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Article
Inferential Interpretations of Many-Valued Logics

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Abstract: Non-Tarskian interpretations of many-valued logics have been widely explored in the logic literature. The development of non-tarskian conceptions of logical consequence set the theoretical foundations for rediscovering well-known (Tarskian) many-valued logics. One may find in distinct authors many novel interpretations of many-valued systems. They are produced through a type of procedure which consists in altering the semantic structure of Tarskian many-valued logics in order to output a non-Tarskian interpretation of these logics. Through this type of transformation the paper explores a uniform way of transforming finitely many-valued Tarskian logics into their non-Tarskian interpretation. Some general properties of carrying out this type of procedure are studied, namely the dualities between these logics and the conditions under which negation-explosive and negation-complete Tarskian logics become non-explosive.

Keywords: many-valued logics; non-tarskian entailment; paraconsistent logics

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1. Introduction

Non-Tarskian interpretations of many-valued logics have been widely explored in the logic literature. The development of non-tarskian conceptions of logical consequence set the theoretical foundations for rediscovering well-known (Tarskian) many-valued logics. One may find in authors such as G. Malinowski ([1]), Shramko and Wansing ([2]) and Cobreros et al. ([3]), many novel interpretations of many-valued systems.

These novel interpretations were created from a revised semantical apparatus first proposed by G. Malinowski [4] in response to Roman Suszko's ideas about the conception of many-valuedness defended by Łukasiewicz. These semantic structures, dubbed *quasi-matrices*, allowed the definition of alternative entailment relations that are different from the usual Tarskian requirements of truth-preservation between premises and conclusion.

In [2,3], the authors explore non-tarskian consequence relations as interpretations of a given Tarskian many-valued logic. In short, this transformation process consists in outputting a non-Tarskian many-valued logic given a Tarskian many-valued logic as input. It is exhibited that this move affects the underlying notion of entailment over the same vocabulary and interpretation of the logical constants. Even though non-Tarskian interpretations of some well-known many-valued logics, like K3, LP, FDE and SIXTEEN, have been explored and discussed by these authors, there is still little discussion over a uniform way of carrying out this procedure to other many-valued logics, as well as over what consequences follow from this type of transformation in many-valued logics. As an example, it is shown in [1] that Łukasiewicz's logic $Ł_3$ becomes paraconsistent after its transformation in Malinowski's style, but questions such as what other many-valued logics are made paraconsistent after the same type of process, what is the standard way (if any) of carrying out such procedure, as well as what are the inferential relation between these logics, are not touched by the author.

The purpose of the present paper is to explore this gap by proposing a uniform way of producing non-Tarskian interpretations of propositional finitely many-valued Tarskian logics endowed with a linearly ordered set of truth-values. By uniform it is meant a procedure applicable to a class of many-valued logics as a way to explore general properties of non-Tarskian logics. As a consequence, it is shown under what conditions a specific class of propositional Tarskian many-valued logic becomes paraconsistent or paracomplete after its transformation, namely, finitely many-valued logics with a linearly ordered set of truth-values and truth-functional semantics. Thus, the topic of the paper aligns with the problem of producing paraconsistent logics from a base class of many-valued systems[1]. The problem of what are the adequate duals for these entailment relations is also explored through distinct dualization procedures and some central properties of these dual logics are exhibited, such as their ability to validate connexive principles relative to implication.

Overview. The paper is structured as follows. Section 2 introduces all notions of entailment to be considered, as well as the notion of semantic revision to be explored throughout the paper. Section 3 introduces the concept of inferentially many-valued logics and its relation to paraconsistency and paracompleteness. We explore under what conditions a finitely many-valued Tarskian logic becomes paraconsistent or paracomplete after its transformation into an inferentially many-valued logic and provide a partial solution to the case of finitely many-valued propositional logics with a linearly ordered sets of truth-values, a problem first raised by Malinowski in [1]. Łukasiewicz's three-valued logic is explored throughout the paper as a illustrative case but shown to generalize to the finitely-valued case. Section 4 introduces two different types of dualization procedure for the semantic structures. These dualization procedures are based on Malinowski ([1]) and Blasio et al. ([7]). After that, in Section 5, we explore the relation between the dual semantic structures and their associated entailment relations. In Section 6, the use of unary operators is introduced as a way of recovering inferences from the Tarskian base logic. The paper ends with final comments and paths for future exploration.

2. Bidimensional Matrices

Let a logic be any structure of kind $\mathcal{L} = \langle Fr, \models \rangle$, where Fr is a non-empty set of formulas understood as the carrier of the algebra freely generated by the set of atoms At over a set of connectives Con, and $\models$ is a SET-FMLA semantic consequence relation (or *entailment relation*). Every entailment relation shall be based on a semantic structure. The semantic structure of Tarskian logics shall be determined by a logical matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \{f_\mu : \mu \in \mathsf{Con}\} \rangle$, where $\mathcal{V}$ is the set of truth-values and $\mathcal{D}$ is a subset of $\mathcal{V}$ called the *designated* set of truth-values. To each logical matrix $\mathcal{M}$ one may associate a semantics Val defined as a set of homomorphic assignments $v : Fr \longrightarrow \mathcal{V}$ [2]. We shall assume every logic considered in the paper comes endowed with a truth-functional semantics, i.e., $v(\mu(\varphi_1, \ldots, \varphi_n)) = f_\mu((v(\varphi_1), \ldots, v(\varphi_n))$ holds for every valuation v and $\mu \in \mathsf{Con}$. Accordingly, given a semantics Val and formulas $\Gamma \cup \{\varphi\} \subseteq Fr$, a **tarskian entailment relation** (hereafter called **t-entailment**) may be introduced as:

$$\Gamma \models^{\mathsf{t}} \varphi \text{ iff } v(\Gamma) \subseteq \mathcal{D} \text{ implies } v(\varphi) \in \mathcal{D}, \text{ for every } v \in Val. \tag{1}$$

As usual, the relation of t-entailment is understood as the preservation of designatedness from premises to conclusion. Entailment relations that preserve values from premises to conclusion shall be called *forwards-preserving*. As proposed by Shramko and Wansing ([2]), dual to a forwards-preserving entailment relation, a notion of preservation of non-designatedness (hereafter called **f-entailment**) from conclusion to premises may be also introduced:

$$\Gamma \models^{\mathsf{f}} \varphi \text{ iff } v(\varphi) \in (\mathcal{V} - \mathcal{D}) \text{ implies } v(\Gamma) \cap (\mathcal{V} - \mathcal{D}) \neq \varnothing, \text{ for every } v \in Val. \tag{2}$$

It is easy to see that $\models^{\mathsf{t}} = \models^{\mathsf{f}}$ in every logical matrix $\mathcal{M}$. This is due the fact that designatedness and non-designatedness are complementary concepts in logical matrices.

The need for treating designatedness and non-designatedness as non-complementary concepts was first explored by G. Malinowski ([4]) as a way of constructing semantical structures capable of accommodating values that are neither designated nor non-designated.

Malinowski's semantical structures, called hereafter **bidimensional matrices**, are any matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_\mu : \mu \in \mathsf{Con}\}\rangle$, where $\mathcal{D}^+$ and $\mathcal{D}^-$ are distinct proper subsets of $\mathcal{V}$. The sets $\mathcal{D}^+$ and $\mathcal{D}^-$ are called, respectively, the *designated* and *antidesignated* sets of truth-values. A bidimensional matrix is said to allow *inferential gaps* whenever $\mathcal{V} \neq (\mathcal{D}^+ \cup \mathcal{D}^-)$ holds, and said to allow *inferential gluts* whenever $\mathcal{D}^+ \cap \mathcal{D}^- \neq \varnothing$ holds. Any bidimensional matrix that allows inferential gaps shall be called a **q-matrix**. Analogously, any bidimensional matrix that allows inferential gluts shall be called a **p-matrix**. It is easy to see that a bidimensional matrix that disallows inferential gaps and gluts coincides with a logical matrix (hereafter called a **t-matrix**).

Given a q-matrix, one may introduce a notion of **quasi-entailment** (or **q-entailment**) in the following way:

$$\varGamma \models^{\mathsf{q}} \varphi \text{ iff } v(\varGamma) \cap \mathcal{D}^- = \varnothing \text{ implies } v(\varphi) \in \mathcal{D}^+, \text{for every } v \in Val. \quad (3)$$

The relation of q-entailment is understood as a kind of reasoning by conjectures, where non-rejected premises lead to accepted conclusions. Analogously, given a q-matrix, one may also introduce a different notion of entailment, originally proposed by Frankowski [8] and dubbed **p-entailment** (for **plausible entailment**):

$$\varGamma \models^{\mathsf{p}} \varphi \text{ iff } v(\varGamma) \subseteq \mathcal{D}^+ \text{ implies } v(\varphi) \notin \mathcal{D}^-, \text{for every } v \in Val. \quad (4)$$

In the way just defined, p- and q-entailment are both based on the idea of rejecting inferential gluts. And whereas q-entailment is understood as a reasoning from non-rejected premises to accepted conclusions, p-entailment is understood as reasoning from accepted premises to non-rejected conclusions. As known in the literature on non-Tarskian logics, while t-entailment validate Reflexivity, Monotonicity and Transitivity (check below), the q-entailment relation does not validate Reflexivity. Furthermore, in our formulation above, p-entailment also validate Reflexivity, Monotonicity and Transitivity.

Reflexivity $\varGamma, \varphi \models \varphi$
Monotonicty If $\varGamma \models \varphi$ then $\varGamma \cup \Delta \models \varphi$
Transitivity If $\Delta \models \varphi$ and $\varGamma \cup \{\varphi\} \models \psi$ then $\varGamma \cup \Delta \models \varphi$

In the following, a formula $\varphi \in Fr$ shall be called an *L-validity* (written $\models \varphi$) in view of any entailment relation in case $v(\varphi) \in \mathcal{D}^+$ for every valuation $v \in Val$. Analogously, it shall be called an *L-falsity* (written $\varphi \models$) in case there is no $v \in Val$ such that $v(\varphi) \in \mathcal{D}^+$.

The paper shall exploit the process of transforming the entailment relation of a base logic $\mathcal{L} = \langle Fr, \models \rangle$ over the set Fr, so as to output a revised logic $\mathcal{L}^{\mathsf{x}} = \langle Fr, \models^{\mathsf{x}} \rangle$. The logic $\mathcal{L}^{\mathsf{x}}$ (also called an x-logic), generated from $\mathcal{L} = \langle Fr, \models \rangle$, shall be called an x-*interpretation* of the logic $\mathcal{L}$. Throughout the paper, the process of generating x-interpretations of a given logic shall be carried out by transforming the semantic structure of a base Tarskian logic into its non-Tarskian counterpart. In sum, given a Tarskian logic $\mathcal{L} = \langle Fr, \models \rangle$ and its associated logical matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \{f_\mu : \mu \in \mathsf{Con}\}\rangle$, we transform $\mathcal{M}$ into its bidimensional form $\mathcal{M}^{\mathsf{q}} = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_\mu : \mu \in \mathsf{Con}\}\rangle$ by rearranging the set $\mathcal{V}$ of truth-values into three disjoint subsets and keeping the set Con of operations unchanged. The logic $\mathcal{L}^{\mathsf{x}} = \langle Fr, \models^{\mathsf{x}} \rangle$, where $\mathsf{x} = \mathsf{q}$ or p, shall be based on the matrix $\mathcal{M}^{\mathsf{q}}$. In the case of q- and p- base logics this shall be done by turning bidimensional matrices into their dual forms, where two conceptions of duality shall be introduced.

3. Inferentially Many-Valued Logics and Paraconsistency

Historically, q-matrices were proposed in reaction to Suszko's thesis, the philosophical claim that every Tarskian logic is (logically) two-valued[3], for their semantics are characterized by the two logical values of designatedness and antidesignatedness, a result known as *Suszko's reduction theorem*. In Suszko's philosophical view, the designated and antidesignated sets of truth-values were responsible for splitting the universe of situations into those that "do obtain" and those that "do not obtain".

As noted by Malinowski, however, if one takes Łukasiewicz's idea of undeterminedness in connection to Suszko's reading of the partitions of the set of truth-values, then one may also consider that true undeterminedness consists not in the addition of a third element in the set of truth-values, but rather in building many-valued structures that allow for the existence of inferential gaps, i.e., values that are neither designated nor antidesignated (and therefore $\mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-) \neq \varnothing$). Accordingly, these semantical structures would allow the existence of situations that neither obtain nor fail to obtain, i.e., undetermined situations.

As a result, a natural way of producing three-valued logics able to avoid Suszko's criticism is through the revision of Tarskian many-valued logics into logics with inferential gaps. For this, in the following I shall call a logic **inferentially many-valued** if its semantic structure is characterized by more than two distinct proper subsets of $\mathcal{V}$[4] Even though other semantic structures can be explored as inferentially many-valued, in what follows I shall focus only on inferentially many-valued logics based on bidimensional matrices. By allowing the type of procedure that outputs an inferentially many-valued interpretation of a given Tarskian base logic, an important problem is to know what consequences follow from this move and what are the correct non-trivial ways of carrying it.

In papers like [1,2], the authors exploit this transformation process, but no general solution is offered as to what causes the gain/lost of certain properties after the output of an x-interpretation of a Tarskian base logic. Moreover, the authors do not account for the problem of producing an effective or uniform way of carrying out the procedure. Therefore, in this section I shall define a uniform way of transforming finitely many-valued Tarskian logics with a linearly ordered set of truth-values into their inferential interpretations. Subsequently, I show why some of these logics become paraconsistent after the procedure.

3.1. Uniform Interpretations

The transition from a many-valued (Tarskian) logic to an inferentially many-valued logic can be carried in several ways. The existence of more than two proper subsets of truth-values allows one to rearrange the truth-values in many distinct ways. As an example, let $\mathcal{L} = \langle Fr, \models \rangle$ be a t-logic associated to the logical matrix $\mathcal{M} = \langle \{0, \frac{1}{2}, 1\}, \{1\} \rangle$. To produce a q-interpretation of $\mathcal{L}$ depends on transforming the logical matrix $\mathcal{M}$ into a q-matrix $\mathcal{M}^{\mathsf{q}}$, for which now, given the existence of three proper subsets of truth-values, there are several different ways of rearranging the truth-values. Therefore one needs to determine which criteria to employ in order to rule out the undesirable options, as well as to fix a uniform manner of carrying this procedure in a non-trivial and well-motivated manner. The notion of uniform transformation is defined below.

In what follows, let $\preceq$ be a reflexive, anti-symmetric and transitive order and let $\mathcal{V}$ be a linearly ordered set under $\preceq$ with more than two elements. Given two distinct sets of truth-values $\mathcal{D}^1$ and $\mathcal{D}^2$, I shall write $\mathcal{D}^1 < \mathcal{D}^2$ to denote that $\mathcal{D}^1$ is strictly smaller than $\mathcal{D}^2$, i.e., $v_1 \preceq v_2$ for every value $v_1 \in \mathcal{D}^1$ and $v_2 \in \mathcal{D}^2$.

Definition 1. *Given a many-valued t-logic $\mathcal{L} = \langle Fr, \models_{\mathcal{M}} \rangle$ and its associated logical matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \{f_\mu : \mu \in Con\} \rangle$, where $(\mathcal{V} - \mathcal{D}) < \mathcal{D}$, the transformation of $\mathcal{M}$ to a bidimensional matrix $\mathcal{M}^{\mathsf{q}} = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_\mu : \mu \in Con\} \rangle$ shall be called* ***uniform*** *iff the following hold:*

(i) $\mathcal{D} = \mathcal{D}^+$,
(ii) $\mathcal{D}^- < \mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-) < \mathcal{D}^+$, *and*
(iii) $\mathcal{D}^-, \mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-), \mathcal{D}^+$ *are non-empty sets.*

The x-logic $\mathcal{L}^{\mathsf{x}} = \langle Fr, \models^{\mathsf{x}} \rangle$ based on the bidimensional matrix $\mathcal{M}^{\mathsf{q}} = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_\mu : \mu \in Con\} \rangle$ shall be called a ***uniform*** x***-interpretation*** *of $\mathcal{L}$.*

Throughout the paper we shall assume that all transformations into a bidimensional matrix preserve the underlying original algebra of the logical matrix and, therefore, its set of valuations. As a result, the paper shall focus only on how the above procedure affects the underlying relation of entailment.

The above type of procedure shall be called uniform due the preservation of Łukasiewicz's intuition over the role of multiple values as undetermined truth-values together with Malinowski's motivations for having a neither designated nor antidesignated set of truth-values. This is achieved by keeping all intermediate truth-values within the set $\mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-)$, a move that gives a new meaning to these values by turning them into inferential gaps (i.e., truth-values able to avoid Suszko's criticism)[5].

It is also worth to note that the uniform transformation of a matrix $\mathcal{M}$ to a bidimensional matrix $\mathcal{M}^{\mathsf{q}}$ does not alter the order of the elements, only its partition set. This allows us to obtain the following:

Fact 1. *Let $\mathcal{M}$ be a logical matrix and $\mathcal{M}^{\mathsf{q}}$ be its* uniform *bidimensional matrix. Given two values v_1 and $v_2 \in \mathcal{V}$ such that $v_1 \preceq v_2$, then $v_1 \preceq v_2$ in $\mathcal{M}^{\mathsf{q}}$.*

Proof. Straight from Definition 1. □

The relation between the original many-valued structures and their bidimensional form can be explored through Humberstone's [10] conception of matrix homomorphisms. The purpose is to extend the concept of homomorphism from algebras to matrices as follows:

Definition 2 (Matrix homomorphism)**.** *Let $\mathbb{V}_1 = \langle \mathcal{V}_1, \{f_\mu : \mu \in Con\}\rangle$ and $\mathbb{V}_2 = \langle \mathcal{V}_2, \{f_\mu : \mu \in Con\}\rangle$ be two algebras of same similarity type as the algebra of formulas.*
Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1^+, \mathcal{D}_1^- \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2^+, \mathcal{D}_2^- \rangle$ be two bidimensional matrices.
Let $f : \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ be a homomorphism from $\mathbb{V}_1$ to $\mathbb{V}_2$.

We say that f is a ***designation-preserving*** *matrix homomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$ if $a \in \mathcal{D}_1^+$ then $f(a) \in \mathcal{D}_2^+$ holds, for every $a \in \mathcal{V}_1$.*

We say that f is a ***undesignation-preserving*** *matrix homomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$ if $a \notin \mathcal{D}_1^+$ then $f(a) \notin \mathcal{D}_2^+$ holds, for every $a \in \mathcal{V}_1$.*

We say that f is a ***strong*** *matrix homomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$ if it is both a designation-preserving and an undesignation-preserving matrix homomorphism fom $\mathcal{M}_1$ to $\mathcal{M}_2$.*

Note that, by definition of uniform transformation, the identity mapping from $\mathcal{V}_1$ into $\mathcal{V}_2$ is a strong matrix homomorphism from a logical matrix $\mathcal{M} = \langle \mathcal{V}_1, \mathcal{D}, \{f_\mu : \mu \in \mathsf{Con}\}\rangle$ into its uniform transformation $\mathcal{M}^{\mathsf{q}} = \langle \mathcal{V}_2, \mathcal{D}^+, \mathcal{D}^-, \{f_\mu : \mu \in \mathsf{Con}\}\rangle$ given that $\mathcal{V}_1 = \mathcal{V}_2$. Furthermore, for matter of clarity and convenience I shall keep writing homomorphisms between logical matrices. The reader may note that any logical matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}\rangle$ can be transformed into its bidimensional equivalent form by setting a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-\rangle$, where $\mathcal{D}^+ = \mathcal{D}$ and $\mathcal{D}^- = (\mathcal{V} - \mathcal{D})$. The following also holds:

Proposition 1. *Let $\mathcal{M}$ be a many-valued matrix and $\mathcal{M}^{\mathsf{q}}$ its uniform* q*-transformation. If there is a designation-preserving homomorphism from $\mathcal{M}$ to $\mathcal{M}^{\mathsf{q}}$, then $\models^t \varphi$ implies $\models^{\mathsf{q}} \varphi$, for every $\varphi \in Fr$.*

Proof. Let $\models^t \varphi$ for an arbitrary $\varphi \in Fr$ and assume f is a designation-preserving homomorphism from $\mathcal{M}$ to $\mathcal{M}^{\mathsf{q}}$. By definition of t-entailment, we know that $v(\varphi) \in \mathcal{D}$ for every $v \in Val$. By definition of designation-preserving homomorphism, it follows that $f(v(\varphi)) \in \mathcal{D}^+$ for every $v \in Val^{\mathsf{q}}$. Therefore, $\models^{\mathsf{q}} \varphi$ holds. □

Corollary 1. *If $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \{f_\mu : \mu \in Con\}\rangle$ and $\mathcal{M}^{\mathsf{q}} = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_\mu : \mu \in Con\}\rangle$ with $\mathcal{D} = \mathcal{D}^+$, then $\models^t \varphi$ implies $\models^{\mathsf{q}} \varphi$.*

Proof. Set $f(v) = v$ for every $v \in \mathcal{D}^+$. By Proposition 1, our desired result follows. □

Proposition 2. *Let $\mathcal{M}$ be a many-valued matrix and $\mathcal{M}^{q}$ its uniform q-transformation. If there is an undesignation-preserving homomorphism from $\mathcal{M}$ to $\mathcal{M}^{q}$, then $\varphi \models^{t}$ implies $\varphi \models^{p}$, for every $\varphi \in Fr$.*

Proof. Set $f(v) = v$ for every $v \notin \mathcal{D}^{+}$. □

Corollary 2. *If $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \{f_\mu : \mu \in Con\}\rangle$ and $\mathcal{M}^{q} = \langle \mathcal{V}, \mathcal{D}^{+}, \mathcal{D}^{-}, \{f_\mu : \mu \in Con\}\rangle$ with $(\mathcal{V} - \mathcal{D}) \subseteq \mathcal{D}^{-}$, then $\varphi \models^{t}$ implies $\varphi \models^{p}$.*

Proof. By Proposition 2. □

In the following, we show that the pair of designation-preserving and undesignation-preserving homomorphisms establish a Galois connection between a logical matrix and its q-transformation[6].

Definition 3 ([11]). *Let $\mathcal{P} = \langle P, \preceq\rangle$ and $\mathcal{Q} = \langle Q, \sqsubseteq\rangle$ be posets. Suppose $f_1 : P \to Q$ and $f_2 : Q \to P$ are a pair of functions between their carrier sets. Then $\langle f_1, f_2\rangle$ is a Galois connection if and only if*

(i) f_1, f_2 are both monotone, and
(ii) for all $p \in P$, $q \in Q$ $p \preceq f_1(f_2(p))$ and $q \sqsubseteq f_1(f_2(q))$.

Theorem 1. *Let $\mathcal{M}^{t} = \langle \mathcal{V}_1, \preceq\rangle$ and $\mathcal{M}^{q} = \langle \mathcal{V}_2, \sqsubseteq\rangle$ be posets. Let also $f_1 : \mathcal{V}_1 \to \mathcal{V}_2$ and $f_2 : \mathcal{V}_2 \to \mathcal{V}_1$ be, respectively, a designation-preserving and an undesignation-preserving homomorphism. Therefore, the pair $\langle f_1, f_2\rangle$ is a Galois connection.*

Proof. It is necessary to show that (i) f_1 and f_2 are both monotone; and
(ii) for all $v_i \in \mathcal{V}_1$, $v_j \in \mathcal{V}_2$, $v_i \preceq f_1(f_2(v_i))$ and $v_j \sqsubseteq f_1(f_2(v_j))$.
In view of Fact 1, f_1 and f_2 are both monotone. For (ii), assume $v_j \preceq f_2(v_j)$. Given the monotonicity of the functions and the definition of designation-preserving homomorphism, the following holds: if $v_1 \preceq v_2$ then $f_1(v_1) \sqsubseteq f_1(v_2)$ for every $v_1, v_2 \in \mathcal{V}_1$. Hence by the transitivity of the orders and our assumption, we obtain the following $v_j \preceq f_1(f_2(v_j))$. The proof of $v_i \sqsubseteq f_1(f_2(v_i))$ follows by analogous reasoning. □

Propositions 1 and 2 show that whereas the uniform transformation of a many-valued t-logic to a q-logic preserves all t-validities, the transformation to a p-logic preserves all *t-falsities*. In the following, I explore Łukasiewicz's three-valued logic and its respective uniform transformation as a way of illustrating some general properties of uniform q-interpretations of Tarskian logics.

Example 1. *Łukasiewicz's 3-valued propositional logic $Ł_3 = \langle Fr, \models^{t}\rangle$ is defined by the matrix $\mathcal{M}_3 = \langle\{0, \frac{1}{2}, 1\}, \{1\}, \{f_\mu : \mu \in \{\neg, \wedge, \vee, \to\}\}\rangle$, where each truth-function f_μ is described by the corresponding table[7], in what follows:*

$f_\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

	$f_\neg$
0	1
$\frac{1}{2}$	$\frac{1}{2}$
1	0

$f_\to$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	1	1

$f_\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

Example 2. *The uniform q-interpretation of $Ł_3$ is defined by the q-matrix $\mathcal{M}_3^{q} = \langle\{0, \frac{1}{2}, 1\}, \{1\}, \{0\}, \{f_\mu : \mu \in \{\neg, \wedge, \vee, \to\}\}\rangle$, where each truth-function f_μ is described as in $\mathcal{M}_3$.*

Where $\mathcal{M}_2 = \langle\{0, 1\}, \{1\}\rangle$ is the classical two-valued matrix, any identity mapping works as a strong matrix homomorphism from $\mathcal{M}_2$ to $\mathcal{M}_3^{q}$. Hence, in accordance with Propositions 1 and 2, the logics $Ł_3^{q} = \langle Fr, \models^{q}\rangle$ and $Ł_3^{p} = \langle Fr, \models^{p}\rangle$ based on $\mathcal{M}_3^{q}$ preserve, respectively, all Ł$_3$-validities and Ł$_3$-falsities. The Table below compares Ł$_3$ to each respective entailment relation in terms of some important properties that display the behavior of the logical constants:

The transformation from Ł$_3^{\mathsf{t}}$ to Ł$_3^{\mathsf{q}}$ deeply affects the behavior of the logical constants (Table 1) . Whereas Ł$_3^{\mathsf{t}}$ is characterized by an explosive negation (line 1), an adjunctive conjunction (line 3), and a detachable conditional (line 4), all these features are lost when moving to its q-interpretation. The transformation to its p-formulation, however, is more gentle and keeps many of these features intact[8]. Moreover, while Ł$_3^{\mathsf{t}}$ is paracomplete (line 2) but not paraconsistent, Ł$_3^{\mathsf{q}}$ is both paracomplete and paraconsistent. In spite of only Ł$_3^{\mathsf{q}}$ be paraconsistent, all logics validate the paradox of the conditional (line 7). In what follows, we investigate the sufficient conditions for the transformation of Tarskian logic in its q-interpretation to output a paraconsistent or a paracomplete logic.

Table 1. Uniform logics.

Properties	$\models^{\mathsf{t}}$	$\models^{\mathsf{q}}$	$\models^{\mathsf{p}}$
1. $\varphi, \neg\varphi \models \psi$	✓	×	✓
2. $\psi \models \varphi \vee \neg\varphi$	×	×	✓
3. $\varphi, \psi \models \varphi \wedge \psi$	✓	×	✓
4. $\varphi \to \psi, \varphi \models \psi$	✓	×	✓
5. $\varphi \to \psi, \psi \to \gamma \models \varphi \to \gamma$	✓	×	✓
6. $\varphi \to \psi \models \neg\psi \to \neg\varphi$	✓	×	✓
7. $\models \neg\varphi \to (\varphi \to \psi)$	✓	✓	✓

3.2. Inferential Paraconsistency

In [4], the transformation from Ł$_3^{\mathsf{t}}$ to Ł$_3^{\mathsf{q}}$ is motivated as an alternative way of producing a paraconsistent interpretation of Ł$_3$[9]. The process of transforming a many-valued t-logic in its paraconsistent q-interpretation is called *inferential paraconsistency*. However, the author does not explore under what conditions a many-valued t-logic becomes paraconsistent after its transformation to a q-logic. For this, I introduce the following characterization of the notion of inferential paraconsistency:

Definition 4. *A logic will be called* ***¬-explosive*** *iff for every formula* $\varphi, \psi \in Fr$, $\varphi, \neg\varphi \models \psi$ *holds.*

Definition 5. *Given a logic* $\mathcal{L} = \langle Fr, \models_{\mathcal{M}} \rangle$, *its associated logical matrix* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \{f_\mu : \mu \in \{\neg\}\} \rangle$ *and semantics* Val, *we shall say that* $\mathcal{L}$ *is* ***inferentially paraconsistent*** *in case there is a set* $\mathcal{I} \subsetneq \mathcal{V}$ *of truth-values and a valuation* $v \in Val$ *such that:* $v(\varphi) \in \mathcal{I}$ *iff* $v(\neg\varphi) \in \mathcal{I}$.

In the context of bi-dimensional matrices, one may fix different choices for the set $\mathcal{I}$. For each choice, different classes of paraconsistent logics arise:

Choice	Class of Logics
$\mathcal{I} = \mathcal{D}^+$	Paraconsistent t-logics
$\mathcal{I} = \mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-)$	Paraconsistent q-logics
$\mathcal{I} = (\mathcal{D}^+ \cap \mathcal{D}^-)$	Paraconsistent p-logics
$\mathcal{I} = \mathcal{D}^-$	Paraconsistent f-logics

Definition 6. *Given a logic* $\mathcal{L} = \langle Fr, \models_{\mathcal{M}} \rangle$, *its associated logical matrix* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \{f_\mu : \mu \in \mu \in \{\neg\}\} \rangle$ *and semantics* Val, *we shall say that* $\mathcal{L}$ *is* ¬-redundant *in case there are values* $a \in \mathcal{D}$ *and* $b \in \mathcal{V} - \mathcal{D}$ *such that* $v(\varphi) = a$ *implies* $v(\neg\varphi) = b$ *for some valuation* $v \in Val$.

For the following we shall assume that the logics considered are not $\neg$-*redundant*, i.e., that for every truth-value $a \in \mathcal{D}$, $v(\varphi) = a$ implies $v(\neg\varphi) \in \mathcal{V} - \mathcal{D}$. It is easy to see that every non $\neg$-redundant t-logic is not paraconsistent. Paraconsistent t-logics depend on the existence of truth-value gluts, i.e. values such that $v(\varphi) \in \mathcal{D}$ and $v(\neg\varphi) \in \mathcal{D}$. We shall also assume that the logics in consideration are finitely many-valued with a linearly ordered set of truth-values and endowed with connectives $\neg$, $\wedge$ and $\vee$ defined as in Example 1. The following lemma guarantees that a non $\neg$-redundant t-logic remains non $\neg$-redundant after its uniform transformation.

Lemma 1. *Where $\mathcal{L}$ is a non $\neg$-redundant n-valued t-logic, its uniform* q*-interpretation $\mathcal{L}^{\mathsf{q}}$ is also not $\neg$-redundant.*

Proof. Let $\mathcal{L}$ be a many-valued non $\neg$-redundant t-logic and assume for a contradiction that $\mathcal{L}^{\mathsf{q}}$ is $\neg$-redundant. Then there is at least one value $a \in \mathcal{D}$ such that (i) $v(\varphi) = a$ and (ii) $v(\neg\varphi) \in \mathcal{D}$. Now, given that $\mathcal{L}^{\mathsf{q}}$ is a uniform q-logic, we know there is a strong matrix homomophism $f : \mathcal{M} \longrightarrow \mathcal{M}^{\mathsf{q}}$. From our assumption, we also know that $v(\varphi) \in \mathcal{D}$ implies $v(\neg\varphi) \in \mathcal{V} - \mathcal{D}$, for every valuation $v \in Val$. Therefore, it also follows that $f(v(\varphi)) \in \mathcal{D}$ implies $f(v(\neg\varphi)) \in \mathcal{V} - \mathcal{D}$, for every $v \in Val^{\mathsf{q}}$, what contradicts (i) and (ii). □

Proposition 3. *Where $\mathcal{L}$ is a $\neg$-explosive and* non $\neg$-redundant *n-valued t-logic, its uniform* q*-interpretation $\mathcal{L}^{\mathsf{q}}$ is inferential paraconsistent iff $\mathcal{L}^{\mathsf{q}}$ is not $\neg$-explosive.*

Proof. Let $\mathcal{L}$ be a $\neg$-explosive many-valued logic and $\mathcal{L}^{\mathsf{q}}$ its associated uniform q-interpretation.

By contraposition, assume $\mathcal{L}^q$ is not $\neg$-explosive. Therefore there are formulas $\varphi, \psi \in Fr$ such that $\varphi, \neg\varphi \not\models^q \psi$. Hence there is a valuation v such that $v(\{\varphi, \neg\varphi\}) \cap \mathcal{D}^- = \varnothing$ and $v(\psi) \notin \mathcal{D}^+$. By Lemma 1, we know $\mathcal{L}^q$ is not $\neg$-redundant, from what follows that either (i) $v(\varphi) \in \mathcal{D}^+$ and $v(\neg\varphi) \in \mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-)$ or (ii) $v(\neg\varphi) \in \mathcal{D}^+$ and $v(\varphi) \in \mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-)$. For both cases, there is at least one value $x \in \mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-)$ such that $v(\varphi) = x$ but $v(\neg\varphi) \notin \mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-)$. Hence $\mathcal{L}^q$ is not inferential paraconsistent.

For the converse, by contraposition again, assume $\mathcal{L}^{\mathsf{q}}$ is not inferential paraconsistent. Then there is a valuation v and a truth-value $x \in \mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-)$ such that $v(\varphi) = x$ but $v(\neg\varphi) \notin \mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-)$ or $v(\neg\varphi) = x$ but $v(\varphi) \notin \mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-)$. From that we obtain one of the following options:

(i) $v(\varphi) \in \mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-)$ and $v(\neg\varphi) \in \mathcal{D}^+$;
(ii) $v(\varphi) \in \mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-)$ and $v(\neg\varphi) \in \mathcal{D}^-$;
(iii) $v(\neg\varphi) \in \mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-)$ and $v(\varphi) \in \mathcal{D}^-$;
(iv) $v(\neg\varphi) \in \mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-)$ and $v(\varphi) \in \mathcal{D}^+$.

By definition of q-entailment, for cases (ii) and (iii) $\mathcal{L}^{\mathsf{q}}$ is not $\neg$-explosive. For cases (i) and (iv) it suffices to set $v(\psi) \notin \mathcal{D}^+$. □

Proposition 3 shows that uniform q-interpretations preserve the non-explosiveness of the Tarskian logics as long as both the Tarskian logic and its q-interpretation are endowed with a non-redundant negation operator.

Example 3. *Examples of many-valued logics that are inferential paraconsistent after their uniform* q*-transformation are Kleene's* K3*, Bochvar's system* B3 *and all finitely-valued Łukasiewicz's logics. (See [12]).*[10]

Example 4. *An example of a logic not inferential paraconsistent after its uniform* q*-transformation is Gödel's logic $G_3 = \langle \{0, \frac{1}{2}, 1\}, \{1\}, \{f_\mu : \mu \in \{\neg, \wedge, \vee, \rightarrow\}\} \rangle$, where negation is defined by the following truth-table:*

	$f_\neg$
0	1
$\frac{1}{2}$	0
1	0

It is easy to see that the uniform transformation to G^q_3 will not alter the explosive character of negation. The bidimensional q*-matrix is not inferential paraconsistent. In fact, all uniform* q*-interpretations of Gödel's logics G_k are not $\neg$-explosive for $\{k \in \mathbb{N} \mid k \geq 2\}$*[11].

Similar results hold for the paracomplete character of negation in q-logics obtained from a Tarskian a base logic.

Definition 7. *A logic will be called **$\neg$-complete** iff for every $\varphi, \psi \in Fr$ for which $\psi \models \varphi \vee \neg\varphi$ holds.*

Definition 8. *Given a logic $\mathcal{L} = \langle Fr, \models_\mathcal{M} \rangle$, its associated logical matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \{f_\mu : \mu \in \{\neg\}\} \rangle$ and semantics Val, we shall say that $\mathcal{L}$ is **inferentially paracomplete** in case there is a set $\mathcal{I} \subsetneq \mathcal{V}$ of truth-values and a valuation v such that: $v(\varphi) \notin \mathcal{I}$ iff $v(\neg\varphi) \notin \mathcal{I}$.*

Once again, in the context of bi-dimensional matrices, one may fix different choices for the set $\mathcal{I}$. This gives rise to distinct classes of paracomplete logics:

Choice	Class of Logics
$\mathcal{I} = \mathcal{D}^+$	Paracomplete t-logics
$\mathcal{I} = \mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-)$	Paracomplete q-logics
$\mathcal{I} = (\mathcal{D}^+ \cap \mathcal{D}^-)$	Paracomplete p-logics
$\mathcal{I} = \mathcal{D}^-$	Paracomplete f-logics

Proposition 4. *Where $\mathcal{L}$ is a $\neg$-complete many-valued* t*-logic, its uniform* q*-interpretation $\mathcal{L}^q$ is inferential paracomplete iff $\mathcal{L}^q$ is not $\neg$-complete.*

Proof. (From l.h.s. to r.h.s.) Assume $\mathcal{L}^q$ is inferential paracomplete. Therefore there is a formula ϕ and a valuation v such that $v(\varphi) \notin \mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-)$ iff $v(\neg\varphi) \notin \mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-)$. Moreover, since v is a total function, we obtain the following options: (i) $\{v(\varphi), v(\neg\varphi)\} \subseteq \mathcal{D}^+$, (ii) $\{v(\varphi), v(\neg\varphi)\} \subseteq \mathcal{D}^-$, and (iii) $v(\varphi) \in \mathcal{D}^+$ and $v(\neg\varphi) \in \mathcal{D}^-$ (or vice-versa) Now, given that $\mathcal{L}^q$ is not $\neg$-redundant, cases (i) and (ii) are excluded. For case (iii), the fact that $f_\vee = max(v(\varphi), v(\neg\varphi))$ together with our definition of uniform logics, we may conclude that $\mathcal{L}^q$ is not $\neg$-complete.
The other direction follows by analogous reasoning. □

Example 5. *Examples of many-valued logics that are inferential paracomplete after their uniform* q*-transformation are also Kleene's $K3$ and all finitely-valued Łukasiewicz's logics.*

3.3. Non-Uniform Interpretations

Other types of interpretations of many-valued logics may be constructed if one is willing to abandon the motivations for uniform interpretations. Non-uniform interpretations of many-valued Tarskian logics were also explored in the logic literature, e.g., the q-interpretation of the nonmonotonic logic LP_m is proposed in [13] due to interest on its connexive principles.

Another non-trivial way of manufacturing paraconsistent q-logics based on many-valued t-logics may be achieved by moving all intermediate values to the designated set. It is easy to see that under this setting p- and q-consequence coincide. In fact, the logic Ł$_3$ with matrix $\mathcal{M}_3 = \langle \{0, \frac{1}{2}, 1\}, \{1, \frac{1}{2}\}, \{0\} \rangle$ was explored by da Costa and Alves ([5]) and shown to preserve the positive fragment of classical logic under specific linguistic extensions. In [1] the author suggests that two modes of paraconsistency may be considered for q-interpretations of Tarskian logics. According to Malinowski, a logic shall be called *paraconsistent* in case $\varphi, \neg\varphi \models (\varphi \equiv \neg\varphi)$ and $\varphi, \neg\varphi \not\models \neg(\varphi \equiv \neg\varphi)$ holds, and *parainconsistent* in case $\varphi, \neg\varphi \not\models (\varphi \equiv \neg\varphi)$ and $\varphi, \neg\varphi \models \neg(\varphi \equiv \neg\varphi)$ holds. The truth-function $f_{\equiv}$ for Ł$_3$ is defined by the following truth-table:

$f_{\equiv}$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
0	0	$\frac{1}{2}$	1

To explore non-uniform interpretations, one may also consider the following variants of Ł$_3$:

the logic Ł$^1 = \langle Fr, \models^{\mathsf{t}} \rangle$ with the matrix $\mathcal{M}_3 = \langle \{0, \frac{1}{2}, 1\}, \{1, \frac{1}{2}\} \rangle$,
the logic Ł$^2 = \langle Fr, \models^{\mathsf{q}} \rangle$ with the matrix $\mathcal{M}_3 = \langle \{0, \frac{1}{2}, 1\}, \{\frac{1}{2}\} \rangle$, and
the logic Ł$^4 = \langle Fr, \models^{\mathsf{q}} \rangle$ with the matrix $\mathcal{M}_3^{\mathsf{q}} = \langle \{0, \frac{1}{2}, 1\}, \{\frac{1}{2}\}, \{0\} \rangle$.

Based on each respective matrix and entailment relation, one obtains the following:

The logics Ł$_3^{\mathsf{q}}$ and Ł^1 (Table 2) are both paraconsistent in the sense of Malinowski. By further abandoning the idea of having the designated value as the maximal element of the set of truth-values, it is possible to define two parainconsistent logics, Ł^2 and Ł^4, by setting $\frac{1}{2}$ as a designated value. The examples above show that there are paraconsistent and parainconsistent logics available not only for q-interpretations of Ł$_3$, but also for different Tarskian interpretations. It is easy to see that $\models_{Ł_3^{\mathsf{q}}} = \models_{Ł^1}$. Furthermore, all logics Ł^1, Ł^2 and Ł^4 enjoy a non-explosive negation. In the next section, we explore other relations between the logics presented so far and their respective duals.

Table 2. Paraconsistent logics.

Properties	**Ł$_3^{\mathsf{t}}$**	**Ł$_3^{\mathsf{q}}$**	**Ł$_3^{\mathsf{p}}$**	**Ł^1**	**Ł^2**	**Ł^4**
Paraconsistent	×	✓	×	✓	×	×
Parainconsistent	×	×	×	×	✓	✓

4. Routes for Dualization

In this paper, the question of how to transform the semantical structure of q- and p-logics shall be explored also in connection to their associated duals. For this we shall fix two specific notions of duality for many-valued structures introduced by Malinowski in ([1]) and Blasio et al ([7]). Each of these authors proposed distinct requirements for duality relative to entailment relations. In the context of Tarskian logics, where entailment is defined as truth-preservation from the premises to the conclusion, the natural dual notion of entailment is that of falsity-preservation from conclusion to the premises. However, this complementarity between truth (designatedness) and non-truth (non-designatedness) is achieved only in the context of logical matrices. When considering bidimensional matrices such as q-matrices, truth and non-truth are not complementary concepts, so that an important problem is to know what should be taken as the adequate dual notion of q- and p-entailment.

Two different strategies for dualization of entailment relations are discussed here. The first follows Malinowski's ([1]) proposal according to which the dual of a consequence relation requires the dualization of the very semantic structure upon which the entailment relation is defined. The latter, proposed by Blasio et al ([7]), is also based on dualizing the semantic structure, but follows a different strategy than the one proposed by Malinowski. We introduce both duality styles in the following.

4.1. Malinowski's Duality

Differently from other approaches that consider duality of entailment through the balance between forward and backward preservation (such as [14]), duality in the sense of [1] is not based on the balance between the designated and the antidesignated sets of truth-values. *Malinowski's dualization* style may be described as follows:

Definition 9. *Given a bidimensional matrix* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}^+ = \{a_1, \ldots, a_m\}, \mathcal{D}^- = \{b_1, \ldots, b_n\}, \{f_\mu : \mu \in \mathsf{Con}\}\rangle$*, where m and n are finite, and its associated entailment relation* $\models^+$*, the dual entailment relation* $\models^-$ *is obtained by producing the dual logical matrix* $\mathcal{M}^d = \langle \mathcal{V}, \mathcal{D}^+ = \{b_1, \ldots, b_n\}, \mathcal{D}^- = \{a_1, \ldots, a_m\}, \{f_\mu : \mu \in \mathsf{Con}\}\rangle$ *through mutual exchange of the sets of truth-values.*

Malinowski's dualization procedure may be described as follows:

1. *Given a bidimensional matrix* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_\mu : \mu \in \mathsf{Con}\}\rangle$ *and a forwards-preserving entailment relation* $\models^+$*,*
2. *Produce the dual matrix* $\mathcal{M}^d$ *by the mutual exchange of the sets of truth-values,*
3. *And define a dual entailment relation* $\models^-$ *based on* $\mathcal{M}^d$ *in the same way as* $\models^+$*.*
4. *The entailment relation* $\models^-$ *is called the dual of* $\models^+$*.*

By following Malinowski's procedure, one may note that in the context of the 2-valued CL matrix, where $\models^+$ is t-entailment and $\models^-$ is f-entailment[12], $\models^+$ and $\models^-$ do not coincide. Whereas $\models^+$ is defined as $\Gamma \models^+ \varphi$ iff $v(\Gamma) \subseteq \{1\}$ implies $v(\varphi) = 1$, $\models^-$ is defined as $\Gamma \models^- \varphi$ iff $v(\Gamma) \subseteq \{0\}$ implies $v(\alpha) = 0$. It is easy to see that $\alpha \vee \neg\alpha$ is a validity in view of $\models^+$, but an antilogy in view of $\models^-$. Therefore, f-entailment (see Section 2) is not the dual of t-entailment in Malinowski's dualization procedure[13].

The following shows that Malinowski's procedure outputs a non-equivalent dual entailment relation in the case of q- and p-entailment.

Proposition 5. *Let* $\mathcal{M} = \langle \{a, b, c\}, \{a\}, \{b\}\rangle$ *be a* q*-matrix, then* $\models^{\mathsf{q}} \neq \models^{\mathsf{q}}_d$ *and* $\models^{\mathsf{p}} \neq \models^{\mathsf{p}}_d$*.*

Proof. Note the dual matrix $\mathcal{M}^d = \langle \{a, b, c\}, \{b\}, \{a\}\rangle$. Now let φ be a validity in view of $\models^{\mathsf{q}}$. By definition of q-entailment, we know that $v(\varphi) = a$, for every $v \in Val$. Therefore, $\not\models^{\mathsf{q}}_d \varphi$. The case for p-entailment follows by analogous reasoning. □

4.2. Blasio's Duality

The second notion of dualization that we shall explore was proposed in [7]. For these authors, the duals of q- and p-entailment should be explored neither in terms of the balance between forwards and backwards preservation, nor in terms of the mutual exchange of the truth-values. Rather, the dual between q- and p- matrices is defined in terms of the balance between inferential gaps and gluts. If q-matrices are characterized by allowing inferential gaps, then their respective duals, p-matrices, ought to be characterized by allowing inferential gluts.

Definition 10. *Given a* q*-matrix* $\mathcal{M}^{\mathsf{q}} = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_\mu : \mu \in \mathsf{Con}\}\rangle$*, where* $\mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-) \neq \varnothing$*, its respective dual is the* p*-matrix* $\mathcal{M}^{\mathsf{p}} = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_\mu : \mu \in \mathsf{Con}\}\rangle$*, where* $\mathcal{D}^+ \cap \mathcal{D}^- \neq \varnothing$*.*

As a result of the above characterization, duality is understood in terms of the relation between the matrices and their entailment relations. The resulting dualization procedure is then described as:

1. *Given a* q*-matrix* $\mathcal{M}$ *and its associated entailment relations* $\models^{\mathsf{q}}$ *and* $\models^{\mathsf{p}}$*,*
2. *Define the dual matrix* $\mathcal{M}^d$ *as the* p*-matrix based on* $\mathcal{M}$*.*
3. *The* q*-entailment relation based on* $\mathcal{M}^d$ *is called the dual* p*-entailment.*
4. *The* p*-entailment relation based on* $\mathcal{M}^d$ *is called the dual* q*-entailment.*

Blasio's style of duality aims to preserve the duality between p- and q-entailment. Therefore, the dualization of the matrix is meant to preserve the balance between the two forms of entailment. If p- and q-entailment are forms of entailment based on matrices that allow inferential gaps, then their respective duals are based on dual matrices that allow inferential gluts. In the following, we explore the relationship between these dual-logics and the ones we introduced before.

5. Uniform Dual Logics and Connexivity

In [13], the authors explored q-interpretations of Tarskian logics due to their ability to alter the behavior of implication and output properties of a connexive implication. The strategy followed by them is based on constructing a q-interpretation over the nonmonotonic logic LP_m. In the following, we show that the logics constructed via Malinowski's or Blasio's style of duality also output logics with a connexive implication when applied to uniform logics endowed with a Łukasiewicz negation and implication.

5.1. Malinowski's Dual Logics

By following Malinowski's dualization procedure described in Section 4.1 to the logic Ł_3^{q}, one obtains the following dual logics:

Definition 11. *The dual logics* Ł_d^{q} *and* Ł_d^{p} *are defined by the dual* q*-matrix* $\mathcal{M}_d^{q} = \langle\{0, \frac{1}{2}, 1\}, \{0\}, \{1\}, \{f_\mu : \mu \in \{\neg, \wedge, \vee, \rightarrow\}\rangle$*, where each truth-function* f_μ *is described as in* $\mathcal{M}_3$*.*

Proposition 6. *There is a strong matrix homomorphism* f *from* $\mathcal{M}_3^{q}$ *to* $\mathcal{M}_d^{q}$*.*

Proof. Let $f(0) = 1$ and $f(1) = 0$. □

Proposition 7. *Given a uniform n-valued* q*-matrix* $\mathcal{M}^{\mathsf{q}} = \langle\mathcal{V}, \mathcal{D}^+, \mathcal{D}^-, \{f_\mu : \mu \in Con\}\rangle$ *(with* $|\mathcal{D}^+| = |\mathcal{D}^-|$*) and its Malinowski dual* $\mathcal{M}^{\mathsf{d}} = \langle\mathcal{V}^d, \mathcal{D}_d^+, \mathcal{D}_d^-, \{f_\mu : \mu \in Con\}\rangle$*, there is a strong matrix homomorphism from* $\mathcal{M}^{\mathsf{q}}$ *to* $\mathcal{M}^{\mathsf{d}}$*.*

Proof. Let $\mathcal{M}^{\mathsf{q}} = \langle\mathcal{V}, \mathcal{D}^+ = \{a_1, \ldots, a_k\}, \mathcal{D}^- = \{b_1, \ldots, b_j\}, \{f_\mu : \mu \in \mathsf{Con}\}\}\rangle$ be a n-valued matrix, where $k = j$ and $j \leq n$. Let also $\mathcal{M}^{\mathsf{d}} = \langle\mathcal{V}^d, \mathcal{D}_d^+ = \{b_1, \ldots, b_j\}, \mathcal{D}^- = \{a_1, \ldots, a_k\}, \{f_\mu : \mu \in \mathsf{Con}\}\rangle$ be its Malinowski-dual.
Let $f_1 : \mathcal{V} \rightarrow \mathcal{V}^d$ be a designation-preserving homomorphism defined as follows: if $a_l \in \mathcal{D}^+$, then $f(a_l) = b_l$, where $1 \leq l \leq k$. Let also $f_2 : \mathcal{V} \rightarrow \mathcal{V}^d$ be an undesignation-preserving homomorphism defined as follows: if $c_l \in \mathcal{V} - \mathcal{D}^+$, then $f(c_l) = c_l$, where $1 \leq l \leq j$.
Given that $k = j$, it follows that f_1 and f_2 are bijective. Finally, by Definition 2, it follows that $f_1 \circ f_2$ is a strong matrix homomorphism. □

Corollary 3. *The pair* $\langle f_1, f_2\rangle$ *is a Galois connection between* $\mathcal{M}^{\mathsf{q}}$ *and* $\mathcal{M}^{\mathsf{d}}$*.*

The dual entailment relations $\models_d^{\mathsf{q}}$ and $\models_d^{\mathsf{p}}$ are based on the duality between designatedness and antidesignatedness. As required by Definition 1, the entailment relation $\models_d^{\mathsf{q}}$ is still understood as a reasoning from non-rejected premises to accepted conclusion. However, due to the exchange of the sets of truth-values in the matrix $\models_d^{\mathsf{q}}$, this amounts to the following:

$$\Gamma \models_d^{\mathsf{q}} \varphi \text{ iff } v(\Gamma) \subseteq \{0, \frac{1}{2}\} \text{ implies } v(\varphi) = 0, \text{ for every } v \in Val. \tag{5}$$

Analogously, the entailment relation $\models_d^{\mathsf{p}}$ may be defined as a form of reasoning from accepted premises to non-rejected conclusions, i.e.,

$$\Gamma \models_d^{\mathsf{p}} \varphi \text{ iff } v(\Gamma) \subseteq \{0\} \text{ implies } v(\varphi) \neq 1, \text{ for every } v \in Val. \tag{6}$$

The effects of revising $Ł_3^q$ to its respective duals are exhibited in the table below:

As exhibited in Table 3 both logics agree with respect to implication in lines 4 and 6. However, whereas the logic $Ł_d^q$ is both paraconsistent and paracomplete, the logic $Ł_d^p$ is not paraconsistent. It is also possible to show that the logic $Ł_d^p$ has a connexive implication.

Table 3. Malinowski's dual logics.

Properties	$Ł_d^p$	$Ł_d^q$
1. $\varphi, \neg\varphi \models \psi$	✓	×
2. $\psi \models \varphi \vee \neg\varphi$	×	×
3. $\varphi, \psi \models \varphi \wedge \psi$	✓	×
4. $\varphi \rightarrow \psi, \varphi \models \psi$	✓	✓
5. $\varphi \rightarrow \psi, \psi \rightarrow \gamma \models \varphi \rightarrow \gamma$	✓	✓
6. $\varphi \rightarrow \psi \models \neg\psi \rightarrow \neg\varphi$	×	×
7. $\models \neg\varphi \rightarrow (\varphi \rightarrow \psi)$	×	×

Definition 12 ([15]). *A unary connective shall be called a **proper negation** iff there are formulas $\varphi, \psi \in Fr$ such that $\varphi \not\models \neg\varphi$ and $\neg\psi \not\models \psi$*[14].

Definition 13 ([15]). *A binary connective $\rightarrow$ is called a **proper implication** iff for every $\varphi, \psi \in Fr$ and $\Delta \subseteq Fr$ the following holds: $\Delta, \varphi \models \psi$ iff $\Delta \models \varphi \rightarrow \psi$.*

Definition 14 ([15]). *A connective $\rightarrow$, in a language with a proper negation, is called a **connexive implication** iff the folowing holds for every $\varphi, \psi \in Fr$: $\neg(\varphi \rightarrow \psi) \models (\varphi \rightarrow \neg\psi)$ and $(\varphi \rightarrow \neg\psi) \models \neg(\varphi \rightarrow \psi)$.*

Remark 1. *The logic $Ł_d^p$ has a connexive implication.*

Proof. To see that the logic has a proper negation, let a valuation v such that $v(\varphi) = 0$. Therefore $v(\neg\varphi) = 1$ and $\varphi \not\models_d^p \neg\varphi$. The case for $\neg\psi \not\models \psi$ follows by analogous reasoning. As for the connexive implication, assume for a contradiction there is a valuation v such that (i) $v(\neg(\varphi \rightarrow \psi)) = 0$ and (ii) $v(\varphi \rightarrow \neg\psi) = 1$. From (i), $f_\neg$ and $f_\rightarrow$, we may obtain $v(\varphi) = v(\psi) = 1$. From (ii) and $f_\rightarrow$, we may have $v(\varphi) = v(\neg\psi) = 1$, what is a contradiction. The case for $(\varphi \rightarrow \neg\psi) \models \neg(\varphi \rightarrow \psi)$ follows by analogous reasoning. □

The reader may note that the logic $Ł_d^p$, however, *does not satisfy either* Aristotle's theses: $\models \neg(\varphi \rightarrow \neg\varphi)$ and $\models \neg(\neg\varphi \rightarrow \varphi)$. This failure of Aristotle's theses is caused by the failure of $\models \varphi \rightarrow \varphi$. On the other hand, the logic $Ł_d^p$ enjoys a proper implication in accordance with Definition 13 and satisfy Boethius's theses in the form $\varphi \rightarrow \neg\psi \models \neg(\varphi \rightarrow \psi)$ and $(\varphi \rightarrow \psi) \models \neg(\varphi \rightarrow \neg\psi)$. All this seems to suggest that connexivity occurs at the level of the consequence relation rather than a connexive interpretation of negation and implication. In the next section, we explore the consequences of following Blasio's dualization style.

5.2. Blasio's Dual Logics

By following Blasio's dualization style to the logic $Ł_3^q$, one obtains the following dual logics:

Definition 15. *The dual logics $Ł_d^q$ and $Ł_d^p$ are defined by a bidimensional matrix with inferential gluts, i.e., the p-matrix $\mathcal{M}^p = \langle\{0, \frac{1}{2}, 1\}, \{1\frac{1}{2}\}, \{\frac{1}{2}, 0\}, \{f_\mu : \mu \in \{\neg, \wedge, \vee, \rightarrow\}\rangle$, where each truth-function f_μ is described as in $\mathcal{M}_3$.*

As required by Definition 10, the dual of p-entailment amounts to a collection of q-statements defined over the matrix $\mathcal{M}^{\mathsf{p}}$. As a result, the entailment relation $\models^{\mathsf{p}}_{d}$ may be defined as follows:

$$\Gamma \models^{\mathsf{p}}_{d} \varphi \text{ iff } v(\Gamma) \cap \mathcal{D}^{-} = \varnothing \text{ implies } v(\varphi) \in \mathcal{D}^{+}, \text{ for every } v \in Val. \quad (7)$$

Analogously, the dual of q-entailment is defined by a collection of p-statements over the matrix $\mathcal{M}^{\mathsf{p}}$.

$$\Gamma \models^{\mathsf{q}}_{d} \varphi \text{ iff } v(\Gamma) \subseteq \mathcal{D}^{+} \text{ implies } v(\varphi) \notin \mathcal{D}^{-}, \text{ for every } v \in Val. \quad (8)$$

As exhibited in Table 4 below, the negation in the logic $Ł^{\mathsf{p}}_{d}$ is a proper negation that is neither explosive nor complete. Although both logics $Ł^{\mathsf{p}}_{d}$ and $Ł^{\mathsf{q}}_{d}$ seem diametrically different, they still agree in line 8. Moreover, one can show that the logic $Ł^{\mathsf{p}}_{d}$ enjoys a connexive implication, but invalidates both Aristotle's theses. In the following, we demonstrate the relation between the dual logics constructed in this section and their respective notions of entailment.

Table 4. Blasio's dual logics.

Properties	$Ł^{\mathsf{p}}_{d}$	$Ł^{\mathsf{q}}_{d}$
1. $\varphi, \neg\varphi \models \psi$	✓	×
2. $\psi \models \varphi \vee \neg\varphi$	✓	×
3. $\varphi, \psi \models \varphi \wedge \psi$	✓	×
4. $\varphi \to \psi, \varphi \models \psi$	✓	×
5. $\varphi \to \psi, \psi \to \gamma \models \varphi \to \gamma$	✓	×
6. $\varphi \to \psi \models \neg\psi \to \neg\varphi$	✓	×
7. $\models \neg\varphi \to (\varphi \to \psi)$	✓	✓

6. Entailment Relations and Their Duals

In what follows, given a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D} \rangle$, let $\mathcal{M}^{q} = \langle \mathcal{V}, \mathcal{D}^{+}, \mathcal{D}^{-} \rangle$ be the canonical bidimensional based on $\mathcal{M}$. Let also $\mathcal{M}^{1} = \langle \mathcal{V}, \mathcal{D}^{-}, \mathcal{D}^{+} \rangle$ be the **Malinowski-dual** built over $\mathcal{M}^{q}$, and $\mathcal{M}^{2} = \langle \mathcal{V}, \mathcal{V} - \mathcal{D}^{+}, \mathcal{V} - \mathcal{D}^{-} \rangle$ be the **BMW-dual** built over $\mathcal{M}^{q}$. For the following result, we shall write $\models^{\mathcal{M}^{x}}_{\mathsf{q}}$ and $\models^{\mathcal{M}^{x}}_{\mathsf{p}}$ to denote the respective q- and p-entailment based on the generalized matrix $\mathcal{M}^{x}$. By combining distinct entailment relations with distinct semantic structures, one can prove the following result.

Proposition 8. *The following equivalences hold:*

(i) $\Gamma \models^{\mathcal{M}^{q}}_{q} \psi$ *iff* $\Gamma \models^{\mathcal{M}^{2}}_{p} \psi$
(ii) $\Gamma \models^{\mathcal{M}^{2}}_{q} \psi$ *iff* $\Gamma \models^{\mathcal{M}^{q}}_{p} \psi$
(iii) $\Gamma \models^{\mathcal{M}^{q}}_{t} \psi$ *iff* $\Gamma \models^{\mathcal{M}^{2}}_{f} \psi$
(iv) $\Gamma \models^{\mathcal{M}^{q}}_{t} \psi$ *iff* $\Gamma \models^{\mathcal{M}^{1}}_{f} \psi$

Proof. (i) Assume $\Gamma \models^{\mathcal{M}^{q}}_{\mathsf{q}} \psi$. Therefore, $v(\Gamma) \cap \mathcal{D}^{-} = \varnothing$ implies $v(\psi) \in \mathcal{D}^{+}$. We have to show that $v(\Gamma) \subseteq \mathcal{D}^{+}$ implies $v(\psi) \notin \mathcal{D}^{-}$. Note that $\mathcal{V} - \mathcal{D}^{-}$ in $\mathcal{M}^{q}$ is the same as $\mathcal{D}^{+}$ in $\mathcal{M}^{2}$ and $\mathcal{D}^{+}$ in $\mathcal{M}^{q}$ is the same as $\mathcal{V} - \mathcal{D}^{-}$ in $\mathcal{M}^{2}$. Therefore, our desired conclusion $\Gamma \models^{\mathcal{M}^{2}}_{\mathsf{p}} \psi$ follows.

The other direction and the other equivalences follow in an analogous manner. For case (iii), let $\Gamma \models^{\mathcal{M}^{2}}_{\mathsf{f}} \psi$ iff $v(\varphi) \in (\mathcal{V} - \mathcal{D}^{-})$ implies $v(\Gamma) \cap (\mathcal{V} - \mathcal{D}^{-}) \neq \varnothing$. Case (iv) follows by similar reasoning. □

Recovery Operators

The use of unary operators for recovering inferences lost by the move to a weaker logic is found in authors such as Bochvar [12] and da Costa [17][15]. In [5], the authors employ linguistic extensions of the logic Ł$_3$ as a way of recovering classical inferences. In the following, we show that the use of recovery operators can be employed as a way of recovering Tarskian inferences that were lost after moving to a q-interpretation.

Definition 16 (Conservative extension)**.** Given two logics $\mathcal{L}_1 = \langle Fr_1, \models_1 \rangle$ and $\mathcal{L}_2 = \langle Fr_2, \models_2 \rangle$, where L_1 and L_2 are their respective languages such that $L_1 \subseteq L_2$, we shall call $\mathcal{L}_2$ a **conservative extension** of $\mathcal{L}_1$ in case the following holds for every $\Gamma \cup \{\varphi\} \subseteq L_1$: $\Gamma \models_1 \varphi$ iff $\Gamma \models_2 \varphi$.

For the following, let $f_\otimes : \mathcal{V} \to \mathcal{V}$ be a truth-function that define the unary propositional operator '$\otimes$' and respects the following truth-conditions:

$v(\varphi)$	$f_\otimes(v(\varphi))$
$\mathcal{D}^+$	$\mathcal{D}^+$
$\mathcal{V} - \mathcal{D}^+$	$\mathcal{D}^-$

Proposition 9. Where $\mathcal{L}_1 = \langle Fr_1, \models_1 \rangle$ and $\mathcal{L}_2 = \langle Fr_2, \models_2 \rangle$ are q-logics based on the matrix $\mathcal{M}^q = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^- \rangle$ and languages $L_1 = \{\neg, \wedge, \rightarrow\}$ and $L_2 = L_1 \cup \{\otimes\}$. The logic $\mathcal{L}_2$ is a conservative extension of $\mathcal{L}_1$.

Proof. From l.h.s. to r.h.s. Recall that monotonicity holds for q-logics. Therefore, it suffices to show that $\models_1 \varphi$ iff $\models_2 \varphi$. The definition of $f_\otimes$ and of q-entailment guarantee that valid formulas are kept valid according to $\mathcal{L}_1$.

From r.h.s. to l.h.s. By contraposition, assume that $\Gamma \not\models^1 \varphi$. Hence there is a valuation $v \in Val$ such that $v(\Gamma) \cap \mathcal{D}^- = \varnothing$ and $\varphi \notin \mathcal{D}^+$. Moreover, given that $L_1 \subseteq L_2$ and both logics are based on $\mathcal{M}^q = \langle \mathcal{V}, \mathcal{D}^+, \mathcal{D}^- \rangle$, it follows that $\Gamma \not\models^2 \varphi$. □

In the context of multiple-valued systems, recovery operators have been used as primitive operators in logics such as Bochvar's B3 or Blaus's three-valued system[16]. A kind of recovery operator that respects the truth-table for $\otimes$ is Bochvar's assertion operator J_1 employed in the three-valued system B3. The truth-conditions for f_{J_1} are given by:

$$f_{J_1}(v(\varphi)) = \begin{cases} 1 & \text{if } v(\varphi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

It easy to see that classical negation may be defined as $\sim \varphi = \neg \otimes \varphi$. This shows that Tarskian inferences involving the $\neg, \wedge, \rightarrow$ may be recovered within its q-interpretation. For the following, let $\otimes\Gamma = \{\otimes\Gamma | \Gamma \in \Gamma\}$.

Proposition 10. *The following equivalences hold:*

(i) $\Gamma \models_t^{\mathcal{M}^q} \psi$ *iff* $\Gamma \models_q^{\mathcal{M}^2} \otimes(\psi)$ *iff* $\Gamma \models_p^{\mathcal{M}^q} \otimes(\psi)$
(ii) $\Gamma \models_t^{\mathcal{M}^q} \psi$ *iff* $\otimes\Gamma \models_p^{\mathcal{M}^2} \psi$ *iff* $\Gamma \models_q^{\mathcal{M}^q} \otimes(\psi)$
(iii) $\Gamma \models_q^{\mathcal{M}^q} \psi$ *iff* $\Gamma \models_t^{\mathcal{M}^1} \otimes(\psi)$

Proof. Similar to Proposition 8. □

7. Final Remarks

The present paper explored a uniform way of transforming Tarskian many-valued logics into their non-Tarskian interpretations. The type of transformation investigated consisted in altering the underlying semantical structure of Tarskian many-valued logics in order to output non-Tarskian interpretations thereof. Some of the effects of carrying out this type of procedure were exhibited, e.g., the conditions under which a finitely many-valued Tarskian logic becomes paraconsistent or paracomplete after its transformation to a q-interpretation were exhibited. Two types of dualization procedure were explored as candidates for the respective dualization of these entailment relations and their associated semantical structures. Connexive features regarding the consequence relation generated by the dual logics are explored and discussed, a feature not available to their base counterparts. Furthermore, the paper paved the way to the study of recovery operators between Tarskian logics and their associated non-Tarskian interpretations. The development of recovery operators for all consequence relations defined within these generalized settings (such as bi-dimensional entailment [7]) remains to be further explored. Furthermore, the effects of these dual structures on other well-known many-valued logics, as well as the philosophical applications of these logics, are also worth investigating.

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Notes

1 A first account on the problem of how to produce paraconsistent systems from many-valued logics was proposed by Newton da Costa and Elias Alves [5] and in the context of non-Tarskian logics by Malinowski in [4]. In [6], the authors develop a more general approach via maximal consistent sets of formulas. In this paper, we follow a different approach by altering the semantic structure of the underlying logic

2 To avoid ambiguity, whenever necessary the set *Val* shall be written with a superscript.

3 This philosophical thesis is employed by Suszko to defend that logical many-valuedness was a conceptual deceipt. For more on Suszko's thesis, see, e.g., [2].

4 The concept of inferential many-valuedness was first introduced by Malinowski ([9]).

5 Other kinds of transformations could be explored by allowing different partitions of the set of truth-values. However, the purpose of this paper is to explore the one established by Definition 1 due to its connection with Malinowski's motivation for keeping the set $\mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-)$ non-empty.

6 The reader may check [11] for an introduction to Galois connections in logic.

7 Truth-tables for $\neg, \wedge, \vee, \rightarrow$ in Łukasiewicz logic are defined as $f_\neg(v(\varphi)) = 1 - v(\varphi)$, $f_\vee(v(\varphi), v(\psi)) = max(v(\varphi), v(\psi))$, $f_\wedge(v(\varphi), v(\psi)) = min(v(\varphi), v(\psi))$ and $f_\rightarrow(v(\varphi), v(\psi)) = min(1, 1 - v(\varphi) + v(\psi))$.

8 Other properties could be obtained by working on FMLA-SET or SET-SET.

9 Other forms of producing paraconsistent logics based on Łukasiewicz's logics are explored in [5].

10 For all finitely-valued Łukasiewicz's logics, note that Definition 1 guarantee all intermediate values inside $\mathcal{V} - (\mathcal{D}^+ \cup \mathcal{D}^-)$.

11 The truth-function for $\neg$ in Gödel's logic may be defined as $f_\neg(v) = 1$, if $v = 0$ and $f_\neg(v) = 0$, if $v > 0$ for $v \in \mathcal{V}$.

12 See page 3 for the definition of t- and f-entailment.

13 In [14], the logics for which $\models^+$ and $\models^-$ do not coincide and their semantics is determined by a n-valued bidimensional q-matrix are called *refined* n-valued logics.

14 The reader may check [16] for a similar account of negation.

15 For a more thorough account of the theory of recovery operators, see [18].

[16] Blau's three-valued system is defined by Łukasiewicz's negation (¬) and conjunction (∧) along with an additional negation (≈). See [12].

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