition a mass centre in S' at that time. By an argument similar to that used on p. 8 for the centre of gravity, we may then conclude that the mixed space time components of Ω_{ik} are zero in the system S', i.e.

$$\Omega'_{\iota 4} = 0, \qquad (28')$$

an equation which may be written in the covariant form

$$\Omega_{ik} u_k = 0, \qquad (30')$$

If we introduce (22′), we get from this equation

$$M_{ik} u_k - x_i (P_k u_k) + P_i (x_k u_k) = 0$$

and by differentiation with respect to τ

$$M_{ik} \dot{u}_k - u_i (P_k u_k) - x_i (P_k \dot{u}_k) + P_i (-c^2 + x_k \dot{u}_k) = 0$$

From this equation we get by multiplication with $\dot{u}_i$

$$P_i \dot{u}_i = \frac{d}{d\tau}(P_i u_i) = 0$$

Therefore,

$$\frac{i}{c}\,\bar{\eta}_i(x_4) = \frac{i}{c}\,\eta_i(x_4) + 2\,\varepsilon_{4k}\int g_k\,x_i\,dv - \varepsilon_{kl}\int x_l\,\frac{\partial g_4}{\partial x_k}\,x_i\,dv. \tag{51}$$

By partial integrations and by means of (11) the last term becomes

$$\varepsilon_{kl}\int x_l\,\frac{\partial g_4}{\partial x_k}\,x_i\,dv = \varepsilon_{kl}\int\left(\frac{\partial g_4\,x_l\,x_i}{\partial x_k} - g_4\,x_l\,\delta_{ik}\right)dv$$

$$= \varepsilon_{4l}\int\frac{\partial\,(g_4\,x_l\,x_i)}{\partial x_4} - \varepsilon_{il}\int g_4\,x_l\,dv$$

$$= \varepsilon_{4l}\int\frac{\partial(g_k\,x_l x_i)}{\partial x_k}\,dv - \frac{i}{c}\,\varepsilon_{ik}\,\eta_k$$

$$= \varepsilon_{4l}\int g_k\,(\delta_{lk}\,x_i + \delta_{ik}\,x_l)\,dv - \frac{i}{c}\,\varepsilon_{ik}\,\eta_k$$

$$= \varepsilon_{4l}\int(g_l\,x_i + g_i\,x_l)\,dv - \frac{i}{c}\,\varepsilon_{ik}\,\eta_k\,.$$

Introducing this into (51) we get by (20) the transformation formula

$$\bar{\eta}_i(x_4) = \eta_i(x_4) + \varepsilon_{ik}\,\eta_k(x_4) + \frac{c}{i}\,\varepsilon_{4k}M_{ik}\,. \tag{52}$$

Now, we have

$$X_i = H^{-1}\eta_i - m_{i4}\,P_4^{-1}$$

$$\bar{P}_4^{-1} = P_4^{-1} - \varepsilon_{4k}\,P_k\,P_4^{-2}$$

$$\bar{m}_{i4} = m_{i4} + \varepsilon_{il}\,m_{l4} + \varepsilon_{4l}\,m_{il}\,.$$

and by means of the equations (23) and (52) we finally get, after some calculation,

$$\bar{X}_i(x_4) = X_i(x_4) + \varepsilon_{ik}\,X_k(x_4) - \varepsilon_{4k}\,X_k(x_4)\,.\,(P_i\,P_4^{-1}). \tag{53}$$

This equation really shows that X_i is a four vector; the unusual last term is due to the peculiar choice $\bar{X}_4 = X_4$ of the time variable.

For $i = 4$ we have, of course,

$$\bar{X}_4(x_4) = X_4(x_4) = x_4$$

and for $i = \iota = 1, 2, 3$ (53) may be written

$$\bar{X}_\iota(x_4) = X_\iota + \varepsilon_{\iota k}\,X_k - \varepsilon_{4k}\,X_k\,.\,\frac{\partial X_\iota}{\partial X_4}$$

since

$$\frac{\partial X_\iota}{\partial X_4} = \frac{\partial X_\iota}{\partial x_4} = P_\iota\,P_4^{-1}\,.$$

4. TRANSFORMATION PROPERTIES OF THE SPACE INTEGRALS IN THE PRECEDING SECTIONS.

The equations (27) may also be written in the fourdimensional form

$$\left.\begin{array}{l} X_i = \dfrac{1}{H}\displaystyle\int h(\mathbf{x}, x_4)\, x_i\, dv - m_{i4} P_4^{-1} = X_i(x_4) \\ i = 1, 2, 3, 4 \end{array}\right\} \qquad (47)$$

since the fourth equation reduces to

$$X_4 = \frac{x_4}{H}\int h dv = x_4. \qquad (48)$$

Although X_i is a four vector, this is not obvious from the expression (47) for X_i. The reason is that (47) gives X_i as function of the time parameter x_4 instead of as function of the invariant proper time τ of the centre of gravity. Nevertheless, the expression (47) is, of course, a covariant expression, and the asymmetry between the space and time variables in (47) is a natural one. It is connected with the physical fact that the region in Minkowski space, in which T_{ik} differs from zero for a finite system, has the form of a tube stretching from $-\infty$ to $+\infty$ in a time-like direction.

Let us consider the transformation of the different terms on the right-hand side of (47) by an infinitesimal Lorentz transformation

$$\left.\begin{array}{l} \bar{x}_i = x_i + \varepsilon_{ik} x_k, \\ \varepsilon_{ik} = -\varepsilon_{ki}. \end{array}\right\} \qquad (49)$$

We shall first consider the transformation of the quantity

$$\eta_i(x_4) = \int h(\mathbf{x}, x_4)\, x_i\, dv. \qquad (50)$$

Since this quantity is a function of the time variable x_4, only, we can expect to find a simple connection between $\bar{\eta}_i(\bar{x}_4)$ and $\eta_i(x_4)$ only for equal values of the time variables, i.e. for $\bar{x}_4 = x_4$. In order to find $\bar{\eta}_i(x_4)$ we shall need the connection between $\bar{h}(\bar{x}_i)$ and $h(x_i)$ for $\bar{x}_i = x_i$, i.e. the so-called "local" variation of h.

On account of (9) we have

$$\bar{g}_4(\bar{x}) = g_4(x) + 2\,\varepsilon_{4k}\, g_k(x)$$

and by use of a Taylor expansion and of (49)

$$\bar{g}_4(x) = g_4(x) + 2\,\varepsilon_{4k}\, g_k(x) - \varepsilon_{kl}\, x_l \frac{\partial g_4}{\partial x_k}.$$

The physical meaning of T_{i4} is given by the equations

$$\frac{1}{\ddot{i}c} T_{i4} = g_i = \left(\mathbf{g}, \frac{\ddot{i}}{c} h\right) \tag{2}$$

where $\mathbf{g}$ and h are the momentum and energy densities, respectively. The last mentioned quantities are Hermitian q-numbers, i.e. $\mathbf{g}^{\dagger} = \mathbf{g}$, $h^{\dagger} = h$. We shall now make the convention that the symbol $\ddot{i}$ introduced to make the Minkowski space formally Euclidean does not change sign when we take the Hermitian conjugate, in contrast to the imaginary unit i introduced by the quantum mechanical commutation relations. We then have

$$g_i^{\dagger} = \left(\mathbf{g}^{\dagger}, \frac{\ddot{i}}{c} h^{\dagger}\right) = \left(\mathbf{g}, \frac{\ddot{i}}{c} h\right) = g_i \tag{3}$$

and g_i and all components T_{ik} are Hermitian.

From (1) it now follows exactly as in the classical theory that the quantities.

$$\left.\begin{aligned} P_i &= \int g_i \, dv = \left(\mathbf{P}, \frac{\ddot{i}}{c} H\right) \\ \text{and} \qquad M_{ik} &= \int (x_i g_k - x_k g_i) \, dv \end{aligned}\right\} \tag{4}$$

are the components of a time independent four-vector and antisymmetrical tensor, respectively. P_i and M_{ik} are Hermitian q-numbers representing the total four momentum vector and the total angular momentum tensor with respect to the arbitrary origin of our space-time coordinates, respectively.

In general, the system will be described by a number of field variables $F(x_i)$ which are q-number functions of the parameters (x_i). For our purpose, we need not make detailed assumptions regarding the field equations or the commutation relations between the field variables. We shall only make the following general assumptions B and C:

B. *By an infinitesimal displacement of the origin of our space-time coordinates corresponding to a transformation*

$$\bar{x}_i = x_i - \varepsilon_i \tag{5}$$

with infinitesimal constants ϵ_i *the field variables* $F(x_i)$ *are transformed into new field variables* $\bar{F}(\bar{x}_i)$ *so that the "local" variation*

$$\delta^* F(x_i) = \bar{F}(x_i) - F(x_i) \tag{6}$$

is given by

$$\delta^* F(x_i) = \varepsilon_k [P_k, F(x_i)]. \tag{7}$$

(53) may thus be written

$$\overline{X}_i(x_4) = X_i + \varepsilon_{ik} X_k - \varepsilon_{4k} X_k \frac{\partial X_i}{\partial X_4}. \qquad (54)$$

If we had taken $\overline{X}_i$ at the time $\overline{X}_4 = \bar{x}_4 = x_4 + \varepsilon_{4k} X_k$ instead of $\overline{X}_4 = \bar{x}_4 = x_4 = X_4$ we would get the usual transformation law for the four vector X_i

$$X_i(\bar{x}_4) = X_i + \varepsilon_{ik} X_k. \qquad (55)$$

The equations (53) and (55) are, thus, completely equivalent. From the point of view of a geometrical representation in the fourdimensional Minkowski space, (55) is the most natural form, since $X_i(\bar{x}_4)$ and X_i may be regarded as coordinates of the same point in this space. For a quantum mechanical system we shall later see, however, that the form (53) or (54) is not only the most natural but the only possible form of the transformation equations.

II.—QUANTUM THEORY.

5. General Assumptions.

Let us now consider an arbitrary quantum mechanical system. In order to make the treatment as general as possible we shall base our discussion on a few fundamental assumptions only. In the first place, we assume that

A. *The system has a symmetrical energy-momentum tensor satisfying the conservation laws*

$$\frac{\partial T_{ik}}{\partial x_k} = 0. \qquad (1)$$

Here, the components $T_{ih} = T_{ih}(x_l)$ are q-numbers which are functions of the space-time coordinates $x_i = (x, y, z, ict)$. The x_i are characteristics of the classically defined frame of reference and are, consequently, throughout to be regarded as c-numbers. They are simply parameters labelling the dynamical variables of the system. The equation (1) is a relation between q-numbers, and, since the dynamical variables are functions of t, we are here working in the so-called Heisenberg picture. It is, of course, at any time possible to go over to the Schrödinger picture, where the states are considered to be variable.

C. *By an infinitesimal homogeneous Lorentz transformation*

$$\left.\begin{aligned} \bar{x}_i &= x_i + \varepsilon_{ik} x_k \\ \varepsilon_{ik} &= -\varepsilon_{ki} \end{aligned}\right\} \tag{15}$$

the "local" variation of $F(x_i)$ is given by

$$\delta^* F(x_i) = \bar{F}(x_i) - F(x_i) = \frac{\varepsilon_{kl}}{2} [M_{kl}, F(x_i)] \tag{16}$$

where M_{kl} is the angular momentum four tensor (4).

The "substantial" variation is in this case, by (10), (15), (16), and (12),

$$\begin{aligned} \delta F &= \delta^* F + \frac{\partial F}{\partial x_k} \varepsilon_{kl} x_l \\ &= \frac{\varepsilon_{kl}}{2} [M_{kl} - (x_k P_k - x_l P_k), F] = \frac{\varepsilon_{kl}}{2} m_{kl}(x_i), F], \end{aligned} \tag{17}$$

where

$$m_{kl}(x_i) = M_{kl} - (x_k P_l - x_l P_k)$$

is the total angular momentum four tensor with respect to the point (x_i) in Minkowski space. The m_{ik} defined by (I. 23) is identical with

$$m_{ik}(x_l) \quad \text{for} \quad x_l = X_l.$$

The formula (16) for the "local" variation holds also for an arbitrary polynomial $f(F(x_i), x_i)$, i.e.

$$\delta^* f = f(\bar{F}(x_i); x_i) - f(F(x_i); x_i) = \frac{\varepsilon_{kl}}{2} [M_{kl}, f]. \tag{18}$$

The transformations of dynamical variables contained in the assumptions **B** and **C** are obviously quantum mechanical contact transformations of the form

$$\bar{f}(x_i) = U f(x_i) U^{-1} \tag{19}$$

where we have written $\bar{f}(x_i)$ for $f(\bar{F}(x_i); x_i)$.

In the case of the transformation (5) we have

$$U = 1 + \frac{\varepsilon_k}{i\hbar} P_k \tag{20 A}$$

and for the Lorentz transformation (15) U is given by

$$U = 1 + \tfrac{1}{2} \frac{\varepsilon_{kl}}{i\hbar} M_{kl}. \tag{20 B}$$

Since P_k and M_{kl} are Hermitian, the operators U in (20A) and (20B) are unitary, i.e.

$$UU^\dagger = U^\dagger U = 1 . \tag{21}$$

If $\mathfrak{F} = \mathfrak{F}(x_4)$ is a covariant quantity of the form

$$\mathfrak{F}(x_4) = \int f(F(x_i);\, x_i)\, dx_1 dx_2 dx_3 \tag{22}$$

so that the corresponding quantity $\bar{\mathfrak{F}}(\bar{x}_4)$ in the system of $\bar{x}_i$-coordinates is

$$\bar{\mathfrak{F}}(\bar{x}_4) = \int f(\bar{F}(\bar{x}_i);\, \bar{x}_i)\, d\bar{x}_1 d\bar{x}_2 d\bar{x}_3$$

the connection between $\bar{\mathfrak{F}}(x_4)$ and $\mathfrak{F}(x_4)$ is given by

$$\bar{\mathfrak{F}}(x_4) = \int f(\bar{F}(\bar{\mathbf{x}},\, x_4);\, \bar{\mathbf{x}},\, x_4)\, d\bar{x}_1 d\bar{x}_2 d\bar{x}_3 = \int f(\bar{F}(x_i),\, x_i)\, dx_1 dx_2 dx_3$$

$$= \int [f(F(x_i),\, x_i) + \delta^* f]\, dv = \mathfrak{F}(x_4) + \int \delta^* f\, dv .$$

Thus, for the transformation (5) we have by (13)

$$\bar{\mathfrak{F}}(x_4) = \mathfrak{F}(x_4) + \varepsilon_k [P_k,\, \mathfrak{F}(x_4)] \tag{13'}$$

and, in the case of the Lorentz transformation (15), by (18)

$$\bar{\mathfrak{F}}(x_4) = \mathfrak{F}(x_4) + \frac{\varepsilon_{kl}}{2}[M_{kl},\, \mathfrak{F}(x_4)] . \tag{18'}$$

6. Commutation Relations.

From the general assumptions **A, B, C** we can now derive commutation relations between all the quantities in which we are here interested. The quantities T_{ik} are functions of the field variables F. These functions cannot depend explicitly on the coordinates x_i, for this would mean an inadmissible inhomogeneity in space and time. Application of (14) on the functions g_i then gives

$$\frac{\partial g_k}{\partial x_i} = [P_i,\, g_k] . \tag{23}$$

By integration over the whole x-space we get for the term on the left-hand side

$$\int \frac{\partial g_k}{\partial x_i}\, dv = \begin{cases} 0 \text{ for } i = 1, 2, 3 \\ \dfrac{\partial}{\partial x_4} \displaystyle\int g_k\, dv = \dfrac{\partial}{\partial x_4} P_k = 0 \quad \text{for } i = 4 \end{cases}$$

on account of P_k being constant in time.

From (23) we, therefore, get

$$[P_i, \int g_k\, dv] = [P_i, P_k] = 0 \tag{24}$$

i.e. all components of P_i commute. Since each P_i is of the form (22), this means by (13′) that P_i is unchanged by a change of origin of our coordinate system.

Further, we get, using (23),

$$\begin{aligned}[P_i, M_{kl}] &= \int (x_k [P_i, g_l] - x_l [P_i, g_k])\, dv \\ &= \int \left(x_k \frac{\partial g_l}{\partial x_i} - x_l \frac{\partial g_k}{\partial x_i} \right) dv \\ &= \int \left(\frac{\partial}{\partial x_i} (x_k g_l - x_l g_k) - \delta_{ik} g_l + \delta_{il} g_k \right) dv \\ &= -\delta_{ik} P_l + \delta_{il} P_k \end{aligned} \tag{25}$$

since

$$\int \frac{\partial}{\partial x_i} (x_k g_l - x_l g_k)\, dv = \begin{cases} 0 \text{ for } i = 1, 2, 3 \\ \dfrac{\partial}{\partial x_4} M_{kl} = 0 \quad \text{for } i = 4. \end{cases}$$

Now, (25) gives

$$\varepsilon_i [P_i, M_{kl}] = -\varepsilon_k P_l + \varepsilon_l P_k$$

and, since M_{kl} is also of the form (22), we see again by (13′) that a displacement of the origin $x_i \to x_i - \epsilon_i$ changes M_{ik} by

$$M_{ik} \to M_{ik} - (\varepsilon_i P_k - \varepsilon_k P_i) \tag{26}$$

This may also be seen directly from the definition (4) of M_{ik}. Further we get from (25)

$$\frac{\varepsilon_{kl}}{2} [M_{kl}, P_i] = \frac{\varepsilon_{il}}{2} P_l - \frac{\varepsilon_{ki}}{2} P_k = \varepsilon_{ik} P_k$$

which by (18′) is seen to be in accordance with the vector character of P_i. Conversely, we could also from the condition that $\overline{P}_i = P_i + \varepsilon_{ik} P_k$ by any infinitesimal Lorentz transformation deduce the commutation relations (25).

Similarly, we get from the condition that the constant M_{ik} transform like the components of a tensor the following commutation relations:

$$[M_{ik}, M_{lm}] = \delta_{il} M_{km} - \delta_{im} M_{kl} - \delta_{kl} M_{im} + \delta_{km} M_{il}. \tag{27}$$

If we now define the proper mass M_0 of the system by the equation

$$P_i P_i = - M_0^2 c^2 \tag{28}$$

it follows at once from (24) and (25) that M_0 commutes with P_i and M_{ik}. We have, thus, found the following commutation relations between P_i, M_{ik} and M_0 from the basic assumptions **A**, **B** and **C**.

$$\left.\begin{aligned} &[P_i, P_k] = 0 \\ &[P_i, M_{kl}] = - \delta_{ik} P_l + \delta_{il} P_k \\ &[M_{ik}, M_{lm}] = \delta_{il} M_{km} - \delta_{im} M_{kl} - \delta_{kl} M_{im} + \delta_{km} M_{il} \\ &[M_0, P_i] = [M_0, M_{ik}] = 0 . \end{aligned}\right\} \tag{29}$$

For i, k, l, m equal to one of the numbers 1, 2, 3 these equations contain the usual commutation relations for angular momenta in quantum mechanics. In particular, we get from the third equation

$$[M_x, M_y] = M_z, \ldots \tag{29'}$$

7. Relative Angular Momentum Tensor.

We shall now try to define the coordinates of the centre of gravity of a general quantum mechanical system in such a way that all relations derived in the classical theory hold as q-number relations in the quantum mechanical case. First of all, we define the relative angular momentum tensor m_{ik} by the relations (I, 32), i.e.

$$m_{ik} = - m_{ki} = M_{ik} + \frac{1}{(M_0 c)^2} (M_{il} P_l P_k - M_{kl} P_l P_i) . \tag{30}$$

By means of the second commutation relation (29) one easily finds, since M_{ik} and P_i are Hermitian,

$$m_{ik}^\dagger = m_{ik}, \tag{31}$$

i.e. m_{ik} is also a Hermitian q-number. Further, we get from (30) and (28) the relation

$$m_{ik} P_k = 0 \tag{32}$$

on the analogy of (I, 29).

If we introduce

$$\begin{aligned} \mathbf{m} &= (m_{23}, m_{31}, m_{12}) \\ \dot{\mathbf{n}} &= (m_{14}, m_{24}, m_{34}) \end{aligned} \tag{33}$$

(32) is equivalent to the equation (I, 31), i.e.

$$\mathbf{n} = - \frac{c}{H} (\mathbf{m} \times \mathbf{P}) . \tag{34}$$

By means of (29) and the defintion (30) we get by a simple calculation the following commutation rules:

$$\left.\begin{aligned}
[P_i\,,\ m_{kl}] &= 0 \\
[M_0\,,\ m_{kl}] &= 0 \\
[m_{ik}\,,\ M_{lm}] &= \delta_{il}\, m_{km} - \delta_{im}\, m_{kl} + \delta_{km}\, m_{il} - \delta_{kl}\, m_{im} \\
[m_{ik}\,,\ m_{lm}] &= \left(\delta_{il} + \frac{P_i P_l}{(M_0 c)^2}\right) m_{km} - \left(\delta_{im} + \frac{P_i P_m}{(M_0 c)^2}\right) m_{kl} \\
&\quad + \left(\delta_{km} + \frac{P_k P_m}{(M_0 c)^2}\right) m_{il} - \left(\delta_{kl} + \frac{P_k P_l}{(M_0 c)^2}\right) m_{im}.
\end{aligned}\right\} \tag{35}$$

The first equation shows that the m_{kl} commute with all components of P_i. In contrast with the tensor M_{ik} the tensor components m_{ik} are, thus, independent of the choice of origin of our coordinate system. The third equation (35) expresses the tensor character of m_{ik}. According to the last equation (35) the components of the relative angular momentum vector do not satisfy the usual commutation relations for angular momenta. We have instead

$$[m_x\,,\ m_y] = m_z + \frac{\mathbf{mP}}{(M_0 c)^2}\, P_z\,, \dots . \tag{36}$$

8. Space-Time Coordinates of the Centre of Gravity.

The space-time coordinates X_i of the centre of gravity must now satisfy equations analogous to (I, 22), i.e.

$$m_{ik} = M_{ik} - (\overline{X_i \,.\, P_k} - \overline{X_k \,.\, P_i}) \tag{37}$$

where the bar over a product of two factors denotes the symmetrical combination

$$\overline{A \,.\, B} = \tfrac{1}{2}(AB + BA)\,. \tag{38}$$

Since X_i must be Hermitian, all terms in (37) are then Hermitian.

It seems now natural to define X_i by an equation analogous to the classical equation (I, 47), namely,

$$X_i = \overline{H^{-1} \cdot \int h x_i \, dv} - m_{i4}\, P_4^{-1}. \tag{39}$$

It is easily seen that (39) satisfies (37). From (4) we get

$$\overline{M_{i4} \,.\, P_4^{-1}} = \overline{H^{-1} . \int h \,.\, x_i \, dv} - x_4 P_i P_4^{-1}$$

and (39) may be written

$$X_i = x_4 \,.\, P_i \,.\, P_4^{-1} + \overline{M_{i4} \,.\, P_4^{-1}} - m_{i4} \,.\, P_4^{-1}\,. \tag{40}$$

Therefore, we get by means of (30)

$$\overline{X_i \,.\, P_k} - \overline{X_k \,.\, P_i} = \overline{(M_{i4} - m_{i4}) \,.\, P_4^{-1} \,.\, P_k} - \overline{(M_{k4} - m_{k4}) \,.\, P_4^{-1} \,.\, P_i}$$
$$= -\frac{1}{(M_0 c)^2} \{\overline{(M_{il}P_lP_4 - M_{4l}P_lP_i) \,.\, P_4^{-1} \,.\, P_k} - \overline{(M_{kl}P_lP_4 - M_{4l}P_lP_k) \,.\, P_4^{-1} \,.\, P_i}\}. \tag{41}$$

If we could disregard the bars, we would then get

$$\overline{X_i \,.\, P_k} - \overline{X_k \,.\, P_i} = -\frac{1}{(M_0 c)^2}(M_{il}P_lP_k - M_{kl}P_lP_i) = M_{ik} - m_{ik} \tag{42}$$

which is just the equation (37). Now, it is obviously allowed to disregard the bars, for we have quite generally

$$\overline{A \,.\, B} = AB - \frac{i\hbar}{2}[A, B]$$

and, since $[M_{il}, P_k]$ is a real polynomial in the commuting variables P_i, the right-hand side of (41) could only differ from the right-hand side of (42) by a purely imaginary function of the (P_i), and since both sides of (42) are real, this function must be identically zero. Thus, the definition (39) of X_i is in accordance with the equation (37).

From (40), (29), and (35) we now get after some calculation

$$\left.\begin{aligned}
[X_i, P_k] &= \overline{[M_{i4}, P_k] \,.\, P_4^{-1}} = \delta_{ik} - \delta_{k4}P_iP_4^{-1} \\
[X_i, M_{kl}] &= -\delta_{ik}X_l + \delta_{il}X_k + \delta_{4k} \,.\, \overline{X_l \,.\, (P_iP_4^{-1})} - \delta_{4l}\overline{X_k \,.\, (P_iP_4^{-1})} \\
[X_i, m_{kl}] &= -\frac{1}{(M_0c)^2}\{m_{ik}P_l - m_{il}P_k + P_im_{k4}P_lP_4^{-1} - P_iP_km_{l4}P_4^{-1}\} \\
[X_i, X_k] &= \frac{1}{(M_0c)^2}(m_{ik} - m_{i4}P_kP_4^{-1} + m_{k4}P_iP_4^{-1}) \\
[X_i, M_0] &= 0\,.
\end{aligned}\right\} \tag{43}$$

For $i = 4$, (39) becomes

$$X_4 = \overline{H^{-1} \,.\, x_4 H} = x_4 \tag{44}$$

so that X_4 is always identical with the c-number x_4. For $i = \iota = 1, 2, 3$ (39) gives the components of the coordinate vector $\mathbf{X}$ of the centre of gravity

$$\mathbf{X} = \overline{H^{-1} \,.\, \int h\mathbf{x}\,dv} - c\,\mathbf{n}\,H^{-1} \tag{45}$$

on the analogy of (I, 27). The covariance of our scheme now requires that X_4 must be a c-number in any system of reference. This seems to be in contradiction with the four vector character of X_i, and it is certainly in contradiction with the Minkowski form (I, 55) of the transformation equations. However, if we consider the form (I, 53) or (I, 54) which, in the classical case, is equivalent to (I, 55), we shall see that it has its quantum mechanical analogue. Since X_i in (39) is built up of quantities of the form (22), we have according to (18′) for an infinitesimal Lorentz transformation (15)

$$\bar{X}_i(x_4) = X_i(x_4) + \frac{\varepsilon_{kl}}{2}[M_{kl}, X_i(x_4)] \tag{46}$$

or, by means of the second equation (43),

$$\bar{X}_i(x_4) = X_i + \varepsilon_{ik} X_k - \varepsilon_{4k} \overline{X_k \,.\, (P_i P_4^{-1})} \tag{47}$$

which is the quantum mechanical analogon of (I, 53).

Further, we get from (40)

$$\frac{\partial X_i}{\partial X_4} = \frac{\partial X_i}{\partial x_4} = P_i P_4^{-1} \tag{48}$$

so that (46) may be written in a form analogous to (I, 54):

$$\bar{X}_i(x_4) = X_i + \varepsilon_{ik} X_k - \varepsilon_{4k} \overline{X_k \,.\, \frac{\partial X_i}{\partial X_4}}. \tag{49}$$

In this sense, we may say that X_i is a four vector also in the quantum mechanical case. Since (46) is a contact transformation, it is clear that the whole scheme is relativistically covariant; in particular, all commutation relations will hold also for the transformed variables.

For the statistical interpretation of our formalism it is also essential that the time variable X_4 is always a c-number, for it has no simple physical meaning to ask for the probability that the clock defining X_4 shows a certain value. On the other hand, it has a well-defined meaning to ask for the probability that any of the components of $\mathbf{X}(x_4) = \mathbf{X}(X_4)$ has a certain value at a definite time $X_4 = x_4$. It thus seems that the geometrical picture of Minkowski, so useful in classical theory, is unsuited to give an adequate representation of relativistic quantum mechanics. Instead, we must use a formalism in which the time variable plays a distinguished rôle. This asymmetry between space and time variables should not be considered to be a defect of the theory, since it is deeply rooted in the physical difference between space and time which was so entirely veiled in the Minkowski representation. This is also seen if we consider a little more closely the notion of proper time of a particle. In the Minkowski representation, the space-time coordinates X_i of a particle are naturally regarded as a function of the proper time τ which

The inverse relation of (52) is

$$\mathbf{m} = \frac{H}{M_0 c^2} \overset{\times}{\mathbf{m}} + \frac{1 - \frac{H}{M_0 c^2}}{P^2} (\overset{\times}{\mathbf{m}} \mathbf{P}) \,.\, \mathbf{P} \tag{59}$$

as is seen by introduction of (59) into the right-hand side of (52). The equation (51′) may then be written

$$\mathbf{X} \times \mathbf{X} = \frac{i\hbar}{(M_0 c)^2} \left(\frac{M_0 c^2}{H} \overset{\times}{\mathbf{m}} + \frac{c^2 (\overset{\times}{\mathbf{m}} \mathbf{P})}{H(H + M_0 c^2)} \,.\, \mathbf{P} \right). \tag{60}$$

In a state corresponding to a definite value j and the mean value zero for $\mathbf{P}$ the "area of uncertainty" in the determination of the position of the centre of gravity is thus of the order $j \,.\, \left(\frac{\hbar}{M_0 c} \right)^2$ and this is quite independent of the possible uncertainties in the value of $\mathbf{P}$.

9. A Centre of Mass with Commuting Coordinates.

We can always work with a representation in which, for instance, the set of quantities M_0, $\mathbf{P}$, m^2, and m_z are diagonal, since all these quantities commute; but we cannot in general replace the $\mathbf{P}$ in this set by $\mathbf{X}$, since the components of $\mathbf{X}$ do not commute neither with $\overset{\times}{\mathbf{m}}$ nor with each other. However, in a definite system of reference it is always possible to define a "point" whose coordinates $\overset{\times}{\mathbf{X}}$ commute with each other and with $\overset{\times}{\mathbf{m}}$. According to (37), we have

$$\mathbf{M} = \mathbf{m} + \mathbf{X} \times \mathbf{P}, \tag{61}$$

i.e. $\mathbf{m}$ is the angular momentum vector with respect to the centre of gravity. Let us now consider a point with respect to which the angular momentum of the system is equal to the vector $\overset{\times}{\mathbf{m}}$ defined in (52). If $\overset{\times}{\mathbf{X}}$ is the coordinate vector of this point, we must have

$$\mathbf{M} = \overset{\times}{\mathbf{m}} + \overset{\times}{\mathbf{X}} \times \mathbf{P}. \tag{62}$$

Comparison of (62) and (61) gives, by means of (52),

$$\begin{aligned} (\overset{\times}{\mathbf{X}} - \mathbf{X}) \times \mathbf{P} = \mathbf{m} - \overset{\times}{\mathbf{m}} &= - \frac{1 - \frac{M_0 c^2}{H}}{P^2} [(\mathbf{m}\mathbf{P})\mathbf{P} - \mathbf{m} P^2] \\ &= - \frac{c^2}{H(H + M_0 c^2)} (\mathbf{m} \times \mathbf{P}) \times \mathbf{P}. \end{aligned} \tag{63}$$

To prove these equations we may use the fact that **P** commutes with all quantities occurring in (52) and (36). We may, therefore, treat **P** as a c-number with definite numerical values, and since (52) has vector form we can choose the x-axis, for instance, in the direction of **P**, without spoiling the generality of our results. Then, we have

$$P_x = P, \quad P_y = P_z = 0,$$
$$\overset{\times}{m}_x = m_x, \quad \overset{\times}{m}_y = \frac{M_0 c^2}{H} m_y, \quad \overset{\times}{m}_z = \frac{M_0 c^2}{H} m_z \tag{54}$$

and the equations (36) reduce to

$$\begin{aligned} [m_x, m_y] &= m_z \\ [m_y, m_z] &= m_x \left(1 + \frac{P^2}{(M_0 c)^2}\right) = m_x \frac{H^2}{(M_0 c)^2} \\ [m_z, m_x] &= m_y . \end{aligned} \tag{55}$$

Therefore, we get

$$\begin{aligned} [\overset{\times}{m}_x, \overset{\times}{m}_y] &= \frac{M_0 c^2}{H} [m_x, m_y] = \frac{M_0 c^2}{H} m_z = \overset{\times}{m}_z \\ [\overset{\times}{m}_y, \overset{\times}{m}_z] &= \left(\frac{M_0 c^2}{H}\right)^2 [m_y, m_z] = \left(\frac{M_0 c^2}{H}\right)^2 m_x \frac{H^2}{(M_0 c^2)^2} = m_x = \overset{\times}{m}_x \\ [\overset{\times}{m}_z, \overset{\times}{m}_x] &= \frac{M_0 c^2}{H} [m_z, m_x] = \frac{M_0 c^2}{H} m_y = \overset{\times}{m}_y . \end{aligned}$$

Thus, we have proved the equation (53) for this special choice of the coordinate system, but, on account of the vector character of the equation (52), the equations (53) must hold generally.

On account of (53), $|\overset{\times}{\mathbf{m}}|^2$ must have the eigenvalues $j(j+1)\hbar^2$ where $2j$ is an even number and any of the components of **m** have the eigenvalues $m.\hbar$ where m can take the values

$$-j, \ -j+1, \ \ldots\ldots \ j-1, \ j.$$

It is easy to see that $\overset{\times}{m}^2 = |\overset{\times}{\mathbf{m}}|^2$ is an invariant; in fact, we have

$$\overset{\times}{m}^2 = m^2 - n^2 = \tfrac{1}{2} m_{ik} m_{ik} . \tag{56}$$

For, from (52), we get

$$\begin{aligned} \overset{\times}{m}^2 &= \left(\frac{M_0 c^2}{H}\right)^2 m^2 + \frac{(\mathbf{m}\,\mathbf{P})^2}{P^2} \left(\left(1 - \frac{M_0 c^2}{H}\right)^2 + 2 \frac{M_0 c^2}{H}\left(1 - \frac{M_0 c^2}{H}\right)\right) \\ &= \left(\frac{M_0 c^2}{H}\right)^2 m^2 + \frac{c^2}{H^2} (\mathbf{m}\,\mathbf{P})^2 \end{aligned} \tag{57}$$

and, on the other hand, we have an account of (34)

$$m^2 - n^2 = m^2 - \frac{c^2}{H^2} (\mathbf{m} \times \mathbf{P})(\mathbf{m} \times \mathbf{P}) = \left(\frac{M_0 c^2}{H}\right)^2 m^2 + \frac{c^2}{H^2} (\mathbf{m}\,\mathbf{P})^2. \tag{58}$$

Thus, since M^0 commutes again with all quantities occurring, we have the following list of commutation relations:

$$\left.\begin{aligned}
&[\overset{x}{X}_\iota, \overset{x}{X}_\kappa] = [P_\iota, P_\kappa] = 0 \\
&[\overset{x}{X}_\iota, P_\kappa] = \delta_{\iota\kappa} \\
&[\overset{x}{X}_\iota, \overset{x}{m}_\kappa] = [P_\iota, \overset{x}{m}_\kappa] = 0 \\
&[M_0, \overset{x}{\mathbf{X}}] = [M_0, \mathbf{P}] = [M_0, \overset{x}{\mathbf{m}}] = 0 \\
&[\overset{x}{m}_x, \overset{x}{m}_y] = \overset{x}{m}_z, \; . \; . \; . \; .
\end{aligned}\right\} \tag{71}$$

Thus, $\overset{x}{\mathbf{X}}$, $\mathbf{P}$, and $\overset{x}{\mathbf{m}}$ have similar properties as coordinates, momenta, and spin of a particle in ordinary quantum mechanics. But, in contrast to the spin, $\overset{x}{\mathbf{m}}$ is a constant of motion also in the relativistic region.

We can now use a representation in which M_0, $\overset{x}{\mathbf{X}}$, m^2, and m_z are on diagonal form. From (64) and (45) we get

$$\overset{x}{\mathbf{X}} = \overline{H^{-1} . \int h \, \mathbf{x} \, dv} - \frac{M_0 c^3}{H(H + M_0 c^2)} \, \mathbf{n} \, . \tag{72}$$

The "point" with the coordinates $\overset{x}{\mathbf{X}}$ corresponds classically to a point on the line joining the centre of gravity and the centre of mass of our system of reference, i.e. it corresponds to a point on the disc of mass centres. But in different systems of reference, the $\overset{x}{\mathbf{X}}$ corresponds to the coordinates of different points on the disc. $\overset{x}{\mathbf{X}}$ increases linearly with the time, and the coefficient of increase is the same as for the centre of gravity: $\mathbf{P}P_4^{-1}$.

10. APPLICATION OF THE GENERAL SCHEME TO DIRAC'S THEORY OF ELECTRONS.

The theory developed in the preceding sections of this chapter may be applied to any system with a well defined energy-momentum tensor, as, for instance, Born's nonlinear electromagnetic fields, meson fields of different types or Dirac's electron-positron fields. If any of these fields are considered separately we meet, however, with the difficulty of the infinite zero point energy which makes an immediate application of the definition (II, 39) of the centre of gravity impossible. This difficulty may be overcome by interchange of certain non-commuting factors in the expression for the energy-momentum. Strictly speaking, this procedure means, however, a change in the system considered.

We may, therefore, put

$$\overset{x}{\mathbf{X}} = \mathbf{X} - \frac{c^2}{H(H + M_0 c^2)} (\mathbf{m} \times \mathbf{P}) = \mathbf{X} + \frac{c}{H + M_0 c^2} \mathbf{n} \qquad (64)$$

on account of (34).

By means of the commutation rules between **X**, H and **n** afforded by the equations (43) and (35), it may be shown by a simple calculation that the components of $\overset{x}{\mathbf{X}}$ all commute

$$[\overset{x}{X}_\iota, \overset{x}{X}_\kappa] = 0 . \qquad (65)$$

Since **n** and H commute with **P**, the commutation relations between $\overset{x}{\mathbf{X}}$ and **P** have the same canonical form as for **X** and **P**, i.e.

$$[\overset{x}{X}_\iota, P_\kappa] = \delta_{\iota\kappa} . \qquad (66)$$

By a rotation of the spatial axes of our coordinate system the components of $\overset{x}{\mathbf{X}}$ will transform like the components of a space vector. The equations (18′) applied to $\overset{x}{\mathbf{X}}$ for the case of a pure spatial rotation then lead to the usual commutation relations for angular momenta and vector components, i.e.

$$[M_x, \overset{x}{X}_x] = 0 , [M_x, \overset{x}{X}_y] = \overset{x}{X}_z , \ldots . \qquad (67)$$

or

$$[\overset{x}{X}_\iota, M_{\kappa\lambda}] = - \delta_{\iota\kappa} \overset{x}{X}_\lambda + \delta_{\iota\lambda} \overset{x}{X}_\kappa \qquad (68)$$

where ι, κ, λ may be any of the numbers 1, 2, 3.

If we now put $\overset{x}{\mathbf{m}} = (\overset{x}{m}_{23}, \overset{x}{m}_{31}, \overset{x}{m}_{12})$ (62) may be written

$$\overset{x}{m}_{\iota\kappa} = M_{\iota\kappa} - (\overset{x}{X}_\iota P_\kappa - \overset{x}{X}_\kappa P_\iota) . \qquad (69)$$

Thus, we get by means of (68), (65), and (66)

$$[\overset{x}{X}_\iota, \overset{x}{m}_{\kappa\lambda}] = - \delta_{\iota\kappa} \overset{x}{X}_\lambda + \delta_{\iota\lambda} \overset{x}{X}_\kappa - \overset{x}{X}_\kappa \delta_{\iota\lambda} + \overset{x}{X}_\lambda \delta_{\iota\kappa} = 0 \qquad (70)$$

i.e. $\overset{x}{\mathbf{X}}$ commutes with all components of $\overset{x}{\mathbf{m}}$.

By a Lorentz transformation the functions u and $\breve{u}$ transform in the well-known way.[7] The quantity

$$\begin{aligned} s_i &= i\varepsilon\breve{u}\gamma_i u = i\varepsilon u^\dagger\gamma_4\gamma_i u = \varepsilon u^\dagger a_i u \\ a_i &= \{\mathbf{a},\ i\}\ , \end{aligned} \tag{80}$$

which satisfies the relation

$$\frac{\partial s_i}{\partial x_i} = 0\ , \tag{81}$$

is a four vector for all Lorentz transformations including spatial reflections but a change of sign of x_4 changes the sign of $\mathbf{s}$ while s_4 is unchanged.

By the method developed by Rosenfeld[8] and Belinfante[9] one obtains for the (symmetrical) energy-momentum tensor the expression

$$\begin{aligned} T_{ik} &= \frac{i\hbar c}{4i}\left(\breve{u}\gamma_k\frac{\partial u}{\partial x_i} + \breve{u}\gamma_i\frac{\partial u}{\partial x_k} - \frac{\partial \breve{u}}{\partial x_i}\gamma_k u - \frac{\partial \breve{u}}{\partial x_k}\gamma_i u\right) \\ &= \frac{\hbar c}{4i}\left(u^\dagger a_k\frac{\partial u}{\partial x_i} + u^\dagger a_i\frac{\partial u}{\partial x_k} - \frac{\partial u^\dagger}{\partial x_i}a_k u - \frac{\partial u^\dagger}{\partial x_k}a_i u\right). \end{aligned} \tag{82}$$

This Hermitian tensor satisfies the equation

$$\frac{\partial T_{ik}}{\partial x_k} = 0 \tag{83}$$

and represents the energy-momentum tensor of a closed system of non-interacting electrons in the original Dirac theory of electrons. In the hole theory of electrons and positrons in the form given by Heisenberg[10] we have to add terms in which the order of the factors in (82) are reversed. We shall later treat this case, but for the moment we consider the case of the original Dirac theory.

The commutation relations for the field variables may be written

$$\begin{aligned} [u(q,t),\ u(q',t)]_+ &= u(q,t)\,u(q',t) + u(q',t)\,u(q,t) = 0 \\ u(q,t)\,u^\dagger(q',t) + u^\dagger(q',t)\,u(q,t) &= \delta(q-q') \end{aligned} \tag{84}$$

where $q = (\mathbf{x}, \zeta)$ is an abbreviation for the set of variables x, y, z, ζ and

$$\delta(q-q') = \delta_{\zeta\zeta'}\delta(\mathbf{x}-\mathbf{x}')\ .$$

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[7] See for instance W. Pauli, Relativistic Field Theories of Elementary Particles, Rev of Modern Physics, Vol. 13, No. 3, p. 203, (1941).

[8] L. Rosenfeld, Mémoires de l'Academie Roy. Belgique 6, 30 (1940).

[9] F. J. Belinfante, Physica 6, 887 (1939); Physica 7, 305 (1940).

[10] W. Heisenberg, Zeits. f. Physik 90, 209; 92, 692 (1934).

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We shall illustrate this by considering the case of Dirac's theory of electrons and positrons. Dirac's field equations may be written in the form

$$\frac{\partial u}{\partial x_0} + \boldsymbol{\alpha} \frac{\partial}{\partial \mathbf{x}} u + i \kappa \beta u = 0 , \tag{73}$$

where $x_0 = ct$, $\kappa = m_0 c/\hbar$ and the $u_\zeta(\mathbf{x}, t)$ are the q-number field variables with four components corresponding to the four values of the index ζ. $\boldsymbol{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$ and β are the usual Hermitian Dirac matrices with four rows and columns. If we put

$$\boldsymbol{\alpha} = \rho_1 \boldsymbol{\sigma}, \quad \beta = \rho_3$$

$$\mathbf{p} = \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}}$$

$$H(q) = c \rho_1 \boldsymbol{\sigma} \mathbf{p} + \rho_3 m_0 c^2$$

the field equation (73) may be written

$$i\hbar \frac{\partial u}{\partial t} = H(q) u . \tag{74}$$

$H(q)$ is an Hermitian operator working on the variables $q = (\mathbf{x}, \zeta)$. Further, defining four quantities γ_i by

$$\gamma_\iota = \frac{\beta \alpha_\iota}{i} , \qquad \gamma_4 = \frac{\ddot{\imath}\beta}{i} \tag{75}$$

(73) takes the form

$$\gamma_i \frac{\partial}{\partial x_i} u + \kappa u = 0 . \tag{76}$$

The γ_i satisfy the relations

$$\gamma_i \gamma_k + \gamma_k \gamma_i = 2\delta_{ik}$$

$$\gamma_\iota^\dagger = \gamma_\iota , \quad \gamma_4^\dagger = - \gamma_4 . \tag{77}$$

Thus if $u^\dagger$ is the Hermitian conjugate of u. (It should be remembered that the symbol $\dagger$ changes the sign of i but not $\ddot{\imath}$) it is seen at once that the adjoint field variable $\breve{u}$ defined by

$$\breve{u} = u^\dagger \gamma_4 \tag{78}$$

satisfies the equation

$$\frac{\partial \breve{u}}{\partial x_i} \gamma_i - \kappa \breve{u} = 0 . \tag{79}$$

where $L(q)$ is a Hermitian operator working on the variables q, the commutation relations (84) give

$$[L,\ u(q,t)] = -\frac{1}{i\hbar} L(q)\, u(q,t)$$

$$[L,\ u^{\dagger}(q,t)] = \frac{1}{i\hbar} u^{\dagger}(q,t)\, L(q). \qquad (94)$$

Since P_i in (90) is of the form (93) the fundamental equation (II, 12) gives

$$\frac{\partial u}{\partial x_i} = [P_i,\ u] = -\frac{1}{i\hbar} p_i(q)\, u$$

in accordance with (86). Further we get, by means of (91), (92) and 94, from (II, 17) for the substantial variation of the function u by an infinitesimal Lorentz transformation

$$\begin{aligned}\delta u &= \frac{\varepsilon_{kl}}{2}[M_{kl} - (x_k P_l - x_l P_k),\ u] \\ &= -\frac{1}{\hbar}\frac{\varepsilon_{kl}}{2}[M_{kl}(q) - (x_k p_l - x_l p_k)]\, u \\ &= \frac{\varepsilon_{kl}}{8}(\gamma_i \gamma_k - \gamma_k \gamma_i)\, u\end{aligned}$$

which is in accordance with the well-known transformation properties of Dirac field functions.

The quantity
$$N = \int u^{\dagger} . u\, dq \qquad (95)$$

is easily seen to commute with any quantity L of the form (93) and we can always use a representation in which N is in diagonal form. In the configuration space representation the state considered is represented by a succession of Schrödinger wave-functions

$$\text{const.}\ \psi(q^{(1)}),\quad \psi(q^{(1)}, q^{(2)}), \ldots \psi(q^{(1)}, \ldots \psi^{(N)}), \qquad (96)$$

corresponding to the different eigenvalues of the quantity N. The submatrix $(N \mid L \mid N)$ of L corresponding to a given eigenvalue N working on the wave-function $\psi(q^{(1)}, \ldots q^{(N)})$ then gives the result

$$\begin{aligned}(N \mid L \mid N)\, \psi(q^{(1)}, \ldots q^{(N)}) &= [L(q^{(1)}) + L(q^{(2)}) + \ldots L(q^{(N)})]\, \psi(q^{(1)}, \ldots q^{(N)}) \\ &= [\sum_{r=1}^{N} L(q^{(r)})]\, \psi(q^{(1)}, \ldots q^{(N)}). \qquad (97)\end{aligned}$$

Similarly we get for the reciprocal quantity L^{-1}

$$(N \mid L^{-1} \mid N)\, \psi(q^{(1)}, \ldots q^{(N)}) = [\sum_{r=1}^{N} L(q^{(r)})]^{-1}\, \psi(q^{(1)}, \ldots q^{(N)}) \qquad (98)$$

If we define a Hermitian operator $p_i(q)$ by the equations

$$p_i(q) \equiv p_i = \begin{cases} \dfrac{\hbar}{i}\dfrac{\partial}{\partial x_\iota} & \text{for } i = \iota = 1, 2, 3 \\ \dfrac{i}{c} H(q) & \text{for } i = 4 \end{cases} \tag{85}$$

we have, on account of the field equation (74),

$$\frac{\partial u}{\partial x_i} = \frac{i}{\hbar} p_i u \tag{86}$$

and (82) may be written

$$T_{ik} = \frac{c}{4}\left[u^\dagger . (a_k p_i + a_i p_k) u + u^\dagger (p_i a_k + p_k a_i) . u\right] . \tag{87}$$

In particular we get for the quantity (II, 2)

$$g_i = \frac{1}{ic} T_{i4} = \frac{1}{4}\left[u^\dagger . \left(p_i + \frac{a_i H}{c}\right) u + u^\dagger \left(p_i + \frac{H a_i}{c}\right) . u\right] . \tag{88}$$

For the total energy momentum vector we then have

$$P_i = \int g_i dV = \frac{1}{2}\int u^\dagger . \left(p_i + \frac{a_i H + H a_i}{c}\right) u dv$$

or since

$$\frac{a_i H + H a_i}{2c} = p_i , \tag{89}$$

$$P_i = \int u^\dagger . p_i u \, dq. \tag{90}$$

Similarly we get, by a simple calculation, for the angular momentum four tensor M_{ik}, defined by (II, 4),

$$M_{ik} = \int (x_i g_k - x_k g_i) \, dv = \int u^\dagger . M_{ik}(q) u dq \tag{91}$$

where the operator $M_{ik}(q)$ is given by

$$M_{ik}(q) = \overline{x_i . p_k(q)} + \overline{x_k . p_i(q)} - \frac{i\hbar}{4}(a_i a_k + a_k a_i)$$

$$= x_i p_k(q) - x_k p_i(q) - \frac{i\hbar}{4}(\gamma_i \gamma_k + \gamma_k \gamma_i). \tag{92}$$

Now, for any Hermitian quantity of the form

$$L = \int u^\dagger . L(q) u \, dq = \int u^\dagger L(q) . u \, dq \tag{93}$$

thus in the configuration space representation

$$\eta_i = \sum_r \overline{x_i^{(r)} . H^{(r)}}. \tag{106}$$

For the space-time coordinates of the centre of gravity, defined by (II, 39), we thus get the following configuration space representation

$$\begin{aligned} X_i &= \overline{H^{-1} . \eta_i - m_{i4} P_4^{-1}} \\ &= [\sum_r H^{(r)}]^{-1} . [\sum_s \overline{x_i^{(s)} . H^{(s)}}] - m_{i4} \frac{c}{i} [\sum_r H^{(r)}]^{-1} \end{aligned} \tag{107}$$

where m_{i4} is given by (104).

In the case of one particle only all the sums in the expressions (100) to (107) reduce to one term, and if we suppress the index $^{(r)}$ we then get

$$\begin{aligned} & P_i = p_i = \left\{\mathbf{p}, \frac{i}{c} H\right\} = \left\{\frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}}, \frac{i}{c} H\right\} \\ & H = c(\boldsymbol{\alpha}\, \mathbf{p}) + \beta m_0 c^2 \\ & a_i = \{\boldsymbol{\alpha}, i\} = \{\rho_1 \boldsymbol{\sigma}, i\} \\ & M_0 = m_0 \\ & M_{ik} = \overline{x_i . p_k} - \overline{x_k . p_i} - \frac{i\hbar}{4}(a_i a_k - a_k a_i) \\ & \mathbf{M} = \mathbf{x} \times \mathbf{p} + \frac{\hbar}{2} \boldsymbol{\sigma} \\ & \mathbf{m} = \frac{\hbar}{2} \boldsymbol{\sigma} + \frac{\hbar}{2 m_0 c} \rho_2 (\boldsymbol{\sigma} \times \mathbf{p}) \\ & \mathbf{n} = -\frac{c}{H}(\mathbf{m} \times \mathbf{p}) = -\frac{\hbar}{2 m_0 c} \rho_3 (\boldsymbol{\sigma} \times \mathbf{p}) \\ & x_i = \{\mathbf{x}, x_4 = ict\} \end{aligned} \tag{108}$$

where $\mathbf{x}$ now means the coordinate vector of the particle and t is the time variable (c-number).

The inner angular momentum $\mathbf{m}$ is thus closely connected with the spin, but it is not identical with the spin vector $\mathbf{s} = \frac{\hbar}{2} \boldsymbol{\sigma}$. This is also apparent from the fact that $\mathbf{m}$ is a constant of the motion (commuting with H) while the spin vector is not constant in time. The invariant $|\mathbf{m}|^2 - |\mathbf{n}|^2$ is identical with the square of the spin vector, i.e.

$$|\mathbf{m}|^2 - |\mathbf{n}|^2 = \tfrac{3}{4}\hbar^2 = |\mathbf{s}|^2. \tag{109}$$

and for two quantities L and M of the type (93) we get

$$(N\,|\,LM\,|\,N)\,\psi(q^{(1)},\dots q^{(N)}) = [\sum_r^N L(q^{(r)})]\,[\sum_{s=1}^N M(q^{(s)})]\,\psi(q^{(1)}\dots q^{(N)}). \qquad (99)$$

In configuration space the quantities (90) and (91) are thus given by the operators

$$\begin{aligned} P_i &= \sum_r p_i^{(r)} \\ M_{ik} &= \sum_r M_{ik}^{(r)} \end{aligned} \qquad (100)$$

with
$$\left.\begin{aligned} p_i^{(r)} &= \left\{\mathbf{p}^{(r)}, \frac{i}{c} H^{(r)}\right\}, \quad H^{(r)} = c\alpha^{(r)}\mathbf{p}^{(r)} + \beta^{(r)} m_0^{(r)} c^2 \\ M_{ik}^{(r)} &= \overline{x_i^{(r)} \cdot p_k^{(r)}} - \overline{x_k^{(r)} \cdot p_i^{(r)}} - \frac{i\hbar}{4}(a_i^{(r)} a_k^{(r)} - a_k^{(r)} a_i^{(r)}) \\ &= x_i^{(r)} p_k^{(r)} - x_k^{(r)} p_i^{(r)} - \frac{i\hbar}{4}(\gamma_i^{(r)}\gamma_k^{(r)} - \gamma_k^{(r)}\gamma_i^{(r)}). \\ x_i^{(r)} &= \{\mathbf{x}^{(r)}, x_4 = ict\} \\ a_i^{(r)} &= \{a_i^{(r)}, i\}. \end{aligned}\right\} \qquad (101)$$

All the quantities (101) are operators working on the variables $q^{(r)} = (\mathbf{x}^{(r)}, \zeta^{(r)})$ of the rth particle with the exception of the quantities $x_4^{(r)} = x_4 = ict$ and $a_4^{(r)} = i$ which are ordinary c-numbers.

In the same way we see that the quantity

$$\xi_i = \int u^\dagger \,.\, x_i u\, dq \qquad (102)$$

in configuration space will be represented by the operator

$$\xi_i = \sum x_i^{(r)}. \qquad (103)$$

For the total proper mass M_0 and relative angular momentum tensor m_{ik}, defined by (II, 28) and (II, 30), respectively, we then get in the configuration space representation

$$\begin{aligned} M_0^2 &= \sum_r m_0^{(r)2} - \frac{1}{c^2}\sum_{r\neq s}\left[\mathbf{p}^{(r)}\mathbf{p}^{(s)} - \frac{H^{(r)}H^{(s)}}{c^2}\right] \\ m_{ik} &= \sum_r M_{ik}^{(r)} + \frac{\sum_{r,s,t} M_{im}^{(r)} p_m^{(s)} p_k^{(t)} - M_{km}^{(r)} p_m^{(s)} p_i^{(t)}}{\sum_r m_0^{(r)2} - \frac{1}{c^2}\cdot\sum_{r\neq s}\left[\mathbf{p}^{(r)}\mathbf{p}^{(s)} - \frac{H^{(r)}H^{(s)}}{c^2}\right]}. \end{aligned} \qquad (104)$$

By means of (88) we obtain for the quantity η_i, defined by (I, 50),

$$\eta_i = \int x_i\, h dv = \tfrac{1}{2}\int (u^\dagger \,.\, x_i H u + u^\dagger H x_i \,.\, u)\, dq = \int u^\dagger \,.\, (\overline{x_i \,.\, H})\, u\, dq \qquad (105)$$

If we define a new spin vector $\overset{x}{\boldsymbol{\sigma}}$ by putting

$$\overset{x}{\boldsymbol{\sigma}} = \frac{2}{\hbar}\overset{x}{\mathbf{m}} = \boldsymbol{\sigma} + (\rho_3 - 1)\left[\boldsymbol{\sigma} - \frac{\mathbf{p}(\boldsymbol{\sigma}\mathbf{p})}{p^2}\right] \tag{116}$$

i.e.

$$\overset{x}{\mathbf{m}} = \frac{\hbar}{2}\overset{x}{\boldsymbol{\sigma}} \tag{117}$$

a simple calculation shows that

$$\overset{x}{\sigma}_x\overset{x}{\sigma}_y = -\overset{x}{\sigma}_y\overset{x}{\sigma}_x = i\overset{x}{\sigma}_z \ldots,$$

$$\overset{x}{\sigma}_x{}^2 = \overset{x}{\sigma}_y{}^2 = \overset{x}{\sigma}_z{}^2 = 1. \tag{118}$$

Thus the components of the vector $\overset{x}{\boldsymbol{\sigma}}$ satisfy the same commutation rules as the Pauli-matrices $\boldsymbol{\sigma}$.

If we further define new variables $\overset{x}{\rho}_1, \overset{x}{\rho}_2, \overset{x}{\rho}_3$ by

$$\overset{x}{\rho}_1 = \rho_1\frac{\boldsymbol{\sigma}\mathbf{p}}{p}, \quad \overset{x}{\rho}_2 = \rho_2\frac{\boldsymbol{\sigma}\mathbf{p}}{p}, \quad \overset{x}{\rho}_3 = \rho_3 \tag{119}$$

we have

$$\overset{x}{\rho}_1\overset{x}{\rho}_2 = -\overset{x}{\rho}_2\overset{x}{\rho}_1 = \rho_1\rho_2 = i\rho_3 = i\overset{x}{\rho}_3, \quad \overset{x}{\rho}_2\overset{x}{\rho}_3 = i\overset{x}{\rho}_1, \quad \overset{x}{\rho}_3\overset{x}{\rho}_1 = i\overset{x}{\rho}_2$$

$$\overset{x}{\rho}_1{}^2 = \overset{x}{\rho}_2{}^2 = \overset{x}{\rho}_3{}^2 = 1$$

and the new variables $\overset{x}{\rho}_i$ commute with the variables $\mathbf{p}$ and $\overset{x}{\boldsymbol{\sigma}}$.
The inverse relations of (116) and (119) are easily seen to be

$$\boldsymbol{\sigma} = \overset{x}{\boldsymbol{\sigma}} + (\overset{x}{\rho}_3 - 1)\left[\overset{x}{\boldsymbol{\sigma}} - \frac{\mathbf{p}(\overset{x}{\boldsymbol{\sigma}}\mathbf{p})}{p^2}\right]$$

$$\rho_1 = \overset{x}{\rho}_1\frac{(\overset{x}{\boldsymbol{\sigma}}\mathbf{p})}{p}, \quad \rho_2 = \overset{x}{\rho}_2\frac{(\overset{x}{\boldsymbol{\sigma}}\mathbf{p})}{p}, \quad \rho_3 = \overset{x}{\rho}_3.$$

In the new variables the Hamiltonian thus takes the simple form

$$H = c\overset{x}{\rho}_1\frac{(\overset{x}{\boldsymbol{\sigma}}\mathbf{p})}{p}(\overset{x}{\boldsymbol{\sigma}}\mathbf{p}) + \overset{x}{\rho}_3 m_0 c^2 = c\overset{x}{\rho}_1 \cdot p + \overset{x}{\rho}_3 m_0 c^2, \tag{120}$$

The coordinates $\overset{x}{\mathbf{X}}$ are increasing linearly in time with the same constant velocity $\frac{c^2\mathbf{p}}{H}$ as the coordinates $\mathbf{X}$ of the centre of gravity, while the variable $\overset{x}{\mathbf{x}}$ in (114) is oscillating with the frequency of Schrödinger's "Zitterbewegung." If ψ is the Schrödinger function in the Heisenberg picture $\psi\,(q)$ is independent of time. We can now always write ψ as a sum of two parts

$$\psi = \psi_+ + \psi_-$$

From (107) we get for the coordinates of the centre of gravity

$$X_\iota = \overline{H^{-1} \cdot \overline{X_\iota} \cdot H} - c n_\iota \cdot H^{-1}$$
$$= X_\iota - \frac{i\hbar c}{2} (a_\iota - H^{-1} c p_\iota) H^{-1} - c n_\iota . H^{-1}$$

or

$$\mathbf{x} = \mathbf{X} + \tilde{\mathbf{x}} \tag{110}$$

with

$$\tilde{\mathbf{x}} = \frac{i\hbar c}{2} (\alpha - H^{-1} c \mathbf{p}) H^{-1} + c \mathbf{n} . H^{-1} . \tag{111}$$

According to (II, 40) the first term in the decomposition of $\mathbf{x}$ in (110) increases linearly with time. The other part $\tilde{\mathbf{x}}$ is oscillating with the frequency of Schrödinger's "Zitterbewegung."

By means of (108)-(111) it is easily verified directly that the commutation relations (II, 29), (II, 35), (II, 43), (II, 50) and (II, 51) are satisfied.

For the quantity $\overset{x}{\mathbf{m}}$ defined by (II, 52) we get after some calculation

$$\overset{x}{\mathbf{m}} = \frac{\hbar}{2} \left\{ \sigma + (\rho_3 - 1) \left[\sigma - \frac{\mathbf{p}(\sigma \mathbf{p})}{p^2} \right] \right. \tag{112}$$

which satisfies the usual commutation rules (II, 53) for an angular momentum. The vector $\overset{x}{\mathbf{m}}$ is a constant of the motion like $\mathbf{m}$ and it has the same magnitude as the spin vector $\mathbf{s} = \frac{\hbar}{2} \sigma$, i.e.

$$| \overset{x}{\mathbf{m}} |^2 = \tfrac{3}{4} \hbar^2 \tag{113}$$

according to (II, 56) and (109).

The coordinates $\overset{x}{\mathbf{X}}$ of the mass centre defined in section 9 are given by (64), thus

$$\overset{x}{\mathbf{X}} = \mathbf{X} + \frac{c}{H + m_0 c^2} \cdot \mathbf{n} = \mathbf{x} - \tilde{\mathbf{x}} + \frac{c}{H + m_0 c^2} \mathbf{n}$$
$$= \mathbf{x} - \frac{i\hbar c}{2} (\alpha - H^{-1} c \mathbf{p}) \cdot H^{-1} - \frac{m_0 c^2}{H(H + m_0 c^2)} c \mathbf{n} \tag{114}$$
$$= \mathbf{x} - \overset{x}{\mathbf{x}} .$$

The components of $\overset{x}{\mathbf{X}}$ commute and satisfy canonical commutation relations with the components of $\mathbf{p}$. Further the components of $\overset{x}{\mathbf{m}}$ commute with $\overset{x}{\mathbf{X}}$ and with $\mathbf{p}$. According to (II, 62) the total angular momentum $\mathbf{M}$ may be written as a sum of two terms

$$\mathbf{M} = \overset{x}{\mathbf{X}} \times \mathbf{p} + \overset{x}{\mathbf{m}} \tag{115}$$

each of which are constants of the motion.

which in the Fourier expansion correspond to positive and negative energies, respectively. In a state described by a pure ψ_+-function or a pure ψ_--function the mean values of $\mathbf{x}$, $\mathbf{X}$ and $\overset{x}{\mathbf{X}}$ are increasing linearly in time with the same constant velocity. The "Zitterbewegung" of the variable $\mathbf{x}$ appears only in the mean values of states which are represented by a super-position of a ψ_+-and a ψ_--function.

Following the prescription of Heisenberg we now pass over to the positron theory by replacing any quantity of the form $u^\dagger \cdot L(q)\, u$ in the original Dirac theory by

$$\tfrac{1}{2}(u^\dagger \cdot Lu - u \cdot u^\dagger L).$$

Here $L = L(q)$ may be any operator working on the variables q.

Similarly we have to replace any quantity $u^\dagger L \,.\, u$ by

$$\tfrac{1}{2}(u^\dagger L \cdot u - Lu \cdot u^\dagger).$$

Thus, we get for instance for the total energy-momentum vector instead of (90),

$$P_i = \tfrac{1}{2}\int (u^\dagger \cdot p_i u - u \cdot u^\dagger p_i)\, dq$$

$$= \tfrac{1}{2}\int (u^\dagger \cdot p_i u - p_i u \cdot u^\dagger)\, dq. \qquad (121)$$

On account of the commutation relations (84) the expression (121) deviates from the quantity (90) by an infinite constant amount

$$-\int \left[\lim_{q' \to q} \tfrac{1}{2} p_i(q)\, \delta(q - q')\right] dq.$$

This is the infinite zero-point energy of the vacuum which is characteristic for the positron theory. A neglect of this infinite constant thus means that we are going back to the original expression (90) for P_i.

For the same reason any of the quantities M_{ik}, η_i, X_i etc. will in the positron theory contain an infinite constant. Neglecting these infinite constants thus formally amounts to a return to the expressions (108)–(119) of the primitive theory. But in the physical interpretation of the formalism there is an essential difference. The Schrödinger wave function of an electron can in the positron theory only be of the form of a ψ_+-function while the state of a positron is always given by a ψ_--function, a super-position of a ψ_+-and a ψ_--function having no meaning in this theory. The mean value of for instance the variable $\mathbf{x}$ in a given electron- or positron-state is thus always increasing linearly with time, while the Schrödinger "Zitterbewegung" is unobservable in such a state.

Added in proof.—In the meantime a paper by M. H. L. Pryce has been published in Proc. Royal Soc., A, vol. **195**, 62 (1948), which contains a number of the results communicated in my Dublin lectures.