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Of madness and many-valuedness: an investigation into Suszko's Thesis

Dissertação de Mestrado apresentada ao Programa de Pós-Graduação em Filosofia para obtenção do título de Mestre em Filosofia.

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Aos meus pais,
sine quibus non.

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"Whatever it is, I'm against it." Groucho Marx

Resumo

A Tese de Suszko é uma posição filosófica acerca da natureza dos múltiplos valores-de-verdade. Formulada pelo lógico polonês Roman Suszko, durante a década de 1970, a tese defende a existência de "apenas dois valores-de-verdade". Tal afirmação diz respeito à concepção de multi-valoração perpetrada pelo lógico Jan Łukasiewicz. Considerado um dos criadores das lógicas multi-valoradas, Łukasiewicz acrescentou, em adição aos valores fregeanos tradicionais de Verdade e Falsidade, um terceiro valor: o Indeterminado. Para ele, seu terceiro valor poderia ser visto como um passo além da dicotomia Aristotélica entre o ser e o não-ser. De acordo com Suszko, as ideias de Łukasiewicz sobre multi-valoração se baseavam em uma confusão entre *valores algébricos* (aquilo que é descrito/denotado por sentenças) e *valores lógicos* (verdade e falsidade). Assim, o terceiro valor-de-verdade criado por Łukasiewicz seria apenas um valor algébrico, isto é, uma possível denotação para uma sentença, mas não um valor lógico genuíno. A tese de Suszko encontra respaldo em um resultado formal conhecido hoje como *Redução de Suszko*, um teorema que afirma que toda lógica tarskiana pode ser caracterizada por uma semântica bivalente. Esta dissertação pretende ser uma investigação da tese de Suszko e de suas implicações. A primeira parte é dedicada às raízes históricas da multi-valoração e introduz as principais motivações de Suszko ao formular a distinção entre valores algébricos e valores lógicos, e assim revelar o caráter duplo dos valores-de-verdade. A segunda parte explora a Redução de Suszko e apresenta seus principais desenvolvimentos; as propriedades das semânticas bivalentes em comparação às semânticas multi-valoradas também são exploradas e discutidas. Por fim, a terceira parte investiga o conceito de valores lógicos dentro do contexto de noções não-tarskianas de consequência lógica; o significado da tese de Suszko dentro desses ambientes também é discutido. Mais ainda, os fundamentos filosóficos das noções de consequências não-tarskianas são discutidos à luz do debate recente sobre pluralismo lógico.

Palavras-chaves: Lógicas multi-valoradas; tese de Suszko; Bivalência; Consequência lógica; Pluralismo lógico.

Abstract

Suszko's Thesis is a philosophical claim regarding the nature of many-valuedness. It was formulated by the Polish logician Roman Suszko during the middle 70s and states the existence of "only but two truth values". The thesis is a reaction against the notion of many-valuedness conceived by Jan Łukasiewicz. Reputed as one of the modern founders of many-valued logics, Łukasiewicz considered a third undetermined value in addition to the traditional Fregean values of Truth and Falsehood. For Łukasiewicz, his third value could be seen as a step beyond the Aristotelian dichotomy of Being and non-Being. According to Suszko, Łukasiewicz's ideas rested on a confusion between algebraic values (what sentences describe/denote) and logical values (truth and falsity). Thus, Łukasiewicz's third undetermined value is no more than an algebraic value, a possible denotation for a sentence, but not a genuine logical value. Suszko's Thesis is endorsed by a formal result baptized as Suszko's Reduction, a theorem that states every Tarskian logic may be characterized by a two-valued semantics. The present study is intended as a thorough investigation of Suszko's thesis and its implications. The first part is devoted to the historical roots of many-valuedness and introduce Suszko's main motivations in formulating the double character of truth-values by drawing the distinction in between algebraic and logical values. The second part explores Suszko's Reduction and presents the developments achieved from it; the properties of two-valued semantics in comparison to many-valued semantics are also explored and discussed. Last but not least, the third part investigates the notion of logical values in the context of non-Tarskian notions of entailment; the meaning of Suszko's thesis within such frameworks is also discussed. Moreover, the philosophical foundations for non-Tarskian notions of entailment are explored in the light of recent debates concerning logical pluralism.

Key-words: Many-valued logics; Suszko's thesis; Bivalence; Entailment; Logical Pluralism.

Contents

Introduction

"Logic is like sword - those who appeal to it shall perish by it."

— Samuel Butler

The roots of many-valuedness may be traced back to the works of Aristotle and even to the earlier debate between the Eleatic and Ephesian schools of philosophy. Those ancient debates were tied to metaphysical queries about the way we conceive and understand the world, and their influence lasted through the development of Medieval and Modern history of philosophy. However, it was only after the establishment of modern symbolic logic that a formal treatment of the notion of many-valuedness became possible. Thus, from the 1920s, through the works of Jan Łukasiewicz, Dmitri Bochvar and Emil Post, many-valued logics were developed and studied from a modern perspective. Łukasiewicz's motivations were related to Aristotle's discussions on the law of excluded middle and the problem of future contingents. In order to evaluate propositions about the future, he added a third undetermined value to his logical system and set the theoretical foundations for what is currently recognized as the field of many-valued logics. Later on, in the middle 70s, the Polish logician Roman Suszko cast doubt on the influential character of Łukasiewicz's works and the very nature of his scientific enterprise. According to Suszko, the many-valued paradigm of logic was nothing but a "humbug" and Łukasiewicz was the "chief perpetrator of a magnificent conceptual deceit". For Suszko, truth values play a double semantic role revealed by a difference between what he calls 'algebraic values' and 'logical values'. The semantic scheme used to express such a dubiety is based on Frege's ideas on sense and reference. The referents of sentences, following Suszko, are situations denoted by algebraic values. Even though sentences can describe/denote more than two situations, they are classified by only two logical values: truth and falsehood. This approach led Suszko to claim that "there are but two logical values", a statement nowadays recognized as Suszko's Thesis. It finds support in a technical result called Suszko's Reduction, a theorem that shows that every Tarskian logic may be characterized by a bivalent semantics. The present dissertation intends to make a thorough assessment of the issues and developments arising from Suszko's ideas. The philosophical concern about the nature of truth values and the contemporary notions of entailment stand as a pivotal motivation for our work. Furthermore, despite of being aware that some issues adressed in this thesis could be explored in different frameworks, like first-order or higher-order logics, we shall restrict our attention to propositional logics.

The present study is divided into three main parts. Part I – On the role of truth-values in logical consequence – has the purpose of introducing Suszko's philosophical ideas concerning the nature of truth-values. The only chapter of Part I tries to recover some important historical elements behind the notion of truth-value since its birth with Gottlob Frege and the classical paradigm of truth-values, until Łukasiewicz's creation of a third non-classical truth-value. By a classical paradigm of truth-values, we mean the assumption that there are only two truth-values, the True and the False. This thesis is often called as the Principle of Bivalence. Despite other authors had also been responsible for considering non-classical truth-values, we focus only on Łukasiewicz's motivation for the sake of a better explanation of Suszko's ideas and its philosophical environment. Thus, the chapter begins with a brief presentation of Frege's conception of truth-values and some central issues that motivated it. After that, the chapter presents Łukasiewicz's reasons for considering a logical system with more than two truth-values. At last, Suszko's criticism to Łukasiewicz's conception of many-valuedness is presented. The main motivations for Suszko's point of view concerning truth-values are exhibited and discussed.

Part II – Suszko's Reduction: in the land of bivaluations – brings the technical results underlying Suszko's Reduction about the division in between algebraic and logical values. The main result of the chapter of Part II is Suszko's Reduction theorem, which has the purpose of showing that Every Tarskian logic is logically two-valued. Thus revealing logical two-valuedness at the core of the Tarskian notion of entailment. Suszko's Reduction is important for establishing the foundations of bivaluations as an adequate semantic tool in comparison to the matrix theory approach to semantics. However, the move from a matrix semantics to a semantics presented in terms of bivaluations will certainly carry some undesired consequences since some of the structural features inherent to matrices are lost. The chapter then continues by discussing how this consequence may be avoided in light of recent contributions given to Suszko's Reduction.

Part III – Beyond Suszko's Reduction – discusses generalizations of Suszko's Thesis by presenting alternative notions of entailment constructed from considering additional logical values beyond truth and falsity. The first chapter begins by introducing Grzegorz Malinowski's ideas about the so-called q-consequence operations. Malinowski is responsible for creating what was baptized by him as *inferential many-valuedness*. Inferential many-valuedness is the kind of many-valuedness obtained by exploring more than two logical values. The addition of a non-classical logical value leaves room for the construction of non-Tarskian notions of entailment. After showing how Suszko's Reduction could be generalized to the case of q-logics, the chapter continues with an exposition of Shramko & Wansing's conception of logical values and how it affects the usual understanding of Suszko's Reduction. Shramko & Wansing's construction redefines the usual notion of logic as characterized by a single consequence relation and propose k-dimensional logics with k consequence relations associated to it.

The last chapter of Part III is a brief commentary upon the issues adressed in the previous chapter. The first section raises the question of what properties should be expected from a logical system and whether or not should logic be recognized as encompassing more than a single consequence relation. The last one asks about the dangers of evading Tarski's world by abandoning logical consequence as truth-preservation. In particular, Jc Beall & Greg Restall's pluralism on logical consequence is discussed in light of some results of the chapter.

Part I

On the role of truth values in logical consequence

1 Truth-values and many-valuedness

Since the development of modern logic and the consolidation of Tarski's approach to the notion of logical consequence, truth values played a major role in systems of logic. In fact, the way truth values are defined and interpreted according to a given logic is central to define its underlying notion of logical consequence. According to [Shramko and Wansing, 2011], "truth values induced a radical rethinking of some central issues in the philosophy of logic and have been put to quite different uses in philosophy and logic". They have been characterized as:

- Primitive abstract objects denoted by sentences in natural and formal languages,
- Abstract entities hypostatized as the equivalence classes of sentences,
- What is aimed at in judgements,
- Values indicating the degree of truth of sentences,
- Entities that can be used to explain the vagueness of concepts,
- Values that are preserved in valid inferences,
- Values that convey information concerning a given proposition.

Roy Cook [Cook, 2009] highlights that truh-values can be understood as proxies for the various relations that can hold between language and the world. The present chapter intends to give an overview on the birth of non-classical truth values from the works of Jan Łukasiewicz. After that, we will discuss the work of some detractors of many-valued logics, paying special attention to the works of Roman Suszko, the Polish logician whose work has inspired a research programme about the meaning and significance of many-valuedness.

In the first section we present Gottlob Frege's conception of truth-values and how it was related to a neo-Kantian perspective on the role of logic as a tool to discover the 'laws of being true'. The second section covers the birth of many-valuedness focusing on Jan Łukasiewicz and the philosophical environment and motivations that led him to the consideration of a third truth-value beyond the dicothomy between truth and falsity. In the following, the third and fourth section aims at exposing Roman Suszko's conception of truth-values and why his ideas cast doubt on the influential character of Łukasiewicz's ideas. The last section intends to be a comparison of Suszko's ideas and other authors who also criticized many-valued logics such as Michael Dumett and Dana

Scott. Three conception of truth-values underlies the whole development of the chapter, Frege's conception of truth-values as abstract entities denoted by sentences, Łukasiewicz's conception of them as degrees of truth, and Suszko's conception of truth-values as values preserved in valid inferences, as well as the admissible referents of sentences.

1.1 Gottlob Frege, truth-values as reference

Truth-values were called into play by G. Frege [Frege, 1892] as objects denoted by sentences. According to Frege, truth values were mere referential objects denoted as values of arguments to which a concept expression apply. [Frege, 1892] became seminal to the development of contemporary philosophy of language. In it, Frege investigated and developed some concepts not thorougly explained in his first masterpiece, the *Begriffschrift* - responsible for launching his famous logicist program. The concepts developed by Frege in those papers were important for formulating his Theory of Meaning for the fragment of language he was concerned since the Begriffschrift.

In the course of investigating the process of grasping the meaning of identity statements, at the core os his account was the distinction between the *sense* (Sinn) and *reference* (Bedeutung) of an expresion[1]. According to him, whereas the names 'Lewis Carroll' and 'Charles Dodgson' may differ in sense, i.e, regarding the cognitive content associated to the expressions, they stand both for the same reference, the actual person corresponding to Charles Dodgson and Lewis Carroll. The sense of an expression is treated as the mode of presentation of its referent, in virtue of which the reference of an expression is denoted.

In Frege's theory of meaning, while the reference of a proper name is the object denoted by it, the reference of a complex saturated expression (sentence) is a truth-value, the *True* or the *False*. Thus sentences are just names that refer to truth-values. Complex expressions are built using proper names as arguments of functional expressions (concepts). Concept expressions appear in the analysis of predicate expressions such as 'is a city', 'is bigger than', 'is a property of' etc. For Frege, all predicates are unsaturated expressions in which the proper names serve as arguments to complete the meaning of the expression. Then, if we have an unsaturated expression like 'is a city', proper names like 'Natal', 'David' and 'Seattle' serve as arguments making the expression saturated and able to denote a truth-value.

According to [Dummett, 1978], Frege's conception of sense and reference lies in the idea that to understand a complete sentence one must think of its truth-value. In

[1] Frege's distinction between sense and reference is the product of other concepts explored by him such as the distinction between the sense, tone and colour of an expression. However, it is not our purpose to make a full presentation of the development of Frege's theory of meaning. For more, see [Dummett, 1981].

this way, the sense of the sentence is a way/procedure to grasp its truth-value. Since the sense of a proper name is a criterion for identifying its referent, the sense of a concept-expression is a way of determining whether or not something satisfies it. Thus, working these two procedures together, we shall have a procedure for determining the truth-value of a sentence. Moreover, since sense is a mode of presentation of the reference, this implies that truth-values could be understood in a myriad of ways. Frege's way out to avoid falling into subjectivism is by placing senses in an objective stance. According to some authors, Frege's solution is in accordance with some Neo-Kantian philosophical thesis; we shall explain this in the following.

Some later philosophers, for instance, Strawson [Strawson, 1950] and Davidson [Davidson, 1969] have advocated the strangeness of the idea of treating truth-values as the reference of sentences. However, Frege's reasons are connected to the philosophical environment of his age, specially his conception of logic as the 'science of most general laws of being true'. For Frege, logic is concerned with truth itself, not truth as a mere property of sentences. According to [Gabriel, 1984] and [Gabriel, 2001], Frege's philosophical positions were influenced by the Southwest german school of Neo-Kantianism, that emerged under the influence of Hermann Lötze. Moreover, the very use of the word truth-value is connected to the pioneers of that tradition, as highlighted by [Gabriel, 2001], "Wilhelm Windelband, the founder and the principal representative of the Southwest school of Neo-kantianism, was actually the first who employed the term 'truth value' ('Warheitswert') in his essay "What is philosophy?" published in 1882 (...)."

As founder of the value-theoretical tradition of the Southwest Neo-Kantian school, Windelband defined Philosophy as a science of universal values. From his point of view, the main task of Philosophy was to establish universal principles for logical, ethical and aesthetical judgement, thus always oriented by a thelos. Following the way paved by the Windelbandian tradition, Frege opened the paper [Frege, 1956] by defining logic in the following manner:

> "The word 'true' indicates the aim of logic as does 'beautiful' that of aesthetics or 'good' that of ethics." [Frege, 1956]

Regarding the roots of that tradition, Gabriel [Gabriel, 2001] highlights that the underlying philosophical position of the Southwest school was based in the reunion of a Platonist and Kantian philosophy that emerged from Lötze's interpretation of Plato. Such a position was called as *transcendental platonism*:

> "Transcendental platonism is platonist because it accepts contents of thinking(thoughts) that are independent of the individual thinking subjects, and it is transcendental (as opposed to transcendent) because the independence is

> not thought of as an ontological one of existence, but a logical one of being valid." [Gabriel, 2001]

According to [Gabriel, 2001], the position of Frege and some Neo-Kantians (like Windelband) could be described as transcendental platonism. Moreover, the above definition could be presented in the form of a transcendental argument which reveals how truth-values appear within the philosophical view purported by the southwest neo-kantian perspective:

> "Logic starts with making a 'distinction of value' between 'truth and untruth'. True and untrue, or false, cannot appear as properties of *processes* of thinking, but only of *contents* of thinking. To think about truth and falsehood necessarily pressuposes – as a condition sine qua non, that is, as a 'condition of possibility' in the Kantian sense – that we have first grasped the same cognitive content and are discussing the same thought. To take this consequence seriously, we have to accept that a thought cannot be a psychological item, because such a view would imply that different individual subjects are not able to participate in the same cognitive content or thought." [Gabriel, 2001]

The position described above is certainly influential to some later thesis defended by Frege, like his notion of sense as the *Gedanke* (thought) expressed by a sentence and his rejection of psychologism. However, to what extent Frege might be considered a Platonist or a Kantian is a question that shall not be adressed in this study. The important point to underline is that such independence of thought suggested above implies an item which we want to value as true or false, an item that is meant as the bearer of a truth-value and cannot have individual psychological existence[2]. This is important to Frege's treatment of the relation between senses and truth-values as reference. However, it is important to remark that despite Frege's being influenced by Windelbrand in choosing the word *Warheitswert* to refer to truth values, he understands it in a different sense from Windelbrand, because Frege treats truth-values as referents of concept expressions. This is a straight consequence of his mathematical approach to language. How it was explained before, given that concept expressions are predicates which, after being applied to singular terms as arguments, produce sentences, then the values of those functions must be the reference of the sentences.

By considering that the range of functions contain typically objects, then the reference of sentences should be objects, as well [3]. The interesting step taken by Frege here was to treat 'the True' and 'the False' as objects and not merely as properties. Frege

[2] Cf. [Gabriel, 2001].
[3] Cf. [Shramko and Wansing, 2011].

understood truth values as *logical objects*, that is, mathematical objects such as numbers, sets and alike. He considered truth-values among the grounding objects for his ontology. The major part of Frege's later work is dedicated to investigating the nature of such logical objects aiming to achieve a rigorous ontological foundation. In this way, those objects were not only abstract, but also possessed a different ontological import in virtue of their primacy[4].

In the next section, we shall expose Jan Łukasiewicz's approach to truth-values. He is responsible for deviating from Frege's conception of truth-values and being able to postulate the need for a third truth-value, beyond the True and False dichotomy.

1.2 Łukasiewicz and *The Possibles*

> "Entre o sim e o não existe um vão."
>
> — Itamar Assumpção

For some authors, a great share of contemporary philosophy emerged from the works of Brentano and his pupils[5]. The development and consolidation of Polish philosophy was no exception, since the most influential figure of the Lvóv-Warsaw School was Kazimierz Twardoswki, one of the three most distinguished students of Brentano (the other two were Alexius Meinong and Edmund Husserl)[6]. Naturally, Twardowski was influenced by the discussions and subjects explored by his intellectual mentor and dedicated his life to the study of ontology and its relation to language and psychology. To some extent, those were the themes explored by the first members of the Lvóv-Warsaw school, moved by a scientific conception of philosophy. However, the school inherited not only the philosophical standpoint from the Brentanian tradition, but the Russellian approach to logic and philosophy as well.

The Russellian tradition was introduced by the first three Polish modern logicians, followers of Russell and Frege: Leon Chwistek, co-inventor of the simple theory of types; Jan Łukasiewicz, one of the founders of many-valued logic; and Stanislaw Lesniewski, famous for his great contributions to nominalistic philosophy of mathematics and to contemporary mereology. Although Meinong is often recognized as one of the founders of the contemporary tradition that seeks to think the nature of non-existent and contradictory objects, according to [Betti, 2011,], "(...) [Twardoswski] was the first philosopher to hold

[4] It is not our concern to adress the question of the adequate meaning of logical objects in Frege's work. It is our only goal to describe a little of the history of the concept of truth and how it is related to the modern notion of truth-value.

[5] Cf. [Dummett, 2014].

[6] Cf. [Perzanowski,].

a theory of intentionality, truth, and predication in which thinking and speaking about non-existents, including contradictions, involves presenting and naming non-existents, including contradictory objects." As we shall see, the struggle for adequate logical tools for dealing with non-existents, as well as contradictory objects, played an important role in the development of many-valued logics.

Some authors, such as [Rescher, 1968] and [Malinowski, 2009], commonly accredit the first discussions related to many-valuedness to the ancient Greek philosophers. Regardless of that, Rescher [Rescher, 1968] locates the "Early History" of many-valued logics within the period from 1875 to 1916. He indicates Hugh MacColl (1837-1909), Charles Peirce (1839-1914) and Nikolai Vasil'ev (1880-1940) as the founding fathers of many-valued logic. However, since few formal developments were made by those authors towards the creation of a many-valued logical system, Rescher calls "The Pioneering Era" of many-valued logics the period from 1920 to 1932, in which first appeared the works of Łukasiewicz and Emil Post. In the present thesis, for the sake of a better explanation of the influences around Suszko's ideas, we shall not talk about Post or any other author from the early history.

1920, the year of the first publication of [Łukasiewicz, 1968], is often mentioned as marking the date of birth of Łukasiewicz's three-valued logic. Albeit, already in 1918, in his farewell speech at Warsaw University, he asserts:

> In 1910 I published a book on the principle of contradiction in Aristotle's work, in which I strove to demonstrate that that principle is not so self-evident as it is believed to be. Even then I strove to construct non-Aristotelian logic, but in vain. Now I believe I have succeeded in this... I have proved that in addition to true and false propositions there are *possible* propositions, to which objective possibility corresponds as a third in addition to being and non-being.(Łukasiewicz apud [Wolenski, 1989], p. 119)

As we can notice in the above paragraph, many-valued logic as conceived by Łukasiewicz was created from a dissatisfaction of having only truth and falsity as primary notions. The motivations found by him comes from investigations about the nature of science, ontology and probability. Among the main reasons for his abandonment of the classical perspective, are[7]:

1) The design of a formal system capable of dealing with the theory of objects proposed by Brentano, Twardowski and Meinong;

2) The problems related to induction and the theory of probability;

[7] Cf. [Malinowski, 2009].

3) The concern with the problem of determinism and its relation to modality.

The first one has direct connections to the issues dealt with by Łukasiewicz in his 1910's book: *On the principle of contradiction in Aristotle*. In it, under the influence of Meinong's theory about the existence of contradictory objects, objects for which an ontological version of the law of non contradiction fails to hold [8], Łukasiewicz had some intuition towards the necessity of values beyond truth and falsity[9]. Furthermore, while contradictory objects, such as Meinong's round square, infringe the law of non-contradiction, Łukasiewicz also considered abstract objects (like the triangle), called by him "incomplete objects", which infringe the law of excluded middle for being free of existence.

As pointed out earlier in this section, Łukasiewicz was a product of two traditions, the philosophical approach pursued by those from the Brentanian tradition and the concern with the ontology of logic and its connection to the world, typical from the Frege-Russellian tradition. It is from such a standpoint that it is possible to understand Łukasiewicz's position regarding the nature of truth values and his creation of the third value. For him, it was clear that the principle of bivalence together with the law of excluded middle made science committed with determinism, because every proposition can only be true or false, specially those about the future since exluded middle ensure the truth of the disjunction despite none of its parts being true[10]. For him, statements about the future do not satisfy the excluded middle since they are neither true nor false. As put by [Simons, 1989], Łukasiewicz's main concern was to make "science free from absolute determinism" and the way to accomplish such a task should involve the causal necessity that pervades scientific prediction. The strive with future contingents and modality stood out as his main drive in order to formulate an adequate semantics for a three-valued logic. According to [Simons, 2014], Łukasiewicz was bothered by the idea of modal logic being trapped into classical bivalent logic. Łukasiewicz's way out of such problem he added a third value, 'the possible', denoted by $\frac{1}{2}$. Thus, Łukasiewicz believed the significance of his three-valued logic was in creating a non-Aristotelian logic.

Łukasiewicz's system Ł$_3$ may be defined in the following way. Let $\mathcal{V} = \{0, \frac{1}{2}, 1\}$ be the set of truth-values, with $\mathcal{D} = \{1\}$ and $\mathcal{U} = \{0, \frac{1}{2}\}$. The elements of $\mathcal{D}$ are called *designated* truth-values and the elements of $\mathcal{U}$ are called *undesignated*. The connectives '$\neg$', '$\rightarrow$' and '$\vee$' are defined by the following truth-tables:

[8] For all a and P: it is not the case that $P(a)$ and $\neg P(a)$.
[9] Cf. [Simons, 1989].
[10] This standpoint on determinism is often called in the literature as *Aristotle's fantasy*.

$\to$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

	$\neg$
0	1
$\frac{1}{2}$	$\frac{1}{2}$
1	0

$\vee$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
1	1	1	1

We say a formula is a tautology if it is always assigned a designated value. In this case, note that, $\alpha \vee \neg\alpha$ and $\neg(\alpha \wedge \neg\alpha)$ are not tautologies in $Ł_3$. Later on, Łukasiewic also exhibited how to extend his system in having finitely or infinitely many truth-values. For him, truth-values within those systems expressed the truth degree of sentences. The idea set the basis for what is nowadays called *Fuzzy logics*.

It is important to mention that Łukasiewicz's conception of truth-value was based on the technical tools explored by the polish logicians, what also moved a mathematical conception of truth-values. According to [Béziau, 2012], this *mathematical conception of truth-values* appears in its complete form in the history of logic only after the development of the notion of logical matrices[11]. In [Béziau, 2012], the author describes the mathematical concept of truth value in the following way:

> "Let us have a look at the MTV [Mathematical concept of Truth Value]. It is a double structure: on the one hand we have an absolutely free algebra, on the other hand, facing it, an algebra of similar type, finite or not, and between them, the central notion establishing relations between mathematical structures, the notion of morphism. The elements of the free algebra are called *propositions* and its functions *connectives*, the elements of the facing structure are called *truth-values* and its functions *truth-functions*, and finally the morphisms between the two structures *valuations*.[Béziau, 2012]"

From this perspective, truth-values are only mathematical elements from an algebra of an adequate similarity type. They are divided by distinguishing proper subsets of them into *designated* and *non-designated*. From this, each morphisms between the structures plays the role of assigning the propositions that receive a designated value and, therefore, have a model, from the ones which does not have a model. The mathematical approach to the concept of truth-value led us to define properties such as truth-functionality, analyticity, and so on, in a proper manner. The development of such mathematical structure of truth values owes a great debt to Boole and Peirce. However, it was Tarski and Lindenbaum the first ones to set the algebra of formulas and truth values in a clear abstract perspective. Their treatment established the foundations for the total abstraction of the concept of logic by looking at the structural features of the notion of entailment.

[11] Yet in this chapter we shall focus only on the philosophical aspects of the concept of truth-value. The mathematical aspects will be explored in the remaining chapters.

Therefore, the connections between truth values and logical consequence were thus properly revealed and treated. In this regard, as we shall discuss in Part III of the present thesis, the notion of logical consequence became prominent in defining what a logical system is.

1.3 Roman Suszko on *algebraic* and *logical* values

Roman Suszko (1919-1979) is one of the most prominent modern logicians from the Lvóv-Warsaw school. He is accredited with an extensive work on abstract logic, on model theory, and on the algebraic tradition initiated with Alfred Tarski. Suszko appeared in Poland's philosophical scenario as a contemporary of Łukasiewicz, therefore during the mix between the Brentanian and the Russellian tradition. Despite the contact of the Brentanian tradition on the philosophical approach pursued by the Lvóv-Warsaw school, the main influences for Suszko, on what regards his treatment of logic and ontology, are the figures of Frege and Wittgenstein.

Suszko's concern with the nature of truth values became central in his work only by the end of his career. The famous paper in which he develops his thoughts on truth values is [Suszko, 1975a], in which he addresses the discussion about the expressive power of so-called non-Fregean logics (NFL), logics characterized by the failure of the so-called *Fregean Axiom (FA)*. The paper is a presentation and development of the most simplified version of NFL, the so-called Sentential Calculus with Identity, (SCI), that is a product of other investigations by Suszko and his co-authors like [Bloom et al., 1972] and [Bloom and Suszko, 1971]. For Suszko, NFL was responsible for continuing Frege's program without the Fregean Axiom, what could be seen as equivalent to "realising Euclid's program without the fifth postulate".

Suszko's conception of truth values and the ontology of logic provided the theoretical motivations for his creation of NFL. His ideas were elaborated in the context of struggling with the Fregean axiom and its implications. He used the expression non-Fregean logics to refer to the class of logics that do not satisfy the following principle:

(FA1) *all true (and, similarly, all false) sentences describe the same,*
that is, have a common referent.

(FA1) is recognized as the *general version of the Fregean Axiom*[12]. Following Frege's ideas, since the referent of sentences are truth values and there are only two of them, then all true sentences denote the truth value True, whilst all false sentences denote the truth value False.

12 Also called the *semantic version of the Fregean Axiom*, [Malinowski and Zygmunt, 1978].

Suszko starts the paper [Suszko, 1975a] by interpreting Frege's semantic scheme in the following way: given a sentence ϕ, we shall call $r(\phi)$ the referent of the sentence, $s(\phi)$ the sense of ϕ, and $t(\phi)$ its logical value. In Frege's theoretical construction, once the reference of saturated sentences are truth values, we have the following conditions on the assignments, given sentences ϕ and ψ:

$$r(\phi) \neq r(\psi) \text{ implies } s(\phi) \neq s(\psi) \tag{1.1}$$

$$t(\phi) \neq t(\psi) \text{ implies } r(\phi) \neq r(\psi) \tag{1.2}$$

The converse of (2), according to Suszko, is one version of the Fregean Axiom, since it tell us that different referents must receive different logical values. The whole point of Suszko's construction with NFL is in negating such idea by allowing that different referents may receive the same logical values. For him, the way to accomplish depends on being able to draw a difference between logical values and the referents of sentences.

By still making use of Frege's terminology of sense and reference, Suszko defined the referent of a sentence ϕ as the *situation* denoted by ϕ and the sense as the *proposition* expressed by ϕ. Based on this, we can mention two reasons for Suszko's rejection of the Fregean Axiom: 1) The fact that equating reference and logical values implies the existence of only two possible situations described by sentences, and 2) It entails the confusion between what sentences describes/denote and their logical values. Therefore, by evading the Fregean Axiom, one would obtain a richer ontology with more than two possible referents.

In order to avoid the problem of having at most two possible situations as reference, caused by the Fregean Axiom, Suszko enriched the language of propositional logic with a new operator '$\equiv$', which is used to assert the identity of situations, i.e, the identity of what the sentences describe. By taking classical propositional logic plus '$\equiv$', we obtain SCI if we impose the following constraints upon '$\equiv$' [13]:

$$\varphi \equiv \varphi \tag{1.3}$$

$$(\varphi \equiv \psi) \Rightarrow \sigma[p \mapsto \varphi] \equiv \sigma[p \mapsto \psi] \tag{FA3}$$

$$(\varphi \equiv \psi) \Rightarrow (\varphi \to \psi) \tag{1.4}$$

[13] The full non-Fregean logic is obtained by constructing a first-order language which includes quantifiers over both individual variables and variables which may be substituted only by sentences (variables running over situations). Cf. [Wójcicki, 1984]. It is not within the scope of the present study to provide a complete description of how Suszko built NFL. The important point to notice is Suszko's rejection of the Fregean Axiom and the philosophical reasons for doing so.

where $\sigma[p \mapsto \alpha]$ denotes the substitution of every p in σ by α. The remaining symbols stands as usual.
The connective "$\equiv$" may also be introduced in the following way:

$$t(\phi \equiv \psi) = 1 \text{ iff } r(\phi) = r(\psi) \tag{1.5}$$

Note that, (1.5) states that $\phi \equiv \psi$ takes the value 1 if and only if ϕ and ψ describe the same situation. One of Suszko's goals in defining '$\equiv$' in the above manner is in avoiding what he calls the *ontological version of the Fregean Axiom*, namely the converse to 1.4:[14].

$$(\phi \leftrightarrow \psi) \Rightarrow (\phi \equiv \psi) \tag{FA2}$$

For Suszko, the problem with (FA2) is in confusing identity of situations and material equivalence. In [Suszko, 1975a], he presented an axiomatic basis for NFL, called $\mathcal{W}$ (for Wittgenstein), and shows how to construct stronger systems by adding different axioms to $\mathcal{W}$ (what gives us the hierarchy of kind $\mathcal{W}$ systems). The strongest system of the hierarchy, called $\mathcal{WF}$, is obtained from $\mathcal{W}$ by adding the ontological version of the Fregean Axiom. Moreover, according to Suszko, the different systems between $\mathcal{W}$ and $\mathcal{WF}$, constructed by imposing different constraints, stands for different "ontological principles concerning the structure of the universe of situations". That is why he believed that (FA) represents a strong ontological constraint, because, as was explained above, it limits the set of possible situations to at most two.

Another interesting point of kind $\mathcal{W}$ systems is in exploring the relation between material equivalence and identity. This relation is a theme well explored in Suszko & Bloom's sentential calculus with identity (SCI). According to them, SCI is the "weakest extensional two-valued logic". Suszko's obsession with an adequate treatment for identity and material equivalence relies on his appreciation of extensionality and his fear of "intensional ghosts". Therefore, in [Suszko, 1975a], he discusses the truth-functional character of identity and even claim that "[identity] is truth functional and, then, coincides with material equivalence, if and only if the Fregean axiom holds".

[14] It is important to highlight the fact that Suszko had different formulations of the Fregean Axiom. In particular, (FA2) appeared first in [Suszko, 1975b]. This version is mentioned and explored by [Caleiro et al., 2003]. Despite the differences in each formulation of the Fregean Axiom, Suszko's motivation for each one is in avoiding the strong ontological constraint implied by (FA1). In [Suszko, 1975b], he claims that Łukasiewicz's logic represents the "true abolition of the Fregean Axiom [(FA3)]". This seems to suggest that Suszko believed that many-valuedness would always imply the failure of (FA3). The claim was analysed and corrected in [Caleiro et al., 2007] by exhibiting a many-valued logic that satisfy (FA3) and showing the condition under which Suszko's claim is correct. Another important point concerns the exact relation between each formulation of (FA). The failure of each formulation of the Fregean Axiom does not seem to characterize the same class of logics. For instance, it is well known that the logics that rejects (FA1) are the ones that do not allow for the formulation of slingshot arguments; the ones that fail (FA3) are in the class of non-algebraizable logics and, moreover, (FA2) is rejected in NFLs. Thus, what is the precise relation between each of those properties and the formulation of (FA) is a question that remains to be further investigated.

On what regards Suszko's conception of reference, it is based on the notion of *states of affairs*, exposed by Wittgenstein in the Tractatus. In [Suszko et al., 1968], he elaborated a system intended as a formalized version of the ontology developed in the Tractatus. (He points out the monograph written by the logician Boguslaw Wolniewicz [Wolniewicz, 1968] as his primary source of inspiration.). By following the Tractarian perspective on reference, Suszko is avoiding Frege's neo-Kantian perspective on truth values. Therefore, Suszko is building the path toward an alternative conception about reference and the role of truth values as semantic entities. According to Malinowski [Malinowski, 2009], "One could say that Suszko's interpretation of truth-values rests on the distinction between two semantic levels: ontological and logical". Thus, Suszko's account tries to separate the ontological world standing as the denotation of sentences from the logical notions of truth and falsity.

According to Suszko's later papers, [Suszko, 1975a] and [Suszko, 1977], $\mathcal{V}$, the set of truth-values, stands as the set of *algebraic* values and each of its elements denotes the possible referents of sentences, i.e., situations. Moreover, the distinction of the algebraic values into two subsets $\mathcal{D}$ and $\mathcal{U}$, called, respectively, *designated* and *undesignated*, represents the two genuine *logical* values, the adequate notions of truth and falsity.

The first paper written by Suszko about the concept of logical values dates from 1957, entitled *Formal theory of logical values*; unfortunately, it never received an English translation [15]. An important consideration made by Suszko regarding the concept of logical value appeared as a definition of logical value in an encyclopedia called *Notions and theorems of elementary formal logic* [Pogorzelski and Pogorzelski, 1994]:

> "Generally speaking, the term logical value in a metalanguage of a certain propositional logic refers to every element of characteristic matrix of that logic or (more generally) an arbitrary element of a universum of an arbitrary logical matrix of the language S_0. In the early stages of the development of modern logic, truth and falseness (which were referred only to propositions or propositional expressions) were called logical values. When it was assumed that a sentence α was true, then it was said that it had a logical value of truth and it was possible to replace it by the symbol 1 or T; when it was false, it was possible to be replaced by the symbol 0 or the sign F (0, 1 — symbols taken from Boolean algebra). However, this point of view was not uniform: as logical values were treated both as truth and falseness (intuitively understood), and the symbols 0, 1, or as symbols of a fixed but arbitrary — false or true sentence. In the last case, it was a linguistic understanding of logical values. " Suszko: *Formal theory of logical values* apud [Pogorzelski and Pogorzelski, 1994]

In the above passage Suszko claims the term logical value refers to every element

[15] Cf. [Béziau, 2012].

of a matrix, what is contradictory to our separation between algebraic and logical values. However, in the same passage, Suszko goes on and expresses dissatisfaction with such use of the term logical value and how it has been treated since the birth of so-called many-valued logics:

> "Good foundation of such a formulation of the notion of logical value was Post's proof showing that classical propositional logic in axiomatic form has two-valued characteristic matrix. However, almost at the same time logics with characteristic matrices of numbers bigger than two occurred — e.g., Łukasiewicz's three-valued logic (the term logic is understood here as set of tautologies of a matrix). This third element of universum of this matrix was traditionally interpreted as possibility.
>
> Logical values have quickly lost their philosophically intuitive interpretations when logics, whose characteristic matrices were n-ary, occurred (with an arbitrary, natural n) or could even be of infinite cardinality.
>
> Obviously, logical values (many-valued logics, in contradistinction to two-valued logic) were — and are — discussed but these logical values have lost their intuitive sense, although an opinion was stated that they could be degress of truthfulness of propositions.
>
> (...)
>
> As the propositional logics and their metatheories were developing, the variety of characteristic models (matrices) and elements of their carriers (from rational numbers to topological spaces) increased and, according to that, it is stated that by logical values are understood elements of a universum of logical matrix, which because of intuitive or philosophical reasons, are identified with the notion of truth, falseness or their variants (necessity, possibility, randomness). However, when such identifications are not made and intentions of founders of a certain logic do not lead to a simple split of the notion of truth or falseness and so making a logic with many logical values instead of two, then a certain logic is not called many-valued and elements of universum of characteristic matrix are not always called logical values. Thus, this notion is not described properly and the fact whether a certain element of characteristic matrix of a certain logic is called logical value or not depends on non-formal means, intuitions or a predilection of a founder of a logic or its main users. " Suszko: *Formal theory of logical values* apud [Pogorzelski and Pogorzelski, 1994]

The above quote clearly states Suszko's dissatisfaction with the lost of the philosophical meaning, or intuition, behind the notion of logical values. The last papers published by him tried to renew the discussion about the philosophical meaning of logical value. Suszko is interested on the way we understand truth values within the mathematical structures we use in mathematical logic. His remark about the good foundation found in Post's result had some influence on his project of showing that every many-valued semantics can be characterized by a bivalent one, a result nowadays called *Suszko's Reduction*. Even though Suszko [Suszko, 1975b] and Malinowski [Malinowski, 1990a] have made some contribution to the way Suszko's Reduction may effectively be carried out, an algorithmic procedure for such a task was exhibited only later on by [Caleiro et al., 2007].

Given the distinction between algebraic and logical values, we see that truth values play a dubious semantic role in a logical system. This ambiguity reveals itself by highlighting the fact that, in Suszko's terminology, the expression truth values denotes different things, e.g, the elements inside the matrix, as well as its partitions. From this, hereafter we shall adopt a terminology to refer to the different modes in which many-valuedness may express itself by using the expressions *referential* and *inferential* many-valuedness. The former is about the cardinality of the designated set of truth values, whereas the latter concerns the number of partitions of the matrix [16]. This difference will be important in order to stress the meaning of Suszko's Thesis and its later developments.

From what was said above, we can conclude that there were two central reasons for Suszko's criticism of the Fregean Axiom: 1) the fact that it collapses algebraic and logical values, 2) that it entails the confusion between identity and material equivalence. Only the former shall be thorougly explored in thein present study in virtue of its direct connection with the formulation of Suszko's Thesis and how it was understood and developed in the literature [17].

1.4 Suszko's Thesis and many-valuedness

During the 1970s, in a talk delivered at the 22^{nd} Conference on History of Logic (Craców), Roman Suszko called into question the nature of the logical enterprise pursued by Polish logic since the 1920s under the influence of Łukasiewicz [Suszko, 1977]. Suszko striked and accused Łukasiewicz to be "the chief perpetrator of a magnificent conceptual deceit lasting out in mathematical logic" and provoked: "how was it possible that the humbug of many logical values persisted over the last fifty years?". Suszko's ideas questioned the notion of many-valuedness proposed by Łukasiewicz.

Based on the above difference between algebraic and logical values, Suszko asked "How could he [Łukasiewicz] confuse truth and falsity with what sentences describe?", the difference that led him to claim "there are but two genuine logical values". His ideas

[16] The expressions referential and inferential many-valuedness were not used by Suszko. Instead, the first authors to use it were Wójcicki and Malinowski.

[17] For more about SCI and its treatment of identity, see [Bloom et al., 1972] and [Bloom and Suszko, 1971].

endorse the fact that, while the classical conception of truth values might be denied at the referential level, i.e., at the level of what sentences describe, it remains true at the inferential level, through the Tarskian notion of inference. The reason for this is in the fact that Tarski's notion of entailment depends only on the dichotomy between the designated and undesignated sets of truth values. Then, once for Suszko the genuine notions of truth and falsity are expressed by logical values, they are our genuine values and it would be a "mad idea" to have more than two of them. This philosophical position concerning the nature of many-valuedness became known and referred to in the literature as *Suszko's Thesis*.

In [Suszko, 1975b], Suszko exhibited a sketch of a bivalent description of Łukasiewicz three-valued logic and suggested that such approach was strong enough to be applied to any many-valued logic. However, the first one to give a first step in showing that Suszko's approach could be applied to any many-valued logic was Malinowski in [Malinowski, 1990a]. Nevertheless, the first ones to show a full description on the procedure of finding an adequate bivalent semantics to any many-valued semantics were Caleiro et al [Caleiro et al., 2003][18].

As was said above, Frege assumed that there were only two referents of sentences: the True and the False. All true sentences denote the True, and all false sentences denote the False. By adopting a Tractarian perspective on reference, Suszko takes a realist stance and builds a richer ontology of situations. Moreover, his division of algebraic and logical values had the importance of defending the idea that sentences assigned the same logical value need not denote the same.

For Suszko, as for Wittgenstein, the world is conceived as the totality of facts (situations). The classification into designated and undesignated values has the purpose of selecting the situations that obtain and the ones that do not. This idea is obviously very close to Wittgenstein's Tractarian construction in dividing the world into *negative* and *positive* facts[19]. According to Suszko, an adequate formalized language to deal with objects and situations must have two types of variables: nominal variables running through the universe of objects and sentential variables running through the universe of situations. This construction set the basis for NFL.

The importance of such concept of reference is in going against the Fregean conception of truth values. Suszko did not take logic as a machinery tool to discover truths about an ontological realm beyond human direct experience. Instead, Suszko believes that formal languages are created from the attempt to grasp some fragments of reality. According to Omyla [Omyla,], for Suszko "the subject-matter of logical investigations are any conceptual structures emerging from the process of world cognition". Moreover, "there is

[18] Here we take a bivalent semantics as any arbitrary family of functions from the set of propositional formulas to the set $\mathcal{V}_2 = \{0, 1\}$. Font [Font, 2009] called such notion of semantics *Susko's semantics*.

[19] In his writings, however, Suszko did not treated the question of what is a negative situation. This was only later explored by Wójcicki in [Wójcicki, 1984].

a structural syntatic framework, by means of which consciousness can grasp reality." Thus the logical structure of a language related to the fragment of reality it tries to formalize is never arbitrary and purely linguistic but is determined by[20]:

(a) the ontological structure of the fragment of reality to which the language refers.
(b) the semantic principles adopted.

Suszko's position is similar to the Wittgensteinean conception of the interaction between reality and the logical structure of language, by which formal languages are able to exhibig the logical form of reality. He takes logic as dealing with some aspects of the logical structure of the world, what is supported by his view on the ontology of the world as the totality of situations. Thus, his realist conception of reference serve as foundation for a relational general theory of reference, where situations are seen as blocks of reality that logical systems tries to grasp. Although he had not entirely developed a theory of situations[21], Suszko was aware of the importance of taking situations as primary ontological entities. In [Suszko, 1994], he took events as objects abstracted from situations and proved that some theories of situations are mutually translatable into theories of events.

Suszko's concern is directed toward drawing a sharp distinction between the levels in which many-valuedness may be expressed by establishing the difference of algebraic and logical values. The notions of algebraic and logical values are useful to relate a rich ontology of situations as referents, and yet avoid clouded concepts related to undetermined truth or falsity. By doing that, Suszko gives support to fundamental notions of truth and falsity and helps to clarify the nature of many-valuedness in logic.

1.5 Michael Dummett and Dana Scott on many-valuedness

Despite the richness of Suszko's thoughts, he was not the first one to contend against many-valuedness. Other authors such as Michael Dummett ([Dummett, 1991] and [Dummett, 1978]) and Dana Scott ([Scott, 1973] and [Scott, 1974]) made, independently, similar criticism regarding the way many-valuedness was treated and defined. In what follows, we present these criticisms and how they relate to each other.

The precise sense in which we claim that Dummett's and Suszko's comments are similar lies on the fact that both of them considered truth and falsiy as concepts expressed by the Tarskian dichotomy of designated and undesignated values. Already in 1959, in [Dummett, 1978] and also [Dummett, 1991], Dummett analyzes the different uses of truth and falsity as performed by some utterances. However, differently from Suszko, Dummett

[20] Cf. [Omyla,].
[21] That would only be accomplished some years later by Barwise and Perry.

is not worried with an ontological principle behind the treatment of truth values, but in setting a in treating the different uses of truth and falsity.

In the preface of [Dummett, 1978], the author says:

> "Truth(...) was a defense of the principle of *tertium non datur*[22] against certain kinds of counterexample; not, of course, that I wanted to contend against uses of 'true' and 'false' under which an utterance could be said to be recognised, in certain senses, as being neither true nor false, so long as the point of using those words in such a way was acknowledged to be only to attain a smoother description of the way the sentential operators worked."

Thus, according to Dummett, we can accept different or intermediate notions of truth and falsity as long as they are used only to get a better description of the connectives. This point is very similar to Suszko's later consideration on algebraic valuations, which are characterized by a homomorphism between the algebra of formulas and the algebra of truth values. The bivalent characterization of the algebra of truth values, of course, would lose its homomorphic structure, thus not possessing a "smooth" description of the connectives. In [Dummett, 1978], in the context of discussing a logic with T, F, X and Y as truth-values, Dummett remarks:

> "Logicians who study many-valued logics have a term which can be employed here: they would say T and X are 'designated' truth-values and F and Y 'undesignated' ones. (In a many-valued logic those formulas are considered valid which have a designated value for every assignment of values to their sentence-letters). The point to observe are just these. (i) The sense of a sentence is determined wholly by knowing the case in which it has a designated value and the cases in wich it has an undesignated one. (ii) Finer distinctions between different designated values or different undesignated ones, however naturally they come to us, are justified only if they are needed in order to give a truth-functional account of the formation of complex statements by means of operators. (iii) In *most* philosophical discussions of truth and falsity, what we really have in mind is the distinction between a designated and an undesignated value, and hence choosing the names 'truth' and 'falsity' for particular designated and undesignated values respectively will only obscure the issue.(...)" [Dummett, 1978]

Dummett then reveal similar ideas to Suszko's distinction on algebraic and logical values. For Dummett, though we can have statements that may seem neither true nor

[22] Dummett understands *tertium non datur* as the following semantic principle: no statement is neither true nor false.

false regarding its content, they are all classified as designated or undesignated. Therefore the genuine conceptions of truth and falsity are expressed in the distinction designated/undesignated. In chapter 2 of [Dummett, 1991], Dummett draws a distinction between *ingredient sense* and *assertoric content* of sentences. A distinction very similar to the difference of algebraic and logical values. Dummett's approach, however, is justified by the content of some utterances, which may be regarded as neither true nor false and, at the formal level, they are described by undetermined truth values only to attain an adequate functional description of the operators. In [Dummett, 1978], the author gives the following example:

> "I once imagined a case in which a language contained a negation operator '$\smile$' which functioned much like our negation save that it made '$\smile (A \to B)$' equivalent to '$A \to\smile B$', where $\to$ is the ordinary two-valued implication. In this case, the truth or falsity of '$\smile (A \to B)$' would not depend solely on the truth or falsity of '$A \to B$', but on the particular way in which '$A \to B$' was true (whether by the truth of both constituents or by the falsity of the antecedent). This would involve the use of three-valued truth tables, distinguishing two kinds of truth. In the same way, it might be necessary to distinguish two kinds of falsity."

L. Humberstone [Humberstone, 1998], explains that the difference between assertoric content and ingredient sense is introduced "in terms of the distinction between knowing the meaning of a statement in the sense of grasping the content of an assertion of it and in the sense of knowing the contribution it makes to determining the content of a complex statement in which it is a constituent." Sentences that are alike regarding the assertoric content (have the same designated value) are true under the same conditions. Therefore, despite sentences being equal regarding their assertoric content (in Suszko's terminology, the logical value), they differ relative to their ingrediente sense (the algebraic value).

As it was said before at the beginning of this section, Dana Scott is another author to make independent similar considerations on many-valuedness. In [Scott, 1974], after some biographical considerations about his relation with Tarski, in adressing the treatment that many-valued logic has received as the natural generalization of matrix semantics, Scott puts hte following question, "Just how many calculi do you want anyway? Multiple-valued logics had not found much of a foundational role, and there did not seem much point in creating new ones." One of his goals in the paper is to discuss whether the matrix method for many-valued logics is actually the correct generalization of the two-valued method, a debate he had partially begun to treat in [Scott, 1973]. In comparing $\{\mathbf{t}, \mathbf{f}\}$-valuations and many-valued truth tables, he remarks, "everyone can understand

$\{\mathbf{t}, \mathbf{f}\}$-valuations, but few — even the creators of the subject — can understand many-valued truth tables".

Aiming to discuss the notion of inference related to Łukasiewicz logic, Scott presents his criticism to many-valued logics in the following way[23]. Take, for instance, Łukasiewicz's 3 valued conjunction with values $\mathcal{V} = \{0, 1, 2\}$, $\mathcal{D} = \{1\}$ defined in the following way :

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

where 2 corresponds to *true* and 0 to *false*. However, in order to avoid calling them *values*, Scott advises to treat them as *types* of sentences. In this way, we could set $S = \{0, 1, 2\}$ and map the set of propositions to their respective types. Moreover, he considers that Łukasiewicz's intuitions about designation are vague and stresses "Is not the division of statements types into the designated and undesignated just a truth-valuation? Of course. So why not call it one?" In this way, there are many valuations that we could consider. We can show them in a table:

V	v_0	v_1
0	**f**	**f**
1	**f**	**t**
2	**t**	**t**

where $V = \{v_0, v_1\}$, consists of two valuations able to distinguish three types of sentences. Each valuation therefore represents one way of designating the elements. From this, we can define an entailment relation in the following way:

$$\Gamma \Vdash \alpha \text{ iff } \Gamma \models_V \alpha, \text{ for all } v \in V. \tag{1.6}$$

and it is straightforward to see that $\Vdash \alpha$ iff $v_0(\alpha) = \mathbf{t}$. For instance, the truth-conditions for conjunction can be described by using the valuations as:

$$v_i(\alpha \wedge \beta) = \mathbf{t} \text{ iff } v_i(\alpha) = \mathbf{t} \text{ and } v_i(\beta) = \mathbf{t} \tag{1.7}$$

$$v_i(\alpha \wedge \beta) = \mathbf{f} \text{ iff } v_i(\alpha) = \mathbf{f} \text{ or } v_i(\beta) = \mathbf{f} \tag{1.8}$$

[23] A first argument presented by Scott against Łukasiewicz's notion of many-valuedness is in claiming that Łukasiewicz's interpretation of truth-values as *probabilities* does not seem adequate since probabilities are not truth-functional, i.e, the probability of a compound proposition may not be a function of the probability of its parts.

where $0 \leq i < 2$. Here the numbers work only as the subscripts for the valuations. After showing this kind of construction, Scott then considers, "it seems much better to consider a variety of *valuations* rather than a variety of 'truth' *values*. Valuations use the ordinary truth-values, **t** and **f**, and sentential 'values' could creep in again (...) as types of sentences as distinguished by the valuations (...)". Scott's construction makes it possible to obtain a bivalent characterization of the original many-valued semantics[24]. This sets the foundations for his criticism to many-valued logics. In the next chapter, we use a similar approach to prove Suszko's reduction theorem, a result that establishes and reveals the bivalent character of many-valued Tarskian systems.

[24] For a more thoroughly discussion on Scott's ideas about consequence relations and many-valuedness, see [Humberstone, 2012].

Part II

Suszko's reduction: in the land of bivaluations

2 Exploring Suszko's reduction

Before starting to list the main points of this chapter, some remarks about the way we shall define logic and the understanding that shall drive the present study might be necessary. Such a conception of logic (see **Definition 5** below) is borrowed from Jean-Yves Béziau from what he baptized as Universal Logic. The history of Universal Logic, understood here as a general theory of logics and not as a universal system of logic (as some might believe) begins with Alfred Tarski. Tarski launched his theory of an abstract consequence operator with the intention of describing the processs of reasoning underlying the *methodology of deductive sciences*. Despite the influential character of Tarski's ideas on the abstract theory of logics, like the one pursued by Brown, Suszko and Bloom [Brown et al., 1973], some other abstract approaches were developed independently, like the idea of sequents by Hertz and Gentzen [1]. The importance of Tarski and the others lies in the fact that they created tools to investigate the notion of deduction apart from the traditional *Hilbert-style proof systems* [2]. Along with that development, several types of semantic tools, such as model theory and matrix theory, were developed to characterize the semantic notion of consequence subjacent to logical systems. This chapter is about how bivaluations are related to matrix theory as an adequate semantic tool to characterize Tarski's notion of consequence.

Bivaluations are not new animals in the logical zoo. We might say that they abound in the logical realm since its modern foundations and they appeared as a semantic tool with the creation of classical logic and its semantics. The aim of this chapter is for it to be an investigation about the range of applicability of bivaluations as an appropriate semantic tool for logical systems by analyzing their fundamental properties in comparison to the usual matrix semantics. The first section is a detailed exposition of the central result that led Suszko to formulate his ideas about algebraic and logical many-valuedness. The result was important in establishing the foundations for studies that sought to develop the theory of bivaluations as a general machinery to construct semantics for different logical systems. On that regard, an important step was taken by Newton da Costa and collaborators in [Loparic and da Costa, 1984], [da Costa and Béziau, 1994] in proposing what is known as the *Theory of Valuations*. Newton da Costa's theory of valuations was intended as an abstract bivalent framework for developing semantics for arbitrary logical systems. The second section aims to be an exposition of the major contributions made to the bivalent reduction procedure initiated by Suszko. Based on Caleiro & Marcos's reduction procedure, we will illustrate how to obtain a bivalent semantic characterization

1 Cf. [Beziau, 2005].

2 A Hilbert-style proof system can be described as a set of axioms endowed with a set of inference rules.

of Gödel's 3-valued logic. We also point out how the procedure can be applied to any finite-valued logic.

2.1 On the meaning of Suszko's reduction

This section introduces some of the main results that laid the foundations for Suszko's Thesis, as well as the major developments based on Suszko's ideas. We begin by presenting the technical machinery that will be useful to prove Suszko's fundamental result, the well-known *Suszko's Reduction theorem* – that states that every Tarskian logic may be characterized by a bivalent semantics. The theorem is responsible for showing that many-valued semantics may be recognized as bivalent semantics in disguise. Afterwards, we will present a sketch of how the technique for Suszko's reduction procedure was improved by Carlos Caleiro & João Marcos in order to extend its range of application and provide a recipe for carrying it out.

Fundamental concepts

We begin by introducing some basic terminology regarding the concept of an abstract algebra.

Definition 1. *An* ***algebraic type*** *is a pair* $\tau = \langle F, \rho \rangle$ *where* $F \neq \emptyset$ *is a set of symbols and* $\rho : F \to \mathbb{N}$ *is the arity function, the map that assigns an arity to each symbol in* F*.*

Definition 2. *An* ***algebra*** *of type* τ *is a structure* $\mathbb{A} = \langle \mathcal{A}, \mathcal{O} \rangle$ *where* $\mathcal{A} \neq \emptyset$ *is the domain of the algebra (the carrier set) and* $\mathcal{O} = \{f_i^{\mathbb{A}}\}_{i \in F}$ *such that for all* $i \in F$*:*

$$\textit{if } \rho(i) = n, \ \textit{then } f_i^{\mathbb{A}} : \mathcal{A}^n \to \mathcal{A} \tag{2.1}$$

Given two algebras $\mathbb{A}$ *and* $\mathbb{B}$*, we say they have the same type in case* $\tau(\mathbb{A}) = \tau(\mathbb{B})$*.*

Definition 3. *The notion of* ***homomorphism*** *between algebras is defined in the following way. Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be two algebras of the same type. We say that* $h : \mathcal{A} \to \mathcal{B}$ *is a homomorphism of* $\mathbb{A}$ *into* $\mathbb{B}$ *if and only if for every n-ary function* $f_i^{\mathbb{A}}$*, we have that*

$$h(f_i^{\mathbb{A}}(a_1, ..., a_n)) = f_i^{\mathbb{B}}(h(a_1), ..., h(a_n)) \tag{2.2}$$

The set of all homomorphisms of $\mathbb{A}$ *into* $\mathbb{B}$ *is denoted by* $Hom(\mathbb{A}, \mathbb{B})$*. If a homomorphism is from a given algebra* $\mathbb{A}$ *into itself* ($h : \mathcal{A} \to \mathcal{A}$)*, then it is called an* ***endomorphism****. The set of all endomorphisms on* $\mathbb{A}$ *is denoted by* $End(\mathbb{A})$*.*

Now we introduce our algebra of formulas, first let $At = \{p_1, p_2, ...\}$ be a denumerable set of atoms, and let $\Sigma = \{\Sigma_n\}_{n \in \mathbb{N}}$ be a propositional signature, where each element from Σ_n has $\rho(\phi) = n$.

Definition 4. *We define the set of formulas For as the algebra freely generated by At over* Σ*, where its set of operations is determined by the connectives from* Σ*.*

In what follows, we introduce the concept of an abstract logic following the approach initiated with Tarski and other logicians such as Bloom, Brown and Suszko [Brown et al., 1973] [3]. Here we depart a little from the perspective of Universal Logic since it is being required some structure on the set of formulas.

Definition 5. *We take an (abstract)* ***logic*** $\mathcal{L}$ *to be a pair* $\mathcal{L} = \langle For, \Vdash \rangle$*, where* $\Vdash$ *is a binary relation between sets of formulas and formulas of* For*. Such* $\Vdash$ *is called a* ***single-conclusion relation****. We also assume that* $\Vdash$ *is a non-trivial consequence relation, i.e, there is some formula* $\phi \in For$ *such that* $\nVdash \phi$*.*

In the following we introduce the notion of a Tarskian consequence relation:

Definition 6. *A* ***tarskian consequence relation*** *is a single-conclusion relation* $\Vdash$ *that has the following properties, for every* $\varphi \in For$ *and every* $\Delta, \Gamma \subseteq For$ *it has the following properties:*

$$\Delta \cup \{\varphi\} \Vdash \varphi \textit{ (Reflexivity)} \tag{2.3}$$

$$\textit{If } \Delta \Vdash \varphi \textit{ then } \Delta \cup \Gamma \Vdash \varphi \textit{ (Monotonicity)} \tag{2.4}$$

$$\textit{If } \Delta \Vdash \varphi \textit{ and for all } \delta \in \Delta,\ \Gamma \Vdash \delta,\ \textit{then } \Gamma \Vdash \varphi \textit{ (Cut for Sets)} \tag{2.5}$$

$$\Gamma \Vdash \varphi \textit{ implies that there is some } \Sigma \in Fin(\Gamma) \textit{ such that } \Sigma \Vdash \varphi \textit{ (Compactness)} \tag{2.6}$$

where $Fin(\Gamma) = \{\Sigma \subseteq \Gamma \mid \Sigma \textit{ is finite}\}$*. Given (Compactness) we may call such consequence relation a* ***finitary*** *Tarskian consequence relation.*

Definition 7. *A consequence relation is called* ***substitution-invariant*** *if the following holds:*

$$\Gamma \Vdash \alpha \textit{ implies } \sigma(\Gamma) \Vdash \sigma(\alpha), \textit{ for all } \sigma \in End(For) \textit{ (Substitution-invariance)} \tag{2.7}$$

where $\sigma(\Gamma) = \{\sigma(\gamma) | \gamma \in \Gamma\}$*.*

Any logic $\mathcal{L}$ endowed with a Tarskian consequence relation shall be called a **Tarskian logic**. In what follows, in order to associate a semantics to the logic $\mathcal{L}$ we define a matrix structure in the following way:

[3] Cf. [Jansana, 2011]

Definition 8. *We call a **logical matrix** an algebra* $\mathbb{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$*, where* $\mathcal{D}$ *is a non-empty proper subset of* $\mathcal{V}$ *and for every* n*-ary connective* c *from* Σ_n*,* $\mathcal{O}$ *includes a corresponding* n*-ary function* f_c*:* $\mathcal{V}^n \to \mathcal{V}$*.*

If the cardinality of $\mathcal{V}$ *is greater than 2, we call* $\mathbb{M}$ *a **many-valued matrix**. The elements of* $\mathcal{V}$ *are called **truth-values**, where the elements of* $\mathcal{D}$ *are **designated** truth values and the elements of* $\mathcal{U} = \mathcal{V} - \mathcal{D}$ *are called **undesignated**.*

Definition 9. *Fix a matrix* $\mathbb{M}$*. Any function* v *from* $\mathcal{L}$ *into* $\mathbb{M}$ *is called a **valuation**. Any set of valuations is called a* n***-valued semantics*** SEM*, where* n *is the cardinality of the set* $\mathcal{V}$*, hereafter denoted by* $|\mathcal{V}|$*. If all valuations are homomorphisms, i.e, respect the following condition for all* $c \in \Sigma_n$ *and* $f_c \in \mathcal{O}$*:*

$$v(c(\varphi_1, ..., \varphi_n)) = f_c(v(\varphi_1), ..., v(\varphi_n)) \tag{2.8}$$

then we say $\mathcal{L}$ *is a **truth-functional** logic.*

If a logic $\mathcal{L}$ is characterized by a n-valued semantics SEM, then we call $\mathcal{L}$ a **many-valued logic**.

Definition 10 (Tarskian entailment relation)**.** *After taking valuations into the semantics* SEM*, a **Tarskian entailment relation** (or semantic consequence) given by* $\models_{\text{SEM}} \subseteq \wp(For) \times For$ *associated to the semantics* SEM *is defined by saying that a formula* ϕ *follows from a set of formulas* Γ *whenever any valuation from* SEM *that assigns a designated value to all formulas from* Γ *also assigns a designated value to* ϕ*. This amounts to:*

$$\Gamma \models_{\text{SEM}} \phi \textit{ iff } v(\Gamma) \subseteq \mathcal{D} \textit{ implies } v(\phi) \in \mathcal{D}, \textit{ for every } v \in \text{ SEM} \tag{2.9}$$

By the above definition, it is possible to prove the following fact:

Remark 1. *Every Tarskian entailment relation has the properties of a Tarskian consequence relation.*

Proof. For (Reflexivity), suppose by reductio that $\alpha \nvDash \alpha$. By definition of Tarskian entailment, there is a valuation $v \in$ SEM such that $v(\alpha) \in \mathcal{D}$ and $v(\alpha) \notin \mathcal{D}$. However, since all valuations are functions, the same valuation can not assign two different values to a given a formula, therefore our desired conclusion follows.

For the case of (Monotonicity), suppose, by contraposition, that $\Delta \cup \Gamma \nvDash_{\text{SEM}} \alpha$. Therefore, from the definition of Tarskian entailment, it follows there is a valuation $v \in$ SEM such that (i) $v(\Delta \cup \Gamma) \subseteq \mathcal{D}$ and (ii) $v(\alpha) \notin \mathcal{D}$. Then, since $\Delta \subseteq \Delta \cup \Gamma$, from (i) we shall conclude that (iii) $v(\Delta) \subseteq \mathcal{D}$. Therefore, from (iii), (ii) and definition of a Tarskian entailment, it follows that $\Delta \not\models_{\text{SEM}} \alpha$.

To prove (Cut for sets), assume by reductio that (i) $\Delta \models_{\text{SEM}} \alpha$, (ii) $\Gamma \models_{\text{SEM}} \delta$ for all $\delta \in \Delta$, and (iii) $\Gamma \not\models_{\text{SEM}} \alpha$. From (iii) and definition of Tarskian entailment, it follows that there is a valuation $v \in \text{SEM}$ such that (iv) $v(\Gamma) \subseteq \mathcal{D}$ and (v) $v(\alpha) \notin \mathcal{D}$. But from (i) and the definition of Tarskian entailment, we know that for every valuation $v \in \text{SEM}$, if $v(\Gamma) \subseteq \mathcal{D}$, then $v(\delta) \in \mathcal{D}$ for all $\delta \in \Delta$. Moreover, from (ii) and the definition of Tarskian entailment, it follows that for every $v \in \text{SEM}$, $v(\Delta) \subseteq \mathcal{D}$ implies $v(\alpha) \in \mathcal{D}$. Therefore since we do have $v(\Gamma) \subseteq \mathcal{D}$ by (iv), it follows that $v(\alpha) \in \mathcal{D}$, what contradicts (v).

For (Substitution-invariance), assume by reductio that (i) $\Gamma \models_{\text{SEM}} \alpha$ and (ii) $\sigma(\Gamma) \not\models_{\text{SEM}} \sigma(\alpha)$. From (ii) and definition of Tarskian entailment, it follows that (iii) there is a valuation $v \in \text{SEM}$ such that $v(\sigma(\Gamma)) \subseteq \mathcal{D}$ and (iv) $v(\sigma(\alpha)) \notin \mathcal{D}$. But from (i), we know that (v) for every valuation $v \in \text{SEM}$, $v(\Gamma) \subseteq \mathcal{D}$ implies $v(\alpha) \in \mathcal{D}$. Since σ is an endomorphism, from (iii) and (iv), we may conclude that $v(\Gamma) \subseteq \mathcal{D}$ and $v(\alpha) \notin \mathcal{D}$, what contradicts (v).

□

We shal define a **s-logic** as a structure $\mathcal{L} = \langle For, \models_{\text{SEM}} \rangle$, where $\models_{\text{SEM}}$ is a Tarskian entailment relation. Given a logic $\mathcal{L}_1 = \langle For, \Vdash \rangle$ and a s-logic $\mathcal{L}_2 = \langle For, \models_{\text{SEM}} \rangle$, we shall say that $\mathcal{L}_2$ is **sound** with respect to $\mathcal{L}_1$ in case $\Vdash \subseteq \models_{\text{SEM}}$; and $\mathcal{L}_1$ is **complete** with respect to $\mathcal{L}_2$ in case $\models_{\text{SEM}} \subseteq \Vdash$. Moreover, if $\Vdash = \models_{\text{SEM}}$, then we say $\mathcal{L}_2$ is an **adequate** semantics for $\mathcal{L}_1$.

In the following, we prove two auxiliary remarks that shall be useful in proving the main theorems of this section:

Consider a family of logics $\mathcal{F} = \{\mathcal{L}_i\}_{i \in I}$ over some fixed set of formulas For. Define the *superlogic* $\mathcal{L}_{\mathcal{F}}$ of this family by taking $\bigcap_{i \in I} \mathcal{L}_i$, namely, $\mathcal{L}_{\mathcal{F}} = \langle For, \bigcap_{i \in I} \Vdash_i \rangle$, where each $\mathcal{L}_i = \langle For, \Vdash_i \rangle$ is a structural Tarskian logic, for $i \in I$.

Remark 2. *The intersection of substitution-invariant Tarkian logics is a substitution-invariant Tarskian logic.*

Proof. Assume $\mathcal{L}_{\mathcal{F}}$ is not substitution-invariant. Then there are formulas $\Gamma \cup \{\alpha\} \subseteq For$ such that (i) $\Gamma \Vdash_{\mathcal{F}} \alpha$ and (ii) $\sigma(\Gamma) \nVdash_{\mathcal{F}} \sigma(\alpha)$ are the case. From (ii), by definition of $\mathcal{L}_{\mathcal{F}}$, we know there is some $\Vdash_k$ $k \in I$, such that (iii) $\sigma(\Gamma) \nVdash_k \sigma(\alpha)$. However, we know that each $k \in I$ is a substitution-invariant Tarskian logic. Therefore, by contraposition and the definition of substitution-invariance in (iii), we have $\Gamma \nVdash_k \alpha$. At last, by definition of $\mathcal{L}_{\mathcal{F}}$, we shall get $\Gamma \nVdash_{\mathcal{L}_{\mathcal{F}}} \alpha$, what is a contradiction with (i).

Assume (Cut for sets) does not hold for $\mathcal{L}_{\mathcal{F}}$. Then there are formulas Δ, Γ, ϕ such that (i) $\Delta \Vdash_{\mathcal{F}} \phi$, (ii) $\Gamma \Vdash_{\mathcal{F}} \delta$, for all $\delta \in \Delta$, and (iii) $\Gamma \nVdash_{\mathcal{F}} \phi$ are the case. From (iii) and definition of $\mathcal{L}_{\mathcal{F}}$, it follows that there is some $k \in I$ such that (iv) $\Gamma \nVdash_k \phi$. However, we know that each $k \in I$ is a Tarskian logic and, by definition of a Tarskian consequence relation, satisfy (Cut for sets). Therefore, by contraposition it follows that either $\Delta \nVdash_k \phi$ or $\Gamma \nVdash_k \delta$, for some $\delta \in \Delta$ are the case. By both cases, by definition of $\mathcal{L}_{\mathcal{F}}$, we shall have a contradiction with (i) or (ii).

Note that analogous reasoning may be employed to prove the other properties of (Reflexivity) and (Monotonicity).

□

While Remark 2 guarantee that the construction of the superlogic inherit all properties of a substitution-invariant Tarskian logic, Remark 3 below guarantee that the procedure used to construct the semantics of a super s-logic is determined by taking the intersection of all inferences from each logic of the index set I.

Remark 3. *The intersection of inferences from a set of entailment relations is equivalent to the relation determined by the union of all their distinct semantics, i.e,*

$$\bigcap_{i \in I} \models_{\mathrm{SEM}^i} = \models_{\bigcup_{i \in I} \mathrm{SEM}^i}$$

Proof. (From r.h.s to l.h.s.)

Take $\langle \Gamma, \phi \rangle \in \bigcap_{i \in I} (\models_{\mathrm{SEM}^i})$. Thus $\langle \Gamma, \phi \rangle \in \models_{\mathrm{SEM}^i}$ for all $i \in I$. It follows that for all $i \in I$ we have (i) $\Gamma \models_{\mathrm{SEM}^i} \phi$, i.e, for all $v \in \mathrm{SEM}^i$, $v(\Gamma) \subseteq \mathcal{D}^i \Rightarrow v(\phi) \in \mathcal{D}^i$. Now fix $v \in \bigcup_{i \in I} \mathrm{SEM}^i$, i.e, $v \in \mathrm{SEM}^i$ for some $i \in I$. Suppose $v(\Gamma) \subseteq \mathcal{D}^i$. By (i), it follows that $v(\phi) \in \mathcal{D}^i$. Therefore, by definition of entailment and the fact that v and i are arbitrary, we have $\Gamma \models_{\bigcup_{i \in I} \mathrm{SEM}^i} \phi$. Thus $\langle \Gamma, \phi \rangle \in \models_{\bigcup_{i \in I} \mathrm{SEM}^i}$.

The proof from l.h.s. to r.h.s. follows in a similar manner.

□

In what follows, we define an an algebraic structure introduced by A. Lindenbaum, the so-called *Lindenbaum matrix*. The main difference of the Lindenbaum matrix to the usual notion of a logical matrix (as in **Definition 8**) is that, in the Lindenbaum matrix we take the very objects in the algebra of formulas *For* as the set of truth values. Then, we use the closure of a given subset of *For* to determine the filter (the designated set of values) of the matrix. In this case, since we are working with the same set of objects and treating them as formulas and truth-values, the semantics of a Lindenbaum matrix is given by a set of endomorphisms. We let the **closure** of a given set of formulas $\Gamma \subseteq For$ be defined as $\Gamma^{\Vdash} = \{\alpha \mid \Gamma \Vdash \alpha\}$

Definition 11. *Given a logic* $\mathcal{L} = \langle For, \Vdash \rangle$*, the closure of a given set of formulas* $\Gamma \subseteq For$ *determines a matrix of the form*

$$\mathbb{M}_\Gamma = \langle For, \Gamma^{\Vdash}, \mathcal{O} \rangle \tag{2.10}$$

called a ***Lindenbaum matrix*** *of* $\mathcal{L}$*. The class of all lindenbaum matrices induced by* For*, that is,* $\{\mathbb{M}_\Gamma : \Gamma \subseteq For\}$ *define what is called the* ***Lindenbaum bundle*** *of* $\mathcal{L}$*.*

From Definition 10 and 11, note that every lindenbaum matrix of a logic $\mathcal{L}$ induces a s-logic $\mathcal{L}_\Gamma = \langle For, \models_\Gamma \rangle$, where $\Delta \models_\Gamma \alpha$ holds if $v(\Delta) \subseteq \Gamma^{\Vdash}$ implies $v(\alpha) \in \Gamma^{\Vdash}$, for all $v \in$ SEM. Therefore, the s-logic $\mathcal{L}_\Gamma$ is determined by the set of valid inferences (the closure set) of Γ. Note, moreover, that by the above definition, based on a Lindenbaum bundle of a given logic $\mathcal{L}$ it is possible to define a super-s-logic to which both Remark 3 and 2 apply.

The forthcoming results are divided into three main theorems. We first shall prove the so-called Wójcicki's reduction, the theorem that states that every tarskian logic is many-valued, such a result was first proved, in its restricted form, by Wójcicki in [Wójcicki, 1970]. After that, we will make use of Wójcicki's reduction to prove Suszko's Reduction, the theorem that states that every many-valued logic can be characterized by a bivalent semantics. The general idea is to show how Suszko's reduction procedure may be applied to a many-valued semantics in order to produce a sound and complete bivalent semantics for it. After that, we shall discuss the fact that Suszko's reduction does not give us a general algorithmic procedure to build the bivalent characterization of a given finite-valued semantics, it only gives us a hint about where on the road we should keep the eye. Thus, in the next section we shall exhibit an improvement of such procedure due to [Caleiro et al., 2007].

For now, we will show Wójcicki's reduction and prove it in a restricted form only for substitution-invariant tarskian logics and in a generalized form for logics without substitution-invariance by following the construction due to [Caleiro et al., 2007] and [Marcos, 2004].

In the following, for the sake of simplicity and convenience to prove the result, based on Definition 12 and the s-logic generated by a Lindenbaum matrix, we use an auxiliar definition:

Definition 12. *Given a logic* $\mathcal{L} = \langle For, \Vdash \rangle$ *and a set of formulas* $\Gamma \subseteq For$*, each Lindenbaum matrix induces a s-logic* $\mathcal{L}_\Gamma = \langle For, \models_\Gamma \rangle$ *such that:*

$$For = \mathcal{V} \tag{2.11}$$

$$\Gamma^{\Vdash} = \mathcal{D} \tag{2.12}$$

$$\Delta \models_{\Gamma} \alpha \textit{ iff } g(\Delta) \subseteq \Gamma^{\Vdash} \textit{ implies } g(\alpha) \in \Gamma^{\Vdash}, \textit{ for all } g \in \text{SEM} \tag{2.13}$$

where each $g : At \to For$ is a ***uniform-substitution****. We write $\phi(p_1, ..., p_n)$ to denote a formula ϕ whose atoms appear among $p_1, ..., p_n$. We use $g(\phi(p_1, ..., p_n))$ to denote the result of applying a substitution g to a formula ϕ by simultaneously replacing the formulas $p_1, ..., p_n$ by $g(p_1), ..., g(p_n)$. We denote by $g(\Gamma)$ the set resulting of applying a substitution $g(\gamma)$ to each $\gamma \in \Gamma$.*

Since the *Lindenbaum bundle* is generated as the collection of all Lindenbaum matrices relative to the set of formulas For. By Definition 13, the lindenbaum bundle will be the Lindenbaum superlogic generated by the intersection of each Lindenbaum s-logic. Note that the semantics of each Lindenbaum s-logic is defined by a collection of substitutions g. Moreover, Remarks 1 and 2 guarantee that the superlogic generated by the Lindenbaum bundle of a given set of formulas For inherit the properties of each Lindenbaum matrix.

The result depends on two important facts about the notion of closure: Fact 1: $\Gamma^{\Vdash} \Vdash \alpha \Leftrightarrow \Gamma \Vdash \alpha$ and Fact 2: For any tarskian logic $\mathcal{L} = \langle For, \Vdash \rangle$, where For is a denumerable set of formulas endowed with a tarskian consequence relation, given arbitrary $\Sigma \cup \Delta \cup \{\varphi\} \subseteq For$, to check whether $\Sigma, \Delta \Vdash \varphi$ hold is equivalent to checking whether $(\forall \delta \in \Delta)\Sigma \Vdash \delta$ implies $\Sigma \Vdash \varphi$.

Fact 1: $\Gamma^{\Vdash} \Vdash \alpha \Leftrightarrow \Gamma \Vdash \alpha$

Proof. Assume $\Gamma \Vdash \alpha$. By definition of closure, we have $\alpha \in \Gamma^{\Vdash}$. Therefore, by (Reflexivity), $\Gamma^{\Vdash} \Vdash \alpha$.
Now assume (i) $\Gamma^{\Vdash} \Vdash \alpha$. By definition of closure, we have (ii) $\Gamma \Vdash \gamma$, for all $\gamma \in \Gamma^{\Vdash}$. By (Cut for sets) in (i) and (ii), we have $\Gamma \Vdash \alpha$.

□

Fact 2: To check whether $\Sigma, \Delta \Vdash \varphi$ hold is equivalent to checking whether $(\forall \delta \in \Delta)\Sigma \Vdash \delta$ implies $\Sigma \Vdash \varphi$

Proof. Take as a premise that $(\forall \delta \in \Delta)\Sigma \Vdash \delta$ implies $\Sigma \Vdash \varphi$. Now suppose (i) $(\forall \delta \in \Delta)\Sigma \Vdash \delta$. From our premise and (i), we may conclude $\Sigma \Vdash \varphi$. Therefore, by using (Monotonicity), we have $\Sigma, \Delta \Vdash \varphi$.
Now take $\Sigma, \Delta \Vdash \varphi$ as a premise and suppose $(\forall \delta \in \Delta), \Sigma \Vdash \delta$. By using (Cut for sets) in our premise and our supposition, we have $\Sigma \Vdash \varphi$. □

From the Facts above and the properties of a Tarskian consequence relation, we prove Wójcicki's Reduction:

Theorem 2.1.1 (Wójcicki's reduction)**.** *Every substitution-invariant Tarskian logic has an adequate semantics.*

Proof. The proof amounts to showing the adequacy between a Tarskian logic and the superlogic generated by its set of formulas. Take a tarskian logic $\mathcal{L} = \langle For, \Vdash \rangle$, fix the Lindenbaum bundle generated by For and let $\mathcal{L}_{\mathcal{F}} = \langle For, \models_{\mathcal{F}} \rangle$ be defined as its associated superlogic.

For **Completeness**, suppose $\Delta \models_{\mathcal{F}} \alpha$. From the definition of $\models_{\mathcal{F}}$, we have (i) $\Delta \models_{\Gamma} \alpha$, for every $\Gamma \subseteq For$. Now suppose $\Delta \subseteq \Gamma^{\Vdash}$. From (i) and Fact 2 about the definition of closure, we have (ii) $(\forall \delta \in \Delta)\Gamma^{\Vdash} \Vdash \delta$ implies $\Gamma^{\Vdash} \Vdash \alpha$. Given (ii), we may apply Fact 1 about closure and conclude (iii) $(\forall \delta \in \Delta)\Gamma \Vdash \delta$ implies $\Gamma \Vdash \alpha$. But we know by Fact 2 that to check (iii) is equivalent to checking $\Gamma, \Delta \Vdash \alpha$. For the particular case in which $\Gamma = \emptyset$, follows the desired conclusion.

For **Soundness**, fix some $\Gamma \subseteq For$ and its associated Lindenbaum s-logic $\mathcal{L}_{\Gamma} = \langle For, \models_{\Gamma} \rangle$. Now suppose (i) $\Delta \Vdash \alpha$ for $\Delta \subseteq \Gamma$ and take (ii) some $g \in$ SEM such that $g(\Delta) \subseteq \mathcal{D}$. Note each $g \in SEM$ is an endomorphism $\sigma \in End(For)$. By substitution-invariance in (i), it follows that (iv) $g(\Delta) \Vdash g(\alpha)$. By (Reflexivity) and the fact that all g is an endomorphism, it follows (v) $\Delta \Vdash g(\delta)$, for all $g(\delta) \in g(\Delta)$. Finally, by (Cut for sets) in (iv) and (v), we shall have $\Delta \Vdash g(\alpha)$. Then, by the definition of closure, it follows $g(\alpha) \in \Delta^{\Vdash}$ and since $\Delta^{\Vdash} \subseteq \Gamma^{\Vdash}$[4], by Definition 12, it follows $g(\alpha) \in \mathcal{D}$. Thus, we have our desired conclusion $\Delta \models_{\Gamma} \alpha$. By construction of $\models_{\mathcal{F}}$ and Remark 3, we shall obtain $\Delta \models_{\mathcal{F}} \alpha$.

□

Corollary 2.1.1.1. *Every substitution-invariant Tarskian logic is n-valued, for* $n = |For|$.

Proof. By the definition of the Lindenbaum matrix, since we have the set For as our set of truth values. □

Corollary 2.1.1.2. *Every substitution-invariant Tarskian logic has a characteristic set of matrices.*

Proof. Trivial. □

An important result related to Corollary 2.1.1.2 was proved in [Shoesmith and Smiley, 1971], where the authors proved that a finitary Tarskian logic is characterized by a single matrix

4 Since $\Delta \subseteq \Gamma$ and (Motonicity).

if the following property holds:

$$\text{If } \Gamma, \Delta \Vdash \phi, Atom(\Gamma \cup \{\phi\}) \cap Atom(\Delta) = \emptyset, \text{ and } \Delta \text{ is consistent then } \Gamma \Vdash \phi \text{ (Uniformity)} \tag{2.14}$$

where consistency means that $\Delta \nVdash \psi$, for some $\psi \in For$. [Marcos, 2009] highlights (Uniformity) as the abstract characteristic property of truth-functional logics. In [Humberstone, 1998], the author calls a *broad many-valued* logic any logic characterized by a set of matrices, while a logic is called *narrow many-valued* if it is characterized by a single matrix. Thus, Corollary 2.1.1.2 shows that every substitution-invariant Tarskian logic is broadly many-valued. As it will be shown, Suszko's Reduction can be applied to many-valued logics on the broad and the narrow sense.

In the following, we prove the generalized version of Wójcicki's reduction due to [Caleiro et al., 2007] and [Marcos, 2004]. Such version of Wójcicki's reduction does not depend on substitution-invariance nor on the algebraic structure of the set of formulas. Their proposal follows the perspective inherited from the so-called Universal Logic.

In what follows, fix some logic $\mathcal{L} = \langle For, \Vdash \rangle$. By making use of Definition 12 and the properties of a Tarskian consequence relation, it is possible to prove the following lemma:

Lemma 2.1.1. *Any matrix from the Lindenbaum bundle is sound for a Tarskian logic.*

Proof. Once we fixed a logic $\mathcal{L} = \langle For, \Vdash \rangle$, take some arbitrary Lindenbaum matrix $\mathcal{L}_\Gamma$, where $\Gamma \subseteq For$, and assume (i) $\Delta \Vdash \alpha$. Now suppose $g(\Delta) \subseteq \mathcal{D}$. Then, by Definition 12-(2.12) above, we have $\Delta \subseteq \Gamma^{\Vdash}$. By (Reflexivity), we get (ii) $(\forall \beta \in \Delta)\Gamma^{\Vdash} \Vdash \beta$. From this, applying (Cut for sets) at (i) and (ii), we get $\Gamma^{\Vdash} \Vdash \alpha$. Therefore from Fact (i), it follows $\Gamma \Vdash \alpha$. Thus $\alpha \in \Gamma^{\Vdash}$ follows by definition of closure. From Definition 12-(2.12), we have $g(\alpha) \in \mathcal{D}$. Finally, since the Lindenbaum matrix comes endowed with a Tarkian relation of entailment, it follows that $\Delta \models_\Gamma \alpha$. □

Lemma 2.1.1 have shown us that for any inference $\Delta \Vdash \alpha$ from a Tarskian logic, there is a Lindenbaum matrix sound for it[5]. From Lemma 2.1.1 and the properties of a Tarskian consequence relation, it is possible to prove the following result with respect to any Tarskian logic[6]:

Theorem 2.1.2 (Generalized Wójcicki's reduction)**.** *Every Tarskian logic is n-valued, for $n = |For|$.*

[5] We have shown that $\Delta \Vdash \alpha \Rightarrow \Delta \models_\Gamma \alpha$, for every $\Gamma \subseteq For$.

[6] Some authors, for instance [Font, 2009], call this structure a pre-logic because of the absence of substitution-invariance.

Proof. Fix a tarskian logic $\mathcal{L} = \langle For, \Vdash \rangle$, take the Lindenbaum bundle generated by its set of formulas and define the logic $\mathcal{L}_{\mathcal{F}} = \langle For, \models_{\mathcal{F}} \rangle$, where $\models_{\mathcal{F}}$ is a Tarskian entailment relation defined as $\bigcap_{\Gamma \subseteq For} \models_{\Gamma}$. To prove Wójcicki's reduction, our goal now is to show soundness and completeness with respect to our fixed Tarskian logic $\mathcal{L}$ and the s-logic $\mathcal{L}_{\mathcal{F}}$.

Soundness: Using Lemma 2.1.1 and the definition of $\mathcal{L}_{\mathcal{F}}$, we have $\Vdash \subseteq \models_{\mathcal{F}}$.

Completeness: Suppose $\Delta \models_{\mathcal{F}} \alpha$. From the definition of $\models_{\mathcal{F}}$, we have (i) $\Delta \models_{\Gamma} \alpha$, for every $\Gamma \subseteq For$. Now suppose $\Delta \subseteq \Gamma^{\Vdash}$. From (i) and Fact 2 about the definition of closure, we have (ii) $(\forall \delta \in \Delta)\Gamma^{\Vdash} \Vdash \delta$ implies $\Gamma^{\Vdash} \Vdash \alpha$. Given (ii), we may apply Fact 1 about closure and conclude (iii) $(\forall \delta \in \Delta)\Gamma \Vdash \delta$ implies $\Gamma \Vdash \alpha$. But we know by Fact 2 that to check (iii) is equivalent to checking $\Gamma, \Delta \Vdash \alpha$. For the particular case in which $\Gamma = \emptyset$, follows the desired conclusion.

□

In order to prove the second theorem, we begin by introducing some other technical tools. Based on Definition 9, let us first denote a n-valued semantics in the following way $\text{SEM} = \{v_i \mid v_i : For \to \mathcal{V}_n, \text{ where } i \in I\}$ given some appropriate index set I, For is the set of formulas of a given logic and $\mathcal{V}_n$ is a set of truth-values with n denoting its cardinality. We shall write SEM_n to denote a many-valued semantics. We define also the function $t : \mathcal{V}_n \to \mathcal{V}_2$, where $\mathcal{V}_2 = \mathcal{D}_2 = \{T\} \cup \mathcal{U}_2 = \{F\}$, in the following manner:

$$t(x) = \begin{cases} T, & \text{if } x \in \mathcal{D} \\ F, & \text{if } x \in \mathcal{U} \end{cases}$$

Then, given a function v and t, we can define a *bivaluation* $b^v = t \circ v$ and collect such bivaluations into the semantics $\text{SEM}_2 = \{b^v \mid v \in \text{SEM}\}$. Hereafter, following [Font, 2009], we shall call any set of bivaluations defined in this general sense a *Suszko's semantics*. The proof of the next theorem makes use of the fact that, since a n-valued semantics is defined as the set of valuations from the set of formulas into the set of truth-values, by composing such valuations with our function t, we can map each designated value to T and each antidesignated value to F. Such composite functions are collected into the semantics SEM_2. Thus, we have a bivalent reduction of the original semantics. Of course, such a procedure must carry some undesired consequences, and those shall be commented upon in the next section. It is important to remark that such construction was exhibited by Suszko already in [Suszko, 1975b]. At last, by making use of Theorem 2.1.2, our goal is to prove the following:

Theorem 2.1.3 (Suszko's reduction)**.** *Every Tarskian logic is* 2*-valued.*

In order to prove Theorem 2.1.3, first note that it reduces to showing that:

$$\Gamma \models_{\text{SEM}_n} \alpha \text{ sse } \Gamma \models_{\text{SEM}_2} \alpha.$$

Proof. (From r.h.s to l.h.s.)

By contraposition, assume $\Gamma \nvDash_{\text{SEM}_n} \alpha$. The definition of entailment implies the existence of a valuation $v \in \text{SEM}_n$ such that $v(\Gamma) \subseteq \mathcal{D}$ and $v(\alpha) \in \mathcal{U}$. Composing t with such valuation will give us $t(v(\Gamma)) \subseteq \{T\}$ and $t(v(\alpha)) = F$. Therefore, there is a bivaluation $b^v \in \text{SEM}_2$ such that $b^v(\Gamma) \subseteq T$ and $b^v(\alpha) = F$. Then, applying the definition of entailment again will give our desired result $\Gamma \nvDash_{\text{SEM}_2} \alpha$.

(From l.h.s to r.h.s.)

By contraposition, assume $\Gamma \nvDash_{\text{SEM}_2} \alpha$. That guarantees the existence of a valuation b such that $b(\Gamma) \subseteq T$ and $b(\alpha) = F$. By definition of SEM_2, there must exist some $v \in \text{SEM}_n$ such that $b = b^v$. So, by definition of b^v, we know that $t(v(\Gamma)) \subseteq \{T\}$ and $t(v(\alpha)) = F$. Therefore, by definition of t, it follows $v(\Gamma) \subseteq \mathcal{D}$ and $v(\alpha) \in \mathcal{U}$, what will guarantee our desired result $\Gamma \nvDash_{\text{SEM}_n} \alpha$. □

Suszko's claims find particular support in Suszko's reduction. According to Suszko, this particular result shows that the Tarskian notion of entailment hides a bivalent character at the division of truth-values in designated and undesignated. However simple and straightforward the bivalent reduction procedure proposed by Suszko may appear, it does not give us any clue about how to construct an adequate bivalued semantics given a many-valued one. Despite Theorem 2.1.3 ensuring the *existence* of an adequate two-valued semantics for any many-valued semantics, the procedure for transforming a many-valued characterization of a semantics into a two-valued one is not at all constructive. In the next section we shall comment some developments originated from such a problem.

2.2 Improving Suszko's reduction

Suszko's reduction, as well as the considerations made by Scott in [Scott, 1974] and exhibited in Section 1.5, gave birth to the systematic study of the relations between bivaluations and other kinds of semantics. Moreover, Newton da Costa et al in [Loparic and da Costa, 1984] proposed a general theory of bivaluations as a general semantic tool for logical systems. The first systematic study of bivaluations and their relation to the abstract perspective of Universal Logic was made by Jean-Yves Béziau in [Béziau, 1998]. The importance of Béziau's approach was in finding the adequate set of bivaluations necessary to characterize a Tarskian operator. Moreover, the abstract setting pursued by him had the importance of revealing that logical two-valuedness does not depend on substitution-invariance since, in contrast to what was pointed out by Marcelo Tsuji in [Tsuji, 1998, p.308], "we may say that Suszko thought that the key to logical two-valuedness rested in the substitution-invariance of the abstract logics".

Although simple as Suszko's reduction may appear, it raises the question of how, given a finite-valued semantic, could we produce the bivalent characterization of it. In [Suszko, 1975b], Suszko presented a sketch of how to obtain a bivalent description of Łukasiewicz three-valued logic. This same procedure was improved in [Malinowski, 1993] by making use of Rosser-Turquette functions. In [Malinowski, 1993], despite presenting the bivalent description of Łukasiewicz's three-valued logic, Malinowski does not exhibit a general procedure behind the result. It is important to remark that the procedure of taking a many-valued semantic and transforming it into a bivalent one is not immune to undesirable consequences. The first of these consequences obtained via the procedure used in Theorem 2.1.3 is that the logic in hand shall lose its truth-functionality.

Now consider Gödel's three-valued logic, which can be formulated by way of:

$$G_3 = \langle \mathcal{V}_3, \mathcal{D}_3, \{f_\neg, f_\to\}\rangle.$$

where $\mathcal{V}_3 = \{0, \frac{1}{2}, 1\}$ and $\mathcal{D}_3 = \{1\}$. The operations over the truth-values can be defined by $f_\to(v(\alpha), v(\beta)) = 1$, if $v(\alpha) \leq v(\beta)$ and $f_\to(v(\alpha), v(\beta)) = v(\beta)$, if $v(\alpha) > v(\beta)$; $f_\neg(v(\alpha)) = 1$, if $v(\alpha) = 0$, and $f_\neg(v(\alpha)) = 0$ otherwise. What give us the following truth-tables:

$\to$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	0	1	1
1	0	$\frac{1}{2}$	1

	$\neg$
0	1
$\frac{1}{2}$	0
1	0

We present also the rules of formation of the set of formulas For of G_3, and its associated measure of complexity of the formulas:

Definition 13. *The set of formulas For is inductively defined in the following way:*
(1) $p \in For$, *for every* $p \in Atom$.
(2) $\phi \in For$, *then* $\neg\phi \in For$.
(3) ϕ, $\psi \in For$, *then* $(\phi \to \psi) \in For$.

Definition 14. *We define the measure of complexity of the formulas of* G_3 *in the following way:*
Let the function $\ell : For \to \mathbb{N}^+$ *be defined as:*
$\ell(\alpha) = 1$, *onde* $\alpha = p$, *tal que* $p \in Atom$.
$\ell(\neg\alpha) = 1 + \ell(\alpha)$.
$\ell(\otimes(\alpha_1, ..., \alpha_n)) = 1 + \ell(\alpha_1) + ... + \ell(\alpha_n)$, *for* $\otimes \in \{\to\}$.

Furthermore, we shall write $\text{SEM}_{G_3} = \{v_i\}_{i \in I}$ to denote the set of valuations from For into $\mathcal{V}_3$. Note that if we use the function $t(x)$, as defined in Section 2.1, over our SEM_{G_3} valuations, for each valuation v we get the following bivaluation $b_v : For \to \{T, F\}$:

$v(\phi)$	$b_v(\phi)$
0	F
$\frac{1}{2}$	F
1	T

Thus, the bivaluations b_v cannot see the difference between the values 0 and $\frac{1}{2}$ and identify them by the same value F. For instance, in the case of negation, if $v(\phi) = \frac{1}{2}$, then $b_v(\phi) = F$ and $b_v(\neg\phi) = F$. However, if $v(\phi) = 0$, then $b_v(\phi) = F$ and $b_v(\neg\phi) = T$. From this, we see that in the bivalent setting acquired from Suszko's reduction we can not compute the value of $\neg\phi$ by just knowing the value of ϕ. Thus implying the lost of truth-functionality. Such a phenomena is a direct consequence of the fact that the function $t(x)$ assign T to all designated values and F to all undesignated values.

Relative to that matter, several important contributions were achieved by Carlos Caleiro, Walter Carnielli, Marcelo Coniglio and João Marcos. In [Caleiro et al., 2003] and [Caleiro et al., 2007], the authors devised an effective algorithmic procedure able to reduce any finite-valued semantic into a bivalent one. Their method uses the very primitive linguistic resources of the logic or a conservative extension thereof to construct a sufficiently expressive language capable of distinguishing each 'algebraic' value within the bivalent semantics. Thus, it shows how any finite many-valued algebraic semantics may be described in terms of only truth and falsehood.

To avoid the problem concerning the b_v valuation of referring to all undesignated values by F and to all designated values by T, thus losing the information about the functional character of the original semantics, Caleiro et al's method establish a way of characterizing each truth-value in a unique form. The core idea is to use the linguistic resources of the logic to *separate* the truth-values located in the same set by the b_v valuations. In our G_3 case, we only need a formula capable of separating 0 and $\frac{1}{2}$ once they have been identified to a single reference F. In the following, we exhibit how Caleiro et al's procedure works for our logic G_3[7].

First, we define a **separator** formula in the folllowing way:

Definition 15 ([Caleiro et al., 2007])**.** *Given $x, y \in \mathcal{V}_3$, we write $x\#y$ and say that x and y are **separated** in case one is designated and the other is undesignated. We say*

[7] It is important to remark that our only goal here is to exhibit how the procedure works with a concrete example. It is not in the scope of the work to work out all the proofs and constructions in [Caleiro et al., 2003].

that a one-variable formula $\theta(p)$ of For ***distinguishes*** *two truth-values x and y if $v(\theta(x))\#v(\theta(y))$. A logic is called* ***separable*** *in case its truth-values are pairwise distinguishable*[8].

In [Caleiro et al., 2013], Proposition 2.9 guarantees that if the original finite-valued logic does not have a separator, then there is a conservative extension of it that does. [9] In case of G_3, we need a separator for the values 0 and $\frac{1}{2}$, as it is possible to note in the following table:

ϕ		$\theta(\phi)$	
0	F	1	T
$\frac{1}{2}$	F	0	F
1	T	0	F

In [Caleiro et al., 2013], the authors describe the algorithm to find an adequate separator for a given a logic in the following way:

1 Decide whether the logic in question is separable. To decide whether a given n-valued logic $\mathcal{L}$ is separable, it suffices to compute the set of all unary functions $f : \mathcal{V}_n \to \mathcal{V}_n$ that are definable by the connectives in Σ.

2 Test each definable function on the pair of values that need separation. The definable truth-functions are precisely those that can be expressed by a formula written with at most one variable.

3 Choose one of the formulas capable of separating the truth-values and use it as the separator.

4 If there are no definable function capable of separating the pair of values that need separation, then build a conservative extension[10].

By carrying the first step of the procedure describe above, since G_3 has three truth-values, we shall obtain the following 27 possible unary functions:

[8] In [Caleiro et al., 2003], the authors highlight that their reduction procedure only works under the assumption of sufficiently expressiveness. A logic is sufficiently expressive if it is possible to define a separator formula and it is genuinely n-valued logic. A logic is said genuinely n-valued if it is characterized only by a n-valued semantics.

[9] Given two languages L_1, L_2 and their respective logics $\mathcal{L}_1$ and $\mathcal{L}_2$, we say $\mathcal{L}_2$ is a conservative extension of $\mathcal{L}_1$ in case $\Gamma \Vdash_{\mathcal{L}_2} \alpha$ iff $\Gamma \Vdash_{\mathcal{L}_2} \alpha$ for every $\Gamma \cup \{\alpha\} \subseteq L_1$.

[10] An example with the use of a conservative extension is exhibited in Section 3.1.1.

ϕ	$f_1 := Id$	f_2	f_3	f_4	f_5	f_6	f_7	f_8	$f_9 := f_\neg$	f_{10}	f_{11}	$f_{12} := f_{\neg\neg}$	f_{13}	f_{14}	f_{15}
0	0	0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1	0	1	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
1	1	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1	0	1	0	$\frac{1}{2}$	$\frac{1}{2}$	1	0	1	0

f_{16}	f_{17}	f_{18}	f_{19}	f_{20}	f_{21}	f_{22}	f_{23}	f_{24}	f_{25}	f_{26}	f_{27}
1	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	1	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	1	0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1

Note that among the 27 unary functions found above, only 12 of them are able to separate the undesignated values 0 and $\frac{1}{2}$, they are the functions f_6, f_8, f_9, f_{10}, f_{17}, f_{19}, f_{20}, f_{21}, f_{23}, f_{25}, f_{26} and f_{27}. Moreover, among the 12, only two of them are definable by a formula with at most one variable, i.e, f_9 and f_{12}. Step 3 of the algorithm tell us to choose one of the two and we will then pick to use f_9 as our separator. For such a case, we have that $v(\neg\phi)(\phi := 0)\#v(\neg\phi)(\phi := \frac{1}{2})$. Thus, $\neg\phi$ is an adequate separator formula to fulfill the conditions of a separator. From this, we have:

ϕ		$\theta(\phi) := \neg\phi$	
0	F	1	T
$\frac{1}{2}$	F	0	F
1	T	0	F

Once we have found our formula separator, the second step of Caleiros et al's reduction procedure is to make use of the separator to express each truth-value individually. Moreover, from now on negation will play a double role as a separator and a unary connective. By using our separator, we can mark/describe each truth-value with a 'binary print' of it. We say a **binary print** is a unique sequence of Ts and Fs used to characterize truth-values separately[11]. The binary prints we get for G_3 are the following:

$$v(\phi) = 0 \text{ iff } b(\phi) = F \ \& \ b(\neg\phi) = T.$$
$$v(\phi) = \tfrac{1}{2} \text{ iff } b(\phi) = F \ \& \ b(\neg\phi) = F.$$
$$v(\phi) = 1 \text{ iff } b(\phi) = T \ \& \ b(\neg\phi) = F.$$

Thus, within our bivalent semantics, we shall talk about the value of the formula and its separator. We only need these two informations to separate all truth-values and refer to them individually. From now on, with our binary prints at hand, the third step of Caleiro et al's reduction procedure amounts to using them to describe the bivalent version of each truth-table.

[11] Cf. [Caleiro et al., 2007].

To help our task, we can shorten the binary prints that refers to 0 and 1. Since there is only one case where $b(\neg\phi) = T$ happens, we need only this information to refer to 0. In the same way, we need only $b(\phi) = T$ to refer to 1. With this in mind, we shall use our binary prints to formulate axioms that describe each line of the truth-tables of G_3. To fully characterize each line of all connectives truth-tables, we must also describe the truth-tables resulting from the separator applied to each connective individually. Take for instance the case of negation, the axioms shall describe the resulting bivalent truth-tables[12]. In what follows, '&' shall signify the metalinguistic 'and', '|' will denote metalinguistic 'or' and '$*$' will denote the metalinguistic absurdity, amounting to an obtainable situation:

ϕ		$\neg\phi$		$\neg\neg\phi$	
0	F	1	T	0	F
$\frac{1}{2}$	F	0	F	1	T
1	T	0	F	1	T

(N1) $b(\neg\phi) = F \Rightarrow b(\phi) = F \mid b(\phi) = T$.
Axiom (N1) describe the second and third rows of the truth-table for negation. Here the condition regarding the first row is already guaranteed by the binary prints, therefore is not necessary a new axiom.

(N2) $b(\neg\neg\phi) = F \Rightarrow b(\neg\phi) = T$.
Axiom (N2) describe the first row of the truth-table for the separator applied to negation.

(N3) $b(\neg\neg\phi) = T \Rightarrow b(\neg\phi) = F$.
Axiom (N3) describe the second and third row of the truth-table for separator applied to negation.

By using only three axioms and our binary prints, we were able to characterize the behavior of negation in Gödel's 3-valued logic. Applying the same procedure to implication will be enough to describe its behavior as well. At last, we shall have the following axioms:

(N1) $b(\neg\phi) = F \Rightarrow b(\phi) = F \mid b(\phi) = T$

(N2) $b(\neg\neg\phi) = F \Rightarrow b(\neg\phi) = T$

(N3) $b(\neg\neg\phi) = T \Rightarrow b(\neg\phi) = F$

(I1) $b(\phi \to \psi) = T \Rightarrow b(\neg\phi) = T \mid b(\psi) = T \mid b(\phi) = b(\neg\phi) = b(\psi) = b(\neg\psi) = F$

(I2) $b(\phi \to \psi) = F \Rightarrow b(\phi) = T \ \&\ b(\psi) = F \mid b(\phi) = b(\neg\phi) = F \ \&\ b(\neg\psi) = T$

[12] Note that those are bivalent 'flattened' versions of the resulting truth-tables since each individual value is determined by a binary print. The resulting structure structure was called by [Skurt, 2011] as bivalent multi-dimensional models.

(I3) $b(\neg(\phi \to \psi)) = T \Rightarrow b(\phi) = T \ \&\ b(\neg\psi) = T \mid b(\phi) = b(\neg\phi) = F \ \&\ b(\neg\psi) = T$

(I4) $b(\neg(\phi \to \psi)) = F \Rightarrow b(\neg\phi) = T \mid b(\psi) = T \mid b(\psi) = b(\neg\psi) = F$

(C1) $b(\phi) = T \ \&\ b(\neg\phi) = T \Rightarrow *$

(C2) $b(\phi) = T \ \&\ b(\phi) = F \Rightarrow *$

The last two axioms, (C1) and (C2), stands for closure conditions representing unobtainable situations. Condition (C2) internalizes the fact that all valuations are functions, while (C1) correspond to the fact that there is no valuation in G_3 that assign a designated value to a formula and its negation, what is an unobtainable situation. Now define SEM_{G_2} as the set of bivaluations respecting all the axioms above. Such kind of semantics was baptized in [Caleiro et al., 2007] as *gentzenian semantics*. A *gentzenian semantics* is a Suszko's semantics respecting a finite number of conditional bivaluations clauses as those above. It is defined in the following way:

Definition 16 ([Caleiro et al., 2007])**.** *A* ***gentzenian semantics*** *for a logic* $\mathcal{L}$ *is an adequate (sound and complete) set of 2-valued valuations* $b : For \to \{T, F\}$ *given by conditional clauses* $(\Phi \to \Psi)$ *where both* Φ *and* Ψ *are (meta)formulas of the form* $\top$ *(top),* $\bot$ *(bottom) or:*

$$b(\phi_1^1) = w_1^1 \& ... \& b(\phi_1^{n_1}) = w_1^{n_1} | .... | b(\phi_m^1) = w_m^1 \& ... \& b(\phi_m^{n_m}) = w_m^{n_m} \tag{2.15}$$

Here, $w_i^j \in \{T, F\}$*, each* ϕ_i^j *is a formula of* $\mathcal{L}$*. The (meta)logic governing these clauses is FOL (further on,* $\to$ *will be used to represent the implication connective from this metalogic).*

Thereby a gentzenian semantics is a Suszko's semantics that comes with a built-in gentzenian presentation. The use of the conditional clauses is important to guarantee a constructive way of transforming the original semantics into a 2-valued one. The clauses has the goal of describing/storing the information obtained from the original many-valued semantics. This feature makes possible to prove, in a constructive manner, that the 2-valued semantics obtained from a finite-valued semantics is sound and complete to the original one. Moreover, our bivalent axioms allows us to extract a set of two-signed tableaux rules for introduction and elimination to every connective.

Since we have defined our two-valued semantics SEM_{G_2} in the gentzenian format, our next step is to show that SEM_{G_2} is sound and complete wih respect to SEM_{G_3}. The proof runs by showing that each valuation from SEM_{G_3} may be interpreted into SEM_{G_2} and vice-versa. Based on that, it is possible to prove that the set of valuations of both semantics validates the same inferences, i.e, $\Gamma \models_{\text{SEM}_{G_3}} \alpha \Leftrightarrow \Gamma \models_{\text{SEM}_{G_2}} \alpha$. But first, we need to prove two auxiliary lemmas:

Lemma 2.2.1. *Given a valuation $b \in \text{SEM}_{G_2}$, it is possible to define an interpretation $v_b \in \text{SEM}_{G_3}$ such that $b(\phi) = T$ iff $v_b(\phi) \in \mathcal{D}$.*

Given a valuation $b \in \text{SEM}_{G_2}$, define:

$$v_b(\phi) = \begin{cases} 0, & \text{if } b(\neg\phi) = T \\ \frac{1}{2}, & \text{if } b(\phi) = F \ \& \ b(\neg\phi) = F \\ 1, & \text{if } b(\phi) = T \end{cases}$$

Proof. The proof is made by induction on the complexity of the formulas.

Basic case:

($\Rightarrow$)

For an atom p. Suppose $b(p) = T$, then by definition of v_b, we have $v_b(p) = 1$. Therefore, $v_b(p) \in \mathcal{D}$.

($\Leftarrow$)

Suppose $v_b(p) \in \mathcal{D}$. Therefore $v_b(p) = 1$. By definition of v_b, it follows $b(\phi) = T$.

Inductive step:

Inductive Hypothesis: for all formulas α and β such that $\ell(\alpha) < \ell(\phi)$ and $\ell(\beta) < \ell(\phi)$, it follows that $b(\alpha) = T$ iff $v_b(\alpha) \in \mathcal{D}$ and $b(\beta) = T$ iff $v_b(\beta) \in \mathcal{D}$.

1.$\phi = \neg\alpha$.

($\Rightarrow$)

Suppose $b(\neg\alpha) = T$, by definition of v_b, we have $v_b(\neg\alpha) = 0$. Then, by definition of $f_\neg$, we have $v_b(\neg\phi) = 1$.

($\Leftarrow$).

Suppose (i) $v_b(\neg\alpha) \in \mathcal{D}$. Since $\ell(\alpha) < \ell(\phi)$, by inductive hypothesis follows that $b(\alpha) = T$ iff $v_b(\alpha) \in \mathcal{D}$. By (i), we know that $v_b(\alpha) \notin \mathcal{D}$. Therefore $b(\alpha) \neq T$ and by contraposition on Axiom (N1), it follows that $b(\neg\alpha) = T$.

2. $\phi = \neg\neg\alpha$.

($\Rightarrow$)

If $b(\neg\neg\alpha) = T$, by using Axiom (N3), we have $b(\neg\alpha) = F$. Therefore, by axiom (N1), we shall have (i) $b(\alpha) = F$ or (ii) $b(\alpha) = T$.

For case (i), note we have $b(\alpha) = b(\neg\alpha) = F$. Therefore, by definition of v_b, we have $v_b(\alpha) = \frac{1}{2}$. So by applying double negation to α, by definition of $f_\neg$, we shall get $v_b(\neg\neg\alpha) = 1$.

For case (ii), since $b(\alpha) = T$, by definition of v_b, we have $v_b(\alpha) = 1$. Again, by applying double negation to α, we shall get $v_b(\neg\neg\alpha) = 1$.

($\Leftarrow$)
Suppose (i) $v_b(\neg\neg\alpha) \in \mathcal{D}$. Since $\ell(\neg\alpha) < \ell(\neg\neg\alpha)$, it follows that $b(\neg\alpha) = T$ iff $v_b(\neg\alpha) \in \mathcal{D}$. Then, from (i) follows that $v_b(\neg\alpha) \notin \mathcal{D}$ and $b(\neg\alpha) \neq T$. Thus, by contraposition on Axiom (N2), we will have $b(\neg\neg\alpha) \neq F$, i.e, $b(\neg\neg\alpha) = T$.

3. $\phi = (\alpha \rightarrow \beta)$.
($\Rightarrow$)
If $b = (\alpha \rightarrow \beta) = T$, by Axiom (I1), we have (i) $b(\neg\alpha) = T$, or (ii) $b(\beta) = T$, or $b(\alpha) = b(\neg\alpha) = b(\beta) = b(\neg\beta) = F$.

For case (i). If $b(\neg\alpha) = T$, then by definition of v_b, we have $v_b(\neg\alpha) = 0$. Therefore, by definition of $f_\rightarrow$, it follows $v_b(\alpha \rightarrow \beta) = T$.

For case (ii). If $b(\beta) = T$, by definition of v_b, that gives us $v_b(\beta) = 1$. Therefore, by definition of $f_\rightarrow$, we shall have $v_b(\alpha \rightarrow \beta) = 1$, i.e, $v_b(\alpha \rightarrow \beta) \in \mathcal{D}$.

For case (iii). If $b(\alpha) = b(\neg\alpha) = b(\beta) = b(\neg\beta) = F$, by definition of v_b, we get $v_b(\alpha) = v_b = (\beta) = \frac{1}{2}$. At last, by definition of $f_\rightarrow$, it follows $v_b(\alpha \rightarrow \beta) = 1$.

($\Leftarrow$)
Suppose $v_b(\alpha \rightarrow \beta) \in \mathcal{D}$. Since $\ell(\alpha) < \ell(\alpha \rightarrow \beta)$ and $\ell(\beta) < \ell(\alpha \rightarrow \beta)$, by inductive hypothesis follows that $b(\alpha) = T$ iff $v_b(\alpha) \in \mathcal{D}$ and $b(\beta) = T$ iff $v_b(\beta) \in \mathcal{D}$. Therefore by contraposition on Axiom (I2), it follows that $b(\alpha \rightarrow \beta) \neq F$, i.e, $b(\alpha \rightarrow \beta) = T$.

4. $\phi = \neg(\alpha \rightarrow \beta)$.

($\Rightarrow$)
Suppose $b(\neg(\alpha \rightarrow \beta)) = T$, by axiom (I3), follows (i) $b(\alpha) = T \& b(\neg\beta) = T$ or (ii) $b(\alpha) = b(\neg\alpha) = F \& b(\neg\beta) = T$.

For case (i), since $b(\alpha) = T \& b(\neg\beta) = T$, by definition of v_b follows $v_b(\alpha) = 1$ and $v_b(\beta) = 0$. Therefore, by definition of $f_\rightarrow$, we shall have $v_b(\alpha \rightarrow \beta) = 0$. Now, using definition of $f_\neg$, follows $v_b(\neg(\alpha \rightarrow \beta)) = 1$.

For case (ii). Since we have $b(\alpha) = b(\neg\alpha) = F \& b(\neg\beta) = F$, by definition of v_b, we have $v_b(\alpha) = \frac{1}{2}$ and $v_b(\beta) = 0$. Therefore, definition of $f_\rightarrow$ will imply $v_b(\alpha \rightarrow \beta) = 0$. At last, from definition of $f_\neg$ we shall have our desired conclusion, $v_b(\neg(\alpha \rightarrow \beta)) = 1$.

($\Leftarrow$)
Suppose (i) $v_b(\neg(\alpha \rightarrow \beta)) \in \mathcal{D}$. Since $\ell(\alpha \rightarrow \beta) < \ell(\neg(\alpha \rightarrow \beta))$, by inductive hypothesis follows that $v_b(\alpha \rightarrow \beta) \in \mathcal{D}$ iff $b(\alpha \rightarrow \beta) = T$. From (i), follows that $b(\alpha \rightarrow \beta) \neq T$, i.e, $b(\alpha \rightarrow \beta) = F$. Therefore, from Axiom (I2), it follows (i) $b(\alpha) = T \& b(\beta) = F$ or (ii) $b(\alpha) = b(\neg\alpha) = F \& b(\neg\beta) = T$. From both cases, by contraposition on Axiom (I4), it follows $b(\neg(\alpha \rightarrow \beta)) \neq F$, i.e, $b(\neg(\alpha \rightarrow \beta) = T$.

□

Lemma 2.2.2. *Given a valuation $v \in \mathrm{SEM}_{G_3}$, it is possible to define an interpretation $b_v \in \mathrm{SEM}_{G_2}$ such that $v(\phi) \in \mathcal{D}$ iff $b_v(\phi) = T$.*

Given a valuation $v \in \mathrm{SEM}_{G_3}$, define:

$$b_v(\phi) = \begin{cases} F, & \text{if } v(\phi) = 0 \text{ or } v(\phi) = \frac{1}{2} \\ T, & \text{if } v(\phi) = 1 \end{cases}$$

Proof. The proof is made by induction on the complexity of the formulas.

Basic case:

($\Rightarrow$)
For an atom p. Suppose $v(p) \in \mathcal{D}$, i.e, $v(p) = 1$. by definition of b_v, it follows $b_v(p) = T$.

($\Leftarrow$) Suppose $b_v(p) = T$. Therefore, by definition of b_v, it follows $v(p) = 1$.

Inductive step:

Inductive Hypothesis: for all formulas α and β such that $\ell(\alpha) < \ell(\phi)$ and $\ell(\beta) < \ell(\phi)$, it follows that $v(\alpha) \in \mathcal{D}$ iff $b_v(\alpha) = T$ and $v(\beta) \in \mathcal{D}$ iff $b_v(\beta) = T$.

1. $\phi = \neg\alpha$.

($\Rightarrow$)
Suppose (i) $v(\neg\alpha) \in \mathcal{D}$. Since $\ell(\alpha) < \ell(\neg\alpha)$, by inductive hypothesis follows that $b_v(\alpha) = T$ iff $v(\alpha) \in \mathcal{D}$. From (i) and $f_\neg$, we know that $v(\alpha) \notin \mathcal{D}$. Therefore $b_v(\alpha) \neq T$ and, by contraposition on Axiom (N1), it follows $b_v(\neg\alpha) \neq F$, i.e, $b_v(\neg\alpha) = T$.

($\Leftarrow$)
Suppose (i) $b_v(\neg\alpha) = T$. By definition of b_v, we have that $v(\neg\alpha) = 1$, i.e, $v(\neg\alpha) \in \mathcal{D}$.

2. $\phi = \neg\neg\alpha$.

($\Rightarrow$)
Suppose (i) $v(\neg\neg\alpha) \in \mathcal{D}$. Since $\ell(\neg\alpha) < \ell(\neg\neg\alpha)$, by inductive hypothesis follows that $v(\neg\alpha) \in \mathcal{D}$ iff $b_v(\neg\alpha) = T$. From (i) and $f_\neg$, we know that $v(\neg\alpha) \notin \mathcal{D}$. Therefore, we have $b_v(\neg\alpha) \neq T$. In this case, by contraposition on Axiom (N2), we shall have $b_v(\neg\neg\alpha) \neq F$, i.e, $b_v(\neg\neg\alpha) = T$.

($\Leftarrow$)
Suppose $b_v(\neg\neg\alpha) = T$. By Axiom (N3), it follows $b_v(\neg\alpha) = F$. Then, by the definition of b_v, it follows that (i) $v(\neg\alpha) = \frac{1}{2}$ or (ii) $v(\neg\alpha) = 0$. For case (i), we shall obtain $b_v(\neg\neg\alpha) = F$, what stands as unobtainable situation by (C2). Therefore, from (ii), by $f_\neg$, it follows that $v(\neg\neg\alpha) = 1$, i.e, $v(\neg\neg\alpha) \in \mathcal{D}$.

3. $\phi = (\alpha \to \beta)$.
($\Rightarrow$)
Suppose (i) $v(\alpha \to \beta) \in \mathcal{D}$. Since $\ell(\alpha) < \ell(\alpha \to \beta)$ and $\ell(\beta) > (\alpha \to \beta)$, by inductive hypothesis it follows that $v(\alpha) \in \mathcal{D}$ iff $b_v(\alpha) = T$ and $v(\beta) \in \mathcal{D}$ iff $b_v(\beta) = T$. Then, by contraposition on Axiom (I2), we shall obtain $b_v(\alpha \to \beta) = T$.

($\Leftarrow$)
Suppose $b_v(\alpha \to \beta) = T$. By Axiom (I1), it follows that $b_v(\neg\alpha) = T$, or $b_v(\beta) = T$ or $b_v(\alpha) = b_v(\neg\alpha) = b_v(\beta) = b_v(\neg\beta) = F$. Therefore, by definition of b_v, we shall obtain $v(\alpha) = 0$, $v(\beta) = 1$ and $v(\alpha) = v(\beta) = \frac{1}{2}$. For all such cases, by $f_\to$, follows $v(\alpha \to \beta) = 1$.

4.$\phi = \neg(\alpha \to \beta)$.

($\Rightarrow$)
Suppose (i) $v(\neg(\alpha \to \beta)) \in \mathcal{D}$. Since $\ell(\alpha \to \beta) < \ell(\neg(\alpha \to \beta))$, by inductive hypothesis follows that $v(\alpha \to \beta) \in \mathcal{D}$ iff $b_v(\alpha \to \beta) = T$. Therefore, from (i) it follows that $b_v(\alpha \to \beta) \neq T$, i.e, $b_v(\alpha \to \beta) = F$. Then, by Axiom (I2), follows $b_v(\alpha) = T \& b_v(\beta) = F$ or $b_v(\alpha) = b_v = F \& b_v(\neg\beta) = T$. Thus, by contraposition on Axiom (I4), we shall obtain $b_v(\neg(\alpha \to \beta)) \neq F$, i.e, $b_v(\neg(\alpha \to \beta)) = T$.

($\Leftarrow$)
Suppose $b_v(\neg(\alpha \to \beta)) = T$. By Axiom (I4), follows (i) $b_v(\alpha) = T \& b_v(\neg\beta) = T$ or (ii) $b_v(\alpha) = b_v(\neg\alpha) = F \& b_v(\neg\beta) = T$. For case (i), by definition of b_v, we shall obtain $v(\alpha) = 1 \& v(\neg\beta) = 1$. By definition of $f_\neg v(\beta) = 0$. Therefore, $v(\alpha \to \beta) = 0$ and $v(\neg(\alpha \to \beta)) = 1$. The same applies to case (ii) and our desired result follows $v(\neg(\alpha \to \beta)) \in \mathcal{D}$.

□

Theorem 2.2.1. SEM_{G_2} *is a sound and complete semantics for* SEM_{G_3}

$$\Gamma \models_{\text{SEM}_{G_3}} \alpha \Leftrightarrow \Gamma \models_{\text{SEM}_{G_2}} \alpha$$

Proof. ($\Rightarrow$)

Suppose (i) $\Gamma \models_{\text{SEM}_{G_3}} \alpha$. Then, by definition of entailment, we know that for every valuation $v \in \text{SEM}_{G_3}$ (ii) $v(\Gamma) \subseteq \mathcal{D}_3$ implies $v(\alpha) \in \mathcal{D}_3$. Now take an arbitrary valuation from $v \in \text{SEM}_{G_3}$ and suppose (iv) $v(\Gamma) \subseteq \mathcal{D}_3$. Thus, from (ii), we have $v(\alpha) \in \mathcal{D}_3$. From this, by Lemma 2.2.2, it is possible to define an interpretation b_v such that $v(\phi) \in \mathcal{D}_3$ iff $b_v(\phi) = T$. Therefore, we have (v) $b_v(\Gamma) \subseteq \{T\}$ and $b_v(\alpha) = T$. Finally, by definition of entailment, we get $\Gamma \models_{\text{SEM}_{G_2}} \alpha$.

($\Leftarrow$)

Suppose (i) $\Gamma \models_{\text{SEM}_{G_2}} \alpha$. Then, by definition of entailment, we know that for every valuation $b \in \text{SEM}_{G_2}$ (ii) $b(\Gamma) \subseteq \{T\}$ implies $v(\alpha) \in \{T\}$. Now take an arbitrary bivaluation from SEM_{G_2} and suppose (iv) $b(\Gamma) \subseteq \{T\}$. Therefore, from (ii), we have that $v(\alpha) = T$. From this, by Lemma 2.2.1, it is possible to define an interpretation v_b such that $b(\phi) \in \{T\}$ iff $v_b(\phi) \in \mathcal{D}_3$. Therefore, we have $v_b(\Gamma) \subseteq \mathcal{D}_3$ and $v_b(\alpha) \in \mathcal{D}_3$. At last, by definition of entailment, it follows that $\Gamma \models_{\text{SEM}_{G_3}} \alpha$. □

The main difference between Theorem 2.1.3 and Theorem 2.2.1 is the constructive character of the latter, acquired by the reduction procedure proposed by Caleiro et al. The use of binary prints as a linguistic resource allows one to obtain a systematic characterization of the original many-valued truth-tables. To summarize, Caleiro et al's procedure is obtained through the following steps:

1 Use the procedure described in this section to find adequate formula separators.

2 Use the separators to define binary prints of each truth-values from the original semantics.

3 Use the binary prints to obtain bivalent axioms describing the behavior of the connectives and the separators applied to the connectives.

4 Define the bivalent semantics SEM_2 as a Suszko's semantics respecting all bivalent axioms obtained in step 3.

The above procedure can be applied to all finite-valued logics. In [Marcelino et al., 2014], the authors extended the range of application of the method to finite-non-deterministic matrices. As was said above, one of the advantages of obtaining a gentzenian semantics from a finite-valued one is that the bivalent axiomatic description allows one to obtain sound and complete two-signed tableaux with respect to the bivalent semantics [13].

[13] For more, see [Caleiro et al., 2007].

Part III

Beyond Suszko's Reduction

3 Logical values

This chapter investigates the notion of logical value. As it was shown in Chapter 1, the terminology was introduced by Roman Suszko in unraveling the double character of truth-values with the distinction in between algebraic and logical values. While algebraic values denote what sentences describe, logical values denote the notions of truth and falsehood. Therefore, algebraic many-valuedness is the kind of many-valuedness obtained by introducing more than two objects in the carrier set of the algebra of truth-values, while logical many-valuedness is obtained by dividing the set of truth-values in more than two parts. Logical many-valuedness was first presented by Grzegorz Malinowski in revealing what was called by him *inferential* many-valuedness. Inferential many-valuedness was created with the intention of increasing the number of logical values in the sense of Suszko. Like Łukasiewicz going beyond the being/non-being dichotomy, inferential many-valuedness goes beyond the designated/undesignated dichotomy by considering alternative ways of classifying the algebraic values. As we shall see, such a framework naturally leaves room for the construction of non-standard definitions of a matrix, as well as non-Tarskian conceptions of entailment.

We start by taking a step beyond Suszko's ideas and considering the introduction of a third logical value. This was proposed in [Malinowski, 1990a] by creating what the author called *quasi*-consequence operations. In his construction, Malinowski abandoned the traditional partition of the set of truth-values in between designated and undesignated by dividing the truth-values between accepted, rejected and neither accepted nor rejected values. In this context, Suszko's thesis may be updated to a 3-valued format. In Section 1 we present q-consequence operations and the class of logics that they give rise to (called q-logics), and prove the reduction theorems related to them. We also exhibit how Caleiro et al's reduction algorithm may be applied to reduce any finite-valued q-logic to a 3-valued semantics. Moreover, based on a dual construction of Malinowski's q-matrices, we shall present another non-Tarskian notion of entailment introduced by Frankowski [Frankowski, 2004], the so-called *plausible*-consequence (shortened p-consequence). Section 2 aims at showing Shramko & Wansing's definition of logical value and how it affects the meaning of Suszko's reduction. Criticisms to Malinowski's notion of inferential many-valuedness and Suszko's thesis will also be presented. The section raises the discussion of what we should expect and consider as an adequate notion of logical consequence and, moreover, why to go beyond the Tarskian realm of consequence. At Section 3, we end up by showing the bi-dimensional notion of entailment called B-entailment, introduced in [Bochman et al., 1998]. By following [Blasio et al., 2014], we aim to show how B-entailment is able to express all conceptions of entailment exposed before and why it is a natural and rich framework to explore different notions of logical consequence.

3.1 G. Malinowski and inferential many-valuedness

In Malinowski [Malinowski, 1990a], the Polish logician Grzegorz Malinowski constructed an example to line-off the limits of Suszko's reduction by changing the underlying definition of matrix and, moreover, constructing a non-Tarskian notion of entailment. Malinowski's motivation lied in increasing the number of logical values, thus evading Suszko's Thesis by considering an additional value, neither designated nor rejected. Therefore, he had the purpose of increasing a third undetermined value at the level of logical values, beyond the traditional idea of designated and rejected, i.e, from Suszko's point of view, of truth and falsity. He baptized this notion of many-valuedness as *inferential many-valuedness*. Since logical values, according to Suszko, are expressed at the level of the division of the set of truth-values, Malinowski's q-entailment makes use of three logical values, thus adding a third 'undetermined' value at the inferential level. It was this kind of consideration that led Malinowski to think of many-valuedness as "expressed in two facets – the referential and the inferential one". In [Malinowski, 2011], the author explains the difference in the following manner: "The first fits the standard approach and it results in multiplication of semantic correlates of sentences(...). The second (...) is a metalogical property of inference and refers to partition of the matrix universe into more than two disjoint subsets, used in the definition of inference". In [Malinowski, 1990b], Malinowski suggests that if, like Suszko, we think of the elements of the matrix as situations, then we must consider three kinds of situations: those which obtain, those which do not obtain and those that are undetermined. Thus, his step towards logical many-valuedness was motivated by Łukasiewicz's ontological basis for considering more than two truth-values in his logical systems.

In what follows, we present Malinowski's technical construction:
Recall Definition 5 from Section 2.1. Let $At = \{p_1, p_2, ...\}$ be a denumerable set of atoms, and let $\Sigma = \{\Sigma_n\}_{n\in\mathbb{N}}$ be a propositional signature, where each Σ_n is a set of connectives of arity n. Let the set of formulas For be defined as the algebra freely generated by At over Σ. We start presenting Malinowski's construction by defining a q-consequence relation:

Definition 17. *A q-**consequence relation** (or Malinowskian consequence relation) is a single-conclusion consequence relation $\Vdash^q$ such that for every $\varphi \in For$ and every $\Delta, \Gamma \subseteq For$ it has the following properties* [1]*:*

$$\text{If } \Delta \Vdash^q \varphi \text{ then } \Delta \cup \Gamma \Vdash^q \varphi \text{ (Monotonicity)} \tag{3.1}$$

$$\Gamma \cup \{\gamma \mid \Gamma \Vdash^q \gamma\} \Vdash^q \phi \text{ implies } \Gamma \Vdash^q \phi \text{ (Weak-cut)} \tag{3.2}$$

[1] Cf. [Shramko and Wansing, 2011].

Therefore we define a q-logic in the following way:

Definition 18. *An (abstract)* ***q-logic*** $\mathcal{L}^q$ *is a pair* $\mathcal{L}^q = \langle For, \Vdash^q \rangle$*, where* $\Vdash^q$ *is a q-consequence relation.*

Definition 19. *A q-consequence relation is called* ***substitution-invariant*** *if the following holds:*

$$\Gamma \Vdash^q \alpha \textit{ implies } \sigma(\Gamma) \Vdash^q \sigma(\alpha), \textit{ for all } \sigma \in End(For) \textit{ (Substitution-invariance)} \tag{3.3}$$

where $\sigma(\Gamma) = \{\sigma(\gamma) | \gamma \in \Gamma\}$.

A q-logic is called **substitution-invariant** if its associated consequence relation is substitution-invariant.

Theorem 3.1.1. *Every Tarskian consequence relation is a q-consequence relation.*

The proof consists in showing that (Reflexivity) + (Cut for sets) implies (Weak-Cut) and (Reflexivity) + (Weak-Cut) + (Monotonicity) implies (Cut for sets). The second part shows the trade-off between (Reflexivity) and (Cut for sets) and why q-consequence operations are presented as a natural generalization for Tarskian consequence operations.

Proof. (Reflexivity) + (Cut for sets) $\Rightarrow$ (Weak-cut)
Assume (i) $\Gamma \cup \{\gamma \mid \Gamma \Vdash \gamma\} \Vdash \phi$. By (Reflexivity), we know that (ii) $\Gamma \Vdash \gamma$, for all $\gamma \in \Gamma$. Therefore, using (Cut for sets), it follows that $\Gamma \Vdash \phi$.

(Reflexivity) + (Weak-cut) + (Monotonicity) $\Rightarrow$ (Cut for sets)
Suppose (i) $\Delta \Vdash \phi$ and (ii) $\Gamma \Vdash \delta$, for all $\delta \in \Delta$. By monotonicity in (i), it follows that (iii) $\Gamma \cup \Delta \Vdash \phi$. By (Reflexivity), we know that $\Gamma \Vdash \gamma$, for all $\gamma \in \Gamma$. But from (ii), it follows that (iv) $\Delta \subseteq \{\gamma \mid \Gamma \Vdash \gamma\}$. Thus, by monotonicity on (i) and (iv) we shall have (v) $\Gamma \cup \{\gamma \mid \Gamma \Vdash \gamma\} \Vdash \phi$ and at last, by (Weak-cut) on (v), it follows that $\Gamma \Vdash \phi$. □

In what follows, we associate to a q-logic $\mathcal{L}^q$ an appropriate semantic structure by defining a q-matrix structure, as proposed by [Malinowski, 1990a], in the following way:

Definition 20. *A* ***logical*** q***-matrix*** *based on* $\mathcal{L}^q$ *is an algebra* $\mathbb{M}^q = \langle \mathcal{V}, \mathcal{D}, \mathcal{R}, \mathcal{O} \rangle$*, where* $\mathcal{D}$ *and* $\mathcal{R}$ *are disjoints non-empty proper subset of* $\mathcal{V}$ *and for every n-ary connective c from* Σ_n*,* $\mathcal{O}$ *includes a corresponding n-ary function* f_c*:* $\mathcal{V}^n \to \mathcal{V}$*.*

The elements of $\mathcal{D}$ *are called* ***accepted*** *truth-values and the elements of* $\mathcal{R}$ *are called* ***rejected*** *truth-values. Note that if* $\mathcal{D}$ *and* $\mathcal{R}$ *are complementary sets, i.e,* $\mathcal{V} = \mathcal{D} \cup \mathcal{R}$*, the q-matrix is reduced to the usual concept of logical matrix (see Definition 8). We also shall use* $\mathcal{N}$ *to denote the set of values that are* ***neither*** *designated nor rejected, i.e,* $\mathcal{N} \stackrel{def}{=} \mathcal{V} - (\mathcal{D} \cup \mathcal{R})$*.*

Thus, in a logical q-matrix we may have three logical values: accepted, rejected and neither accepted nor rejected. Malinowski considered such logical values as expressing the notions of truth, falsity and undeterminedness. His approach was based on Łukasiewicz's idea that acceptance and rejection need not be complementary[2]. According to Malinowski, the basic assumption that acceptance and rejection need not to be complementary reflected "the mathematical practice that treats some auxiliary assumptions as mere hypothesis rather than axioms. These assumptions may be accounted for by deduction (or not), which results in their justified further occurrence in the place of conclusions"[3]. Thus q-consequence operations would reflect the scientific modus operandi of working with hypothesis and evaluating their consequences. From such a point of view, the demise of Reflexivity seems natural once we do not want that a scientific hypothesis is used to justify itself.

With the notion of a q-matrix in hand, we can now introduce a semantics:

Definition 21. *Fix a q-matrix $\mathbb{M}^q$. The set of homomorphisms (also called* ***valuations****) from $\mathcal{L}^q$ into $\mathbb{M}^q$ is called a n-**valued semantics** SEM^q, where n is the cardinality of $\mathcal{V}$, hereafter denoted by $|\mathcal{V}|$.*

After collecting valuations and introducing the semantics, a q-**entailment** relation (or quasi entailment) given by $\models^q_{\mathrm{SEM}} \subseteq \wp(For) \times For$ associated to the semantics SEM^q can be defined by saying that a formula α follows from a set of formulas Γ if all valuations from SEM^q that do not assign a rejected value to any formula $\gamma \in \Gamma$ imply that α takes a designated value. Formally:

Definition 22. *Malinowskian relation of* q***-entailment****:*

$$\Gamma \models^q_{\mathrm{SEM}^q} \alpha \textit{ iff } v(\Gamma) \cap \mathcal{R} = \emptyset \textit{ implies } v(\alpha) \in \mathcal{D}, \textit{ for every } v \in \mathrm{SEM}^q \textit{ and } \Gamma \cup \{\alpha\} \subseteq For \tag{3.4}$$

This way, differently from the Tarskian relation of entailment, where a valid inference means the preservation of truth from all premises to the conclusion, in the relation of q-entailment valid inference means that an accepted conclusion follows from a non-rejected set of premises.

Theorem 3.1.2. *Every relation of q-entailment has the properties of a q-consequence operation.*[4]

Proof. For the case of (Monotonicity), suppose, by contraposition, that $\Delta \cup \Gamma \not\models^q_{\mathrm{SEM}^q} \alpha$. Therefore, from the definition of q-entailment, it follows there is a valuation $v \in \mathrm{SEM}^q$

2 Cf. [Malinowski, 1990a].
3 Cf [Malinowski, 2007].
4 To see that reflexivity does not hold, let α be an atom and consider a valuation v such that $v(\alpha) \in \mathcal{N}$. Since α is neither rejected nor accepted according to v, we have that $\alpha \not\models^q \alpha$.

such that (i) $v(\Delta \cup \Gamma) \cap \mathcal{R} = \emptyset$ and (ii) $v(\alpha) \notin \mathcal{D}$. Then, since $\Delta \subseteq \Delta \cup \Gamma$ we shall have (iii) $v(\Delta) \cap \mathcal{R} = \emptyset$. Therefore, from (iii), (ii) and the definition of a Malinowskian entailment, it follows that $\Delta \nvDash^q_{\mathrm{SEM}^q} \alpha$.

For (Weak-cut), by contraposition, suppose $\Gamma \nvDash^q_{\mathrm{SEM}^q} \alpha$. Therefore, from the definition of q-entailment, it follows that there is a valuation $v \in \mathrm{SEM}^q$ such that (i) $v(\Gamma) \cap \mathcal{R}$ and (ii) $v(\alpha) \notin \mathcal{D}$. Using (i), (ii), (monotonicity) proven above and definition of q-entailment, we shall obtain $\Gamma \cup \{\gamma \mid \Gamma \Vdash^q \gamma\} \nvDash^q_{\mathrm{SEM}^q} \alpha$.

□

We shal define a **s-q-logic** as a structure $\mathcal{L}^q = \langle For, \vDash^q_{\mathrm{SEM}^q} \rangle$, where $\vDash^q_{\mathrm{SEM}^q}$ is a Malinowskian relation of q-entailment. Given a logic $\mathcal{L}^q_1 = \langle For, \Vdash^q \rangle$ and a s-q-logic $\mathcal{L}^q_2 = \langle For, \vDash^q_{\mathrm{SEM}^q} \rangle$, we shall say that $\mathcal{L}^q_2$ is **sound** with respect to $\mathcal{L}^q_1$ in case $\Vdash^q \subseteq \vDash^q_{\mathrm{SEM}^q}$; and $\mathcal{L}^q_1$ is **complete** with respect to $\mathcal{L}^q_2$ in case $\vDash^q_{\mathrm{SEM}^q} \subseteq \Vdash^q$. Moreover, if $\Vdash^q = \vDash^q_{\mathrm{SEM}^q}$, then we say $\mathcal{L}^q_2$ is an **adequate** semantics for $\mathcal{L}^q_1$.

We prove now an auxiliary remark useful to prove the main results of this section:

Consider a family of q-logics $\mathcal{F} = \{\mathcal{L}^q_i\}_{i \in I}$ over some fixed set of formulas For. Define the *superlogic* $\mathcal{L}^q_{\mathcal{F}}$ of this family by taking $\bigcap_{i \in I} \mathcal{L}^q_i$, that is, $\mathcal{L}^q_{\mathcal{F}} = \langle For, \bigcap_{i \in I} \Vdash^q_i \rangle$, where each $\mathcal{L}^q_i = \langle For, \Vdash^q_i \rangle$ is a structural q-logic, for $i \in I$.

Remark 4. *The intersection of substitution-invariant q-logics is a substitution-invariant q-logic.*

Proof. Assume $\mathcal{L}^q_{\mathcal{F}}$ is not substitution-invariant. Then there are formulas $\Gamma \cup \{\alpha\} \subseteq For$ such that (i) $\Gamma \Vdash^q_{\mathcal{F}} \alpha$ and (ii) $\sigma(\Gamma) \nVdash^q_{\mathcal{F}} \sigma(\alpha)$ are the case. From (ii), by definition of $\mathcal{L}^q_{\mathcal{F}}$, we know there is some $\Vdash^q_k$ $k \in I$, such that (iii) $\sigma(\Gamma) \nVdash^q_k \sigma(\alpha)$. However, we know that each $k \in I$ is a substitution-invariant q-logic. Therefore, by contraposition and the definition of substitution-invariance in (iii), we have $\Gamma \nVdash^q_k \alpha$. At last, by definition of $\mathcal{L}^q_{\mathcal{F}}$, we shall get $\Gamma \nVdash^q_{\mathcal{L}_{\mathcal{F}}} \alpha$, what is a contradiction with (i).

Analogous reasoning may be employed to prove the other properties of (Monotonicity) and (Weak-Cut).

□

In what follows, we make use of the notion of a Lindenbaum matrix used in Section 3.1 to prove Wójcicki's reduction (Theorem 2.1.2). Here we adapt it for q-logics and prove Malinowski's reduction. For the sake of defining our Lindenbaum q-matrix in an adequate way, we need to separate the set of formulas in pairs of mutually disjoints subsets.

Definition 23. *Let $DS(For)$ be the class of all pairs of mutually disjoints subsets of For:*

$$DS(For) = \{\langle \Sigma, \Delta \rangle : \langle \Sigma, \Delta \rangle \in \wp(For) \times \wp(For) \text{ and } \Sigma \cap \Delta = \emptyset\} \tag{3.5}$$

Definition 24 ([Malinowski, 1990a])**.** *Let $Q \subseteq DS(For)$. We shall say that a pair $\langle \Sigma, \Delta \rangle$ is q-**closed** relative to $\Gamma \subseteq For$ provided that:*

$$\Delta = \bigcap\{\Delta : \langle \Sigma, \Delta \rangle \in Q \text{ and } \Gamma \cap \Sigma = \emptyset\} \tag{3.6}$$

$$\Sigma = For - (\Gamma \cup \Delta) \tag{3.7}$$

*A set $Q \subseteq DS(For)$ is called a Q-**closure system** whenever $Q = \{\langle \Sigma, \Delta \rangle \in DS(For) : \langle \Sigma, \Delta \rangle$ is q-closed relative to some $\Gamma \subseteq For\}$.*

In what follows, we will define a Lindenbaum q-matrix based on the relative q-closure to a given set of formulas:

Definition 25. *Given a logic $\mathcal{L}^q = \langle For, \Vdash^q \rangle$, the pair $\langle \Sigma, \Delta \rangle$ q-closed relative to a given set of formulas $\Gamma \subseteq For$ determines a q-matrix of the form*

$$\mathbb{M}^q_\Gamma = \langle For, \mathcal{D}, \mathcal{R} \rangle, \text{ where} \tag{3.8}$$

$$\mathcal{D} = \Delta \tag{3.9}$$

$$\mathcal{R} = \Sigma \tag{3.10}$$

*called a **Lindenbaum q-matrix** of $\mathcal{L}^q$. The class of all Lindenbaum q-matrices induced by For, that is, $\{\mathbb{M}^q_\Gamma : \Gamma \subseteq For\}$ define what is called the **Lindenbaum bundle** of $\mathcal{L}^q$.*

Given the fact that the Lindenbaum q-matrices were defined in relation to a pair $\langle \Sigma, \Delta \rangle$ which is q-closed relative to $\Gamma \subseteq For$, we let the **q-closure** of a set of formulas Γ be defined as $\Gamma^{\Vdash^q} = \bigcap\{\Delta_i : \langle \Sigma, \Delta \rangle \in Q$ and $\Gamma \cap \Sigma = \emptyset\}$. The definition is in accordance with the notion of q-entailment since it guarantee that all premises are non-rejected. Again, for the sake of convenience, we shall present the Lindenbaum q-matrix in the form of its associated s-q-logic:

Definition 26. *Given a q-logic $\mathcal{L}^q = \langle For, \Vdash^q \rangle$ and a set of formulas $\Gamma \subseteq For$, we shall call a **Lindenbaum q-matrix** the s-q-logic $\mathcal{L}^q_\Gamma = \langle \mathbb{M}^q_\Gamma, \models^q_\Gamma \rangle$, where $\mathbb{M}_\Gamma = \langle For, \mathcal{D}, \mathcal{R} \rangle$ is such that:*

$$For = \mathcal{V} \tag{3.11}$$

$$\mathcal{D} = \Gamma^{\Vdash^q} \tag{3.12}$$

$$\mathcal{R} = For - (\Gamma \cup \Gamma^{\Vdash^q}) \tag{3.13}$$

$$\Delta \models^q_\Gamma \alpha \textit{ iff } g(\Delta) \cap \mathcal{R} = \emptyset \textit{ implies } g(\alpha) \in \mathcal{D}, \textit{ for all } g \in \mathrm{SEM}^q, \tag{3.14}$$

*where each g is a **uniform-substitution** as in Definition 12 (Section 2.1).*

In what follows, by making use of the tools constructed above, we prove the following theorems:

Theorem 3.1.3. *[Malinowski, 1990a] Every substitution-invariant q-logic has an adequate semantics.*

Proof. Fix a q-logic $\mathcal{L}^q = \langle For, \Vdash^q \rangle$, take the Lindenbaum bundle generated by For and define the super s-q-logic $\mathcal{L}^q_{\mathcal{F}} = \langle For, \models^q_{\mathcal{F}} \rangle$, by taking $\models^q_{\mathcal{F}}$ as $\bigcap_{\Gamma \subseteq For} \models^q_\Gamma$.

For **Completeness**, take some pair $\langle \Sigma, \Delta \rangle$ Q-closed relative to some $\Gamma \subseteq For$ and the Lindenbaum q-matrix $\mathcal{L}^q_\Gamma$ generated from Γ. Suppose $\Delta \models^q_{\mathcal{F}} \alpha$. From the definition of $\models^q_{\mathcal{F}} \alpha$, we have $\Delta \models_\Gamma \alpha$, for every $\Gamma \subseteq For$. Now assume (i) $\Delta \subseteq \Gamma^{\Vdash}$. By definition of q-closure, we know that (ii) $\Gamma \cap \Sigma = \emptyset$. Since every valuation of a Lindenbaum q-matrix is an endomorphism, from (ii) and the definition of a Lindenbaum q-matrix, we obtain (iii) $g(\Gamma) \cap \mathcal{R} = \emptyset$. Moreover, given our hypothesis and the definition of q-entailment, we know that, for every valuation $v \in \mathrm{SEM}^\Gamma$, $v(\Gamma) \cap \mathcal{R} = \emptyset$ implies $v(\alpha) \in \mathcal{D}$. Therefore, from (iii), it follows that $g(\alpha) \in \mathcal{D}$. Additionally, by definition of a Lindenbaum q-matrix, $\alpha \in \Gamma^{\Vdash}$, i.e, $\alpha \in \bigcap \Delta$. Then it follows that $\Delta \cup \{\delta | \Delta \Vdash^q \delta\} \Vdash^q \alpha$. Thus, by (Weak-cut) we shall obtain $\Delta \Vdash^q \alpha$.

For **Soundness**, suppose (i) $\Delta \Vdash^q \alpha$ and select some logic $\mathcal{L}_\Gamma$ from $\mathcal{L}_{\mathcal{F}}$. Assume g to be a valuation such that (ii) $g(\Delta) \cap \mathcal{R} = \emptyset$. Then, by definition of a Lindenbaum q-matrix, we shall have that (iii) $g(\Delta) \subseteq (\Gamma \cup \Gamma^{\Vdash})$. From (i), substitution-invariance and the fact that all valuations of a Lindenbaum q-matrix are endomorphisms, it follows that (v) $g(\Delta) \Vdash^q g(\alpha)$. Therefore, from (iii), (iv) and monotonicity, we obtain (v) $\Gamma, \Gamma^{\Vdash} \Vdash^q g(\alpha)$. Moreover, using (Weak-cut) in (v), it follows that $\Gamma \Vdash^q g(\alpha)$. Therefore, by definition of closure and of Lindenbaum q-matrix, we have $g(\alpha) \in \mathcal{D}$. Finally, by definition of q-entailment, our desired conclusion $\Delta \models_\Gamma \alpha$ follows.

□

Corollary 3.1.3.1. *Every substitution-invariant q-logic is n-valued, for $n = |For|$*

Proof. By construction of the Lindenbaum q-matrices. □

Corollary 3.1.3.2. *Every substitution-invariant q-logic is characterized by a set of matrices.*

Proof. By construction of the super s-q-logic. □

Until the present date the author is not aware of any contribution that shows the property that characterizes the existence of a single q-matrix adequate to a given q-logic.

We now employ the same construction used to prove Theorem 2.1.3 in order to show a generalized version of the same theorem. This actualized version shall be called Malinowski's Reduction and it shows that every q-logic may be charaterized by a trivalent semantics. Based on Definition 30, let us denote a n-valued semantics in the following way SEM_n^q, where n denote the cardinality of the set $\mathcal{V}^q$ and $\mathcal{V}^q$ is the carrier set of the associated q-matrix. Analogous to the function $t(x)$ that we used to prove Suszko's Reduction, we now define the function $f : \mathcal{V}_n \to \mathcal{V}_3$, where $\mathcal{V}_3 = \mathcal{D} = \{T\} \cup \mathcal{R} = \{F\}$, in the following way:

$$f(x) = \begin{cases} T, & \text{if } x \in \mathcal{D} \\ F, & \text{if } x \in \mathcal{R} \\ N, & \text{if } x \in \mathcal{V} - (\mathcal{D} \cup \mathcal{R}) \end{cases}$$

Now given a valuation v, we can define a *trivaluation* $t_v = f \circ v$ and collect such trivaluations into the semantics $\mathrm{SEM}_3^q = \{t_v \mid v \in \mathrm{SEM}_n^q\}$.

Theorem 3.1.4 (Malinowski's reduction – [Malinowski, 1994]). *Every substitution-invariant q-logic is 2-valued or 3-valued.*

$$\textit{Proof.}\ \Gamma \models^q_{\mathrm{SEM}_n^q} \alpha \Leftrightarrow \Gamma \models^q_{\mathrm{SEM}_3^q} \alpha$$

($\Leftarrow$)

By contraposition, suppose (i) $\Gamma \not\models^q_{\mathrm{SEM}_n^q} \alpha$. Therefore, by definition of q-entailment, there is a valuation $v \in \mathrm{SEM}_n^q$ such that (ii) $v(\Gamma) \cap \mathcal{R}_n = \emptyset$ and (iii) $v(\alpha) \notin \mathcal{D}_n$. By composing f with such v, we shall get $f(v(\Gamma)) \cap \{F\} = \emptyset$ and either $f(v(\alpha)) = F$ or $f(v(\alpha)) = N$. Therefore, there is a trivaluation $t_v \in \mathrm{SEM}_3^q$ such that $t_v(\Gamma) \cap F = \emptyset$ and either $t_v(\alpha) = \{F\}$ or $t_v(\alpha) = N$. In both cases, by applying the definition of q-entailment, it follows that $\Gamma \not\models^q_{\mathrm{SEM}_3^q} \alpha$.

($\Rightarrow$)

By contraposition, suppose (i) $\Gamma \not\models^q_{\mathrm{SEM}_3^q} \alpha$. Therefore, by the definition of q-entailment, there is a valuation $t \in \mathrm{SEM}_3^q$ such that (ii) $t(\Gamma) \cap \mathcal{R}_3 = \emptyset$ and (iii) $t(\alpha) \notin \mathcal{D}_3$. By definition of SEM_3^q, there must exist some $v \in \mathrm{SEM}_n^q$ such that $t = t_v$. Thus by definition of t_v, it follows that $f(v(\Gamma)) \subseteq \mathcal{V}_3 - \{F\}$ and $t(v(\alpha)) \neq T$. By definition of f, we shall have $v(\Gamma) \cap \mathcal{R}_n = \emptyset$ and $v(\alpha) \notin \mathcal{D}_n$. Then, from the definiton of q-entalment, it follows that $\Gamma \not\models^q_{\mathrm{SEM}_n^q} \alpha$.

□

The above theorem is a generalized version of Suszko's reduction. The 3-valued reduction is obtainable due to the tripartitioned character of the q-matrices. In [Malinowski, 2011] and [Malinowski, 1994], Malinowski shows how to increase the number of partitions of the matrix universe for more than three and define alternative notions of entailment based on them, althoug he does not show how to obtain alternative abstract notions of consequence in each case. From this approach, we see that Suszko's Reduction is applied only to the referential nature of many-valuedness as discussed in Chapter 1. The inferential nature of many-valuedness can evade bivalence by making use of Malinowski's ideas about the existence of a class of 'undetermined' values. This makes possible to import Suszko's Thesis in a generalized version for q-logics.

3.1.1 Reducing Q-logics

In this section, we show how to use the same algorithmic procedure proposed at [Caleiro et al., 2007] in order to reduce a q-logic to a three-valued semantics. The underlying strategy is kept all the same as illustrated in Section 2.2, the only difference is that now we must construct a three-valued semantics defined by three-valued axioms describing the behavior of each operator from the original logic.

Consider now the q-version of Gödel's four-valued logic:

$$G_4^q = \langle \mathcal{V}_4, \mathcal{D}_4, \mathcal{R}_4, \{f_\neg, f_\rightarrow\}\rangle.$$

where $\mathcal{V}_4 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, with $\mathcal{D}_4 = \{1\}$ and $\mathcal{R}_4 = \{0\}$. The operations over the truth-values can be defined as $f_\rightarrow(v(\alpha), v(\beta)) = 1$, if $v(\alpha) \leq v(\beta)$ and $f_\rightarrow(v(\alpha), v(\beta)) = v(\beta)$, if $v(\alpha) > v(\beta)$; $f_\neg(v(\alpha)) = 1$, if $v(\alpha) = 0$, and $f_\neg(v(\alpha)) = 0$ otherwise. Then the operations give us the following truth-tables:

$\rightarrow$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
0	1	1	1	1
$\frac{1}{3}$	0	1	1	1
$\frac{2}{3}$	0	$\frac{1}{3}$	1	1
1	0	$\frac{1}{3}$	$\frac{2}{3}$	1

	$\neg$
0	1
$\frac{1}{3}$	0
$\frac{2}{3}$	0
1	0

Note that the function $f(x)$ as defined in the previous section gives us the following assignment:

ϕ	$f(\phi)$
0	F
$\frac{1}{3}$	N
$\frac{2}{3}$	N
1	T

The step-by-step described by the end of Section 2.2 tell us to first look for adequate separators. However, for the sake of economy, we shall not list all unary truth-functions and search the separators among them (as it was done in Section 2.2), instead we shall use a conservative extension of the logic by adding an adequate separator. Let us first recall the definition of a separator:

Definition 27 ([Caleiro et al., 2007]). *Let x_1, $x_2 \in \mathcal{V}_4$. We say that x_1 and x_2 are **separated**, and we write $x_1 \# x_2$, in case x_1 and x_2 belong to different classes of truth-values. We say that a one-variable formula $\theta(p)$ of For **distinguishes** two truth-values x and y if $v(\theta(x)) \# v(\theta(y))$. A logic is called **separable** in case its truth-values are pairwise distinguishable*

Therefore the 3-valued description of the q-version of Gödel's four-valued logic depends on being able to separate the values inside the set $\mathcal{N} = \mathcal{V} - (\mathcal{D} \cup \mathcal{R})$. For that sake, define as a separator the formula $\theta(\phi) \stackrel{def}{=} \phi \leftrightarrow \frac{2}{3}$, where $(\phi \leftrightarrow \psi) = (\phi \to \psi) \wedge (\psi \to \phi)$[5]. We shall then obtain the following truth-table:

ϕ		$\theta(\phi)$	
0	F	0	F
$\frac{1}{3}$	N	$\frac{1}{3}$	N
$\frac{2}{3}$	N	1	T
1	T	$\frac{2}{3}$	N

The third step of the algorithmic reduction procedure tell us to use the separator to obtain the 'ternary prints' of each truth-value of $\mathcal{V}_4$, where the ternary prints are three-valued characterizations of the truth-values. Based on the truth-table above, we shall have:

$$v(\phi) = 0 \text{ iff } t(\phi) = F \ \& \ t(\theta(\phi)) = F$$
$$v(\phi) = \tfrac{1}{3} \text{ iff } t(\phi) = N \ \& \ t(\theta(\phi)) = N$$
$$v(\phi) = \tfrac{2}{3} \text{ iff } t(\phi) = N \ \& \ t(\theta(\phi)) = T$$
$$v(\phi) = 1 \text{ iff } t(\phi) = T \ \& \ t(\theta(\phi)) = N$$

[5] Here we are building a linguistic extension of the logic since we have only '$\neg$' and '$\to$' as our primitive operators. The '$\wedge$' works in the usual way as the $min(v(\alpha), v(\beta))$. Theorem 3.1.5 below will also guarantee that such a construction results in a conservative extension of the logic.

Given our ternary prints, the next step is to describe every line of the truth-tables by means of the conditions for truth, falsity and undeterminedness. Observe that in this case, since we have three-values, we shall have more unobtainable conditions corresponding to impossible assignments. So we will naturally have more closure conditions, in this case, we shall obtain the conditions (C1)-(C6) below, where (C6) internalizes the fact that every valuation is a function and (C1)-(C5) forbids the other possible combinations for the three truth-values. At last, the following axioms are obtained:

(N1) $t(\neg\phi) = T \Rightarrow t(\phi) = F \;\&\; t(\theta(\phi)) = F$

(N2) $t(\neg\phi) = F \Rightarrow t(\phi) = N \;\&\; t(\theta(\phi)) = N \mid t(\phi) = N \;\&\; t(\theta(\phi)) = T \mid t(\phi) = T \;\&\; t(\theta(\phi)) = N$

(N3) $t(\theta(\neg\phi)) = F \Rightarrow t(\neg\phi) = F$

(N4) $t(\theta(\neg\phi)) = N \Rightarrow t(\neg\phi) = T$

(I1) $t(\phi \to \psi) = T \Rightarrow t(\phi) = F \mid t(\psi) = T \mid t(\theta(\phi)) = T \;\&\; t(\theta(\psi)) = T \mid t(\theta(\phi)) = T \;\&\; t(\psi) = t(\theta(\psi)) = N \mid t(\phi) = t(\theta(\phi)) = t(\psi) = t(\theta(\psi)) = N$

(I2) $t(\phi \to \psi) = F \Rightarrow t(\theta(\phi)) = T \;\&\; t(\psi) = F \mid t(\phi) = t(\theta(\phi)) = N \;\&\; t(\psi) = F \mid t(\phi) = T \;\&\; t(\psi) = F$

(I3) $t(\phi \to \psi) = N \Rightarrow t(\phi) = t(\theta(\phi)) = N \;\&\; t(\theta(\psi)) = T \mid t(\phi) = T \;\&\; t(\theta(\psi)) = T \mid t(\phi) = T \;\&\; t(\psi) = t(\theta(\psi)) = N$

(I4) $t(\theta(\phi \to \psi)) = T \Rightarrow t(\theta(\phi)) = t(\phi) = N \;\&\; t(\theta(\psi)) = T \mid t(\phi) = T \;\&\; t(\theta(\psi)) = T$

(I5) $t(\theta(\phi \to \psi)) = F \Rightarrow t(\theta(\phi) = T \;\&\; t(\psi) = F \mid t(\phi) = t(\theta(\phi)) = N \;\&\; t(\psi) = F \mid t(\phi) = T \;\&\; t(\psi) = F$

(I6) $t(\theta(\phi \to \psi)) = N \Rightarrow t(\phi) = F \mid t(\psi) = T \mid t(\psi) = t(\theta(\psi)) = N \mid t(\theta(\phi)) = t(\theta(\psi)) = T$

(C1) $t(\phi) = T \;\&\; t(\theta(\phi)) = T \Rightarrow *$

(C2) $t(\phi) = T \;\&\; t(\theta(\phi)) = F \Rightarrow *$

(C3) $t(\phi) = F \;\&\; t(\theta(\phi)) = T \Rightarrow *$

(C4) $t(\phi) = F \;\&\; t(\theta(\phi)) = N \Rightarrow *$

(C5) $t(\phi) = N \;\&\; t(\theta(\phi)) = F \Rightarrow *$

(C6) $t(\phi) = X \;\&\; t(\neg\phi) = X \Rightarrow *$, for $X \in \mathcal{V}_3$

Now define $\text{SEM}_{G_3^q}$ as the set of trivaluations respecting all axioms above. As in the example exhibited in Section 2.2, the following lemmas are guaranteed:

Lemma 3.1.1. *Given a valuation* $v \in \text{SEM}_{G_4^q}$*, it is possible to define an interpretation* $t_v \in \text{SEM}_{G_3^q}$ *such that* $v(\phi) \in \mathcal{D}$ *iff* $t_v(\phi) = T$ *and* $v(\phi) \in \mathcal{R}$ *iff* $t_v(\phi) = F$*.*

Given a valuation $v \in \text{SEM}_{G_4^q}$, define:

$$t_v(\phi) = \begin{cases} F, & \text{if } v(\phi) = 0 \\ N, & \text{if } v(\phi) = \frac{1}{3} \text{ or } v(\phi) = \frac{2}{3} \\ T, & \text{if } v(\phi) = 1 \end{cases}$$

Proof. The proof is made by induction on the complexity of the formulas.

Basic case:

($\Rightarrow$)
For an atom p. Suppose $v(p) \in \mathcal{D}$ or $v(p) \in \mathcal{R}$. For case (i), we have $v(p) = 1$. Thus, by definition of t_v, it follows $t_v(p) = T$. For case (ii), we have that $v(p) = 0$. Therefore, by definition of t_v, we shall obtain $t_v(p) = F$.

($\Leftarrow$)
Now suppose (i) $t_v(p) = T$ or (ii) $t_v(p) = F$. For case (i), by definition of t_v, we have $v(p) = 1$, i.e, $v(p) \in \mathcal{D}$. For case (ii), by definition of t_v, it follows $v(p) = 0$, i.e, $v(p) \in \mathcal{R}$.

Inductive step:
Inductive Hypothesis: for all formulas α and β such that $\ell(\alpha) < \ell(\phi)$ and $\ell(\beta) < \ell(\phi)$, it follows that $t_v(\alpha) = T$ iff $v(\alpha) \in \mathcal{D}$ or $t_v(\alpha) = F$ iff $v(\alpha) \in \mathcal{R}$, and $t_v(\beta) = T$ iff $v(\beta) \in \mathcal{D}$ or $t_v(\beta) = F$ iff $v(\beta) \in \mathcal{R}$.

1. $\phi = \neg\alpha$
($\Rightarrow$)
Suppose (i) $v(\neg\alpha) \in \mathcal{D}$ or (ii) $v(\neg\alpha) \in \mathcal{R}$. Since $\ell(\alpha) < \ell(\neg\alpha)$, by inductive hypothesis follows (iii) $v(\alpha) \in \mathcal{D}$ iff $t_v(\alpha) = T$ and (iv) $v(\alpha) \in \mathcal{R}$ iff $t_v(\alpha) = F$.

From (i), (iii) and definition of $f_\neg$, we have that $v(\alpha) \notin \mathcal{D}$. Therefore, by contraposition on (iii), it follows that $t_v(\alpha) \neq T$, i.e, (v) $t_v(\alpha) = F$ or (vi) $t_v(\alpha) = N$. For case (vi), by the closure condition (C5), it follows (vii) $t_v(\alpha) = N \& t_v(\theta(\alpha)) = N$ or (viii) $t_v(\alpha) = N \& t_v(\theta(\alpha)) = T$. For both cases, by contraposition on Axiom (N2), it follows that $t_v(\neg\alpha) \neq F$, i.e, (ix) $t_v(\neg\alpha) = T$ or (x) $t_v(\neg\alpha) = N$. From closure condition (C6) in (xi) and (vi), we have only (ix), that is our desired result. For case (v). Since we have (iv), it follows that $v(\alpha) \in \mathcal{R}$. Therefore $v(\alpha) = 0$ and, by definition of $f_\neg$, we shall obtain $v(\neg\alpha) = 1$. Finally, by definition of t_v, we will have $t_v(\neg\alpha) = T$.

For our supposition (ii), since we have (iv), by definition of $f_\neg$ and contraposition, we shall obtain $t_v(\alpha) \neq F$, i.e, (x) $t_v(\alpha) = T$ or (xi) $t_v(\alpha) = N$. For case (x), since we have (iii), by modus ponens follows that $v(\alpha) \in \mathcal{D}$, i.e, $v(\alpha) = 1$. Thus, by definition of $f_\neg$, we have $v(\neg\alpha) = 0$ and, by definition of t_v, $t_v(\neg\alpha) = F$. For case (xi), by definition of t_v, it follows $v(\alpha) = \frac{1}{3}$ or $v_t(\alpha) = \frac{2}{3}$. Therefore, from definition of $f_\neg$, we shall obtain $v(\neg\alpha) = 0$. At last, by definition of t_v, our desired result $t_v(\neg\alpha) = F$ follows.

($\Leftarrow$)
Suppose (i) $t_v(\neg\alpha) = T$ or (ii) $t_v(\neg\alpha) = F$.
For case (i), from Axiom (N1), it follows $t_v(\alpha) = F$. Thus, by definition of t_v, we shall obtain $v(\alpha) = 0$. Finally, by definition of $f_\neg$, it follows $v(\neg\alpha) = 1$, i.e, $v(\neg\alpha) \in \mathcal{D}$.

For case (ii), by Axiom (N2), we shall obtain (i) $t_v(\alpha) = N$ or $t_v(\alpha) = T$. Therefore from definition of t_v, it follows (i) $v(\alpha) = \frac{1}{3}$, (ii) $v(\alpha) = \frac{2}{3}$ or (iii) $v(\alpha) = 1$. From this, by definition of $f_\neg$, for all cases we shall obtain our desired conclusion $v(\neg\alpha) = 0$, i.e, $v(\neg\alpha) \in \mathcal{R}$.

2. $\phi = (\theta(\neg\alpha))$
($\Rightarrow$)
Suppose (i) $v(\theta(\neg\alpha)) \in \mathcal{D}$ or (ii) $v(\theta(\neg\alpha)) \in \mathcal{R}$. Since $\ell(\neg\alpha) < \ell(\theta(\neg\alpha))$, by indutive hypothesis follows (iii) $t_v(\neg\alpha) = T$ iff $v(\neg\alpha) \in \mathcal{D}$ and (iv) $t_v(\neg\alpha) = F$ iff $v(\neg\alpha) \in \mathcal{R}$. Note case (i) is done since there is no case where $v(\theta(\neg\alpha)) = 1$.

For case (ii), by definition of θ, we have that $v(\neg\alpha) \in \mathcal{R}$. Thus, by modus ponens on (iv), it follows $t_v(\neg\alpha) = F$. Finally, by contraposition on Axiom (N4), we shall obtain $t_v(\theta(\neg\alpha)) \neq N$ and since there is no case that $t_v(\theta(\neg\alpha)) = T$, our desired conclusion $t_v(\theta(\neg\alpha)) = F$ follows.

($\Leftarrow$)
Suppose (i) $t_v(\theta(\neg\alpha)) = T$ or (ii) $t_v(\theta(\neg\alpha)) = F$. Since there is no axiom for case (i), by definition of t_v, it follows that $v(\theta(\neg\alpha)) = 1$, i.e, $v(\theta(\neg\alpha)) \in \mathcal{D}$.

For case (ii), by Axiom (N3), it follows $t_v(\neg\alpha) = F$. Therefore, by definition of t_v, we have $v(\neg\alpha) = 0$. Thus, by definition of $\theta(\phi)$, our desired conclusion $v(\theta(\neg\alpha)) = 0$, i.e, $v(\theta(\neg\alpha)) \in \mathcal{R}$ follows.

The remaining cases for $\phi = (\alpha \to \beta)$ and $\phi = (\theta(\alpha \to \beta))$ runs in a similar manner.

□

Lemma 3.1.2. *Given a valuation* $t \in \mathrm{SEM}_{G_3^q}$*, it is possible to define an interpretation* $v_t \in \mathrm{SEM}_{G_4^q}$ *such that* $v_t(\phi) \in \mathcal{D}$ *iff* $t(\phi) = T$ *and* $v_t(\phi) \in \mathcal{R}$ *iff* $t(\phi) = F$*.*

Given a valuation $t \in \mathrm{SEM}_{G_3^q}$, define:

$$v_t(\phi) = \begin{cases} 0, & \text{if } t(\phi) = F \ \& \ t(\theta(\phi)) = F \\ \frac{1}{3}, & \text{if } t(\phi) = N \ \& \ t(\theta(\phi)) = N \\ \frac{2}{3}, & \text{if } t(\phi) = N \ \& \ t(\theta(\phi)) = T \\ 1, & \text{if } t(\phi) = T \ \& \ t(\theta(\phi)) = N \end{cases}$$

Proof. The proof is made by induction on the complexity of the formulas.

Basic case:

($\Rightarrow$)
For an atom p, suppose (i) $t(p) = T$ or (ii) $t(p) = F$. For case (i), by closure conditions (C1) and (C2), it folows $t(\theta(p)) = N$. Therefore, by definition of v_t, we have $v_t(p) = 1$, i.e, $v_t \in \mathcal{D}$. For case (ii), by closure conditions (C3) and (C4), it follows that $t(\theta(p)) = F$. Then by definition of v_t, it follows our desired conclusion $v_t(p) = 0$, i.e, $v_t \in \mathcal{R}$.

($\Leftarrow$)
Suppose (i) $v_t(p) \in \mathcal{D}$, i.e, $v_t(p) = 1$ or (ii) $v_t(p) \in \mathcal{R}$, i.e, $v_t(p) = 0$. For case (i), by definition of v_t, our desired result follows $t(p) = T$. For case (ii), by definition of v_t again, we shall have $t(p) = F$.

Inductive step:
Inductive Hypothesis: for all formulas α and β such that $\ell(\alpha) < \ell(\phi)$ and $\ell(\beta) < \ell(\phi)$, it follows that $t(\alpha) = T$ iff $v_t(\alpha) \in \mathcal{D}$ or $t(\alpha) = F$ iff $v_t(\alpha) \in \mathcal{R}$, and $t(\beta) = T$ iff $v_t(\beta) \in \mathcal{D}$ or $t(\beta) = F$ iff $v_t(\beta) \in \mathcal{R}$.

1. $\phi = \neg\alpha$
($\Rightarrow$)
Suppose (i) $t(\neg\alpha) = T$ or (ii) $t(\neg\alpha) = F$.
For case (i), from Axiom (N1), it follows $t(\alpha) = F \ \& \ t(\theta(\alpha) = F$. Thus, by definition of v_t, we shall obtain $v_t(\alpha) = 0$. Finally, by definition of $f_\neg$, it follows $v_t(\neg\alpha) = 1$, i.e, $v_t(\neg\alpha) \in \mathcal{D}$.

For case (ii), by Axiom (N2), we shall obtain (i) $t(\alpha) = N \ \& t(\theta(\alpha)) = N$, (ii) $t(\alpha) = N \ \& \ t(\theta(\alpha)) = T$ or (iii) $t(\alpha) = T \ \& \ t(\theta(\alpha)) = N$. Therefore from definition of v_t, it follows (i) $v_t(\alpha) = \frac{1}{3}$, (ii) $v_t(\alpha) = \frac{2}{3}$ or (iii) $v_t(\alpha) = 1$. From this, by definition of $f_\neg$, for all cases we shall obtain our desired conclusion $v_t(\neg\alpha) = 0$, i.e, $v_t(\neg\alpha) \in \mathcal{R}$.

($\Leftarrow$)
Suppose (i) $v_t(\neg\alpha) \in \mathcal{D}$ or (ii) $v_t(\neg\alpha) \in \mathcal{R}$. Since $\ell(\alpha) < \ell(\neg\alpha)$, by inductive hypothesis follows that (iii) $t(\alpha) = T$ iff $v_t(\alpha) \in \mathcal{D}$ and (iv) $t(\alpha) = F$ iff $v_t(\alpha) \in \mathcal{R}$.

From (i), (iii) and definition of $f_\neg$, we have that $v_t(\alpha) \notin \mathcal{D}$. Therefore, by contraposition on (iii), it follows that $t(\alpha) \neq T$, i.e, (v) $t(\alpha) = F$ or (vi) $t(\alpha) = N$. For case (vi), by the closure condition (C5), it follows (vii) $t(\alpha) = N\&t(\theta(\alpha)) = N$ or (viii) $t(\alpha) = N\&t(\theta(\alpha)) = T$. For both cases, by contraposition on Axiom (N2), it follows that $t(\neg\alpha) \neq F$, i.e, (ix) $t(\neg\alpha) = T$ or (x) $t(\neg\alpha) = N$. From closure condition (C6) in (xi) and (vi), we have only (ix), our desired result. For case (v). Since we have (iv), it follows that $v_t(\alpha) \in \mathcal{R}$. Therefore $v_t(\alpha) = 0$ and, by definition of $f_\neg$, we shall obtain $v_t(\neg\alpha) = 1$. Finally, by definition of v_t, we will have $t(\neg\alpha) = T$.

For our supposition (ii), since we have (iv), by definition of $f_\neg$ and contraposition, we shall obtain $t(\alpha) \neq F$, i.e, (x) $t(\alpha) = T$ or (xi) $t(\alpha) = N$. For case (x), since we have (iii), it follows that $v_t(\alpha) \in \mathcal{D}$, i.e, $v_t(\alpha) = 1$. Thus, by definition of $f_\neg$, we have $v_t(\neg\alpha) = 0$ and $t(\neg\alpha) = F$. For case (xi), by closure condition (C5), we have $t(\alpha) = N \ \& \ t(\theta(\alpha)) = N$ or $t(\alpha) = N \ \& \ t(\theta(\alpha)) = T$. Then, by definition of v_t, it follows $v_t(\alpha) = \frac{1}{3}$ or $v_t(\alpha) = \frac{2}{3}$. From definition of $f_\neg$, we shall obtain $v_t(\neg\alpha) = 0$. Therefore, by definition of v_t, our desired result $t(\neg\alpha) = F$ follows.

2. $\phi = \theta(\neg\alpha)$
($\Rightarrow$)
Suppose (i) $t(\theta(\neg\alpha)) = T$ or (ii) $t(\theta(\neg\alpha)) = F$. Since there is no axiom for case (i), by definition of v_t, it follows that $v_t(\theta(\neg\alpha)) = 1$.

For case (ii), by Axiom (N3), it follows $t(\neg\alpha) = F$. Therefore, by definition of v_t, we have $v_t(\neg\alpha) = 0$. Thus, by definition of $\theta(\phi)$, our desired conclusion $v_t(\theta(\neg\alpha)) = 0$ follows.

($\Leftarrow$)
Suppose (i) $v_t(\theta(\neg\alpha)) \in \mathcal{D}$ or (ii) $v_t(\theta(\neg\alpha)) \in \mathcal{R}$. Since $\ell(\neg\alpha) < \ell(\theta(\neg\alpha))$, by indutive hypothesis follows (iii) $t(\neg\alpha) = T$ iff $v_t(\neg\alpha) \in \mathcal{D}$ and (iv) $t(\neg\alpha) = F$ iff $v_t(\neg\alpha) \in \mathcal{R}$. Note case (i) holds since there is no case where $v_t(\neg\alpha)) = 1$.

For case (ii), by definition of v_t, we have that $v_t(\neg\alpha) \in \mathcal{R}$. Thus, by modus ponens on (iv), it follows $t(\neg\alpha) = F$. Finally, by contraposition on Axiom (N4), we shall obtain $t(\theta(\neg\alpha)) \neq N$ and since there is no case for $t(\theta(\neg\alpha)) = T$, our desired conclusion $t(\theta(\neg\alpha)) = F$ follows.

The remaining cases for $\phi = (\alpha \to \beta)$ and $\phi = (\theta(\alpha \to \beta))$ runs in a similar manner.

□

The proof of the above lemmas is made by following reasoning analogous to the one employed in Section 2.2. Based on that, as in the case of Theorem 2.2.1, the following is a corollary:

Theorem 3.1.5. $\mathrm{SEM}^q_{G^q_4}$ *is a sound and complete semantics for* $\mathrm{SEM}^q_{G^q_3}$

$$\Gamma \models^q_{\mathrm{SEM}_{G^q_4}} \alpha \Leftrightarrow \Gamma \models^q_{\mathrm{SEM}_{G^q_3}} \alpha$$

Proof. ($\Rightarrow$)

Suppose (i) $\Gamma \models^q_{\mathrm{SEM}_{G^q_4}} \alpha$. Then, by definition of q-entailment, we know that (ii) for every $v \in \mathrm{SEM}_{G^q_4}$, if $v(\Gamma) \cap \mathcal{R}_4 = \emptyset$, then $v(\alpha) \in \mathcal{D}_4$. Now take an arbitrary valuation from $v \in \mathrm{SEM}_{G^q_4}$ and suppose (iv) $v(\Gamma) \cap \mathcal{R}_4 = \emptyset$. From Lemma 3.1.1, it is possible to define an interpretation t_v such that $t_v(\Gamma) \cap \{F\} = \emptyset$. Moreover, from (iv) and (ii), we shall obtain $v(\alpha) \in \mathcal{D}_4$. Therefore, by using Lemma 3.1.1, it follows $t_v(\alpha) = T$. Finally, by definition of q-entailment, we shall have $\Gamma \models^q_{\mathrm{SEM}_{G^q_3}} \alpha$.

($\Leftarrow$)
Suppose (i) $\Gamma \models^q_{\mathrm{SEM}_{G^q_3}} \alpha$. Then, by definition of q-entailment, we know that (ii) for every $t \in \mathrm{SEM}_{G^q_3}$, if $t(\Gamma) \cap \{F\} = \emptyset$, then $t(\alpha) = T$. Now take an arbitrary valuation from $t \in \mathrm{SEM}_{G^q_3}$ and suppose (iv) $t(\Gamma) \cap \{F\} = \emptyset$. From Lemma 3.1.2, it is possible to define an interpretation v_t such that $v_t(\Gamma) \cap \mathcal{R} = \emptyset$. Moreover, from (iv) and (ii), we shall obtain $t(\alpha) = T$. Therefore, by using Lemma 3.1.2 again, it follows $v_t(\alpha) \in \mathcal{D}$. Finally, by definition of q-entailment, we shall have $\Gamma \models^q_{\mathrm{SEM}_{G^q_4}} \alpha$.

□

Theorem 3.1.5 is a constructive version of Malinowski's reduction. The reduction made in this section suggest that three-valued semantics might be expressed in a gentzenian format likewise the two-valued semantics. The structure guaranteed by a gentzenian three-valued format would allow the construction of three-signed tableaux rules able to characterize finite-valued q-logics. The results suggests that the same algorithmic procedure proposed by Caleiro et al may be updated to a three-valued format in claming that any finite-valued q-logic may be characterized by a three-valued semantics. In the next section, we exhibit another logically three-valued class of logics, proposed as the dual to q-logics, the so-called p-logics.

3.1.2 Plausible entailment

In [Frankowski, 2004], Szymon Frankowski presented a different version of the q-matrix proposed by Malinowski and introduced a consequence relation considered the natural dual of q-consequence operations, the so-called *p-consequence operations*.

In what follows, we will show the technical construction behind p-consequence:

Definition 28. *A p-**consequence relation** is a single-conclusion consequence relation such that for every $\varphi \in For$ and every $\Delta, \Gamma \subseteq For$ it has the following properties* [6]*:*

$$\Delta \cup \{\varphi\} \Vdash^p \varphi \text{ (Reflexivity)} \tag{3.15}$$

$$\text{If } \Delta \Vdash^p \varphi \text{ then } \Delta \cup \Gamma \Vdash^q \varphi \text{ (Monotonicity)} \tag{3.16}$$

We define a p-**logic** as $\mathcal{L}^p = \langle For, \Vdash^p \rangle$, where $\Vdash^p$ is a p-consequence relation. Remark 3.1.1 has shown us why (Weak-Cut) fails for p-logics, in the presence of (Weak-Cut), p-consequence relations are reduced to Tarskian consequence relations. This highlight the fact that q- and p-logics are constructed by trading (Reflexivity) for (Weak-Cut).

Definition 29. *A **logical** p-**matrix** based on $\mathcal{L}^p$ is an algebra $\mathbb{M}^p = \langle \mathcal{V}, \mathcal{R}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{D}$ and $\mathcal{R}$ are non-empty subsets of $\mathcal{V}$, with $\mathcal{V} = \mathcal{D} \cup \mathcal{R}$, and for every n-ary connective c from Σ, $\mathcal{O}$ includes a corresponding n-ary function f_c: $\mathcal{V}^n \to \mathcal{V}$.*

*Call $\mathcal{A} \stackrel{def}{=} (\mathcal{D} - \mathcal{R})$ the **accepted** set of values and $\mathcal{R}$ the **rejected** set. Note that if $\mathcal{D} \cap \mathcal{R} = \emptyset$, the p-matrix is reduced to the usual concept of matrix (Definition 8). We also shall use $\mathcal{B}$ to denote the set of **both** designated and rejected values, i.e, $\mathcal{B} \stackrel{def}{=} \mathcal{D} \cap \mathcal{R}$.*

Definition 30. *Any valuation from $\mathcal{L}^p$ into a p-matrix induces a relation of plausible-**entailment** (hereafter p-**entailment**) or Frankowskian entailment, defined in the following way:*

$$\Gamma \models^p_{\mathrm{SEM}^p} \alpha \text{ iff } v(\Gamma) \subseteq \mathcal{A} \text{ implies } v(\alpha) \notin \mathcal{R}, \tag{3.17}$$

for every $v \in \mathrm{SEM}^p$, where SEM^p is any set of valuations (homomorphisms) v from For into a p-matrix.

plausible-entailment was introduced with the intention of formalizing a kind of reasoning that, from accepted premises follows non-rejected conclusions. The use of the term plausible indicates the idea of approximate. His intuitions on the plausibility relation was inspired in Ajdukiewicz's project of a Pragmatic Logic. At the core of Ajdukewicz's project is the idea of analyzing formal and informal argumentation.

By making use of the same tools explored in the previous section, it is possible to prove Wójcicki's Reduction for p-matrices[7] and Frankowski's Reduction:

Theorem 3.1.6 (Frankowski's reduction)**.** *Every substitution-invariant p-logic is 2-valued or 3-valued.*

[6] Cf. [Frankowski, 2004]
[7] Cf. [Malinowski, 2011].

3.2 Shramko & Wansing beyond inferential many-valuedness

In [Wansing and Shramko, 2008] and [Shramko and Wansing, 2011], the authors analyse Suszko's thesis from a different angle in comparison to Malinowski's approach. The importance of Shramko & Wansing's approach is in raising the question of what exactly should characterize a logical value. Therefore, by aiming to propose a new definition of logical value, they criticize Malinowski's notion of logical many-valuedness. Based on an alternative definition of logical value, they attack the meaning of Suszko's reduction and propose what should be recognized as an adequate logically two-valued logic.

Take some set of truth-values $\mathcal{V} = \mathcal{D} \cup \mathcal{U}$, with $\mathcal{D} \cap \mathcal{U} = \emptyset$, where $\mathcal{D}$ is the usual set of designated values and $\mathcal{U}$ is the set of undesignated values. Remember from Section 1.2 and 1.3 that Suszko's conception of logical values is defined as the subsets of the set of algebraic values, and therefore $\mathcal{D}$ stands for the logical value true, while $\mathcal{U}$ stands for the logical value false. Shramko & Wansing start by calling attention to the fact that the logical value *true* is determined by the specification of a set of designated algebraic values and a corresponding notion of entailment associated to it. Thus, a formula α follows from a set of premises Γ if and only if the conclusion is designated whenever all the premises are. In such a context, a valid inference is one that preserves truth from the premises to the conclusion. Let us call ***t*-entailment** such notion of logical consequence .

Now, since falsity is determined by the complement of the designated set, called by us undesignated, its associated notion of consequence as preservation of falsity must be read in the opposite direction, i.e, from the conclusion to at least one of the premises. Thus if the conclusion is false this implies that at least one of the premises is also false. Such notion of inference is called by them ***f*-entailment** and is understood as the backwards preservation of undesignated values. Therefore, according to Shramko & Wansing, our traditional division of the set of truth-values gives rise to two notions of entailment, on one hand, the membership in $\mathcal{D}$ (t-entailment) is preserved from the premises to the conclusion; on the other hand, the membership in $\mathcal{U}$ (f-entailment) is preserved from the premises to the conclusion, what gives us the very same notion of t-entailment.[8]

Shramko & Wansing goes on and point out, "since $\mathcal{D}$ is uniquely determined by its complement and vice-versa, logical two-valuedness is, in fact, reduced to logical *mono-*

[8] Define:
$\Gamma \models_t \alpha$ iff for all v, $(v(\Gamma) \subseteq \mathcal{D}$ implies $v(\alpha) \in \mathcal{D})$
and
$\Gamma \models_f \alpha$ iff for al v, $(v(\alpha) \in \mathcal{U}$ implies $v(\Gamma) \cap \mathcal{U} \neq \emptyset)$
Then, note that:
$\Gamma \models_f \alpha$ iff $(v(\alpha) \in \mathcal{U}$ implies $v(\Gamma) \cap \mathcal{U} \neq \emptyset)$
$\Gamma \models_f \alpha$ iff $(v(\alpha) \notin \mathcal{U}$ or $v(\Gamma) \cap \mathcal{U} \neq \emptyset)$
$\Gamma \models_f \alpha$ iff $(v(\Gamma) \cap \mathcal{U} \neq \emptyset$ or $v(\alpha) \notin \mathcal{U})$
$\Gamma \models_f \alpha$ iff $(v(\Gamma) \cap \mathcal{U} = \emptyset$ implies $v(\alpha) \notin \mathcal{U})$
Then, since $\mathcal{V} - \mathcal{U} = \mathcal{D}$ and v is a total function, we have:
$\Gamma \models_t \alpha$ iff $v(\Gamma) \subseteq \mathcal{D}$ implies $v(\alpha) \in \mathcal{D}$. Therefore, $\models_f \, = \, \models_t$.

valuedness if there is just one entailment relation defined as truth preservation from the premises to the conclusion". Consequently, the two entailment relations above, when associated to each logical value, are equivalent. In this context, since Suszko's reduction rests on the partition of the set of truth-values, Shramko & Wansing attack it by claiming that Suszko's reduction is actually a reduction into a mono-valuedness since each logical value imply the same notion of entailment. This criticism will serve as a key to Shramko & Wansing's definition of logical value.

Before exhibiting Shamko & Wansing's concept of logical value, let us first take a look at how they criticize Malinowki's q-consequence. If we do not treat falsity as a mere abbreviation for 'non-truth' and if we correspondingly distinguish not only a set $\mathcal{D}$ of designated algebraic values but also a set $\mathcal{R}$ of 'rejected' algebraic values such that $\mathcal{V} \neq \mathcal{D} \cup \mathcal{R}$, therefore f-entailment may be different from t-entailment, since preservation of designatedness from the premises to the conclusion is not the same as preservation of rejectedness from conclusion to the premises. Thereby, Shramko & Wansing rightly remark about Malinowki's construction:

> "This approach [q-matrices] may be viewed as taking 'true' and 'false' to be expressions that give rise to contrary instead of contradictory pairs of sentences. As such, the pair 'true' versus 'false' is reflected by the contrary pairs 'designated' versus 'antidesignated' and 'accepted' versus 'rejected'. Admitting algebraic values that are neither designated nor antidesignated amounts to admitting, in addition to the logical values *true* and *false*, the third logical value *neither true nor false*. In other words, being false is distinguished from not being true." [Shramko and Wansing, 2011]

Since q-matrices open the room for the admission of three logical values, accepted, rejected and neither, we can think of at least three different entailment relations, one as preservation of accepted values, one as preservation of rejected values and other as preservation of neither accepted, nor rejected values. For Shramko & Wansing there is no good reason to prefer one notion of logical consequence over the other. Thus, they propose an increase not only in the number of logical values, but also in the number of entailment relations associated to a given logic. For this purpose, they present a different way of understanding the notion of a *logical value* by stating:

> "If logic is thought of as the theory of valid inferences, then a *logical value* may be seen as a value that is used to define in a canonical way an entailment relation on a set of formulas. By a canonical definition of entailment we mean a definition of entailment as a relation that (in the single conclusion case) preserves membership in a certain set of algebraic values, either from the

premises to the conclusion inferences or from the conclusion to the premises. Such a relation will be Tarskian (...). Two logical values are independent of each other iff the canonically defined entailment relations associated with these values are distinct." [Shramko and Wansing, 2011]

Given what was stressed above, the authors differ from Malinowski's conception of what a logical value is. While Malinowski takes logical values as merely the partitions of the set of algebraic values, according to Shramko & Wansing a logical value is used to define a canonical notion of entailment and is independent insofar it defines also an independent entailment relation. These considerations led them to criticize Malinowski by claiming that Malinowski's approach is concerned with defining a *single* entailment relation which does not need to be canonically defined. By Shramko & Wansing's criteria, Malinowski does not treat truth and falsity as logical values, since they are both used only to define a single notion of consequence, given by the q-entailment. They still complement "if the idea of entailment as the preservation of a logical value is given up, then entailment, will not, in general, be a Tarskian relation(...) We want inferability to preserve!". Moreover, following Malinowski's criteria, we could say that any subset of algebraic values is a logical value. On the other hand, for Shramko & Wansing, a subset of algebraic values only deserve to be called a logical value as long as it implies a canonical notion of entailment that is independent from all the others. Therefore a logical value is defined by the entailment relation induced by it.

In [Shramko and Wansing, 2007] and [Shramko and Wansing, 2011], the authors go on and propose a different notion of a logical system which allows for many logical values, associated with various corresponding consequence relations, each one being defined by the Tarskian properties of entailment. In what follows, we will show Shramko & Wansing's construction. Let us first introduce the notion of k-dimensional logic:

Definition 31. *A* ***Tarskian k-dimensional logic*** *(Tarskian k-logic) is a $k+1$ tuple $\mathcal{L} = (For, \Vdash_1, ..., \Vdash_k)$ such that:*
(i) For is a denumerable set of formulas,
(ii) for every $i \leq k$, $\Vdash_i \subseteq \wp(For) \times For$, where $k \geq 2$ with $k \in \mathbb{N}$, and
(iii) every relation $\Vdash_i$ is a single conclusion Tarskian consequence relation. [9]

Definition 32. *A k-dimensional logic is called* ***substitution-invariant*** *if the following holds for each $\Vdash^i$:*

$$\Gamma \Vdash^i \alpha \textit{ implies } \sigma(\Gamma) \Vdash^i \sigma(\alpha), \textit{ for all } \sigma \in End(For) \textit{ (Substitution-invariance)} \quad (3.18)$$

where $\sigma(\Gamma) = \{\sigma(\gamma) | \gamma \in \Gamma\}$.

[9] (Recall Definition 7 at Section 2.1).

We associate to a Tarskian k-dimensional logic $\mathcal{L}$ a semantic structure by defining a k-dimensional matrix structure, proposed in [Shramko and Wansing, 2011], in the following manner:

Definition 33. *A* ***logical*** k***-dimensional matrix*** $\mathbb{M}^k$ *based on* $\mathcal{L}$ *is a tuple* $\mathbb{M}^k = \langle \mathcal{V}, \mathcal{D}_1, ..., \mathcal{D}_k, \mathcal{O} \rangle$*, where each* $\mathcal{D}_i$ *are pairwise distinct subsets of* $\mathcal{V}$ *(the set of truth-values) with* $1 \leq i \leq k$*, and for every* n*-ary connective* c *from* Σ_n*,* $\mathcal{O}$ *includes a corresponding* n*-ary function* $f_c \colon \mathcal{V}^n \to \mathcal{V}$*.*

We say that a semantics SEM is any collection of homomorphisms from a set of formulas For into $\mathbb{M}^k$. When necessary, we shall use again the subscript SEM_n in order to refer to the cardinality of the set of truth-values. Given the definition of a k-dimensional matrix, we introduce the associated notion of entailment:

Definition 34. *For each set* $\mathcal{D}_i$ *we have an associated canonical semantic consequence relation:*

$$\Gamma \models^i_{\text{SEM}} \alpha \textit{ iff } v(\Gamma) \subseteq \mathcal{D}_i \textit{ implies } v(\alpha) \in \mathcal{D}^i, \tag{3.19}$$

for every $v \in \text{SEM}$ and each $\mathcal{D}_i \in \mathbb{M}^k$, with $1 \leq i \leq k$.

Each distinguished set of truth-values in the k-dimensional matrix induces an associated notion of entailment. Also, each of these entailment relations is defined by the preservation of algebraic values from premises to the conclusion[10].

In the following we employ Lindenbaum matrices for k-dimensional matrices in order to prove yet another generalization of Wójcicki's reduction theorem. Remember that the closure of a set of formulas Γ is defined by $\Gamma^{\Vdash^i} = \{\alpha \mid \Gamma \Vdash^i \alpha\}$.

Consider a family of Tarskian k-dimensional logics $\mathcal{F} = \{\mathcal{L}_i\}_{i \in I}$ over some fixed set of formulas For. Define the *superlogic* $\mathcal{L}_\mathcal{F}$ of this family by considering $\mathcal{L}_\mathcal{F} = \langle For, \bigcap_{i \in I} \Vdash^1_i, ..., \bigcap_{i \in I} \Vdash^k_i \rangle$, where each $\mathcal{L}_i = \langle For, \Vdash^1_i, ..., \Vdash^k_i \rangle$ is a structural Taskian k-dimensional logic, for $i \in I$.

Remark 5. *The intersection of substiution-invariant Tarkian* k*-dimensional logics is a substitution-invariant Tarskian* k*-dimensional logic.*

Proof. Straight forward by the same reasoning employed to prove Remark 2. □

In the following we introduce the notion of a Lindenbaum k-dimensional matrix in order to prove a generalization of Wójcicki's Reduction for k-dimensional logics.

---

[10] In [Shramko and Wansing, 2007] and [Shramko and Wansing, 2007], the authors also consider the possibility of defining the entailment relation as preservation of values from conclusion to the premises, i.e, $v(\alpha) \in \mathcal{D}_i \Rightarrow v(\Gamma) \cap \mathcal{D}_i \neq \emptyset$, for every $v \in \text{SEM}$. However, without loss of generality, we shall use here only the direction as preservation of values from premises to the conclusion.

Definition 35. *Given a Tarskian k-dimensional logic* $\mathcal{L} = \langle For, \Vdash^1, ..., \Vdash^k \rangle$ *and a set of formulas* $\Gamma \subseteq For$, *we shall call a a* ***Lindenbaum*** k***-dimensional matrix*** $\mathcal{L}_\Gamma$ *the s-k-dimensional logic* $\mathcal{L}_\Gamma = \langle For, \models^1_\Gamma, ..., \models^k_\Gamma \rangle$ *such that:*

$$\mathcal{V} = For \tag{3.20}$$

$$\mathcal{D}^i = \Gamma^{\Vdash^i}, \text{ for } 1 \leq i \leq k. \tag{3.21}$$

$$\Delta \models^i_\Gamma \alpha \text{ iff } v(\Delta) \subseteq \Gamma^{\Vdash^i} \text{ implies } v(\alpha) \in \Gamma^{\Vdash^i}, \text{ for all uniform-substitution } g \in \mathrm{SEM}. \tag{3.22}$$

We call the ***Lindenbaum bundle*** *of* $\mathcal{L}$ *the set of all Lindenbaum k-dimensional matrices over a given* For, *that is,* $\{\mathcal{L}_\Gamma : \Gamma \subseteq For\}$.

Theorem 3.2.1. *Every Tarskian k-dimensional logic is n-valued.*

Proof. The proof run by following the same reasoning employed to prove Theorem 2.1.1. □

Suszko's Reduction, as proved in Theorem 2.1.3, may be applied to the case of Tarskian k-dimensional logics by following the same reasoning. Of course, in this case, the bivalent reduction must be carried out for each entailment relation $\models^k_i$, for $1 \leq i \leq k$. Thus, Suszko's Reduction can be applied to each dimension of a k-dimensional logic. However, seen as a whole, the reduction of a k-dimensional logic must not be logically two-valued. In [Shramko and Wansing, 2011], the authors present some examples of natural k-dimensional logics. One of them is based on a trilattice structure and proposed as a natural generalization of Belnap's four-valued logic.

3.2.1 Reducing Tarskian k-dimensional logics

In this section we show how to employ Caleiro et al's reduction procedure, exhibited in Section 2.2, in order to reduce a Tarskian k-dimensional logic. The very same idea used in Section 2.2 and 3.2 shall be carried out with the only difference that now the m-valued algebraic semantics shall be characterized by a n-valued semantics, where n is the number of partitions of the truth-values set, i.e, the number of logical values and, of course, $m > n$.

Consider now a k-dimensional version of Łukasiewicz's four-valued logic:

$$Ł^k_4 = \langle \mathcal{V}_4, \mathcal{D}^1_4, \mathcal{D}^2_4, \mathcal{D}^3_4, \{f_\neg, f_\to\} \rangle.$$

where $\mathcal{V}_4 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, with $\mathcal{D}_4^1 = \{0\}$, $\mathcal{D}_4^2 = \{\frac{1}{3}, \frac{2}{3}\}$ and $\mathcal{D}_4^3 = \{1\}$. The operations over the truth-values can be defined by $f_{\to}(v(\alpha), v(\beta)) = 1$, if $v(\alpha) \leq v(\beta)$ and $f_{\to}(v(\alpha), v(\beta)) = (1 - v(\alpha)) + v(\beta)$, if $v(\alpha) > v(\beta)$; $f_{\neg}(v(\alpha)) = 1 - v(\alpha)$. What give us the following truth-tables:

$\to$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
0	1	1	1	1
$\frac{1}{3}$	$\frac{2}{3}$	1	1	1
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1	1
1	0	$\frac{1}{3}$	$\frac{2}{3}$	1

	$\neg$
0	1
$\frac{1}{3}$	$\frac{2}{3}$
$\frac{2}{3}$	$\frac{1}{3}$
1	0

We define the semantics $\text{SEM}_{Ł_4^k}$ as the set of valuations $v : For \to \mathcal{V}_4$. Now define the function $z : \mathcal{V}_4 \to \mathcal{V}_3$, where $\mathcal{V}_3 = \{A, B, C\}$ and $\mathcal{D}_3^1 = \{A\}$, $\mathcal{D}_3^2 = \{B\}$, $\mathcal{D}_3^3 = \{C\}$, in the following way:

$$z(x) = \begin{cases} A, & \text{se } x \in \mathcal{D}^1 \\ B, & \text{se } x \in \mathcal{D}^2 \\ C, & \text{se } x \in \mathcal{D}^3 \end{cases}$$

Thus, given a valuation v and the function z, we can define a *g-valuation* $g_v = z \circ v$ and collect such g-valuations into the semantics $\text{SEM}_{Ł_3^k} = \{g_v \mid v \in \text{SEM}_{Ł_4^k}\}$. The use of the z function shall gives us the following:

ϕ	$z(\phi)$
0	A
$\frac{1}{3}$	B
$\frac{2}{3}$	B
1	C

Therefore the reduction of this k-dimensional version of Łukasiewicz's four-valued logics depends on being able to separate the values $\frac{1}{3}$ and $\frac{2}{3}$ inside the set $\mathcal{D}^2$. Now take as a separator the formula $\theta(\phi) \stackrel{def}{=} \phi \to \frac{1}{3}$. Then we shall obtain the following truth-table:

ϕ		$\theta(\phi)$	
0	A	1	C
$\frac{1}{3}$	B	1	C
$\frac{2}{3}$	B	$\frac{1}{3}$	B
1	C	$\frac{1}{3}$	B

Based on that, we shall obtain the following prints of each truth-value from $\mathcal{V}_4$:

$$v(\phi) = 0 \text{ iff } g(\phi) = A \ \& \ g(\theta(\phi)) = C$$
$$v(\phi) = \tfrac{1}{3} \text{ iff } g(\phi) = B \ \& \ g(\theta(\phi)) = C$$
$$v(\phi) = \tfrac{2}{3} \text{ iff } g(\phi) = B \ \& \ g(\theta(\phi)) = B$$
$$v(\phi) = 1 \text{ iff } t(\phi) = C \ \& \ t(\theta(\phi)) = B$$

Again, after defining the set of axioms for $\text{SEM}_{\text{Ł}_3^k}$, it is possible to prove the following Lemmas:

Lemma 3.2.1. *Given a valuation $v \in \text{SEM}_{\text{Ł}_4^k}$, it is possible to define an interpretation $g_v \in \text{SEM}_{\text{Ł}_3^k}$ such that $v(\phi) \in \mathcal{D}_4^1$ iff $g_v(\phi) \in \mathcal{D}_3^1$, and $v(\phi) \in \mathcal{D}_4^2$ iff $g_v(\phi) \in \mathcal{D}_3^2$, and $v(\phi) \in \mathcal{D}_4^3$ iff $g_v(\phi) \in \mathcal{D}_3^3$.*

Lemma 3.2.2. *Given a valuation $g \in \text{SEM}_{\text{Ł}_3^k}$, it is possible to define an interpretation $v_g \in \text{SEM}_{\text{Ł}_4^k}$ such that $g(\phi) \in \mathcal{D}_3^1$ iff $v_g(\phi) \in \mathcal{D}_4^1$, and $g(\phi) \in \mathcal{D}_3^2$ iff $v_g(\phi) \in \mathcal{D}_4^2$, and $g(\phi) \in \mathcal{D}_3^3$ iff $v_g(\phi) \in \mathcal{D}_4^3$.*

Theorem 3.2.2. $\text{SEM}_{\text{Ł}_4^k}$ *is a sound and complete semantics for* $\text{SEM}_{\text{Ł}_3^k}$

$$\Gamma \models^i_{\text{SEM}_{\text{Ł}_4^k}} \alpha \Leftrightarrow \Gamma \models^i_{\text{SEM}_{\text{Ł}_3^k}} \alpha$$

Proof. ($\Rightarrow$) Suppose (i) $\Gamma \models^i_{\text{SEM}_{\text{Ł}_4^k}} \alpha$. Then, by definition of entailment, we know that for every valuation $v \in \text{SEM}_{\text{Ł}_4^k}$ (ii) $v(\Gamma) \subseteq \mathcal{D}_4^i$ implies $v(\alpha) \in \mathcal{D}_4^i$. Now take an arbitrary valuation from $v \in \text{SEM}_{\text{Ł}_4^k}$ and suppose (iv) $v(\Gamma) \subseteq \mathcal{D}_4^i$. Thus, from (ii), we have $v(\alpha) \in \mathcal{D}_4^i$. From this, by Lemma 3.2.1, it is possible to define an interpretation g_v such that $v(\phi) \in \mathcal{D}_4^i$ iff $g_v(\phi) \in \mathcal{D}_3^i$. Therefore, we have (v) $g_v(\Gamma) \subseteq \mathcal{D}_3^i$ and $g_v(\alpha) \in \mathcal{D}_3^i$. Finally, by definition of entailment, we get $\Gamma \models_{\text{SEM}_{\text{Ł}_3^k}} \alpha$.

($\Leftarrow$) Suppose (i) $\Gamma \models^i_{\text{SEM}_{\text{Ł}_3^k}} \alpha$. Then, by definition of entailment, we know that for every valuation $g \in \text{SEM}_{\text{Ł}_3^k}$ (ii) $g(\Gamma) \subseteq \mathcal{D}_3^i$ implies $g(\alpha) \in \mathcal{D}_3^i$. Now take an arbitrary valuation from $g \in \text{SEM}_{\text{Ł}_3^k}$ and suppose (iv) $g(\Gamma) \subseteq \mathcal{D}_3^i$. Thus, from (ii), we have $g(\alpha) \in \mathcal{D}_3^i$. From this, by Lemma 3.2.2, it is possible to define an interpretation v_g such that $g(\phi) \in \mathcal{D}_3^i$ iff $v_g(\phi) \in \mathcal{D}_4^i$. Therefore, we have (v) $v_g(\Gamma) \subseteq \mathcal{D}_4^i$ and $v_g(\alpha) \in \mathcal{D}_4^i$. Finally, by definition of entailment, it follows that $\Gamma \models_{\text{SEM}_{\text{Ł}_4^k}} \alpha$.

□

The above results had shown that, by using the reduction exhibited, it is possible to reduce a k-dimensional logic to a n-dimensional logic, where $n < k$ and $n =$ the number of logical values of the original logic. Therefore the logic characterized by n-values has a lesser number of dimensions and validates the same set of inferences as the original logic. For future work, it remains to be shown whether there is any example of a k-dimensional logic that could not be reduced to a lesser number of dimensions.

It is also important to remark that by abandoning Shramko & Wansing's restriction that entailment must be about preservation of logical values, it leaves room for k-dimensional logics where each dimension may have a different notion of entailment. Moreover, by merging q-matrices and k-dimensional logics we can think of different versions for the k-dimensional version of Łukasiewicz's four valued logic exhibited above. For instance, each dimension of the k-matrix could be split into designated and rejected, what would give us the possibility of having the first dimension determined by a q-consequence, the second dimension determined by a tarskian consequence, and so on. Such approach would represent different points of view internalized in the same logic/structure. In the following section, we illustrate of a notion of entailment able to express such idea by encompassing a two-dimensional notion of logical consequence.

3.3 The bi-dimensional notion of entailment

In this section we illustrate the bi-dimensional of entailment proposed by Alexander Bochman in [Bochman et al., 1998] and recently revived and improved by [Blasio et al., 2014][11]. The bi-dimensional notion of entailment is a rich framework able to encompass a multiplicity of consequence relations. In what follows, we present its construction:

Definition 36. *A* ***2-dimensional matrix*** *is a tuple* $\mathbb{B}^M = \langle \mathcal{V}, \mathcal{R}, \mathcal{D}, \mathcal{O} \rangle$*, where* $\mathcal{V}$ *is a non-empty set,* $\mathcal{D}$ *and* $\mathcal{R}$ *are subsets of* $\mathcal{V}$*.*

$\mathcal{D}$ *is also called the* ***accepted*** *set of values,* $\mathcal{R}$ *is called the* ***rejected*** *set, the set* $\mathcal{B} \overset{def}{=} \mathcal{D} \cap \mathcal{R}$ *is called the set of* ***both*** *accepted and rejected values and the set* $\mathcal{N} \overset{def}{=} \mathcal{V} - (\mathcal{D} \cup \mathcal{R})$ *is called the set of* ***neither*** *accepted nor rejected values.*

We say a semantics SEM is any collection of valuations from the set of formulas *For* into a 2-dimensional matrix. Accordingly, any set of valuations induces an associated bi-dimensional notion of entailment in the following way:

Definition 37. *Any collection of valuations from* $\mathcal{L}$ *into a bi-dimensional matrix induces a bi-dimensional notion of entailment, defined in the following way:*

$$\frac{\alpha}{\gamma}\Big|\frac{\delta}{\beta} \text{ iff for every } v \in \text{SEM}^B,\ v(\gamma) \in \mathcal{D} \Rightarrow v(\delta) \in \mathcal{D} \text{ and } v(\beta) \in \mathcal{R} \Rightarrow v(\alpha) \in \mathcal{R}.$$

Thus, a bi-dimensional s-logic is defined as $\mathcal{L}_B = \langle For, \frac{\cdot}{\cdot}|\frac{\cdot}{\cdot} \rangle$, where *For* is a set of formulas endowed with a 2-dimensional entailment relation. Moreover, because of the interaction between accepted and rejected values a bi-dimensional logic is able to express all notions of entailment exhibited before:

- $\gamma \models^t \delta$ iff there is no $v \in$ SEM such that, $v(\gamma) \in \mathcal{D}$ and $v(\delta) \notin \mathcal{D}$

[11] Something similar was also presented in [Muskens et al., 1999].

- $\alpha \models^f \beta$ iff there is no $v \in \text{SEM}$ such that, $v(\alpha) \notin \mathcal{R}$ and $v(\beta) \in \mathcal{R}$
- $\alpha \models^q \delta$ iff there is no $v \in \text{SEM}$ such that, $v(\alpha) \notin \mathcal{R}$ and $v(\delta) \notin \mathcal{D}$
- $\gamma \models^p \beta$ iff there is no $v \in \text{SEM}$ such that, $v(\gamma) \in \mathcal{D}$ and $v(\beta) \in \mathcal{R}$

Each of these are expressed in one point of the relation:

- $\frac{\cdot}{\gamma}\Big|\frac{\delta}{\cdot}$ - t-entailment.
- $\frac{\alpha}{\cdot}\Big|\frac{\cdot}{\beta}$ - f-entailment.
- $\frac{\alpha}{\cdot}\Big|\frac{\delta}{\cdot}$ - q-entailment.
- $\frac{\cdot}{\gamma}\Big|\frac{\cdot}{\beta}$ - p-entailment.

Thus, the corresponding notion of consequence relation shall have the properties defined for each direction of the relation. B-entailment has been proposed by [Blasio and Marcos, 2013] as a logically four-valued notion of entailment. Moreover, in [Blasio and Marcos, 2013], it is shown how it may serve as an useful tool to reconstruct Belnap's four-valued logic.

4 Final remarks

4.1 What, then, should we expect from a logical system?

In introductory logic manuals it is not rare to find the claim that logic is the study of "what logically follows from what" [1]. Let us call this position as the *folk view* on logic. According to such view, the fundamental role of logic is to line off what conclusions should we accept from a given set of premises. Thus, logic is concerned with argument *validity*, i.e, with being able to distinguish valid arguments from invalid ones. For [Hofweber, 2014], this feature of logic is linked to one notion of logic, as the study of formally valid inferences and logical consequence. In the previous chapters, we have seen different conceptions of logical consequence, each of them formulated with different motivations and purposes. That amounts to saying there are several ways of understanding the "follows from" in the folk view.

Another common feature related to the idea that logic is the study of formally invalid inference is that logic is about "the structure of arguments, not the content". This thesis has been addressed as the *form versus matter distinction*. It has its roots in Aristotle and, in modern formal logic, has been understood in different forms. According to Novaes [Novaes, 2012], "the form versus matter distinction is to be applied to objects such as arguments so as to outline what is distinctively *logical* about them – which is associated to their *formal* aspects – as opposed to their merely material aspects. Thus seen, the form versus matter distinction has the responsibility of demarcating what is logical from what is not logical (...), and of grounding the validity of valid arguments."

If we restrict our look to the consequence operator associated to a given logic, to explore the 'form' means to explore the so-called structural properties and its relation to substitution-invariance. Despite the Tarskian consequence operator has been extensively studied in the literature, there is still little consensus on what properties a consequence operator should have in order to its associated structure deserve the name *logic*. In the present study, we have shown motivations for the development of non-reflexive and non-transitive logics. However, the question of whether they should be called logics was not adressed in full detail here, the motivation for such logics was studied in the scope of the problem of logical many-valuedness and its meaning within Suszko's work.

On what regards the "matter" of arguments, the challenge of studying it also led to what is nowadays called *defeasible reasoning*. Reasoning is called defeasible when the cor-

[1] Cf. [Barwise et al., 2000], [Gensler, 2002].

responding argument is rationally compelling but not deductively valid[2]. It is well-known that the class of non-monotonic logics is related to defeasible reasoning and its formal study. Therefore this class of non-Tarskian logics has found a good motivation for its role, and challenged the claim that logic is about the structure and not the content. Therefore, the question of what grounds the validity of arguments deserves further investigation in view of the proliferation of non-Tarskian systems.

Non-reflexive logics as those proposed by Malinowski and presented in Chapter 3 leave open the question of what other philosophical motivations could be given to them. The notion of q-consequence was a result of the search for generalizing Suszko's Thesis and exploring it beyond logical two-valuedness. Despite Malinowski's methodological motivation for q-logics as expressing the modus operandi underlying scientific reasoning, it is not clear what specific kind of phenomena would q-logics prove to be helpful in explaining/modelling. Moreover, other logics have been presented as modelling the process of construction of scientific knowledge. Thus, the relation among such logics and q-logics remains as a topic for future exploration.

Shramko & Wansing's k-dimensional logics, as well as Marcos & Blasio's presentation of b-logics, raises the question of what should be recognized as an abstract logical system. To put it short, should logic be a set of formulas endowed with a single consequence relation, or with several consequence relations? Or, maybe, with a many-dimensional consequence relation? In this regard, we may say that we can sharpen the folk view by saying that *logic is about what follows from what (**in a given dimension/perspective**).*

A further development of the aforementioned sharpened view depends on approaching the issue not only inside the context of logical many-valuedness, but also facing it in the context of pluralism regarding logical consequence. The intimate connection between logical many-valuedness and the notions of entailment related to them proved in the present study to be a good motivation for a logical pluralism that not only considers the plurality emerging from different logical systems and their underlying language [3], but also emerging from different notions of entailment. From such a point of view, the question of what happens beyond the Tarskian truth-preserving realm of entailment deserves closer inspection.

2 Cf. [Koons, 2014].

3 In [Cook, 2010], the author presents several types of pluralism of this kind and discusses how they relate to relativism.

4.2 If truth-preservation is dethroned, what role is left for it?

> "Never accept a dogma that cannot withstand a good joke."
>
> — Gabriela Mistral

It is well-known that some arguments that may be valid in one logic could be rejected in another one. One example is disjunctive syllogism in classical and paraconsistent logics. For some logical pluralists, the classical and paraconsistent logicians has different views on logical consequence. Pluralism about logical consequence is the view that there is more than one way of defining logical consequence. One of the reasons for such is the fact that terms like 'valid' or 'follows from' may be defined in more than one way, as it was done during the development of this thesis. A famous defense of logical pluralism regarding logical consequence is made in [Beall and Restall, 2006].

Beall & Restall's logical pluralism, intended as a pluralism regarding logical consequence, lies in defending a plurality of logical consequences as long as they can be expressed/fit into the following schema:

Generalized Tarski Thesis (GTT):
An argument is valid$_x$ if and only if in every case$_x$ in which the premises are true, so is the conclusion.

where each expression case$_x$ in the (GTT) can be made more precise in at least three ways which result in different extensions for 'valid'. For example, by case one might mean a first-order interpretation of a Tarskian model or even a situation or a possible-world. Other alternatives include consistent or incomplete interpretations of the sort used in the models theories for intuitionistic and paraconsistent logic. Different choices for the interpretation of 'case' will result different precisifications of the (GTT) analysis of logical consequence, which may result in different relations of logical consequence. Beall & Restall's defense of (GTT) rests on the idea that logical consequence is determined by three main features: necessity (the truth of the premises in a valid argument necessitates the truth of the conclusion), formality (valid arguments are so in virtue of their logical form), and normativity (it is irrational to reject a valid argument). According to them, those features imply the fact that logical consequence should be truth-preserving and, therefore, is a reflexive, monotonic and transitive relation.

Regarding the problem as to whether non-reflexive or non-transitive accounts of logical consequence should be taken as legitimate, Beall & Restall explain their view with the following remark:

> "The given kinds of non-transitive or irreflexive systems of 'logical consequence' are logics by courtesy and by family resemblance, where the courtesy is granted via analogy with logics *properly* so called. Non-transitive or non-reflexive systems of 'entailment' may well model interesting phenomena, but they are not accounts of *logical consequence*. One must draw the line somewhere and, pending further argument, we (defeasibly) draw it where we have. We require transitivity and reflexivity in logical consequence. We are pluralists. It does not follow that absolutely *anything goes*." [Beall and Restall, 2006, p. 91]

Beall & Restall's defense of reflexive and transitive accounts of logical consequence seems legitimate insofar as they match their requirement for necessity. However, it is not at all clear that "[absolutely] anything goes" beyond the Tarskian account. Section 3.1 of the present thesis has shown that non-reflexive logics are constructed through a natural generalization of Tarskian logics. Moreover, since every q-logic can be characterized by a three-valued model (see Theorem 3.1.4), thus having the two-valued models as its subclass, a question that seems reasonable is to know whether it is possible to have a generalized version of (GTT) (a Malinowskian version of it), thus extending its range of precisifications and making Beall & Restall's logical pluralism even more plural and able to encompass non-reflexive logics. Of course, such a move may imply the demise/weakening of some of the conditions required by Beall & Restall, but it remains to be investigated whether that is enough to claim the non-legitimacy of non-reflexive accounts of logical consequence.

A very recent attack made to Beall & Restall's logical pluralism can be found at [Beziau, 2014]. According to the author, "logical pluralism is the defense that there are various logics, it is not a general theory of logics. The logical pluralist does not make the distinction between reasoning and the theory of reasoning; both are put in the same bag. Beall and Restall uses the distinction between *Logic* and *logic* in the following way: **Logic names the discipline**, and **logic names a logical system**. According to this view there is no clear distinction between a logic system and the reasoning it is describing. Futhermore, logic as a discipline does not explicitly appear as a systematic study of logical systems."

According to Béziau, logical pluralism as proposed by Beall & Restall is not interested in understanding the relation between different logical systems by combining, translating and comparing them, but to defend that each one has its legitimacy and that one is not necessarily better than the other. Thus the view defended by Beall & Restall is quite distant from reality since some of the logics accepted by the (GTT) schema might disagree between them. Therefore, the peaceful view promoted by Beall & Restall's logical pluralism is inaccurate.

Béziau's project of Universal Logic has the goal of promoting a general framework capable of studying the reasoning behind each logic. From such a point of view, there seems to be no reason why not to accept non-Tarskian notions of logical consequence as formal tools able to express specific kinds of reasoning. In [Estrada-Gonzalez, 2014], the author summarizes Béziau's view found in some of its papers in the following way:

P1 Virtually every theorem, principle for connectives, principle for the consequence relation, etc, let us call them collectively 'properties of a logic', has been thrown out or, at least, challenged.

P2 The outcome of such droppings and challenges have been regarded as logics.

P3 If the properties $P_1, ..., P_n$ of a logic can be dropped or challenged, an additional property P_m also can be dropped or challenged and the result will still count as logic.

P4 The situation is analogous to the case of algebra, where an algebraic structure needs not to satisfy any property in particular.

P5 Hence, a relation of logical consequence can be defined with no reference to a particular property of a logic.

So, differently of Beall & Restall's pluralism, Béziau's view is that logic need not be reflexive or transitive since it is not grounded on any particular principle or law. Therefore, the kind of pluralism entailed by Universal Logic does not commit one to any particular notion of logical consequence. All the requirements postulated by Beall & Restall, such as necessity, normativity and formality, are ruled out in Béziau's account. Regardless of such matters, a problem that deserves to be adressed is to know whether any structure with a non-trivial relation between formulas deserves to be called a logic. From such a point of view, Béziau's account seems closer to a form of relativism than Beall & Restall's.

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