Social Preference Under Twofold Uncertainty

Philippe Mongin (CNRS & HEC Paris)
and
Marcus Pivato (Université de Cergy-Pontoise)*

November 2, 2017

Abstract

We investigate the conflict between the *ex ante* and *ex post* criteria of social welfare in a new framework of individual and social decisions, which distinguishes between two sources of uncertainty, here interpreted as an objective and a subjective source respectively. This framework makes it possible to endow the individuals and society not only with *ex ante* and *ex post* preferences, as is usually done, but also with interim preferences of two kinds, and correspondingly, to introduce interim forms of the Pareto principle. After characterizing the *ex ante* and *ex post criteria*, we present a first solution to their conflict that extends the former as much possible in the direction of the latter. Then, we present a second solution, which goes in the opposite direction, and is also maximally assertive. Both solutions translate the assumed Pareto conditions into weighted additive utility representations, and both attribute to the individuals common probability values on the objective source of uncertainty, and different probability values on the subjective source. We discuss these solutions in terms of two conceptual arguments, i.e., the by now classic *spurious unanimity* argument and a novel informational argument labelled *complementary ignorance*. The paper complies with the standard economic methodology of basing probability and utility representations on preference axioms, but for the sake of completeness, also considers a construal of objective uncertainty based on the assumption of an exogeneously given probability measure.

Keywords: *Ex ante* social welfare; *ex post* social welfare; objective *versus* subjective uncertainty; objective *versus* subjective probability; Pareto principle; separability; Harsanyi social aggregation theorem; spurious unanimity; complementary ignorance.
JEL classification: D70; D81.

*This research was pursued while the first author visited Wissenschaftskolleg zu Berlin. The second author was supported by NSERC grant #262620-2008 and by Labex MME-DII (ANR11-LBX-0023-01).

1 Introduction

Any normative analysis of collective decisions under uncertainty must confront an old and unresolved problem: the conflict between the *ex ante* and *ex post* criteria of social welfare. This paper proposes new solutions to this problem, which are based on a distinction between two sources of uncertainty. In our framework, agents may hold different beliefs about one source while holding the same beliefs about the other. Before explaining what difference this twofold uncertainty makes, we restate the conflict in its classical form.

The *ex ante social welfare criterion* assumes that the individuals form preferences over social uncertain prospects according to some normative decision theory – typically that of subjective expected utility – and it applies the Pareto principle to these *ex ante* individual preferences, thus following an *ex ante* version of the principle. In contrast, the *ex post social welfare criterion* assumes that society itself forms preferences over social prospects according to the normative decision theory under consideration, while it endows the individuals only with state-by-state preferences. It then applies the Pareto principle statewise to these *ex post* individual preferences, thus following an *ex post* version of the principle.[1] If all agents satisfy the axioms of subjective expected utility (SEU), or even weaker axioms, then the *ex ante* and *ex post* criteria are incompatible. Hence a trilemma: we must abandon the underlying decision theory, the *ex ante* social welfare criterion, or the *ex post* social welfare criterion.

This clash between the *ex ante* and *ex post* social welfare criteria has long been recognized, although the problem has been formulated in several different ways. The early statements by Starr (1973) and Hammond (1981, 1983) belonged to traditional welfare economics, and envisaged only two extreme solutions to the conflict, i.e., endorsing one of the two criteria and rejecting the other, with an overall preference for the *ex post* criterion. ? Contradicts page 16
Mongin's (1995) abstract formulation in terms of Savage's (1972) SEU postulates avoided the domain-specific assumptions made by the welfare economists, thus sharpening the conflict, while also clarifying the role of probability or utility dependencies in this conflict. This axiomatic approach also facilitated comparison with Harsanyi's (1955) Social Aggregation Theorem, which famously says that, if both individuals and society form their preferences over social lotteries according to von Neumann-Morgenstern (VNM) theory, and the social preferences satisfy the Pareto principle, then society's preferences can be represented by a weighted ("utilitarian") sum of individual utility representations.[2] As the Pareto principle applies here both *ex ante* (to lotteries) and *ex post* (to final outcomes), Harsanyi's assumptions contain all the ingredients of the two welfare criteria, and his weighted sum formula seems to contradict the claim that the two criteria are incompatible. However, the assumption of a common lottery set amounts to imposing identical probabilities on the individuals and society, an extreme case of those probabilistic dependencies which Mongin's

[1]Note the difference between a social welfare *criterion* and the corresponding *Pareto principle*. There is more to the the *ex ante* (*ex post*) social welfare criterion than just the *ex ante* (*ex post*) Pareto principle, because a criterion also decides where rationality assumptions apply (to the individuals or society).

[2]We do not claim that such weighted sums of VNM utilities have a genuine utilitarian interpretation. Harsanyi took this for granted, but Sen famously denied it, and the debate is still unsettled. See Mongin and Pivato (2016*a*) for a review, and Fleurbaey and Mongin (2016) for a new defence of Harsanyi's position.

axiomatic treatment covers.

The present paper will also exploit the fact that the conflict between the *ex ante* and *ex post* social welfare criteria vanishes when probabilities are identical, but in a much more general fashion than Harsanyi. In our treatment, there are *two* sources of uncertainty. Informally, let us call them O and S (we will be more formal later). One possible interpretation of O and S is as *objective* and *subjective* sources of uncertainty, respectively. This interpretation motivates the different axiomatic treatment we give to O and S in the hypotheses of our main results (summarized below). This interpretation is then born out in the conclusions of these theorems, which impute probabilistic beliefs to the individuals and the social observer. As we shall see, these beliefs must be identical with respect to O, but they are allowed to be heterogeneous with respect to S, as is consistent with the objective/subjective interpretation.

However, this distinction between objective and subjective uncertainty is merely an *interpretation* of our formalism; we do not articulate any substantive philosophical theory of this distinction in the paper, nor do our results depend upon such a substantive theory. For our purposes, the distinction simply means that there is some exogenous reason why all agents have the same mental model or attitude about the O-uncertainty, whereas there is no reason for such unison with respect to the S-uncertainty. Here, we are deliberately vague about the nature of these "mental models" or "attitudes", because we do *not* assume that agents are SEU-maximizers; the aforementioned probabilistic beliefs emerge only in the *conclusions* of our theorems, not their hypotheses. We follow the standard methodology of taking preferences over prospects to be the only axiomatic primitives. Thus, our treatment generalizes Harsanyi's in two ways, i.e., both by reducing the scope of his common probability assumption and by endogenizing it in the representation theorem.

A key step in our formalism is the introduction of *conditional preferences*. For both society and the individuals, we posit preferences conditional on O and preferences conditional on S, each obeying distinctive decision-theoretic properties. This, in turn, leads us to consider new, *interim* forms of the Pareto principle, in addition to the classic *ex ante* and *ex post* forms. By simultaneously varying the forms of the Pareto principle and the decision-theoretic assumptions on conditionals, we obtain a rich set of theoretical possibilities. The paper explores these possibilities to discover relevant compromises between the *ex ante* and the *ex post* social welfare criteria. The most interesting ones are those which capitalize on one of the two criteria and absorb as much of the contents of the other criterion as is possible without reducing to one of the classic impossibility theorems.

There are two such optimal compromises in the paper. The first (Theorem 4) combines the *ex ante* criterion in full with a partial version of the *ex post* criterion. In this case, we deduce that all individuals and society assign identical probabilities to O, but regarding S, individuals can assign different probabilities, while society merely aggregates their preferences without forming any probabilities of its own. We paraphrase this by saying that the *ex ante* criterion holds for both O and S, but the *ex post* criterion only holds for O. This makes society's *ex post* preferences *state-dependent* – i.e., the *ex post* social welfare function varies according to which state is realized – a flexibility that several writers have judged attractive and indeed offered as a way out of the conflict between the two criteria

(Mongin, 1998; Chambers and Hayashi, 2006; Keeney and Nau, 2011).

The second optimal compromise (Theorem 7) encapsulates the *ex post* criterion in full and a partial version of the *ex ante* criterion, which amounts to replacing the full *ex ante* Pareto principle with the *interim* Pareto principle associated with O. Once again, we deduce that all individuals and society assign identical probabilities to O, while individuals can assign different probabilities to S. Meanwhile, society forms probabilities on S that are unrelated to the individuals' beliefs. Again, we say that the *ex post* criterion holds for both O and S, but the *ex ante* criterion only holds for O. This solution is reminiscent of the elegant compromise proposed in an influential paper by Gilboa, Samet and Schmeidler (2004), who apply the *ex ante* Pareto principle only when the individuals attribute identical probabilities to the events underlying the prospects they compare. However, we justify our restriction of the *ex ante* Pareto principle in terms of an *exogenous* distinction between two sources of uncertainty, whereas Gilboa, Samet and Schmeidler justified it *endogeneously*, accepting at face value *any* identity of probabilistic beliefs between agents, without examining the reason for this identity. As we shall see in Section 7, this purely endogenous approach can lead to problems.

A bit of flattery never hurt.

Besides axiomatizing these two compromises, the paper critiques them. Two main arguments are involved in this critique, the *spurious unanimity objection*, which was first developed by Mongin (1997) and has become fairly well accepted today, and the *complementary ignorance objection*, which we introduce in this paper. Our *ex ante*-based compromise falls prey to the spurious unanimity objection, as would any solution that retains the *ex ante* Pareto principle, but this defect can be traded off against the advantage of having a state-dependent *ex post* social welfare function at one's disposal. We use the novel complementary ignorance argument to compare our *ex post*-based compromise with Gilboa, Samet and Schmeidler's. Ours connects society's probabilities with those of the individuals regarding only the "objective" uncertainty O; as to "subjective" uncertainty S, we allow for totally unrelated individual and social treatments. By contrast, in Gilboa, Samet and Schmeidler, society's probabilities are a weighted sum of the individual probabilities – what is called elsewhere the *linear pooling rule*. The linear pooling rule has already been criticized, and our complementary ignorance objection undermines it further. We would argue that it is better not to have any pooling rule at all than to use an incorrect one. This critique also applies to some works inspired by Gilboa, Samet and Schmeidler (2004), which refine their version of the *ex ante* Pareto principle or generalize their SEU assumptions, but still derive the linear pooling rule.

The paper is organized as follows. Section 2 introduces the framework, and the various decision-theoretic and Pareto conditions. As a preliminary for the more original results to come, Section 3 provides minimal axiomatizations of the *ex ante* social welfare criterion (Proposition 1), the *ex post* social welfare criterion (Proposition 2), and a first reconciliation of the two on a weak logical basis (Proposition 3). Section 4 proposes our first optimal compromise, which extends the full *ex ante* criterion as much as possible in the direction of the *ex post* criterion (Theorem 4, with Proposition 5 and Corollary 6 serving as points of comparison). Section 5 proposes our second optimal solution, which symmetrically extends the *ex post* criterion as much as possible in the direction of the *ex ante* criterion (Theorem 7). Section 6 provides Theorem 4 and Theorem 7 with variants in which objective uncertainty is captured by an exogeneously given probability on one of the two sources. This is added for readers who might not accept our preference-based distinction between objective * and subjective uncertainty. The purely conceptual Section 7 discusses our solutions and that of Gilboa, Samet and Schmeidler (2004) in the light of the spurious unanimity and complementary ignorance problems. Section 8 reviews the more recent literature.

Appendix A contains a table summarizing the main results of the paper. Appendix B * contains technical background, and Appendix C contains the proofs of all our results.

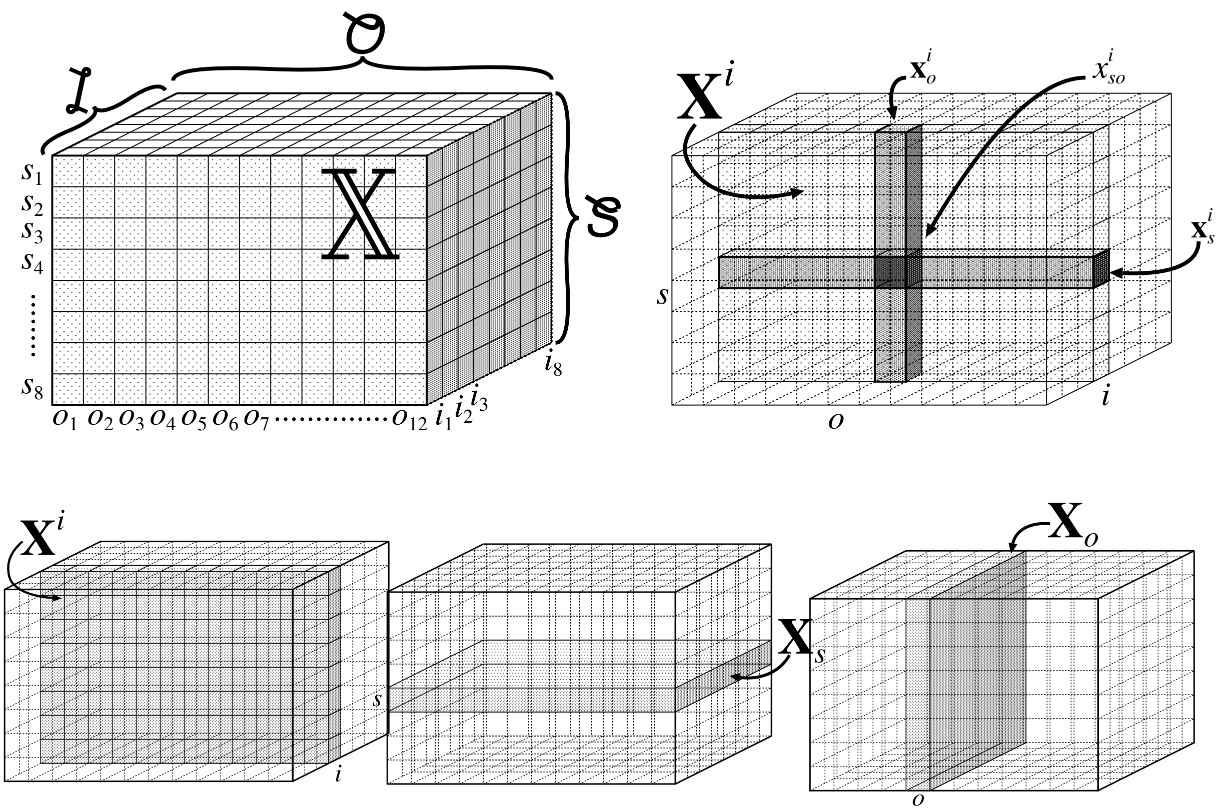

Figure 1: *Top left:* A completely uncertain social prospect $\mathbb{X}$. *Bottom left:* A completely uncertain individual prospect $\mathbf{X}^i$. *Top right:* A subjectively uncertain prospect $\mathbf{x}^i_o$ and an objectively uncertain individual prospect $\mathbf{x}^i_s$. *Bottom centre:* An objectively uncertain social prospect $\mathbf{X}_s$. *Bottom right:* A subjectively uncertain social prospect $\mathbf{X}_o$.

2 The framework

Uncertain Prospects. We assume that states of the world are pairs (s, o), where $s \in \mathcal{S}$ and $o \in \mathcal{O}$ represent two distinctive sources of the uncertainty. As discussed in the intro- *

duction, we interpret $\mathcal{S}$ as the state space of a subjective source and $\mathcal{O}$ as the state space of an objective source. Although this is but one interpretation, and finds its justification mainly in the *conclusions* of our results, we will adopt this terminology throughout the paper, for ease of expression.

We assume that $\mathcal{S}$ and $\mathcal{O}$ are finite with $|\mathcal{S}|, |\mathcal{O}| \geq 2$. Denote by $\Delta_{\mathcal{S}}$ and $\Delta_{\mathcal{O}}$ the sets of probability vectors on $\mathcal{S}$ and $\mathcal{O}$, respectively. We assume that the individuals i belong to a finite set $\mathcal{I}$ with $|\mathcal{I}| \geq 2$, and that each individual i and society face uncertain prospects. In the present framework, these can be *completely uncertain* (when both s and o are unknown), or *interim uncertain*, with the latter being either *subjectively uncertain* (o is fixed and s is unknown), or *objectively uncertain* (s is fixed and o is unknown).

We think of prospects in the usual way, as mappings from states of the world to consequences. We express consequences directly in terms of payoff numbers x^i_{so} for the individuals. For all $i \in I$, $s \in S$, $o \in O$, we assume that x^i_{so} varies across $\mathbb{R}$. We discuss domain assumptions at the end of this section. We leave it for the interpretation to decide whether the x^i_{so} numbers represent *physical* payoffs (levels of consumption in a good) or *subjective* payoffs (utility values in some metric).

We define a *completely uncertain* prospect:

- in the case of an individual $i \in \mathcal{I}$, as a matrix $\mathbf{X}^i = (x^i_{so})_{s\in\mathcal{S},o\in\mathcal{O}} \in \mathbb{R}^{\mathcal{S}\times\mathcal{O}}$;
- in the case of society, as a three-dimensional array, $\mathbb{X} = (x^i_{so})^{i\in\mathcal{I}}_{s\in\mathcal{S},o\in\mathcal{O}} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}$.

(See the left panels in Figure 1).[3] We define an *objectively uncertain* prospect (respectively, a *subjectively uncertain* prospect):

- in the case of an individual $i \in \mathcal{I}$, as a vector $\mathbf{x}^i_s = (x^i_{so})_{o\in\mathcal{O}} \in \mathbb{R}^{\mathcal{O}}$ for some fixed $s \in \mathcal{S}$ (respectively, $\mathbf{x}^i_o = (x^i_{so})_{s\in\mathcal{S}} \in \mathbb{R}^{\mathcal{S}}$ for some fixed $o \in \mathcal{O}$);
- in the case of society, as a matrix $\mathbf{X}_s = (x^i_{so})^{i\in\mathcal{I}}_{o\in\mathcal{O}} \in \mathbb{R}^{\mathcal{I}\times\mathcal{O}}$ for some $s \in \mathcal{S}$ (respectively, $\mathbf{X}_o = (x^i_{so})^{i\in\mathcal{I}}_{s\in\mathcal{S}} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}}$for some fixed $o \in \mathcal{O}$).

(See the centre and right panels of Figure 1.) We refer to prospects in the last two classes as *interim* prospects. Notice that they are labelled according to the nature of the *unresolved* uncertainty; i.e., an *objective* prospect is one in which o is unknown and s known (and vice versa for a *subjective* prospect). When uncertainty is completely resolved, an individual i faces a scalar $x^i_{so} \in \mathbb{R}$, while society faces a vector $\mathbf{x}_{so} = (x^i_{so})^{i\in\mathcal{I}} \in \mathbb{R}^{\mathcal{I}}$.

Preferences. We assume that both the individuals and society assess completely uncertain prospects in terms of *ex ante* preference relations, denoted by $\succeq^i$ for $i \in \mathcal{I}$ and $\succeq$ for society; these are the only primitives data in our model. *Throughout the paper, we take $\succeq^i$ and $\succeq$ to be continuous weak orders.* Thus, these relations can be represented by continuous real-valued utility functions on their respective domains.

[3]The order in which s and o enter the notation is purely conventional. We do not mean to suggest that the s-uncertainty is resolved before the o-uncertainty.

The other preference relations of this paper are *conditional relations* induced by either the $\succeq^i$ or $\succeq$. There are six of them to consider: $\succeq_s^i$, $\succeq_o^i$ and $\succeq_{so}^i$ for individual i, and $\succeq_s$, $\succeq_o$ and $\succeq_{so}$ for society. While $\succeq_{so}^i$ and $\succeq_{so}$ make *ex post* comparisons, $\succeq_s^i$, $\succeq_s$, $\succeq_o^i$ and $\succeq_o$ make *interim* comparisons, which are specific to the twofold uncertainty framework. As usual, conditional preferences are defined by restricting unconditional preferences to prospects that vary only along the component of interest. It will be enough if we formalize this definition for representative conditionals. An abstract definition covering all cases at once appears in the Appendix. For all $\mathbf{x}_s^i, \mathbf{y}_s^i \in \mathbb{R}^{\mathcal{O}}$, the conditional $\succeq_s^i$ is defined as follows:

> $\mathbf{x}_s^i \succeq_s^i \mathbf{y}_s^i$ if and only if $\mathbf{X}^i \succeq^i \mathbf{Y}^i$ for some $\mathbf{X}^i, \mathbf{Y}^i \in \mathbb{R}^{\mathcal{S}\times\mathcal{O}}$ such that $\mathbf{x}_s^i$ and $\mathbf{y}_s^i$ are the (vector-valued) s-components of $\mathbf{X}^i$ and $\mathbf{Y}^i$, and $\mathbf{X}^i$ and $\mathbf{Y}^i$ coincide on all other components.

(A definition for $\succeq_o^i$ follows *mutatis mutandis*.) For all $\mathbf{X}_s, \mathbf{Y}_s \in \mathbb{R}^{\mathcal{I}\times\mathcal{O}}$, the conditional $\succeq_s$ is defined as follows:

> $\mathbf{X}_s \succeq_s \mathbf{Y}_s$ if and only if $\mathbb{X} \succeq \mathbb{Y}$ for some $\mathbb{X}, \mathbb{Y} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}$ such that $\mathbf{X}_s$ and $\mathbf{Y}_s$ are the (matrix-valued) s-components of $\mathbb{X}$ and $\mathbb{Y}$, and $\mathbb{X}$ and $\mathbb{Y}$ coincide on all other components.

(A definition for $\succeq_o$ follows *mutatis mutandis*.) For all $\mathbf{x}_{so}, \mathbf{y}_{so} \in \mathbb{R}^{\mathcal{I}}$, the conditional $\succeq_{so}$ is defined as follows:

> $\mathbf{x}_{so} \succeq_{so} \mathbf{y}_{so}$ if and only if $\mathbb{X} \succeq \mathbb{Y}$ for some $\mathbb{X}, \mathbb{Y} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}$ such that $\mathbf{x}_{so}$ and $\mathbf{y}_{so}$ are the (vector-valued) (s, o)-components of $\mathbb{X}$ and $\mathbb{Y}$, and $\mathbb{X}$ and $\mathbb{Y}$ coincide on all other components.

We discuss the remaining case of $\succeq_{so}^i$ below.

Properties of conditionals. The conditional relations $\succeq_s^i$ etc. defined in the previous subsection are complete, but not necessarily transitive. To make them transitive, we must make an extra assumption: *separability*; this says in effect that the defining preference comparisons do not depend on the chosen alternatives, provided these alternatives coincide outside the components of interest. To illustrate with the first example, $\succeq^i$ is said to be *separable in s* if, for all $\mathbf{X}^i, \mathbf{Y}^i, \widetilde{\mathbf{X}}^i, \widetilde{\mathbf{Y}}^i \in \mathbb{R}^{\mathcal{S}\times\mathcal{O}}$, if $\mathbf{X}^i$ coincides with $\widetilde{\mathbf{X}}^i$ on all components except s, and $\mathbf{Y}^i$ with $\widetilde{\mathbf{Y}}^i$ on all components except s, then the following equivalence holds:

$$\mathbf{X}^i \succeq^i \mathbf{Y}^i \quad \text{if and only if} \quad \widetilde{\mathbf{X}}^i \succeq^i \widetilde{\mathbf{Y}}^i.$$

If $\succeq^i$ is separable in s, then the conditional relation $\succeq_s^i$ is transitive, hence a *bona fide* ordering. The converse also holds. Appendix B restates this basic fact in general form.

As to the conditional relations $\succeq_{so}^i$, we will assume that they compare real numbers according to the natural ordering of these numbers. That is, for all $(s, o) \in \mathcal{S}\times\mathcal{O}$, all $i \in \mathcal{I}$ and all $x_{so}^i, y_{so}^i \in \mathbb{R}$,

$$x_{so}^i \succeq_{so}^i y_{so}^i \quad \text{if and only if} \quad x_{so}^i \geq y_{so}^i. \tag{1}$$

This is consistent with the payoff interpretation of these numbers, and it automatically makes $\succeq^i_{so}$ transitive.

For all other conditional relations, we do *not* generally assume transitivity (or the equivalent property of separability). Our results crucially depend on selecting *which* conditionals are transitive. For ease of expression, when a conditional has this property, whether by way of assumption or by way of conclusion, we say that its source relation *induces a preference ordering*. Thus, "$\succeq^i$ induces an interim preference ordering $\succeq^i_s$" means that $\succeq^i_s$ is transitive, or equivalently, that $\succeq^i$ is separable in s; likewise,"$\succeq$ induces an interim preference ordering $\succeq_{so}$" means that $\succeq_{so}$ is transitive, or equivalently, that $\succeq$ is separable in (s, o), and similarly with the other cases.

We always make such transitivity assumptions *uniformly* across the uncertainty type. That is, we take the ordering property of conditionals to hold either for *all* $(s, o) \in \mathcal{S} \times \mathcal{O}$ or for none, either for *all* $s \in \mathcal{S}$ or for none, either for *all* $o \in \mathcal{O}$ or for none. Thus, we will simply say, "$\succeq^i$ induces interim preference orderings $\succeq^i_s$" without adding the implied "for all $s \in \mathcal{S}$"; and similarly with the other cases.[4]

When $\succeq^i$ or $\succeq$ induces conditional preference orderings of some type, we may, by a separate decision, require that these preferences be *identical* across the given type, in which case we say that they are *invariant*. Thus, we may assume not only that $\succeq^i$ induces interim preference orderings $\succeq^i_s$ (for all $s \in \mathcal{S}$), but also that $\succeq^i_s = \succeq^i_{s'}$ for all $s, s' \in \mathcal{S}$; we may assume not only that $\succeq$ induces *ex post* preference orderings $\succeq_{so}$ (for all $(s, o) \in \mathcal{S} \times \mathcal{O}$), but also that $\succeq_{so} = \succeq_{s'o'}$ for all $s, s' \in \mathcal{S}$, and all $o, o' \in \mathcal{O}$; and similarly with the other cases. Note that the $\succeq^i_{so}$ preferences are automatically invariant, by statement (1).

The intended meaning of these invariance assumptions should be clear. If, for instance, $\succeq^i$ induces invariant interim $\succeq^i_s$ preferences, this means that the resolution of the s-uncertainty has no influence on i's preferences over interim prospects that depend on the o-uncertainty, or that, for all decision purposes, i regards s as being *uninformative* about o. If $\succeq$ induces invariant *ex post* preferences $\succeq_{so}$, this means that society has *state-independent ex post* preferences, a standard assumption in decision theory.

For suitable ordering and invariance assumptions put on the conditionals, our framework will give rise to SEU representations for the individuals, society, or both of them. Section 8 will compare this implied representation theorem with the classic ones by Savage (1972) and Anscombe and Aumann (1963), but we would like to emphasize that, despite its finiteness assumptions, which liken it to Anscombe and Aumann's, our framework leans more towards Savage's, since we eschew *any* probabilistic primitives in our axioms, and work with a pure uncertainty framework.

Pareto conditions. In the standard framework of social preference under uncertainty, social preference is subjected to Pareto conditions defined either *ex ante* or *ex post*. But the twofold uncertainty framework introduces more options. Here, the *ex ante* condition applies to completely uncertain social prospects, the *ex post* condition applies to fully

[4]The requirement that separability conditions hold across the uncertainty type means that they are equivalent to *dominance conditions* for the given type; see Appendix B.

resolved social prospects, and two newly defined *interim* conditions apply to *s-resolved social prospects* and *o-resolved social prospects*. Formally:

- $\succeq$ satisfies the *ex ante* Pareto principle if for all $\mathbb{X}, \mathbb{Y} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}$:
  if $\mathbf{X}^i \succeq^i \mathbf{Y}^i$ for all $i \in \mathcal{I}$, then $\mathbb{X} \succeq \mathbb{Y}$; if, in addition, $\mathbf{X}^i \succ^i \mathbf{Y}^i$ for some $i \in \mathcal{I}$, then $\mathbb{X} \succ \mathbb{Y}$.

- $\succeq$ satisfies the *ex post* Pareto principle if, for all $(s, o) \in \mathcal{S} \times \mathcal{O}$, and all $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^{\mathcal{I}}$:
  if $x^i \geq y^i$ for all $i \in \mathcal{I}$, then $\mathbf{x} \succeq_{so} \mathbf{y}$; if, in addition, $x^i > y^i$ for some $i \in \mathcal{I}$, then $\mathbf{x} \succ_{so} \mathbf{y}$;

- $\succeq$ satisfies the *objective interim* Pareto principle if for all $s \in \mathcal{S}$, and all $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{\mathcal{I}\times\mathcal{O}}$:
  if $\mathbf{x}^i \succeq^i_s \mathbf{y}^i$ for all $i \in \mathcal{I}$, then $\mathbf{X} \succeq_s \mathbf{Y}$; if, in addition, $\mathbf{x}^i \succ^i_s \mathbf{y}^i$ for some $i \in \mathcal{I}$, then $\mathbf{X} \succ_s \mathbf{Y}$;

- $\succeq$ satisfies the *subjective interim* Pareto principle if for all $o \in \mathcal{O}$, and all $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}}$:
  if $\mathbf{x}^i \succeq^i_o \mathbf{y}^i$ for all $i \in \mathcal{I}$, then $\mathbf{X} \succeq_o \mathbf{Y}$; if, in addition, $\mathbf{x}^i \succ^i_o \mathbf{y}^i$ for some $i \in \mathcal{I}$, then $\mathbf{X} \succ_o \mathbf{Y}$.

These interim Pareto principles are called *objective* or *subjective*, depending on whether the remaining uncertainty concerns the objective or subjective source. That is, they draw their denomination from the the prospects they handle, not from the conditioning variable. Note that all our forms of the Pareto principle except for the *ex ante* one are defined in terms of binary relations rather than preference orderings. This makes the Pareto conditions logically independent of the decision-theoretic conditions discussed above.

Domain assumptions. In this paper, we assume maximal domains of objects; all are indeed of the form $\mathbb{R}^l$ for some l. This runs afoul of feasibility considerations. It may well be that not every array, matrix or vector of payoff values x^i_{so} can be obtained by a feasible prospect; and that not every vector of payoff values $\mathbf{x}_{so}$ can be obtained as an *ex post* social outcome. But these domain assumptions are just for mathematical simplicity; it would be possible to replace them by more realistic ones. The proofs of this paper use a mathematical theory of separability, developed in Mongin and Pivato (2015), which allows for domains that are *not* Cartesian products, but only satisfy certain connectedness properties (convex sets being a particular case).[5] We could have applied this theory here in full generality, but we refrained, in order not to add complexity to an already rich formalism.

3 The *ex ante* and *ex post* criteria of social welfare

The first result of this section axiomatically characterizes the *ex ante* social welfare criterion. As usual, this is done by assuming the *ex ante* Pareto principle and decision-theoretic

[5] This builds on key earlier papers by Wakker (1993), Segal (1992), Chateauneuf and Wakker (1993).

conditions on individual preferences that secure an SEU representation for them. In the twofold uncertainty framework, the latter conditions can be stated economically. It is enough to require that each individual have well-defined *invariant* interim preferences for each type of uncertainty. This SEU representation theorem will occur repeatedly in the * paper. As to the social *ex ante* preferences, they are simply represented by a function that is increasing with the SEU individual representations, a direct translation of the selected form of the Pareto principle.

Proposition 1 *Suppose that* (A1) *for all* $i \in \mathcal{I}$, *the individual preferences* $\succeq^i$ *induce interim preference orderings* $\succeq^i_s$ *and* $\succeq^i_o$, *and* (A2) *both families of orderings are invariant. Suppose also that* (A3) $\succeq$ *satisfies the* ex ante *Pareto principle.*

Then, for all $i \in \mathcal{I}$, *there are strictly positive probability vectors* $\mathbf{p}^i \in \Delta_{\mathcal{S}}$ *and* $\mathbf{q}^i \in \Delta_{\mathcal{O}}$, *and an increasing continuous utility function* u^i *on* $\mathbb{R}$, *such that the* ex ante *individual* * *preference* $\succeq^i$ *admits the following SEU representation:*

$$U^i(\mathbf{X}) \quad := \quad \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} q^i_o \, p^i_s \, u^i(x_{so}), \quad \textit{for all} \;\; \mathbf{X} \in \mathbb{R}^{\mathcal{S}\times\mathcal{O}}. \tag{2}$$

Moreover, there is a continuous increasing function F *on the range of the vector-valued function* $(U^i)_{i\in\mathcal{I}}$ *such that* $\succeq$ *is represented by the ex ante social welfare function*

$$W_{\mathrm{xa}}(\mathbb{X}) \quad := \quad F\big(\big[U^i(\mathbf{X}^i)\big]_{i\in\mathcal{I}}\big), \quad \textit{for all} \;\; \mathbb{X} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}. \tag{3}$$

In these representations, for all $i \in \mathcal{I}$, *the probability vectors* $\mathbf{p}^i$ *and* $\mathbf{q}^i$ *are unique, and* u_i *is unique up to positive affine transformations, while* F *is unique up to continuous increasing transformations.*

Each SEU representation U^i builds upon two probability functions $\mathbf{p}^i$ and $\mathbf{q}^i$, which represent i's beliefs about $\mathcal{S}$ and $\mathcal{O}$, respectively. Given the multiplicative form $q^i_o \, p^i_s$, the events in $\mathcal{S}\times\mathcal{O}$ associated with s values and those associated with o values are *stochastically independent* according to i. This entails that i would not revise the probability values for * one class of events upon learning which event of the other class of events occurs. Indeed, * for any $o \in \mathcal{O}$, the o-conditional preference $\succeq^i_o$ has SEU representation

$$U^i_o(\mathbf{x}) \quad := \quad \sum_{s\in\mathcal{S}} p^i_s \, u^i(x_s), \quad \text{for all} \;\; \mathbf{x} \in \mathbb{R}^{\mathcal{S}}, \tag{4}$$

independent of o; meanwhile, for any $s \in \mathcal{S}$, the s-conditional preference $\succeq^i_s$ has SEU representation

$$U^i_s(\mathbf{x}) \quad := \quad \sum_{o\in\mathcal{O}} q^i_o \, u^i(x_o), \quad \text{for all} \;\; \mathbf{x} \in \mathbb{R}^{\mathcal{O}}, \tag{5}$$

independent of s. This stochastic independence property warrants the interpretation Section 2 proposed for invariance assumptions. What is missing from the theorem is a semantic distinction between the two sources of uncertainty. Their symmetric treatment certainly does not permit interpreting one as being objective and the other as being subjective.

The second result of this section axiomatizes the *ex post* social welfare criterion. As usual, this is done by selecting the *ex post* form of the Pareto principle and reserving decision-theoretic conditions for society. The conclusions deliver a SEU representation for social preference and make the *ex post* social welfare function increasing in individual *ex post* utilities.

Proposition 2 *Suppose that* (B1) *the social preference $\succeq$ induces interim preference orderings $\succeq_s$ and $\succeq_o$, and* (B2) *both families of orderings are invariant. Suppose also that* (B3) *$\succeq$ satisfies the* ex post *Pareto principle.*

Then, the ex post *social preference relations $\succeq_{so}$ are orderings and they are invariant, and there is a continuous and increasing representation W_{xp} for them. There also exist strictly positive probability vectors $\mathbf{p} \in \Delta_{\mathcal{S}}$ and $\mathbf{q} \in \Delta_{\mathcal{O}}$ such that the* ex ante *social preference $\succeq$ has the following SEU representation:* *

$$W_{\mathrm{xa}}(\mathbb{X}) \quad := \quad \sum_{s \in \mathcal{S}} \sum_{o \in \mathcal{O}} p_s q_o W_{\mathrm{xp}}(\mathbf{x}_{so}), \quad \textit{for all } \ \mathbb{X} \in \mathbb{R}^{\mathcal{I} \times \mathcal{S} \times \mathcal{O}}. \tag{6}$$

In this representation, $\mathbf{p}$ and $\mathbf{q}$ are unique, and the ex post *social welfare function W_{xp} is unique up to positive affine transformations.*

The probabilities $\mathbf{p}$ and $\mathbf{q}$ that appear here, again in multiplicative form, belong to society exclusively. Those of the individuals – if any – are left unspecified, since the only individual data are the orderings $\succeq^i$ and $\succeq^i_{so}$, nothing being said on the other conditionals. Like the individuals in Proposition 1, society regards the two sources of uncertainty as being informationally unrelated, and as in this proposition, it is not possible to distinguish between the two sources semantically.

We close this section by stating the conflict between the *ex ante* and *ex post* social welfare criteria in a form adapted to the present framework. This statement draws on an earlier result from Mongin and Pivato (2015), and like that result, improves on the classic impossibilities by *not* making SEU theory part of its assumptions. These consist of the *ex ante* Pareto principle, which is only one component of the *ex ante* criterion, and the decision-theoretic requirement that society have state-independent *ex post* social preferences, which is only one logical implication of the *ex post* criterion. *

Proposition 3 *Suppose that the social preference $\succeq$ induces invariant* ex post *preference orderings $\succeq_{so}$ and the* ex ante *Pareto principle holds.*

Then, there are a strictly positive probability vector $\boldsymbol{\pi} \in \Delta_{\mathcal{S} \times \mathcal{O}}$, and for all $i \in \mathcal{I}$, continuous and increasing utility functions u^i on $\mathbb{R}$ such that the ex ante *social preference $\succeq$ admits the following subjective expected utility representation:*

$$W_{\mathrm{xa}}(\mathbb{X}) \quad = \quad \sum_{s \in \mathcal{S}} \sum_{o \in \mathcal{O}} \pi_{so} W_{\mathrm{xp}}(\mathbf{x}_{so}), \quad \textit{for all } \ \mathbb{X} \in \mathbb{R}^{\mathcal{I} \times \mathcal{S} \times \mathcal{O}},$$

with

$$W_{\mathrm{xp}}(\mathbf{x}_{so}) = \sum_{i \in \mathcal{I}} u^i(x^i_{so}), \textit{for all } \ \mathbf{x}_{so} \in \mathbb{R}^{\mathcal{I}}.$$

As a result, the ex post Pareto principle holds, and for all $i \in \mathcal{I}$, $\succeq^i$ admits a SEU representation:

$$U^i(\mathbf{X}) \quad := \quad \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} \pi_{so}\, u^i(x_{so}), \quad \textit{for all } \ \mathbf{X} \in \mathbb{R}^{\mathcal{S}\times\mathcal{O}}.$$

In these representations, $\boldsymbol{\pi}$ is unique, and the u^i are unique up to a positive affine transformation with a common multiplier.

A classic formulation of the clash between the *ex ante* and *ex post* social welfare criteria assumes both, and deduces that all individuals and society must share the *same* probabilities —an implausible scenario. Proposition 3 reproduces this impossibility theorem in our twofold uncertainty framework. Our strategy now will consist in enriching each criterion with as much of the other as is possible without triggering the impossibility. Section 4 proceeds from the *ex ante* criterion, and Section 5 from the *ex post* criterion.

4 An *ex ante*-oriented reconciliation

The following theorem enriches the *ex ante* social welfare criterion of Proposition 1 with one significant component of the *ex post* criterion. Among the decision-theoretic conditions that the *ex post* criterion puts on social preference, the theorem retains all of those relative to the o-uncertainty, but only part of those relative to the s-uncertainty. The *ex post* Pareto principle does not need to be assumed, because it logically follows from these premises. The end of the section explains that no more of the *ex post* criterion can be added without precipitating an impossibility.

Theorem 4 *Take the full set of assumptions for the ex ante criterion in Proposition 1, i.e.,* (A1), (A2) *and* (A3). *Among the assumptions for the ex post criterion in Proposition 2, take* (B1)*, i.e., that $\succeq$ induces interim social preference orderings $\succeq_s$ and $\succeq_o$, but suppose only that the interim preferences $\succeq_o$ are invariant, a weakening of* (B2).

Then, for all $i \in \mathcal{I}$, the SEU representations (2) *of Proposition 1 for* ex ante *individual preferences hold with $\mathbf{q}^1 = \ldots = \mathbf{q}^n = \mathbf{q}$, i.e., for all $i \in \mathcal{I}$,*

$$U^i(\mathbf{X}) \quad = \quad \sum_{o\in\mathcal{O}} \sum_{s\in\mathcal{S}} q_o\, p^i_s\, u^i(x_{so}), \quad \textit{for all } \ \mathbf{X} \in \mathbb{R}^{\mathcal{S}\times\mathcal{O}}. \tag{7}$$

Furthermore, the ex ante *social preference $\succeq$ is now represented by the additive* ex ante *social welfare function*

$$W_{\mathrm{xa}}(\mathbb{X}) \quad := \quad \sum_{i\in\mathcal{I}} U_i \quad = \quad \sum_{i\in\mathcal{I}} \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} q_o\, p^i_s\, u^i(x^i_{so}), \quad \textit{for all } \ \mathbb{X} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}. \tag{8}$$

Finally, for all states $(s, o) \in \mathcal{S}\times\mathcal{O}$, $\succeq$ induces a state-dependent ex post *social preference ordering $\succeq_{so}$ represented by*

$$W_{\mathrm{xp},s}(\mathbf{x}) \quad := \quad \sum_{i\in\mathcal{I}} p^i_s\, u^i(x^i), \quad \textit{for all } \ \mathbf{x} \in \mathbb{R}^{\mathcal{I}}. \tag{9}$$

Thus, (B3) *(the* ex post *Pareto principle) holds. For any $i \in \mathcal{I}$, the individual conditional preferences $\succeq_o^i$ and $\succeq_s^i$ have SEU representations given by formulae* (4) *and* (5)*, respectively.* *
Meanwhile, the conditional social preference $\succeq_o$ and $\succeq_s$ are represented by (weighted) sums of the individual representations in (4) *and* (5)*; thus, both the subjective and objective interim Pareto principles are satisfied. Finally, for any $s \in \mathcal{S}$, the s-conditional social preference $\succeq_s$ also admits an SEU representation in terms of the probability vector $\mathbf{q}$ and the utility function $W_{\mathrm{xp},s}$ defined in* (9)*.*

In these representations, $\mathbf{q}$ and each $\mathbf{p}^i$ are unique, while the utility functions u^i are unique up to a positive affine transformation with a common multiplier.

Note to Philippe: Note an important addition to this theorem: in addition to the *objective* Pareto principle, the *subjective* Pareto principle also holds. To see this, observe that for any $o \in \mathcal{O}$, formula (8) yields the following (state-independent) "utilitarian" representation for $\succeq_o$:

$$\text{for all } \mathbf{X} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}}, \quad W_{\mathrm{subj}}(\mathbf{X}) \quad = \quad \sum_{i\in\mathcal{I}} \sum_{s\in\mathcal{S}} p_s^i\, u^i(x_s^i) \quad = \quad \sum_{i\in\mathcal{I}} U_{\mathrm{subj}}^i(\mathbf{x}^i),$$

where, for all $i \in \mathcal{I}$, we define

$$U_{\mathrm{subj}}^i(\mathbf{x}) \quad := \quad \sum_{s\in\mathcal{S}} p_s^i\, u^i(x_s), \quad \text{for all } \mathbf{x} \in \mathbb{R}^{\mathcal{S}},$$

which is an SEU representation of the individual o-conditional preference $\succeq_o^i$ (independent of o). Thus, according to our terminology on page 9, the *subjective* Pareto principle holds. On the other hand, for any $s \in \mathcal{S}$, formula (8) also yields the following (state-dependent) "utilitarian" representation for $\succeq_s$:

$$\text{for all } \mathbf{X} \in \mathbb{R}^{\mathcal{I}\times\mathcal{O}}, \quad W_s(\mathbf{X}) \quad = \quad \sum_{o\in\mathcal{O}} \sum_{i\in\mathcal{I}} q_o\, p_s^i\, u^i(x_o^i) \quad = \quad \sum_{i\in\mathcal{I}} p_s^i\, U_{\mathrm{obj}}^i(\mathbf{x}^i),$$

where, for all $i \in \mathcal{I}$, we define

$$U_{\mathrm{obj}}^i(\mathbf{x}) \quad := \quad \sum_{o\in\mathcal{O}} q_o\, u^i(x_s), \quad \text{for all } \mathbf{x} \in \mathbb{R}^{\mathcal{O}},$$

which is an SEU representation of the individual o-conditional preference $\succeq_o^i$ (independent of o). Thus (as Theorem 4 originally asserted), the *objective* Pareto principle also holds. Finally, I think it is important to include in the theorem the fact that $\succeq_s$ also admits an SEU representation:

$$W_s(\mathbf{X}) \quad = \quad \sum_{o\in\mathcal{O}} \sum_{i\in\mathcal{I}} q_o\, p_s^i\, u^i(x_o^i) \quad = \quad \sum_{o\in\mathcal{O}} q_o\, W_{\mathrm{xp},s}(\mathbf{x}_o), \quad \text{for all } \mathbf{X} \in \mathbb{R}^{\mathcal{I}\times\mathcal{O}},$$

Theorem 4 strengthens Proposition 1 in several ways. First of all, from the individuals' SEU representations, we see that each can entertain idiosyncratic probabilistic beliefs $\mathbf{p}^i$ on $\mathcal{S}$, but they must adopt common probabilistic beliefs $\mathbf{q}$ on $\mathcal{O}$. From society's utility * representation, we see that it entertains probabilistic beliefs on $\mathcal{O}$ (also given by $\mathbf{q}$), but no such beliefs about $\mathcal{S}$. This different treatment of the two sources of uncertainty arguably justifies our interpretation of one as being *subjective* and the other as being *objective*.

Second, Theorem 4 turns the unspecified social welfare function W_{xa} of Proposition 1 into a weighted sum of individual expected utilities, as in Harsanyi's (1955) Social Aggregation Theorem. However, unlike in this famous result, the identity of probabilistic beliefs is here *derived*, not merely assumed, and moreover, it only holds for the *objective* *
uncertainty, as was just emphasized.

The W_{xa} function also delivers weighted sum representations for conditional social preferences, but with distinctive properties. The representations of the interim social preferences $\succeq_o$:

$$\sum_{i\in\mathcal{I}}\sum_{s\in\mathcal{S}} p_s^i u^i(x_{so}^i)$$

are independent of o. But the representation (9) of the *ex post* social preferences $\succeq_{so}$, and *
the resulting representation

$$\sum_{i\in\mathcal{I}}\sum_{o\in\mathcal{O}} q_o p_s^i u^i(x_{so}^i)$$

of the interim social preferences $\succeq_s$ do depend on s. To assume or derive *state-dependence* of social preferences is one way of reconciling the *ex ante* and *ex post* criteria of social welfare (see Mongin (1998), Chambers and Hayashi (2006) and Keeney and Nau (2011)). This reconciliation is not merely formal, but can be defended normatively. It seems desirable that society could adjust the individuals' weights in the utility sum according to *
which state of the world is realized, and state-dependent social preferences offer such a flexibility. However, in a standard SEU framework, this resolution leaves society without any probabilistic beliefs at all, and some have objected to it for this reason. Theorem 4 circumvents the objection by deriving a social probability for the objective uncertainty and *
none for the subjective uncertainty. *

An important feature of the last two utility sums is that state-dependence occurs through the factors p_s^i. This means that the individuals who assigned s the highest probability *ex ante* are those who are most favoured by the social welfare function *ex post*. This *
is an inevitable consequence of adopting the full *ex ante* Pareto axiom in an environment where individuals have different beliefs. *Ex ante* Pareto entails "nonpaternalism": if some individuals wish to make what others deem to be foolish bets, then society must let them, as long as the result is *ex ante* Pareto-improving. *Ex post*, there will be transfers from the losers to the winners of these bets, and society must endorse these transfers (to avoid an intertemporal inconsistency in social preferences). To ensure such endorsement, the *ex post* social welfare function must give more weight to the winners than the losers in each state. In other words, in each state of nature, it must give more weight to whichever individuals were prescient enough to assign that state *higher* subjective probabilities *ex ante*. We do not claim this is attractive as a normative principle. But this *is* the *de facto* normative principle underlying (unregulated) financial markets and betting markets, which can be seen as institutional embodiments of the *ex ante* Pareto principle.

In our framework, the *ex post* influence of p_s^i is unavoidable unless one is prepared also to make the $\succeq_s^i$ state-dependent and thus lose the individual probabilities on s. We pursue this alternative line in the following variation of Theorem 4.

Proposition 5 *Make the same assumptions as in Theorem 4, but drop* (A2)*. Then there is a strictly positive probability vector* $\mathbf{q} \in \Delta_{\mathcal{O}}$*, and for all* $i \in \mathcal{I}$*, there are increasing continuous utility functions* u^i_s *on* $\mathbb{R}$*, such that instead of* (7)*, the individual* ex ante *preference* $\succeq^i$ *has the following additive representation:*

$$U^i(\mathbf{X}) \quad := \quad \sum_{s\in\mathcal{S}}\sum_{o\in\mathcal{O}} q_o\, u^i_s(x_{so}), \quad \text{for all } \mathbf{X} \in \mathbb{R}^{\mathcal{S}\times\mathcal{O}}. \tag{10}$$

Furthermore, instead of (8)*, the* ex ante *social preference* $\succeq$ *is now represented by the additive* ex ante *social welfare function*

$$W_{\mathrm{xa}}(\mathbb{X}) \quad := \quad \sum_{i\in\mathcal{I}} U^i \quad = \quad \sum_{i\in\mathcal{I}}\sum_{s\in\mathcal{S}}\sum_{o\in\mathcal{O}} q_o u^i_s(x^i_{so}), \quad \text{for all } \mathbb{X} \in\mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}. \tag{11}$$

Finally, for all states $(s,o) \in \mathcal{S}\times\mathcal{O}$*,* $\succeq$ *induces a state-dependent* ex post *social preference ordering* $\succeq_{so}$ *represented by*

$$W_{\mathrm{xp},s}(\mathbf{x}) \quad := \quad \sum_{i\in\mathcal{I}} u^i_s(x^i), \quad \text{for all } \ \mathbf{x} \in \mathbb{R}^{\mathcal{I}}. \tag{12}$$

In these representations, $\mathbf{q}$ *is unique, while the utility functions* u^i_s *are unique up to a positive affine transformation with a common multiplier.*

The SEU representation obtained for the *ex ante* individual preferences entails representations for the interim individual preferences. For any $s \in \mathcal{S}$, $\succeq^i_s$ admits an SEU representation:

$$U^i_s(\mathbf{x}) \quad := \quad \sum_{o\in\mathcal{O}} q_o\, u^i_s(x_o), \quad \text{for all } \mathbf{x} \in \mathbb{R}^{\mathcal{O}}. \tag{13}$$

This differs from the earlier SEU representation (5) in two ways: the probabilistic belief $\mathbf{q}$ is now common to all agents, but the utility function u^i_s now depends on the state s. Meanwhile, for any $o \in \mathcal{O}$, $\succeq^i_o$ no longer admits an SEU representation like (4), but only the additive representation:

$$U^i_o(\mathbf{x}) \quad := \quad \sum_{s\in\mathcal{S}} u^i_s(x_s), \quad \text{for all } \mathbf{x} \in \mathbb{R}^{\mathcal{S}}. \tag{14}$$

Notice the contrast: the U^i_s representation depends on s, but the U^i_o representation does not depend on o, which establishes that the $\succeq^i_o$ are in fact invariant. The *ex post* social preference $\succeq_{so}$ is represented by the social welfare function (12), which is is state-dependent, as in Theorem 4, but it avoids the *ex post* use of probabilities, unlike Theorem 4. Some will view this as an advantage, but the cost should also be clear, i.e., a less informative representation of the individuals' *ex ante* preferences, since they do not assign probabilities to subjective uncertainty.

Finally, we show that Theorem 4 adds as much as possible of the *ex post* criterion to the *ex ante* criterion. By assuming the full force of (B2), one derives a social probability $\mathbf{p}$ also on the subjective source, and this forces the individuals to align their probabilities $\mathbf{p}_i$ on this $\mathbf{p}$ — a conclusion that reproduces the unpalatable conclusion of Proposition 3.

Corollary 6 *Make the same assumptions as in* Theorem 4 *except that* (B2) *now holds in full, i.e., both the* $\succeq_o$ *and the* $\succeq_s$ *are invariant. Then, the representations of Theorem 4 hold with a common probability vector* $\mathbf{p} \in \Delta_{\mathcal{S}}$ *such that* $\mathbf{p}_1 = ... = \mathbf{p}_n = \mathbf{p}$. *

All in all, Theorem 4 appears to be a welcome improvement on the position of those writers, in the early welfare economics controversy, who bluntly adopted the *ex ante* criterion and rejected the *ex post* one. Among the more recent participants, Hild, Jeffrey and Risse (2003) and Risse (2003) have made a sophisticated case for the *ex ante* Pareto principle against the *ex post* one. In effect, they argue that social and individual preferences are always *ex ante*. The distinction between final consequences and uncertain prospects is a matter of convention; a more refined analysis of these consequences would reveal that they define yet another class of uncertain prospects. By focusing on this particular class, the *ex post* Pareto principle makes an arbitrary restriction to the *ex ante* principle, while being open to the same difficulties; hence it should be avoided. This troubling argument connects with worries that Savage once expressed on the relevance of his representation theorem.[6] However, the same argument does not have the same implication here and in Savage. While it may encourage the adoption of the *ex ante* criterion when the only other choice is the *ex post* criterion, it would rather push in favour of synthetic solutions when these become available, as in the twofold uncertainty framework. ? Contradicts page 2

5 An *ex post*-oriented reconciliation

Reversing the direction of reasoning of Section 4, the next theorem enriches the *ex post* criterion with some of the contents of the *ex ante* criterion. From the latter, it retains all decision-theoretic conditions put on individual preferences, but substitutes the *ex ante* Pareto principle with the less demanding objective interim Pareto principle. As shown at the end of this section, an impossibility results from strengthening these assumptions. *

Theorem 7 *Take the full set of assumptions for the* ex post *criterion in Proposition 2, i.e.,* (B1), (B2) *and* (B3). *Among the assumptions for the* ex ante *criterion in Proposition 1, take* (A1) *and* (A2). *Finally, suppose that the objective interim Pareto principle holds.* *

Then, there exist strictly positive probability vectors $\mathbf{p} \in \Delta_{\mathcal{S}}$ *and* $\mathbf{q} \in \Delta_{\mathcal{O}}$, *and for all* $i \in \mathcal{I}$, *there exist strictly positive probability vectors* $\mathbf{p}^i \in \Delta_{\mathcal{S}}$ *and continuous and increasing utility functions* u^i *on* $\mathbb{R}$, *with the following properties. For all* $i \in \mathcal{I}$, *the* ex ante *individual preferences* $\succeq^i$ *have the SEU representation:*

$$U^i(\mathbf{X}) \quad := \quad \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} p_s^i \, q_o \, u^i(x_{so}), \quad \textit{for all } \ \mathbf{X} \in \mathbb{R}^{\mathcal{S}\times\mathcal{O}}, \tag{15}$$

while the ex ante *social preference* $\succeq$ *has the SEU representation:*

$$W_{\mathrm{xa}}(\mathbb{X}) \quad = \quad \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} p_s \, q_o \, W_{\mathrm{xp}}(\mathbf{x}_{so}), \quad \textit{for all } \ \mathbb{X} \in \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}}. \tag{16}$$

[6]See Savage's (1972) analysis of "small worlds" and the problem they raise for his SEU theory.

Furthermore, there is a vector of positive weights $\mathbf{r} = (r^i)_{i\in\mathcal{I}}$ *such that the* ex post *social welfare function* $W_{\rm xp}$ *has the additive form*

$$W_{\rm xp}(\mathbf{x}) \quad := \quad \sum_{i\in\mathcal{I}} r^i\, u^i(x^i), \quad \textit{for all} \ \ \mathbf{x}\in\mathbb{R}^{\mathcal{I}}. \tag{17}$$

In these representations, the vectors $\mathbf{p}$, $\mathbf{q}$ *and* $\mathbf{r}$ *are unique, and the utility functions* u^i *are unique up to positive affine representations with a common multiplier.*

As in the analysis of Theorem 4 in Section 4, the conclusions should be compared with those of the base-line statement, which in this case is Proposition 2. First, while Proposition 2 said nothing of the decision theory satisfied by the individuals, we now see that their preferences $\succeq^i$ have SEU representations. As in Theorem 4, these SEU representations endow the individuals with identical probabilities on $\mathcal{O}$ and idiosyncratic probabilities on $\mathcal{S}$, thereby justifying our interpretation of the two sources of uncertainty as being objective and subjective, respectively. However, Theorem 4 and 7 involve different epistemic attitudes on the part of society. Whereas society did not form probabilistic beliefs about $\mathcal{S}$ in Theorem 4, it now entertains such beliefs. But a common feature is that society does not in any way restrict the heterogeneity of individual beliefs about $\mathcal{S}$.

Compared to both Proposition 2 and Theorem 4, a key additional hypothesis of Theorem 7 is the objective interim Pareto principle. This principle is consistent with our interpretation of $\mathcal{O}$ as a source of "objective" uncertainty: arguably, it is appropriate to apply the Pareto principle to $\mathcal{O}$-contingent prospects because any uncertainty about these prospects is evaluated the same way by all individuals —hence there can be no possibility of "spurious unanimity". Both this principle and the *ex post* Pareto principle translate into weighted sums of individual utilities in the conclusions of Theorem 7. That is, the *ex post* social utility $W_{\rm xp}$ (which Proposition 2 did not determine) turns out to be the weighted sum (17) of *ex post* individual utilities. Meanwhile, for each given s, the objective interim social welfare function is given by

$$W_{\rm obj}(\mathbf{X}) \quad := \quad \sum_{o\in\mathcal{O}} q_o\, W_{\rm xp}(\mathbf{x}_o), \quad \text{for all } \mathbf{X}\in\mathbb{R}^{\mathcal{O}\times\mathcal{I}}. \tag{18}$$

By substituting (17) into (18), it is easily checked that $W_{\rm obj}$ is a weighted sum of the individuals' objective interim expected utilities. That is,

$$W_{\rm obj}(\mathbf{X}) \quad = \quad \sum_{i\in\mathcal{I}} r^i \sum_{o\in\mathcal{O}} q_o\, u^i(x^i_o), \quad \text{for all } \mathbf{X}\in\mathbb{R}^{\mathcal{O}\times\mathcal{I}}.$$

No such "utilitarian" decomposition exists for the *subjective* interim social welfare function

$$W_{\rm subj}(\mathbf{X}) \quad := \quad \sum_{s\in\mathcal{S}} p_s\, W_{\rm xp}(\mathbf{x}_s), \quad \text{for all } \mathbf{X}\in\mathbb{R}^{\mathcal{S}\times\mathcal{I}}.$$

Nor does such a decomposition exist for the *ex ante* social utility, which reflects the absence of the subjective interim and *ex ante* Pareto principles. Of course, replacing $W_{\rm xp}(\mathbf{x})$ by formula (16) in the SEU formula (17) delivers weighted sums of u^i values, but since society uses its own probability $\mathbf{p}$, these formulas do *not* add up individual SEU representations.

Third, as with Theorem 4, any further attempt at reconciling the two criteria would precipitate an impossibility. Indeed, adding the *ex ante* Pareto principle collapses the individual probabilities p_s^i onto the social ones p_s, the same undesirable consequence as in Corollary 6. We actually prove the result in slightly stronger form.

Corollary 8 *Under the assumptions of Theorem 7, the following three conditions are equivalent:* (a) *for all $s \in \mathcal{S}$, $\succeq_s$ satisfies the subjective interim Pareto principle;* (b) *for all $i \in \mathcal{I}$, $\mathbf{p}^i = \mathbf{p}$;* (c) *the* ex ante *Pareto principle holds.*

The solution offered by Theorem 7 should be compared with that of Gilboa, Samet and Schmeidler (2004) (hereafter GSS), which has recently attracted significant attention. In the same Savage framework as Mongin (1995), GSS assume the *ex post* criterion in full and the *ex ante* criterion in part. They limit the *ex ante* Pareto principle to comparisons of social prospects which do not involve probabilistic disagreements between the individuals. From this, they are able to conclude that (i) society's *ex post* preference is represented by a weighted sum of individual *ex post* utility functions, and (ii) society's probability equals a weighted average of individual probabilities, which is usually called the *linear pooling rule* in statistics and the management literature.[7] Importantly, their assumptions do *not* entail that society's SEU representation can be represented in terms of the individuals' SEU representations, let alone by a weighted sum of them. By excluding this, they evade the overdetermination explained in Mongin (1995).

The significant dissimilarities in technical frameworks may obfuscate the comparison between the GSS conclusions and those of Theorem 7. On the one hand, Savage's infinitely divisible probability space offers mathematical resources we do not have here; on the other hand, by taking our probability space to be a product space, we are able to distinguish two sources of uncertainty, a possibility that Savage does not have. However, glossing over these dissimilarities, we may conclude that the GSS representations consist of the same statements as those of Theorem 7, plus the linear pooling rule:

$$\mathbf{p} \quad = \quad \sum_{i \in \mathcal{I}} a^i \mathbf{p}^i \qquad \text{for some constants } a^i \geq 0. \tag{19}$$

Whether this rule is really an advantage is a central concern of the Section 7.

6 Simplified versions of the theorems

For the sake of completeness, we introduce two variants of our main results. In both Theorem 4 and Theorem 7, we proved that the individuals' interim preferences $\succeq_s^i$ admit SEU representations and these representations involve a uniquely defined common probability vector $\mathbf{q}$ on $\mathcal{O}$. Now we will *assume* these properties and by suitably adapting the assumptions of Theorems 4 and 7, recover their other conclusions. The reason for so weakening our

[7] See e.g. the surveys by Genest and Zidek (1986) and Clemen and Winkler (2007).

results is to answer the possible objection that they do not warrant a semantic distinction between objective and subjective uncertainty. Arguably, what makes an objective source different from a subjective one is that the former, but not the latter, is amenable to *a preexisting probability measure*. The rationale for such a measure, which we will here also denote by $\mathbf{q}$, is to be found in one of the philosophical construals of objective probability. For instance, $\mathbf{q}$ may represent that part of the uncertainty which involves repetitive phenomena and can be subjected to knowable empirical frequencies; other less familiar construals are available.[8] Without digging into the philosophical question of how to interpret $\mathbf{q}$, we may explore the mathematical consequences of supposing that the individuals take it as a *datum* in their decision processes. This is the object of the present variants. Formally, we define condition (OBJ) as follows: *

> There exist $\mathbf{q} \in \Delta_{\mathcal{O}}$ and, for all $i \in \mathcal{I}$, continuous and increasing utility functions u^i on $\mathbb{R}$ such that, for all $i \in \mathcal{I}$ and all $s \in \mathcal{S}$, $\succeq_s^i$ admits the SEU representation $\sum_{o \in \mathcal{O}} q_o u^i(x_o)$, for all $\mathbf{x} \in \mathbb{R}^{\mathcal{O}}$.

As (OBJ) preempts some of the conclusions of Theorem 4 and Theorem 7, it makes some of their assumptions redundant, and they need reshaping as follows.

Theorem 9 *For the* ex ante *criterion, assume* (A1), (A3), (OBJ) *and that, for all* $i \in \mathcal{I}$, *the individual preferences* $\succeq^i$ *induce interim preference orderings* $\succeq_s^i$*and these orderings are invariant. For the* ex post *criterion, take the same assumptions as in Theorem 4.* *

Then, for all $i \in \mathcal{I}$, *there is a strictly positive probability vector* $\mathbf{p}^i \in \Delta_{\mathcal{S}}$ *such that the* ex ante *individual preferences* $\succeq^i$ *admit the SEU representation* (7) *from Theorem 4. Likewise, The* ex ante *social preference* $\succeq$ *and* ex post *social preference* $\succeq_{so}$ *admit representations* (8) *and* (9)*, and satisfy the other conclusions of Theorem 4. In particular,* (B3) *again holds, along with the objective and subjective interim Pareto principles.* * * *

In these representations, each $\mathbf{p}^i$ *is unique, while the utility functions* u^i *are unique up to a positive affine transformation with a common multiplier.*

Theorem 10 *For the* ex post *criterion, take the same assumptions as in Theorem 7. For the* ex ante *criterion, assume* (OBJ) *and that, for all* $i \in \mathcal{I}$, *the individual preferences* $\succeq^i$ *induce interim preference orderings* $\succeq_s^i$*and these orderings are invariant. As in Theorem 7, assume the objective interim Pareto principle.*

Then, for all $i \in \mathcal{I}$, *there exist probability vectors* $\mathbf{p}^i \in \Delta_{\mathcal{S}}$ *and continuous and increasing utility functions* u^i *on* $\mathbb{R}$, *such that the* ex ante *individual preferences* $\succeq^i$ *admit the SEU representation* (15) *from Theorem 7. Likewise, the* ex ante *social preferences* $\succeq$ *and the* ex post *social preferences* $\succeq_{so}$ *admit representations* (16) *and* (17)*, and satisfy the same conclusions as in Theorem 7.* *

In these representations, the vectors $\mathbf{p}$ *and* $\mathbf{r}$ *are unique, and the utility functions* u^i *are unique up to positive affine representations with a common multiplier.*

[8]For the frequency-based and other objective philosophical interpretations of probabilities, see the classic treatment by Fine (1973) and the more recent one by Gillies (2000).

These variants are enlightening, albeit of lesser mathematical interest than the original *
results. They show that our reconciliation of the *ex ante* and *ex post* social welfare criteria *
does not depend on a preference-based axiomatization of individual and social beliefs. *

7 Spurious unanimity and complementary ignorance

Let us first consider the *ex ante*-oriented compromise embodied in Theorem 4. As it retains the *ex ante* social welfare criterion in full, it is open to the objection of *spurious unanimity* that is now often raised against this criterion, and more specifically, against the *ex ante* Pareto principle. Mongin (1997) was the first to develop this objection in detail and coin an expression for it. To make our argument self-contained, we briefly restate it in terms of a simple example adapted from the initial paper.

Imagine the members of society are spread out in two areas, an island and a mainland, with the island being rich, sparsely populated, and beautifully preserved, and the mainland being poor, densely populated, and disfigured by industrialization. Being worried that the island is lacking sufficient public services and the mainland is lacking recreation areas, the Government considers connecting one to the other by a bridge, and being democratically inspired, it organizes a public hearing. Given the relatively high toll that will have to be paid by users of the bridge, it is not clear whether the flow will go from the little populated but rich island, or from the heavily populated but poorer mainland. (But we assume that the bridge is financially feasible, whichever the direction of the main flow of users.) As it happens, the Islanders think that the former consequence is more probable than the latter, while the Mainlanders have the opposite belief. It is also the case that both communities are self-concerned, so that the Islanders value the former consequence more than the latter, while the Mainlanders have the opposite preference. Given these data, SEU and even more general decision theories predict that the *two groups will both support the project*. If the Government takes the *ex ante* Pareto principle seriously, it will push the project forward. However, this would be a dubious decision to make. The two groups are unanimous in preferring the bridge, but spuriously, since they are in fact *twice opposed* - i.e., in their utility and probability comparisons – and their disagreements just cancel out in the SEU or related calculation. Arguably, the Government should make an exception to the consumer sovereignty doctrine, and refrain from endorsing unanimous preferences in those cases in which individuals so strongly differ from each other.

The bridge example takes the direction of the flow to be subjectively, not objectively uncertain. In the present notation, it would be evaluated in terms of the $\mathbf{p}^i$, not the $\mathbf{q}$. Thus, the example is damaging for any solution that retains the *ex ante* Pareto principle, as is the case with our first optimal compromise, and even for any solution that would just retain the subjective interim Pareto principle. This very serious defect must however be weighted against the advantage we mentioned earlier of endowing society with a state-dependent representation of its *ex post* preferences. Suppose the latter preferences must evaluate the income distribution a few years ahead, and suppose also there are sufficiently wide disagreements on the future level of GNP for this level to qualify as subjective uncertainty; then, the formula of Theorem 4 permits adjusting the income distribution to the realized level, as seems highly sensible. If one is worried that the more prescient (in the formula, those i with higher p^i_s) will get a greater distributive share, one can retreat to the less constraining formula of Proposition 5.

Gilboa, Samet and Schmeidler (2004) have taken up the spurious unanimity objection, using the different example of a duel, and answered it by an original solution that amounts to restricting the *ex ante* Pareto principle.[9] Formally, they introduce the family $\mathcal{F}$ of all events on whose probabilities all individuals agree, and they require society to respect only those unanimous comparisons which take place between social prospects that are *measurable* with respect to $\mathcal{F}$. For instance, a social prospect that is constant on the cells of a partition, and such that each cell receives the same probabilities from all individuals, satisfies the measurability restriction. This is tailor-made to avoid the bridge and duel examples, since it precludes probabilistic disagreement from cancelling utility disagreement.

However, as a simple example will now show, GSS's elegant resolution can go wrong in cases where the individuals have private information. Consider a society made out of two individuals, Alice and Bob, a partitition of the set of states $\mathcal{S}$ into three events E_1, E_2, E_3, and two prospects, f and g, which we describe in terms of the utility values that each of the three events brings to Alice and Bob. We assume that they share the *same* utility function, and initially have the *same* probabilistic beliefs, as in the next table.

	E_1	E_2	E_3
Alice's and Bob's utilities for f	1	0	1
Alice's and Bob's utilities for g	0	1	0
Alice's and Bob's probabilities	0.49	0.02	0.49

Initially, Alice and Bob both assign an SEU of 0.98 to f and an SEU of 0.02 to g, so that they both prefer f over g. Thus, GSS's restricted *ex ante* Pareto principle says that society should also prefer f over g.

To make the example more concrete, suppose a card has been drawn at random from a well-shuffled deck containing only one Joker. Let E_1 denote the event "Red Suit" (i.e. Hearts or Diamonds), Let E_3 denote the event "Black Suit" (i.e. Clubs or Spades), and let E_2 denote the event "Joker". Thus, f is a bet on "Suit Card", whereas g is a bet on "Joker". Alice and Bob will equally split the winnings of either bet. Given the (unanimous) probabilities and utility assignments in the table, both Alice and Bob prefer to bet on "Suit Card" rather than "Joker".

Now, suppose Alice privately receives information that the card is *not* from a black suit —equivalently, she observes the event $E_1 \cup E_2$. Meanwhile Bob privately learns that the card is *not* from a red suit —equivalently, he observes the event $E_2 \cup E_3$. For whatever reason, suppose Alice and Bob do not communicate their information to each other. Now,

[9]If two individuals agree to fight a duel, this is presumably because they have both opposite beliefs (on who will win) and opposite preferences (on who should survive). This example questions the optimality of mutually agreed transactions, whereas the bridge example questions the optimality of procuring a unanimously desired public good.

after Bayesian updating, they will reach the following probabilities and SEU values:

	$P(E_1)$	$P(E_2)$	$P(E_3)$	$SEU(f)$	$SEU(g)$
Alice	0.96	0.04	0	0.96	0.04
Bob	0	0.04	0.96	0.96	0.04

From this table, we construct the family $\mathcal{F}$ of events on the probabilities of which Alice and Bob agree. Both put $P(E_1 \cup E_3) = 0.96$ and $P(E_2) = 0.04$, so $\mathcal{F} = \{\varnothing, E_1 \cup E_3, E_2, \mathcal{S}\}$. The prospects f and g are constant on $E_1 \cup E_3$ and constant on E_2, hence measurable with respect to $\mathcal{F}$, so GSS's restricted *ex ante* Pareto principle says that that f should be socially preferred to g. However, if Alice knows that the card is not black, and Bob knows that the card is not red, then only one possibility remains: it must be the Joker. Thus, society should prefer g to f, contrary to the GSS conclusion. If Alice and Bob unanimously prefer f over g, this is only because each one has information the other one lacks; let us say they are in a state of *complementary ignorance*.

Besides GSS's restricted *ex ante* Pareto principle, the complementary ignorance problem plagues the linear pooling rule (19), which they recover in their conclusions. When presented in this way, the example may become of interest to the statistical and management literatures. It is well-known that linear pooling does not commute with Bayesian updating; that is, depending on whether one first applies Bayesian revision to individual probabilities and then averages them, or first averages them and then applies Bayesian revision, one ends up with different collective beliefs.[10] Since either method seems *prima facie* plausible, this observation is usually viewed as an embarrassment for the linear pooling rule. In the cards example, if Alice and Bob merely *average* their posterior probabilities, they will obtain the incorrect value $P(E_2) = 0.04$. On the other hand, and if they first average their prior probabilities and then revise this average by E_2, they will get $P(E_2) = 1$, which is the right answer.

The problem with the latter option is that it may not be available. It could well be that society can observe no more than Alice and Bob's posterior beliefs, or perhaps even no more than their betting attitudes. If this happens to be the case, the society will have no way of aggregating priors. Instead, it should extract whatever information it can from the posteriors. In the above example, from the posteriors *alone* we can deduce E_2. In sumary, once informational arguments are fully taken aboard, one may have to reject not only the linear pooling rule, but also the second-best rules that come to the mind in its stead.[11]

At this point, one might object that the linear pooling rule and the *ex ante* Pareto principle (or its GSS restriction) were never intended to cover the case of changing or asymmetric information. In other words, both are stated under the implicit proviso that the individuals' probabilities are *priors*, unlike the *posteriors* of the last table. However, the distinction between prior and posterior is partly a matter of convention. Depending on what one considers to be background knowledge and what one considers to be information,

[10]In technical jargon, the linear pooling rule is not "externally Bayesian". This is an old observation; see, e.g., Genest and Zidek (1986) and Clemen and Winkler (2007)).

[11]Mongin (1997) has a related urn example against the linear pooling rule. In this example, the social observer can infer what each of two individuals has observed by knowing their revised probability values.

a prior in one context becomes a posterior in another. Leaving aside the hypothetical constructs of the "original position" or the "veil of ignorance", which are usually understood in terms of pure priors, the individual probabilities relevant to normative economics are typically posteriors.[12] Instead of the prior *versus* posterior distinction, we would privilege that between those posteriors which can be analyzed so as to reveal information, as in the Alice and Bob example, and those which cannot.

All this is to say that to derive the linear pooling rule, as GSS do, is a mixed blessing. In the end, we view it as an *advantage* that Theorem 7 involves no connection between the social probability and the individual ones. To see that there is indeed *no* connection, suppose that, for any choice of u^i, the $\mathbf{p}^i$ are all the same, but differ from $\mathbf{p}$; the assumptions of the theorem can easily be satisfied under this supposition. Admittedly, Theorem 7 is not as assertive as it could be, and it does not answer the informational problems surrounding the linear pooling rule. But at least it does not endorse this dubious rule, and in the Alice and Bob example, does not produce an incorrect social belief.

8 Comparisons with the literature

In this comparative section, we first relate our decision-theoretic apparatus to the standard systems of Savage (1972) and Anscombe and Aumann (1963) for SEU theory, and then move to an overview of the solutions recently proposed to reconcile the *ex ante* and *ex post* social welfare criteria.

When we say that $\succeq^i$ or $\succeq$ induces conditional preference orderings of some type, we are in effect applying Savage's *sure-thing principle* – (P2) in his list of postulates – but in a significantly weaker form. While (P2) requires conditionals on *all* possible events to be preference orderings, we restrict this requirement to events of the form s, o, and (s, o). As for our invariance properties, they correspond to Savage's *event-independent preference* condition – (P3) in his list – and are neither weaker nor stronger than this condition. Under (P3), only preferences over constant prospects are event-independent, and this holds for all possible non-null events, whereas invariance here bears on all prospects, but only for the limited class of events we consider.[13]

Unlike Savage, and like Anscombe and Aumann in their variant of Savage's system, we have a finite set of states of the world and a structured set of consequences. However, we merely take these consequences to be real numbers, and only require *ex post* individual preferences to be monotonic, whereas Anscombe and Aumann endow their consequences with the highly specific form of lotteries, and require *ex post* individual preferences to satisfy the VNM axioms. They have often been criticized for already introducing probability values and expected utility representations in an axiomatic context where these two components should be derived fully. Mongin and Pivato (2015) have proposed a system for finite state spaces that avoids this discrepancy, and the SEU theory of the present paper

[12] While not the case in Harsanyi and Rawls, some recent constructs of the "original position" or the "veil of ignorance" do allow for private information (Nehring, 2004; Chambers and Hayashi, 2014).

[13] Savage's other event-independence condition, i.e. (P4), is not needed in the present framework. On the respective contributions of (P3) and (P4), see, e.g., Karni's (2014) review of SEU theories.

follows this line. Notice that the variants of Section 6 technically comply neither with * Savage's nor with Anscombe and Aumann's system. They rely on a preexisting probability vector $\mathbf{q}$, in contradistinction with the former, but they avoid any reference to lotteries, in contradistinction with the latter.

The conflict between the *ex ante* and *ex post* social welfare criteria has attracted much attention recently. The common strategies are to weaken either the *ex ante* Pareto principle, as in Gilboa, Samet and Schmeidler (2004), or the SEU assumptions (whether on society or the individuals), while preserving the *ex post* Pareto principle. Along the first road, Nehring (2004) and Chambers and Hayashi (2014) have found a new impossibility theorem. They assume that agents have private information, and restrict the *ex ante* Pareto principle to situations where it is common knowledge that one prospect *ex ante* Pareto-dominates another. If society satisfies statewise dominance, even this restricted variant leads to an undesirably strong conclusion: the agents must share a common prior on the common knowledge events. Chambers and Hayashi further show that *ex ante* social welfare is a weighted sum of individual expected utilities, whereas Nehring assumes this. These results refine those of Harsanyi (1955) and Mongin (1995).

Others use both strategies at the same time. For example, Qu (2017) considers the possibility that both society and the individuals conform to the *maximin expected utility* (MEU) theory of Gilboa and Schmeidler (1989), a generalization of SEU where each agent is described by a *set* of probabilistic beliefs. Since Qu operates in the Anscombe and Aumann (1963) framework, which draws an exogenous distinction between objective and subjective uncertainty, he can deploy a Pareto principle for objective uncertainty that bears some analogy with our objective interim Pareto principle. He also restricts the *ex ante* Pareto principle to a *Common Taste* version, which regulates comparisons between prospects f and g only when the individuals have unanimous *ex post* preferences over every possible outcome arising from f or g, with comparisons being performed on the certainty equivalents of such prospects (or convex combinations thereof). Qu shows that society satisfies his two Pareto principles if and only if *ex post* social welfare takes the form of a weighted sum of individual utilities and the social set of probabilities $\mathcal{P}$ is a convex combination of the individuals' sets. The first conclusion is identical to that of Gilboa, Samet and Schmeidler (2004) and ours, while the second generalizes the linear pooling rule (19) of the former.[14] *

Alon and Gayer (2016) assume Savage's SEU theory for the individuals, and put axioms on society that endow it with a MEU representation. They strengthen GSS's restricted *ex ante* Pareto principle to a *Consensus Pareto* version, which says that if all individuals (according to their own probabilistic beliefs) deem that prospect f yields a higher SEU than prospect g *for every individual*, then society should prefer f over g. This excludes the spurious unanimity implication of the *ex ante* Pareto principle, while accommodating some situations where individuals have different beliefs. Alon and Gayer show that society satisfies Consensus Pareto if and only if *ex post* social welfare takes the usual weighted sum form and the social probability set is included in the convex hull of the individual

[14] In a variant replacing MEU by the more general *Choquet expected utility* (CEU) of Schmeidler (1989), Qu derives the same *ex post* social welfare function, and another generalization of the linear pooling rule.

probability measures, a conclusion comparable to Qu (2017) and more recently, Billot and *
Qu (2017).

In a framework with a finite state space and given *ex post* individual utilities, somewhat analogous to that of Mongin and Pivato (2015), Hayashi and Lombardi (2016) take up the analysis of *ex ante* and *ex post* social rules under the assumptions of MEU theory. In contrast with most papers discussed here, the authors are aiming at maxmin "egalitarianism" rather than weighted sum "utilitarianism", and this leads them to axiomatize two novel criteria. In the former, which is *ex ante* oriented, society pays special attention to those individuals who have minimal *ex ante* utility given their sets of probabilistic beliefs; in the latter, which is *ex post* oriented, society pays special attention to those individuals who have minimal *ex post* utility, and makes its comparisons in terms of its own set of probabilistic beliefs. The paper belongs to the growing literature on egalitarianism under uncertainty, a literature space prevents us from expanding on here; see the recent survey articles by Mongin and Pivato (2016*b*) and Fleurbaey (2017) for more information.

Using the Anscombe-Aumann framework, Danan et al. (2016) suppose that society and each individual have partial orders $\succeq$ and $\succeq_i$ that admit representations in the sense of Bewley (2002), i.e., there are sets $\mathcal{P}$ and $\mathcal{P}_i$ of probability distributions such that $f \succeq g$ (resp. $f \succeq_i g$) if and only if f yields at least as high an expected utility as g according to *all* elements in $\mathcal{P}$ (resp. in $\mathcal{P}_i$). The authors refer to these partial orders as *unambiguous preference relations* and show that those of society satisfy an *ex ante* Pareto principle relative to those of the individuals if and only if conclusion (i) holds, and $\mathcal{P}$ is included in the intersection of the $\mathcal{P}_i$. In the particular case of SEU theory, the individuals' unique probability measure is the same for all individuals and society, which also satisfies SEU theory; this recovers one of the classic impossibilities in the Anscombe-Aumann framework. Danan et al. also consider Common Taste Pareto and show that society's unambiguous
preference relation satisfies this condition if and only if *ex post* social welfare is a weighted *
sum of individual utilities, and $\mathcal{P}$ is included in the convex hull of the unions of the $\mathcal{P}_i$. They also provide a solution to an impossibility theorem of Gajdos, Tallon and Vergnaud (2008), in which a society of ambiguity-sensitive agents is susceptible to a phenomenon of *spurious hedging*, analogous to spurious unanimity.[15]

The positive results in the three aforementioned papers have many attractive features, but they are all vulnerable to the problem of complementary ignorance from Section 7. Admittedly, the MEU and Bewley theories are much more flexible than SEU theory (which corresponds to the case of a singleton probability set). But in our example, Alice and Bob have the same utility function, so the Consensus Pareto condition of Alon and Gayer (2016) and the Common Taste Pareto condition of Qu (2017) and Danan et al. (2016) reduce to the *ex ante* Pareto principle, and the results of these three papers force society to obey SEU theory, as in Gilboa, Samet and Schmeidler (2004). Hence the social probabilities are
given by the linear pooling rule (19), which we have explained founders upon the problem *
of complementary ignorance.

Billot and Vergopoulos (2016) have devised a framework in which neither spurious unanimity nor complementary ignorance can arise. They endow each individual with a

[15]This is explained most clearly in the preprint version Danan et al. (2015) .

personalized state space and a personalized consequence set, and society with a state space and a consequence set that are simply Cartesian products of these spaces. Assuming SEU theory for individuals and society, Billot and Vergopoulos show that the latter satisfies a set of three Pareto conditions if and only if the *ex post* social welfare admits an additive representation, and the social probability measure is the *product* of the individual ones, an interesting alternative to the linear pooling rule. Multiplicative rules satisfy the condition alluded to in the Section 7 of being "externally Bayesian", that is of being indifferent to the order between the aggregative step and Bayesian revision. However, the chosen framework presupposes that individuals face *independent risks*, as in standard insurance markets. This makes it appropriate for some policy applications, but not for all. In the bridge example, the risk faced by the Islanders is not independent of the risk faced by the Mainlanders. In the duel example, the risks faced by the duellists are not independent either.

Complementing these axiomatic endeavours, other papers usefully investigate the same issues in relation to financial markets. Standard economic theory generally endorses transactions on these markets by assuming the *ex ante* Pareto principle, but uncertainty raises spurious unanimity objections here as it does elsewhere. Thus, Posner and Weyl (2013), Blume et al. (2015) and others identify *purely speculative* transactions with those driven by different beliefs, and argue for public regulation in this case. Defining a new form of Paretian comparison, Gilboa, Samuelson and Schmeidler (2014) say that a prospect f *No-Betting Pareto* (NBP) *dominates* another prospect g if there exists some probability measure p such that f yields at least as high an expected utility as g for every agent, according to p. They show that NBP-dominance holds if and only if, for any weighted sum of *ex post* utilities, g does not statewise dominate f. They explore the consequences for financial markets of restricting the *ex ante* Pareto principle to NBP-dominance comparisons, an analysis continued by Gayer et al. (2014). But NBP is still vulnerable to the problem of complementary ignorance. *

Instead of Pareto dominance, Brunnermeier, Simsek and Xiong (2014) strengthen the concept of Pareto *inefficiency*, by defining a prospect f to be *belief-neutral inefficient* if, for every probability measure p arising from a convex combination of the individual ones, there is some prospect yielding a higher p-expected utility for every agent than f. They propose to use this criterion to identify speculative transactions, and recommend regulatory scrutiny for these. They also propose a second criterion, which is based on a utilitarian-style social welfare function, and is related to the Bewley preferences considered by Danan et al. (2016). Like the authors cited in the previous paragraph, they argue forcefully and convincingly that unrestricted speculation in financial markets can destroy social welfare.

9 Conclusion

By enriching the underlying model of uncertainty, we have identified intermediate positions * between the classical *ex ante* and *ex post* social welfare criteria. Whereas some of the recent literature has departed from SEU theory, we have instead chosen to enhance this theory * with a distinction it does not normally explore: that between two sources of uncertainty, here interpreted as being objective and subjective respectively. This semantic distinction emerges in the representation theorems through the difference between multiple probability assessments for one source and a single probability assessment for the other.

The intermediate positions we have uncovered do not fully resolve the underlying trade-off: one must still choose between a primarily *ex ante* and a primarily *ex post* solution. But, as we have shown, the former (Theorem 4) can reap the benefits of an *ex post* state-dependent social welfare function without depriving society of all probabilistic beliefs, and the latter (Theorem 7) can extend some way in the direction of *ex ante* Paretianism, while avoiding the trap of spurious unanimity. Unlike many previously proposed solutions in the literature, our second solution also avoids the trap of *complementary ignorance*, whose introduction in Section 7 is another important contribution of this paper.

We feel that the theoretical potential of the twofold uncertainty framework is not exhausted. Elsewhere, it has served to introduce a preference axiomatization, in a Bayesian theorist's style, of the property of stochastic independence (Mongin, 2017). It may guide other foundational explorations in probability theory, in particular a more thorough discussion of objective probability than that provided here. We also feel that the complementary ignorance objection calls for more attention, as it suggests that collective beliefs should take into account not merely individual beliefs, but also their origin and justification; this opens new avenues for aggregation theory.

Appendices

Appendix A contains a table summarizing our results. Appendix B reviews technical background about separability. Appendix C contains the proofs of all results in the paper.

Appendix A: Table of results

	Proposition 1		Proposition 2		Proposition 3		**Theorem 4**		Proposition 5		Corollary 6		**Theorem 7**		Corollary 8	
	Hypotheses	Conclusions	Hypotheses	Conclusions	Hypotheses	Conclusions	Hypotheses	Conclusions	Hypotheses	Conclusions	Hypotheses	Conclusions	Hypotheses	Conclusions	Hypotheses	Conclusions
Individuals																
Ex ante SEU?		Y				Y		Y				Y		Y		Y
s-conditional prefs?	y					Y	y		y		y		y		y	
Are they invariant?	y						y				y		y		y	
s-conditional SEU?		Y				Y		Y		Y		Y		Y		Y
o-conditional prefs?	y					Y	y		y		y		y		y	
Are they invariant?	y						y			Y	y		y		y	
o-conditional SEU?		Y				Y		Y				Y		Y		Y
Social preferences																
Ex ante Pareto?	y				y		y		y		y				(y)	Y[g]
Objective Pareto?						Y		Y		Y		Y	y		y	
Subjective Pareto?						Y		Y		Y		Y			y[g]	(Y)
Ex post Pareto?			y			Y		Y		Y		Y	y		y	
Ex ante SEU?				Y		Y						Y		Y		Y
Same beliefs?[a]						☠						☠				☠
Ex ante utilitarian?[c]						Y		Y		Y		Y				Y
s-conditional prefs?			y			Y	y		y		y		y		y	
Are they invariant?			y								y		y		y	
s-conditional SEU?				Y		Y		Y		Y		Y		Y		Y
Same beliefs on $\mathcal{O}$?[a]						Y		Y		Y		Y		Y		Y
Interim utilitarian?[d]						Y		Y		Y		Y		Y		Y
o-conditional prefs?			y			Y	y		y		y		y		y	
Are they invariant?			y				y		y		y		y		y	
o-conditional SEU?				Y		Y						Y		Y		Y
Same beliefs on $\mathcal{S}$?[a]						☠						☠				☠
Interim utilitarian?[e]						Y		Y		Y		Y				Y
Ex post social prefs?[b]			y		y			Y		Y		Y	y		y	
Invariant?				Y	y							Y		Y		Y
Ex post SWF?				Y		Y		Y		Y		Y		Y		Y
State-independent?				Y		Y						Y		Y		Y
Ex post utilitarian?[f]						Y		Y		Y		Y		Y		Y

Notes. "y" means "yes" for a hypothesis, "Y" means "yes" for a conclusion, and "☠" indicates a "yes" that could be interpreted as an impossibility theorem. For ease of reading, "no" is always indicated by a blank entry in the table.

(a) "Same beliefs" means that all individual beliefs are identical to the social beliefs.

(b) "*Ex post* social prefs" means that the *ex ante* social preferences satisfy *statewise dominance* with respect to these *ex post* social preferences.

(c, d, e, f) Here, we mean "utilitarian" only in the *formal* sense that the social utility function is a (weighted) sum of individual utility functions. In (c), this refers to *ex ante* utility functions, in (d) it refers to s-conditional utility functions, in (e) it refers to o-conditional utility functions, and in (f) it refers to *ex post* utility functions.

(g) Given the other hypotheses of Corollary 8, *ex ante* Pareto and subjective interim Pareto are logically equivalent. For simplicity, we have presented subjective interim Pareto as a "hypothesis" and *ex ante* Pareto as a "conclusion", but their roles could be reversed.

Appendix B: Technical background

We begin by restating the definition of a conditional relation in terms of its master relation, and the separability property that turns a conditional relation into an ordering.

Suppose that a weak preference ordering R is defined on a product set $\mathcal{X} = \prod_{\ell \in \mathcal{L}} \mathcal{X}_\ell$, where $\mathcal{L}$ is a finite set of indexes. Take a subset of indexes $\mathcal{J} \subseteq \mathcal{L}$ and its complement $\mathcal{K} := \mathcal{L} \backslash \mathcal{J}$. Denote the subproduct sets $\prod_{\ell \in \mathcal{J}} \mathcal{X}_\ell$ and $\prod_{\ell \in \mathcal{K}} \mathcal{X}_\ell$ by $\mathcal{X}_\mathcal{J}$ and $\mathcal{X}_\mathcal{K}$, respectively. By definition, the *conditional induced by* R *on* $\mathcal{J}$ is the relation $\mathsf{R}_\mathcal{J}$ on $\mathcal{X}_\mathcal{J}$ thus defined: for all $\xi_\mathcal{J}, \xi'_\mathcal{J} \in \mathcal{X}_\mathcal{J}$,

$$\xi_\mathcal{J} \; \mathsf{R}_\mathcal{J} \; \xi'_\mathcal{J} \quad \text{if and only if} \quad \text{for some} \xi_\mathcal{K} \in \mathcal{X}_\mathcal{K}, (\xi_\mathcal{J}, \xi_\mathcal{K}) \; \mathsf{R} \; (\xi'_\mathcal{J}, \xi_\mathcal{K}).$$

We denote the conditional $\mathsf{R}_{\{\ell\}}$ by R_ℓ. By a well-known fact, the conditional $\mathsf{R}_\mathcal{J}$ is an ordering if and only if R is *separable in* $\mathcal{J}$, that is: for all $\xi_\mathcal{J}, \xi'_\mathcal{J} \in \mathcal{X}_\mathcal{J}$ and $\xi_\mathcal{K}, \xi'_\mathcal{K} \in \mathcal{X}_\mathcal{K}$,

$$(\xi_\mathcal{J}, \xi_\mathcal{K}) \; \mathsf{R} \; (\xi'_\mathcal{J}, \xi_\mathcal{K}) \quad \text{if and only if} \quad (\xi_\mathcal{J}, \xi'_\mathcal{K}) \; \mathsf{R} \; (\xi'_\mathcal{J}, \xi'_\mathcal{K}).$$

In this case, we may also say that $\mathcal{J}$ is a R-*separable*. Clearly, separability in $\mathcal{J}$ entails that R is *increasing* with $\mathsf{R}_\mathcal{J}$, that is: for all $\xi_\mathcal{J}, \xi'_\mathcal{J} \in \mathcal{X}_\mathcal{J}$ and $\xi_\mathcal{K} \in \mathcal{X}_\mathcal{K}$,

$$\text{if} \xi_\mathcal{J} \; \mathsf{R}_\mathcal{J} \; \xi'_\mathcal{J}, \text{ then} (\xi_\mathcal{J}, \xi_\mathcal{K}) \; \mathsf{R} \; (\xi'_\mathcal{J}, \xi_\mathcal{K}),$$

and if the $\mathsf{R}_\mathcal{J}$-comparison is in fact strict, so is the resulting R-comparison. Conversely, if $\mathsf{R}_\mathcal{J}$ is some ordering on $\mathcal{X}_\mathcal{J}$, the property that R on $\mathcal{X}$ is increasing with $\mathsf{R}_\mathcal{J}$ entails that R is weakly separable in $\mathcal{J}$.[16]

This apparatus can be applied by taking the $\mathcal{X}_\ell$ sets to be copies of $\mathbb{R}$, and suitably fixing the relation R and the indexing sets $\mathcal{L}$ and subsets $\mathcal{J} \subset \mathcal{L}$. For example, for any $i \in \mathcal{I}$ and $s \in \mathcal{S}$, to translate the statement, "$\succeq^i$ induces a conditional preference $\succeq^i_s$" into a statement about separability, we take $\mathsf{R} = \succeq^i$, $\mathcal{L} = \{(s, o) \mid s \in \mathcal{S}, o \in \mathcal{O}\}$ and $\mathcal{J} = \{(s, o) \mid o \in \mathcal{O}\}$.

[16]For these definitions and basic facts, see Fishburn (1970), Keeney and Raiffa (1976), and Wakker (1989). What is called *separable* here is sometimes called *weakly separable* elsewhere.

All the other "conditional preference" statements introduced in Section 2 can be translated in the same way. Recall that we assume throughout that conditional preference orderings exist across the uncertainty type or not at all (i.e., for all s or none, etc). It then follows from the equivalence between separability and the increasing property just said that our decision-theoretic assumptions could be restated in terms of *dominance*. For example, the statement "$\succeq^i$ induces conditional preferences $\succeq^i_s$" is equivalent to asserting that $\succeq^i$ satisfies dominance with respect to the $(\succeq^i_s)_{s\in\mathcal{S}}$.

Let $\mathcal{X}$ specifically be an open box in $\mathbb{R}^{\mathcal{L}}$, i.e., $\mathcal{X} = \prod_{\ell\in\mathcal{L}} \mathcal{X}_\ell$, where the $\mathcal{X}_\ell$ are open intervals. An ordering $\succeq$ on $\mathcal{X}$ has an *additive representation* if it is represented by a function $U : \mathcal{X} \longrightarrow \mathbb{R}$ of the form

$$U(\mathbf{x}) \quad := \quad \sum_{\ell\in\mathcal{L}} u_\ell(x_\ell), \tag{B1}$$

where $u_\ell :\ \mathcal{X}_\ell \longrightarrow \mathbb{R}$, $\ell \in \mathcal{L}$.

Let us say that $\mathcal{J} \subseteq \mathcal{L}$ is *strictly $\succeq$-essential* if, for all $\mathbf{x} \in \mathcal{X}$, there exist $\mathbf{y}, \mathbf{y}' \in \mathcal{X}$ such that $(y_\ell)_{\ell\in\mathcal{K}} = (y'_\ell)_{\ell\in\mathcal{K}} = (x_\ell)_{\ell\in\mathcal{K}}$, and $\mathbf{y} \succ \mathbf{y}'$. In words, we can create a strict preference by only manipulating the $\mathcal{J}$ coordinates, while keeping the $\mathcal{K}$ coordinates fixed at given values. We now record two classic results due to Gorman (1968*a*,*b*) and Debreu (1960)), respectively. If every subset $\mathcal{J} \subseteq \mathcal{L}$ is $\succeq$-separable, we say that $\succeq$ is *totally separable.*

Lemma B1 *Let $\succeq$ be a continuous order on an open box $\mathcal{X} \subseteq \mathbb{R}^{\mathcal{L}}$. Let $\mathcal{J}, \mathcal{K} \subseteq \mathcal{L}$ be two $\succeq$-separable subsets of indexes, such that $\mathcal{J} \cap \mathcal{K} \neq \emptyset$. Suppose that $\mathcal{J}$, $\mathcal{K}$, and $\mathcal{J} \cap \mathcal{K}$ are all strictly $\succeq$-essential. Then:*

(a) *$\mathcal{J} \cup \mathcal{K}$ is $\succeq$-separable.*

(b) *$\mathcal{J} \cap \mathcal{K}$ is $\succeq$-separable.*

Lemma B2 *If $\succeq$ is a continuous, totally separable order on an open box $\mathcal{X} \subseteq \mathbb{R}^{\mathcal{L}}$, and $\succeq$ is increasing in every coordinate, then $\succeq$ has an additive utility representation* (B1). *Furthermore, the functions $\{u_\ell\}_{\ell\in\mathcal{L}}$ in this representation are unique up to positive affine transformations* (PAT) *with a common multiplier.*

We will now adapt these results to our framework. Suppose $\mathcal{L} = \mathcal{I} \times \mathcal{S} \times \mathcal{O}$, and let $\succeq$ be a preference order on $\mathcal{X} = \mathbb{R}^{\mathcal{L}}$. For any $i \in \mathcal{I}$, we will say that $\succeq$ is *separable in i* if $\{i\} \times \mathcal{S} \times \mathcal{O}$ is $\succeq$-separable. Likewise, for any $s \in \mathcal{S}$ (resp. $o \in \mathcal{O}$), say $\succeq$ is *separable in s* (resp. *separable in o*) if $\mathcal{I} \times \{s\} \times \mathcal{O}$ (resp. $\mathcal{I} \times \mathcal{S} \times \{o\}$) is $\succeq$-separable.

Proposition B3 *Take $\mathcal{L} = \mathcal{I} \times \mathcal{S} \times \mathcal{O}$, with $|\mathcal{I}|, |\mathcal{S}|, |\mathcal{O}| \geq 2$, and $\mathcal{X} = \mathbb{R}^{\mathcal{L}}$, viewing elements $\mathbb{X} \in \mathcal{X}$ as arrays $[x^i_{so}]^{i\in\mathcal{I}}_{s\in\mathcal{S},o\in\mathcal{O}}$. If a continuous order $\succeq$ on $\mathcal{X}$ is increasing in every coordinate, and is separable in each $i \in \mathcal{I}$, each $s \in \mathcal{S}$, and each $o \in \mathcal{O}$, then it admits an additive utility representation $U : \mathcal{X} \longrightarrow \mathbb{R}$ of the form*

$$U(\mathbf{x}) \quad := \quad \sum_{i\in\mathcal{I}} \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} u^i_{so}(x^i_{so}),$$

where each u^i_{so} is a continuous, increasing function from $\mathbb{R}$ to $\mathbb{R}$. Furthermore, the utility functions $\{u^i_{so}\}^{i\in\mathcal{I}}_{s\in\mathcal{S},o\in\mathcal{O}}$ are unique up to PAT with a common multiplier.

Proof. (Sketch)

Since $\succeq$ is increasing in every coordinate, every subset of $\mathcal{L}$ is strictly $\succeq$-essential. For all $i \in \mathcal{I}$, $s \in \mathcal{S}$, and $o \in \mathcal{O}$, the subsets $\{i\} \times \mathcal{S} \times \mathcal{O}$, $\mathcal{I} \times \{s\} \times \mathcal{O}$, and $\mathcal{I} \times \mathcal{S} \times \{o\}$ are $\succeq$-separable, by hypothesis. Thus, Lemma B1 says that the (nonempty) intersections of these sets are $\succeq$-separable, as are their unions. At this point, by further applications of Lemma B1, we can show that every two-element subset of $\mathcal{L}$ is $\succeq$-separable; from there, it can be shown that *every* subset of $\mathcal{L}$ is $\succeq$-separable. In other words, $\succeq$ is totally separable. By hypothesis, $\succeq$ is increasing in every coordinate. Thus, we can apply Lemma B2 to get the additive representation. □

We now specialize the basic sets differently. Take $\mathcal{L} = \mathcal{J} \times \mathcal{K}$ with $|\mathcal{J}|, |\mathcal{K}| \geq 2$, and $\mathcal{X} = \mathbb{R}^{\mathcal{L}}$, viewing elements $\mathbf{X} \in \mathcal{X}$ as matrices $[x_k^j]_{k\in\mathcal{K}}^{j\in\mathcal{J}}$, with $j \in \mathcal{J}$ indexing the rows and $k \in \mathcal{K}$ indexing the columns. Alternatively, we can think of $\mathbf{X}$ as a $\mathcal{J}$-indexed array of row vectors $\mathbf{x}^j := [x_k^j]_{k\in\mathcal{K}} \in \mathbb{R}^{\mathcal{K}}$, or as a $\mathcal{K}$-indexed array of columns vectors $\mathbf{x}_k := [x_k^j]^{j\in\mathcal{J}} \in \mathbb{R}^{\mathcal{J}}$. Now consider a continuous ordering $\succeq$ on $\mathcal{X}$. Here are three axioms that $\succeq$ might satisfy.

Coordinate Monotonicity: For all $\mathbf{X}, \mathbf{Y} \in \mathcal{X}$, if $x_k^j \geq y_k^j$ for all $(j,k) \in \mathcal{J} \times \mathcal{K}$, then $\mathbf{X} \succeq \mathbf{Y}$. If, in addition, $x_k^j > y_k^j$ for some $(j,k) \in \mathcal{J} \times \mathcal{K}$, then $\mathbf{X} \succ \mathbf{Y}$.

Row Preferences: For each column $j \in \mathcal{J}$, $\succeq$ is separable in $\{j\} \times \mathcal{K}$.

Column Preferences: For all rows $k \in \mathcal{K}$, $\succeq$ is separable in $\mathcal{J} \times \{k\}$.

Define $\succeq^j$ and $\succeq_k$ to be the conditional relations of $\succeq$ on j and k, respectively. It follows from Row Preferences that the $\succeq^j$ are orders on $\mathbb{R}^{\mathcal{K}}$, and from Column Preferences that the $\succeq_k$ are orders on $\mathbb{R}^{\mathcal{J}}$. Moreover, $\succeq$ is increasing with respect to each of these conditional relations . The next two axioms force the conditional orders to be invariant.

Invariant Row Preferences: Row Preferences holds, and there is an ordering $\succeq^{\mathcal{J}}$ on $\mathcal{Y}^{\mathcal{K}}$ such that $\succeq^j = \succeq^{\mathcal{J}}$ for all $j \in \mathcal{J}$.

Invariant Column Preferences: Column Preferences holds, and there is an ordering $\succeq_{\mathcal{K}}$ on $\mathcal{Y}^{\mathcal{J}}$ such that $\succeq_k = \succeq_{\mathcal{K}}$ for all $k \in \mathcal{K}$.

These five axioms draw their use from the following proposition, which the proofs in Appendix C will repeatedly use. (Each of these proofs will involve two of the sets $\mathcal{I}$, $\mathcal{S}$, $\mathcal{O}$ taking the place of the abstract indexing sets $\mathcal{J}$ and $\mathcal{K}$.)

Proposition B4 (a) *Suppose a continuous preference order $\succeq$ on $\mathcal{X} = \mathbb{R}^{\mathcal{L}}$ satisfies Coordinate Monotonicity, Row Preferences and Column Preferences. Then for all $j \in \mathcal{J}$ and $k \in \mathcal{K}$, there is an increasing, continuous function $v_k^j : \mathbb{R} \longrightarrow \mathbb{R}$, such that $\succeq$ is represented by the function $W : \mathcal{X} \longrightarrow \mathbb{R}$ defined by:*

$$W(\mathbf{X}) := \sum_{k\in\mathcal{K}} \sum_{j\in\mathcal{J}} v_k^j(x_k^j).$$

In this representation, the functions v_k^j are unique up to PAT with a common multiplier.

(b) *Assume* Invariant Column Preferences *instead of* Column Preferences*, holding the other conditions the same as in part (a). Then there is a strictly positive probability vector $\mathbf{p} \in \Delta_{\mathcal{K}}$, and for all $j \in \mathcal{J}$, there is an increasing, continuous function $u^j : \mathbb{R} \longrightarrow \mathbb{R}$, such that $\succeq$ is represented by the function $W : \mathcal{X} \longrightarrow \mathbb{R}$ defined by:*

$$W(\mathbf{X}) \ := \ \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}} p_k \, u^j(x_k^j).$$

In this representation, the probability vector $\mathbf{p}$ is unique, and the functions u^j are unique up to PAT with a common multiplier.

(c) *Assume* Invariant Row Preferences *instead of* Row Preferences*, holding the other conditions the same as in part (b). Then there is an increasing, continuous function $u : \mathbb{R} \longrightarrow \mathbb{R}$ and strictly positive probability vectors $\mathbf{q} \in \Delta_{\mathcal{J}}$ and $\mathbf{p} \in \Delta_{\mathcal{K}}$ such that $\succeq$ is represented by the function $W : \mathcal{X} \longrightarrow \mathbb{R}$ defined by*

$$W(\mathbf{X}) \ := \ \sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}} q^j \, p_k \, u(x_k^j).$$

In this representation, the probability vectors $\mathbf{q}$ and $\mathbf{p}$ are unique, and the function u is unique up to a PAT.

Proof. See Mongin and Pivato (2015). Part (a) follows from Proposition 1(b). Part (b) follows from Theorem 1(c,d), and part (c) from Corollary 1(c,d). The axioms of that paper are stated differently, because the domain considered there is not necessarily a Cartesian product. □

Appendix C: Proofs of the results of the paper

Our twofold uncertainty framework may seem to raise the worrying possibility that conditional orderings depend on how they are induced; e.g., that $\succeq_{so}$, as directly induced by $\succeq$, differs from $\succeq_{so}$, as induced by the ordering $\succeq_s$ induced by $\succeq$, or from $\succeq_{so}$, as induced by the ordering $\succeq_o$. However, such a discrepancy cannot occur; the different forms of conditionalization "commute" with one another . We skip the purely formal proof. In the next lemma and elsewhere, we will repeatedly use this *commutativity of conditionalization.*

Lemma C1 *Let $\succeq$ be a continuous order on $\mathbb{R}^{\mathcal{I} \times \mathcal{S} \times \mathcal{O}}$.*

(a) *If $\succeq$ induces interim preferences $\succeq_s$ and $\succeq_o$, then it also induces* ex post *preferences $\succeq_{so}$.*

(b) *If, moreover, the interim preferences $\succeq_o$ are invariant, then for any given s, $\succeq_s$ induces invariant* ex post *preferences $\succeq_{so}$.*

(c) *If, moreover, the interim preferences $\succeq_o$ and $\succeq_s$ are both invariant, then the* ex post *preferences $\succeq_{so}$ are invariant.*

Proof. Let $(s, o) \in \mathcal{S} \times \mathcal{O}$. For all $o \in \mathcal{O}$, let $\mathcal{J}_o := \{(i', s', o);\ i' \in \mathcal{I} \text{ and } s' \in \mathcal{S}\}$. Then $\mathcal{J}_o$ is a $\succeq$-separable subset of $\mathcal{I} \times \mathcal{S} \times \mathcal{O}$, because, by hypothesis, $\succeq$ induces interim preferences $\succeq_o$. Similarly, for all $s \in \mathcal{S}$, let $\mathcal{K}_s := \{(i', s, o');\ i' \in \mathcal{I} \text{ and } o' \in \mathcal{O}\}$; this is a $\succeq$-separable subset of $\mathcal{I} \times \mathcal{S} \times \mathcal{O}$, because $\succeq$ induces interim preferences $\succeq_s$. The nonempty intersection $\mathcal{I}_{so} := \mathcal{J}_o \cap \mathcal{K}_s$ is $\succeq$-separable by Lemma B1(b), meaning that $\succeq$ induces *ex post* preferences $\succeq_{so}$.

Adding the assumption that the interim preferences $\succeq_o$ induced by $\succeq$ are invariant, we fix s and consider any pair $o \neq o'$. By commutativity of conditonalization , we can regard the *ex post* preferences $\succeq_{so}$ and $\succeq_{so'}$ as being induced by $\succeq_o$ and $\succeq_{o'}$, respectively. But $\succeq_o = \succeq_{o'}$, so that $\succeq_{so} = \succeq_{so'}$, and now regarding these *ex post* preferences as being induced by $\succeq_s$, we conclude that this ordering induces invariant *ex post* preferences.

Now we add the assumption that the interim preferences $\succeq_s$ induced by $\succeq$ are invariant, fix o and consider any pair $s \neq s'$. By symmetric reasoning, we conclude that $\succeq_{so} = \succeq_{s'o}$. The two paragraphs together prove that, for all $o, o' \in \mathcal{O}$ and $s, s' \in \mathcal{S}$, $\succeq_{so} = \succeq_{s'o'}$, meaning that $\succeq$ induces invariant *ex post* preferences. $\square$

Proof of Proposition 1. Let $\mathcal{J} := \mathcal{S}$ and $\mathcal{K} := \mathcal{O}$. We will check which of the axioms of Appendix B apply to the ordering $\succeq^i$, for any $i \in \mathcal{I}$. Coordinate Monotonicity holds because $\succeq^i$ induces preference orderings $\succeq^i_{so}$ that coincide with the natural ordering of real numbers, by statement (1). As the $\succeq^i_s$ (resp. the $\succeq^i_o$) are invariant, Invariant Row Preferences (resp. Invariant Column Preferences) holds. Thus, Proposition B4(c) yields the SEU representation (2) for $\succeq_i$. Since $\succeq$ has a numerical representation that is increasing with the $\succeq^i$ by the *ex ante* Pareto principle, the social representation (3) follows. The uniqueness condition for F is obvious, and the other uniqueness statements follow from Proposition B4(c). $\square$ *

Proof of Proposition 2. By Lemma C1(c), the assumption that $\succeq$ induces invariant interim preferences of both kinds guarantees that $\succeq$ also induces invariant *ex post* preferences $\succeq_{\text{xp}}$ on $\mathbb{R}^{\mathcal{I}}$. These preferences inherit the continuity of $\succeq$ and the *ex post* Pareto principle makes them increasing in every coordinate. Thus, each of them is represented by a continuous and increasing function $v : \mathbb{R}^{\mathcal{I}} \longrightarrow \mathbb{R}$.

To any $\mathbb{X} \in \mathbb{R}^{\mathcal{I} \times \mathcal{S} \times \mathcal{O}}$, we associate the element $\widetilde{\mathbf{X}} \in \mathbb{R}^{\mathcal{S} \times \mathcal{O}}$ whose (s, o) component is $\widetilde{x}_{so} := v(\mathbf{x}_{so})$. The function $V : \mathbb{R}^{\mathcal{I} \times \mathcal{S} \times \mathcal{O}} \to \mathbb{R}^{\mathcal{S} \times \mathcal{O}}$ defined by $V(\mathbb{X}) := \widetilde{\mathbf{X}}$ is continuous and increasing in each component. By these two properties, the image set $\widetilde{\mathcal{X}} := V(\mathbb{R}^{\mathcal{I} \times \mathcal{S} \times \mathcal{O}})$ is a set of the form $\mathcal{Y}^{\mathcal{S} \times \mathcal{O}}$, where $\mathcal{Y} := v(\mathbb{R}^{\mathcal{I}})$ is an open interval.

Define an ordering $\widetilde{\succeq}$ on $\widetilde{\mathcal{X}}$ by the condition that for all $\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}} \in \widetilde{\mathcal{X}}$, if $\widetilde{\mathbf{X}} = V(\mathbb{X})$ and $\widetilde{\mathbf{Y}} = V(\mathbb{Y})$, then

$$\widetilde{\mathbf{X}} \mathrel{\widetilde{\succeq}} \widetilde{\mathbf{Y}} \quad \text{if and only if} \quad \mathbb{X} \succeq \mathbb{Y}. \tag{C1}$$

(To see that $\widetilde{\succeq}$ is mathematically well-defined by (C1), suppose $V(\mathbb{X}) = \widetilde{\mathbf{X}} = V(\mathbb{X}')$ for some $\mathbb{X}, \mathbb{X}' \in \mathcal{X}$. Then for all $(s,o) \in \mathcal{S} \times \mathcal{O}$, we have $v(\mathbf{x}_{so}) = v(\mathbf{x}'_{so})$, and hence $\mathbf{x}_{so} \approx_{\rm xp} \mathbf{x}'_{so}$. Thus $\mathbb{X} \approx \mathbb{X}'$, because $\succeq$ is increasing relative to $\succeq_{\rm xp}$.) In terms of the Appendix B, putting $\mathcal{J} := \mathcal{S}$ and $\mathcal{K} := \mathcal{O}$, we conclude that $\widetilde{\succeq}$ is continuous and satisfies Invariant Row Preferences and Invariant Column Preferences, and Coordinate Monotonicity, by using the respective properties that $\succeq$ is continuous, induces invariant interim orderings $\succeq_s$, and induces invariant interim orderings $\succeq_o$, and induces invariant *ex post* orderings $\succeq_{\rm xp}$. Thus, Proposition B4(c) yields strictly positive probability vectors $\mathbf{p} \in \Delta_{\mathcal{S}}$ and $\mathbf{q} \in \Delta_{\mathcal{O}}$, and a continuous increasing function $u : \mathbb{R} \longrightarrow \mathbb{R}$, such that $\widetilde{\succeq}$ is represented by the function $\widetilde{W} : \widetilde{\mathcal{X}} \longrightarrow \mathbb{R}$ defined by

$$\widetilde{W}(\widetilde{\mathbf{X}}) := \sum_{s \in \mathcal{S}} \sum_{o \in \mathcal{O}} q_o \, p_s \, u(\widetilde{x}_{so}).$$

Now, putting $W_{\rm xa}(\mathbb{X}) := \widetilde{W} \circ V(\mathbf{X})$ for all $\mathbf{X} \in \mathbb{R}^{\mathcal{I} \times \mathcal{S} \times \mathcal{O}}$, and $W_{\mathbf{xp}}(\mathbf{x}) := u \circ v(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^{\mathcal{I}}$, we obtain the desired representations. The uniqueness properties are those of Proposition B4(c).[17] $\square$

Proof of Proposition 3. First we show that $\succeq$ is increasing in every coordinate. Let $(i,s,o) \in \mathcal{I} \times \mathcal{S} \times \mathcal{O}$. Statement (1) implies that $\succeq^i$ is increasing with respect to the coordinate $x^i_{s,o}$. By the *ex ante* Pareto principle, $\succeq$ is also increasing with respect to $x^i_{s,o}$.

The result now follows from Theorem B4(b), by setting $\mathcal{J} := \mathcal{I}$ and $\mathcal{K} := \mathcal{S} \times \mathcal{O}$. *Ex ante* Pareto then becomes Row Preferences, while the existence of invariant *ex post* preferences yields Invariant Column Preferences. Meanwhile, $\succeq$ satisfies Coordinate Monotonicity by the previous paragraph. $\square$

Proof of Theorem 4. First note that $\succeq$ is increasing in every coordinate, by exactly the same argument as the first paragraph in the proof of Proposition 3. Next, since the $\succeq^i$ relations are orderings and the *ex ante* Pareto principle makes $\succeq$ increasing with them, $\succeq$ is separable in each $i \in \mathcal{I}$. As $\succeq$ induces interim preferences of both types, $\succeq$ is also separable in each $s \in \mathcal{S}$ and $o \in \mathcal{O}$. It then follows from Proposition B3 that, for all $(i,s,o) \in \mathcal{I} \times \mathcal{S} \times \mathcal{O}$, there exist continuous and increasing functions $u^i_{so} : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\succeq$ is represented by the function $W_{\rm xa} : \mathbb{R}^{\mathcal{I} \times \mathcal{S} \times \mathcal{O}} \longrightarrow \mathbb{R}$ defined by

$$W_{\rm xa}\,(\mathbb{X}) \quad := \quad \sum_{i \in \mathcal{I}} \sum_{s \in \mathcal{S}} \sum_{o \in \mathcal{O}} u^i_{so}(x^i_{so}). \tag{C2}$$

Furthermore, the u^i_{so} are unique up to positive affine transformations (PAT) with a common multiplier. We can fix any $\mathbb{Y} \in \mathbb{R}^{\mathcal{I} \times \mathcal{S} \times \mathcal{O}}$ and add constants to these functions so as to ensure that $u^i_{so}(y^i_{so}) = 0$ for all $(i,s,o) \in \mathcal{I} \times \mathcal{S} \times \mathcal{O}$.[18] For convenience, fix some $\overline{y} \in \mathbb{R}$, and suppose that $y^i_{so} = \overline{y}$ for all $(i,s,o) \in \mathcal{I} \times \mathcal{S} \times \mathcal{O}$.

[17] Proposition B4(c) is stated for $\mathbb{R}^{\mathcal{J} \times \mathcal{K}}$, but it carries through to subsets $Y^{\mathcal{J} \times \mathcal{K}} \subseteq \mathbb{R}^{\mathcal{J} \times \mathcal{K}}$, when these are open and take the form of a product of intervals.

[18] To avoid burdening notation, we refer to the original and translated functions by the same symbol. This convention is applied throughout the proofs.

For all $i \in \mathcal{I}$, equation (C2) implies that the preference ordering $\succeq^i$ can be represented by the function $U^i : \mathbb{R}^{\mathcal{S}\times\mathcal{O}} \longrightarrow \mathbb{R}$ defined by

$$U^i(\mathbf{X}) \quad := \quad \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} u^i_{so}(x_{so}) \ . \tag{C3}$$

From Proposition 1 , there are continuous increasing utility functions $\tilde{u}^i : \mathbb{R} \longrightarrow \mathbb{R}$, and two strictly positive probability vectors $\mathbf{p}^i \in \Delta_{\mathcal{S}}$ and $\mathbf{q}^i \in \Delta_{\mathcal{O}}$, such that $\succeq^i$ is represented by the function $U^i : \mathbb{R}^{\mathcal{S}\times\mathcal{O}} \longrightarrow \mathbb{R}$ defined by

$$U^i(\mathbf{X}) \quad := \quad \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} q^i_o\, p^i_s\, \tilde{u}^i(x_{so}). \tag{C4}$$

Furthermore, in this representation, $\mathbf{p}^i$ and $\mathbf{q}^i$ are unique, and $\tilde{u}^i$ is unique up to PAT. By adding a constant, we ensure that $\tilde{u}^i(\overline{y}) = 0$.

From the uniqueness property applied to (C3) and (C4), there exist constants $\alpha^i > 0$ and $\beta^i \in \mathbb{R}$ such that :

$$u^i_{so}(x) = \alpha^i\, q^i_o\, p^i_s\, \tilde{u}^i(x) + \beta^i, \text{ for all } \ (s,o) \in \mathcal{S} \times \mathcal{O}. \tag{C5}$$

Substituting $x = \overline{y}$ into (C5) leads to $\beta^i = 0$. Then substituting (C5) (for all $i \in \mathcal{I}$) into the representation (C2) yields:

$$W_{\text{xa}}\ (\mathbb{X}) \quad = \quad \sum_{i\in\mathcal{I}} \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} \alpha^i\, q^i_o\, p^i_s\, \tilde{u}^i(x^i_{so}). \tag{C6}$$

For given $s \in \mathcal{S}$ in this representation, we obtain a representation $V_s : \mathbb{R}^{\mathcal{I}\times\mathcal{O}} \longrightarrow \mathbb{R}$ of the interim preference $\succeq_s$ on $\mathbb{R}^{\mathcal{I}\times\mathcal{O}}$:

$$V_s(\mathbf{X}) \quad := \quad \sum_{i\in\mathcal{I}} \sum_{o\in\mathcal{O}} \alpha^i\, q^i_o\, p^i_s\, \tilde{u}^i(x^i_o). \tag{C7}$$

Let $\mathbf{Y}_s := (\overline{y}, \ldots, \overline{y}) \in \mathbb{R}^{\mathcal{I}\times\mathcal{O}}$; then $V_s(\mathbf{Y}_s) \ = 0$.

Let us now put $\mathcal{J} := \mathcal{I}$ and $\mathcal{K} := \mathcal{O}$, and check which axioms in Appendix B the interim preference $\succeq_s$ satisfies. This is a continuous ordering by the continuity of $\succeq$. By the representation (C7) , $\succeq_s$ is separable in each $\{i\} \times \mathcal{O}$ and each $\mathcal{I} \times \{o\}$, and increasing in every coordinate , and thus satisfies Row Preferences, Column Preferences, and Coordinate Monotonicity. As $\succeq$ induces invariant $\succeq_o$, Lemma C1(b) entails that the induced preferences $\succeq_{so}$ are invariant, meaning that the stronger axiom of Invariant Column Preferences holds. Hence, Proposition B4(b) yields a strictly positive probability vector $\mathbf{r}_s \in \Delta_{\mathcal{O}}$, and for all $i \in \mathcal{I}$, continuous, increasing utility functions $\widehat{u}^i_s : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\succeq_s$ is represented by the function $\widehat{V}_s : \mathbb{R}^{\mathcal{I}\times\mathcal{O}} \longrightarrow \mathbb{R}$ defined by

$$\widehat{V}_s(\mathbf{X}) \quad := \quad \sum_{i\in\mathcal{I}} \sum_{o\in\mathcal{O}} r_{so}\, \widehat{u}^i_s(x^i_o). \tag{C8}$$

In this representation, $\mathbf{r}_s$ is unique and the functions $\widehat{u}^i_s$ are unique up to PAT with a common multiplier. We add constants to ensure that $\widehat{u}^i_s(\overline{y}) = 0$ for all $i \in \mathcal{I}$. It follows that $\widehat{V}_s(\mathbf{Y}_s) = 0$.

From the uniqueness property applied to (C7) and (C8), there exist $\gamma_s > 0$ and $\delta_s \in \mathbb{R}$ such that $\widehat{V}_s = \gamma_s V_s + \delta_s$. Substituting $\mathbf{Y}_s$ leads to $\delta_s = 0$. Since this holds for all $s \in \mathcal{S}$, we can conclude that

$$\gamma_s \, r_{so} \, \widehat{u}^i_s \quad = \quad \alpha^i \, q^i_o \, p^i_s \, \tilde{u}^i, \text{ for all } (i, s, o) \in \mathcal{I} \times \mathcal{S} \times \mathcal{O} \tag{C9}$$

Let us now fix i and s in these equations. All the coefficients are positive and the increasing functions $\widehat{u}^i_s$ and $\tilde{u}^i$ are nonzero for some $y^* \in \mathbb{R}$. Thus we can derive the relations:

$$\frac{r_{so}}{q^i_o} \quad = \quad \frac{\alpha^i \, p^i_s \, \tilde{u}^i(y^*)}{\gamma_s \, \widehat{u}^i_s(y^*)}, \quad \text{for all } o \in \mathcal{O}. \tag{C10}$$

The right-hand side of (C10) does not depend on o. Thus, the left-hand side must also be independent of o, which means that the vectors $\mathbf{q}^i$ and $\mathbf{r}_s$ are scalar multiples of one another. Thus, since they are probability vectors, we have $\mathbf{q}^i = \mathbf{r}_s$. Since this holds for all i and s, we can drop the indexes. Denote the common probability vector by $\mathbf{q}$. Substituting $\mathbf{q}$ into (C6) and defining $u^i := \alpha^i \, \tilde{u}^i$, we get the formula (8) of the theorem. The other parts readily follow.

Proof of Proposition 5. As in the proof of Proposition 3, $\succeq$ is increasing in every coordinate. As in the proof of Theorem 4, $\succeq$ is separable in each i in $\mathcal{I}$ (by the *ex ante* Pareto principle), each s in $\mathcal{S}$, and each o in $\mathcal{O}$ (because it induces interim preferences of both kinds). Thus, Proposition B3 yields continuous, increasing functions $w^i_{so} : \mathbb{R} \longrightarrow \mathbb{R}$ for all $i \in \mathcal{I}$, $s \in \mathcal{S}$, and $o \in \mathcal{O}$ (unique up to PAT with common multiplier) such that $\succeq$ has an additive representation $W : \mathbb{R}^{\mathcal{I}\times\mathcal{S}\times\mathcal{O}} \longrightarrow \mathbb{R}$ of the form

$$W(\mathbb{X}) \quad := \quad \sum_{i\in\mathcal{I}} \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} w^i_{so}(x^i_{so}). \tag{C11}$$

Fix some $\overline{y} \in \mathbb{R}$. By adding constants to these functions, we can assume without loss of generality that, for all $i \in \mathcal{I}$ and all $(s, o) \in \mathcal{S} \times \mathcal{O}$,

$$w^i_{so}(\overline{y}) = 0. \tag{C12}$$

Now, for any fixed o in $\mathcal{O}$, the representation (C11) yields the following representation $W_o : \mathbb{R}^{\mathcal{I}\times\mathcal{S}} \longrightarrow \mathbb{R}$ for $\succeq_o$:

$$W_o(\mathbf{X}) \quad := \quad \sum_{i\in\mathcal{I}} \sum_{s\in\mathcal{S}} w^i_{so}(x^i_s). \tag{C13}$$

However, the interim preferences $\succeq_o$ are invariant. Thus, if we fix $\overline{o} \in \mathcal{O}$, then for any other $o \in \mathcal{O}$, the uniqueness results yield some constants $\overline{q}_o > 0$ and $r^i_{so} \in \mathbb{R}$ such that for all $i \in \mathcal{I}$ and $s \in \mathcal{S}$, $w^i_{so} = \overline{q}_o \, w^i_{s\overline{o}} + r^i_{so}$. Since equation (C12) implies that $r^i_{so} = 0$, we get

$$w^i_{so} \quad = \quad \overline{q}_o \, w^i_{s\overline{o}}. \tag{C14}$$

Let $Q := \sum_{o\in\mathcal{O}} \overline{q}_0$ and $q_o := \overline{q}_o/Q$ for all $o \in \mathcal{O}$; then $\mathbf{q} := (q_o)_{o\in\mathcal{O}} \in \Delta_{\mathcal{O}}$. Define $u_s^i := Q\, w_{s\overline{o}}^i$ for all $i \in \mathcal{I}$ and $s \in \mathcal{S}$. Then equation (C14) becomes

$$w_{so}^i \quad = \quad q_o\, u_s^i. \tag{C15}$$

Substituting into (C11), we get the additive representation (11).

Fix $i \in \mathcal{I}$. Let $\mathcal{J} := \mathcal{S}$ and $\mathcal{K} := \mathcal{O}$; then in the terminology of Proposition B4, $\succeq^i$ satisfies Row Preferences and Column Preferences because it admits interim preferences $\succeq_s^i$ and $\succeq_o^i$. Meanwhile, $\succeq^i$ satisfies Coordinate Monotonicity by statement (1). Thus, Proposition B4(a) says that there exist continuous, increasing functions $v_{so}^i : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\succeq^i$ has the additive representation:

$$V^i(\mathbf{X}) \quad = \quad \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} v_{so}^i(x_{so}), \quad \text{for all } \mathbf{X} \in \mathbb{R}^{\mathcal{S}\times\mathcal{O}}. \tag{C16}$$

By adding constants to these functions if necessary, we can assume without loss of generality that $v_{so}^i(\overline{y}) = 0$ for all $s \in \mathcal{S}$ and $o \in \mathcal{O}$. For all $s \in \mathcal{S}$, formula (C16) yields the following additive representation for the interim preference $\succeq_s^i$:

$$V_s^i(\mathbf{x}) \quad = \quad \sum_{o\in\mathcal{O}} v_{so}^i(x_o), \quad \text{for all } \mathbf{x} \in \mathbb{R}^{\mathcal{O}}. \tag{C17}$$

Likewise, for all $o \in \mathcal{O}$, the interim preference $\succeq_o^i$ has additive representation:

$$V_o^i(\mathbf{x}) \quad = \quad \sum_{s\in\mathcal{S}} v_{so}^i(x_s), \quad \text{for all } \mathbf{x} \in \mathbb{R}^{\mathcal{S}}. \tag{C18}$$

By comparing (10) with (C16), applying the usual uniqueness results, and invoking (C12) and (C15), we conclude that there is some constant $C_i > 0$ such that

$$q_o\, u_s^i \;=\; C_i\, v_{so}^i, \quad \text{for all } s \in \mathcal{S} \text{ and } o \in \mathcal{O}. \tag{C19}$$

For any $s \in \mathcal{S}$, substituting (C19) into (C17) multiplied by C_i, we obtain the SEU representation (13) for $\succeq_s^i$. Likewise, for any $o \in \mathcal{O}$, substituting (C19) into (C18) multiplied by C_i/q_o, we obtain the additive representation (14) for $\succeq_o^i$. Finally, because we have the representations (13) and (14) for $\succeq_s^i$ and $\succeq_o^i$, we are justified in interpreting the representation (10) as a state-dependent SEU representation for $\succeq^i$, as claimed. □

Proof of Theorem 7. For each $i \in \mathcal{I}$, $\succeq^i$ satisfies the assumptions of Proposition 1. Thus, by the argument used to prove this proposition, we conclude that there exist a continuous increasing utility function $u^i : \mathbb{R} \longrightarrow \mathbb{R}$, and strictly positive probability vectors $\mathbf{p}^i \in \Delta_{\mathcal{S}}$ and $\mathbf{q}^i \in \Delta_{\mathcal{O}}$, such that $\succeq^i$ is represented by the function $U^i : \mathbb{R}^{\mathcal{S}\times\mathcal{O}} \longrightarrow \mathbb{R}$ defined by

$$U^i(\mathbf{X}) \quad := \quad \sum_{s\in\mathcal{S}} \sum_{o\in\mathcal{O}} q_o^i\, p_s^i\, u^i(x_{so}),$$

and the u^i are unique up to PAT with a common multiplier. This establishes the SEU representation (15). Fix $\overline{x} \in \mathbb{R}$. By adding constants, we ensure that $u^i(\overline{x}) = 0$ for all $i \in \mathcal{I}$.

Meanwhile, Proposition 2 yields strictly positive probability vectors $\mathbf{p} \in \Delta_{\mathcal{S}}$ and $\mathbf{q} \in \Delta_{\mathcal{O}}$, and a continuous increasing function $W_{\text{xp}} : \mathbb{R}^{\mathcal{I}} \longrightarrow \mathbb{R}$, such that $\succeq$ is represented by the function $W_{\text{xa}} : \mathcal{X} \longrightarrow \mathbb{R}$ defined by

$$W_{\text{xa}}(\mathbb{X}) \quad := \quad \sum_{s\in\mathcal{S}}\sum_{o\in\mathcal{O}} p_s\, q_o\, W_{\text{xp}}(\mathbf{x}_{so}),$$

where $\mathbf{p}$ and $\mathbf{q}$ are unique, and W_{xp} is unique up to PAT. This establishes the SEU representation (16). Let $\overline{\mathbf{x}} := (\overline{x}, \ldots, \overline{x})$. By adding a constant, we ensure that $W_{\text{xp}}(\overline{\mathbf{x}}) = 0$.

Now let $\mathcal{J} = \mathcal{I}$ and $\mathcal{K} = \mathcal{O}$ and consider how the axioms of Appendix B apply to $\succeq_s$ for any given $s \in \mathcal{S}$, recalling that these interim social preferences are well-defined and invariant (i.e. independent of s) . The objective interim Pareto principle makes $\succeq_s$ separable in each $i \in \mathcal{I}$, so that Row Preferences holds. By Proposition 2, the *ex post* social preferences $\succeq_{so}$ are well-defined and invariant, so that Invariant Column Preferences holds. Then, by Proposition B4(b), there exist a probability vector $\widetilde{\mathbf{q}} \in \Delta_{\mathcal{O}}$, and for all $i \in \mathcal{I}$, continuous increasing functions v^i such that $\succeq_s$ is represented by the function $W : \mathbb{R}^{\mathcal{I}\times\mathcal{O}} \longrightarrow \mathbb{R}$ defined by

$$W(\mathbf{X}) \quad := \quad \sum_{i\in\mathcal{I}}\sum_{o\in\mathcal{O}} \widetilde{q}_o\, v^i(x^i_{so}),$$

where $\widetilde{\mathbf{q}}$ is unique and the v^i are unique up to PAT with a common multiplier. The same representation holds for all $s \in \mathcal{S}$. Adding a constant, we ensure that $v^i(\overline{x}) = 0$ for all $i \in \mathcal{I}$.

We now show that $\mathbf{q} = \widetilde{\mathbf{q}}$. By fixing $s \in \mathcal{S}$ and applying the representation W_{xa} to elements $\mathbb{X}$ whose components for $s' \neq s$ are fixed at some values, we obtain a new representation for $\succeq_s$ and reduce it to the representation just obtained in terms of W by the standard uniqueness property. That is, there exist constants $\alpha > 0$ and β such that

$$\sum_{o\in\mathcal{O}} q_o\, W_{\text{xp}}(\mathbf{x}_o) \quad = \quad \alpha \sum_{i\in\mathcal{I}}\sum_{o\in\mathcal{O}} \widetilde{q}_o\, v^i(x^i_o) + \beta, \quad \text{for all } \ \mathbf{X} \in \mathbb{R}^{\mathcal{I}\times\mathcal{O}}.$$

Substituting $x^i_o = \overline{x}$ for all $i \in \mathcal{I}$ and $o \in \mathcal{O}$ leads to $\beta = 0$. Now fixing o and putting $x^i_{o'} = \overline{x}$ for all $o' \neq o$ leads to the equation:

$$W_{\text{xp}}(\mathbf{x}_o) \quad = \quad \frac{\widetilde{q}_o}{q_o} \sum_{i\in\mathcal{I}} \alpha\, v^i(x^i_{so}), \quad \text{for all } \ \mathbf{x}_o \in \mathbb{R}^{\mathcal{I}}.$$

Since this holds for all $o \in \mathcal{O}$, the two probability vectors $\mathbf{q}$ and $\widetilde{\mathbf{q}}$ are proportional, hence equal. Hence

$$W_{\text{xp}}(\mathbf{x}_o) \quad = \quad \sum_{i\in\mathcal{I}} \alpha\, v^i(x^i_{so}), \quad \text{for all } \ \mathbf{x}_o \in \mathbb{R}^{\mathcal{I}}. \tag{C20}$$

and the invariant conditional preference $\succeq_s$ is represented by the function $\widetilde{W} : \mathbb{R}^{\mathcal{I}\times\mathcal{O}} \longrightarrow \mathbb{R}$ defined by

$$\widetilde{W}(\mathbf{X}) \quad := \quad \sum_{i\in\mathcal{I}}\sum_{o\in\mathcal{O}} q_o\,\alpha\,v^i(x^i_{so}).$$

We now use a similar argument to show that $\mathbf{q} = \mathbf{q}^i$ for all $i \in \mathcal{I}$. Fixing $i \in \mathcal{I}$ and $s \in \mathcal{S}$, we can obtain a representation for the invariant interim preferences $\succeq^i_s$ in two ways: first, from $\widetilde{W}$ by applying this representation to elements of $\mathbb{R}^{\mathcal{I}\times\mathcal{O}}$ whose components for $i' \neq i$ are fixed at some values (because $\succeq_s$ satisfies the objective interim Pareto principle), and second, from U^i by applying this representation to elements of $\mathbb{R}^{\mathcal{S}\times\mathcal{O}}$ whose components for $s' \neq s$ are fixed at some values. By the standard uniqueness property, there exist $\gamma^i_s > 0$ and δ^i_s such that

$$\sum_{o\in\mathcal{O}} q_o\alpha v^i(x_o) \quad = \quad \gamma^i_s \sum_{o\in\mathcal{O}} q^i_o\, p^i_s\, u^i(x_o) + \delta^i_s, \quad \text{for all } \ \mathbf{x} \in \mathbb{R}^{\mathcal{O}}. \tag{C21}$$

Substituting $x_o = \overline{x}$ into (C21) leads to $\delta^i_s = 0$. Fix $o \in \mathcal{O}$. Put $x_{o'} = \overline{x}$ for all $o' \neq o$ leads to the equation:

$$\frac{q_o}{q^i_o}\alpha v^i(x) \quad = \quad \gamma^i_s\, p^i_s\, u^i(x), \quad \text{for all } \ x \in \mathbb{R}\ . \tag{C22}$$

The right-hand side of (C22) is independent of o. Thus, the probability vectors $\mathbf{q}$ and $\mathbf{q}^i$ are proportional, hence equal, and thus

$$\alpha v^i(x) \quad = \quad \gamma^i_s\, p^i_s\, u^i(x), \quad \text{for all } \ x \in \mathbb{R}. \tag{C23}$$

Equation (C23) holds for all $s \in \mathcal{S}$. Hence, for all $i \in \mathcal{I}$, the product $r^i := \gamma^i_s\, p^i_s$ is independent of s; note that $r^i > 0$. Equation (C23) now says $\alpha v^i = r^i\, u^i$. Substituting this into the representation (C20) yields the representation (17) for $\succeq_{\text{xp}}$. This completes the proof.

Proof of Theorems 9 and 10. Adapt the proofs Theorems 4 and 7, respectively. □ *

References

Alon, Shiri, and Gabrielle Gayer. 2016. "Utilitarian preferences with multiple priors." *Econometrica*, 84(3): 1181–1201.

Anscombe, Francis J., and Robert J. Aumann. 1963. "A definition of subjective probability." *Ann. Math. Statist.*, 34: 199–205.

Bewley, Truman. 2002. "Knightian decision theory. Part I." *Decisions in economics and finance*, 25(2): 79–110.

Billot, Antoine, and Vassili Vergopoulos. 2016. "Aggregation of Paretian preferences for independent individual uncertainties." *Social Choice and Welfare*, 47(4): 973–984.

Billot, Antoine, and Xiangyu Qu. 2017. "Utilititarian Aggregation with Reasonably Heterogeneous Beliefs." Working paper, Université Panthéon-Assas (Paris 2).

Blume, Lawrence E, Timothy Cogley, David A Easley, Thomas J Sargent, and Viktor Tsyrennikov. 2015. "A case for incomplete markets." Reihe Ökonomie, Institut für Höhere Studien.

Brunnermeier, Markus, Alp Simsek, and Wei Xiong. 2014. "A welfare criterion for models with distorted beliefs." *The Quarterly Journal of Economics*, 129(4): 1753–1797.

Chambers, Christopher, and Takashi Hayashi. 2014. "Preference aggregation with incomplete information." *Econometrica*, 82(2): 589–599.

Chambers, Christopher P., and Takashi Hayashi. 2006. "Preference aggregation under uncertainty: Savage vs. Pareto." *Games Econom. Behav.*, 54: 430–440.

Chateauneuf, Alain, and Peter Wakker. 1993. "From local to global additive representation." *J. Math. Econom.*, 22: 523–545.

Clemen, Robert T., and Robert L. Winkler. 2007. "Aggregating probability distributions." In *Advances in Decision Analysis.* , ed. W. Edwards, R. Miles and D. von Winterfeldt, 154–176. Cambridge, UK:Cambridge University Press.

Danan, Eric, Thibault Gajdos, Brian Hill, and Jean-Marc Tallon. 2015. "Aggregating tastes, beliefs, and attitudes under uncertainty." (published as Danan et al. 2016).

Danan, Eric, Thibault Gajdos, Brian Hill, and Jean-Marc Tallon. 2016. "Robust social decisions." *American Economic Review*, (to appear). (earlier circulated as Danan et al. 2015).

Debreu, Gerard. 1960. "Topological methods in cardinal utility theory." In *Mathematical methods in the social sciences 1959.* 16–26. Stanford, Calif.:Stanford Univ. Press.

Fine, T. 1973. *Theories of Probability: An Examination of Foundations.* New York:Academic Press.

Fishburn, Peter C. 1970. *Utility theory for decision making. Publications in Operations Research, No. 18*, New York:John Wiley & Sons Inc.

Fleurbaey, Marc. 2017. "Welfare economics, risk and uncertainty." *Canadian Journal of Economics*, (to appear).

Fleurbaey, Marc, and Philippe Mongin. 2016. "The utilitarian relevance of the social aggregation theorem." *American Economic Journal (Microeconomics)*, 8: 289 – 306.

Gajdos, Thibault, Jean-Marc Tallon, and Jean-Christophe Vergnaud. 2008. "Representation and aggregation of preferences under uncertainty." *J. Econom. Theory*, 141: 68–99.

Gayer, Gabrielle, Itzhak Gilboa, Larry Samuelson, and David Schmeidler. 2014. "Pareto efficiency with different beliefs." *Journal of Legal Studies*, 43(S2): S151–S171.

Genest, Christian, and James V. Zidek. 1986. "Combining probability distributions: a critique and an annotated bibliography." *Statist. Sci.*, 1: 114–148.

Gilboa, Itzhak, and David Schmeidler. 1989. "Maxmin expected utility with nonunique prior." *J. Math. Econom.*, 18(2): 141–153.

Gilboa, Itzhak, Dov Samet, and David Schmeidler. 2004. "Utilitarian aggregation of beliefs and tastes." *Journal of Political Economy*, 112: 932–938.

Gilboa, Itzhak, Larry Samuelson, and David Schmeidler. 2014. "No-betting Pareto dominance." *Econometrica*, 82: 1405–1442.

Gillies, D. 2000. *Philosophical Theories of Probability.* London:Routledge.

Gorman, W.M. 1968*a*. "Conditions for additive separability." *Econometrica*, 36: 605–609.

Gorman, W.M. 1968*b*. "The structure of utility functions." *Review of Economic Studies*, 35: 367–390.

Hammond, Peter J. 1981. "Ex ante and ex post welfare optimality under uncertainty." *Economica*, 48: 235–250.

Hammond, Peter J. 1983. "Utilitarianism, uncertainty and information." In *Utilitarianism and Beyond.* , ed. A. Sen and B. Williams, 85–102. Cambridge:Cambridge University Press.

Harsanyi, John C. 1955. "Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility." *Journal of Political Economy*, 63: 309–321.

Hayashi, T., and M. Lombardi. 2016. "Fair social decision under uncertainty and responsibility for beliefs." Adam Smith Business School, Glasgow.

Hild, Matthias, Richard Jeffrey, and Mathias Risse. 2003. "Flipping and ex post aggregation." *Social Choice and Welfare*, 20: 267–275.

Karni, Edi. 2014. "Axiomatic foundations of expected utility and subjective probability." In *Handbook of the Economics of Risk and Uncertainty.* , ed. Mark J. Machina and W. K. Viscusi, 1–39.

Keeney, Ralph L., and Howard Raiffa. 1976. *Decisions with multiple objectives: preferences and value tradeoffs.* New York (Republished by Cambridge University Press, 1993):John Wiley & Sons.

Keeney, Ralph L., and R. Nau. 2011. "A theorem for Bayesian group decisions." *Journal of Risk and Uncertainty*, 43: 1–17.

Mongin, Philippe. 1995. "Consistent Bayesian aggregation." *J. Econom. Theory*, 66: 313–351.

Mongin, Philippe. 1997. "Spurious unanimity and the Pareto principle." THEMA, Université de Cergy-Pontoise (working paper). Reissued as LSE-CPNSS working paper (2005).

Mongin, Philippe. 1998. "The paradox of the Bayesian experts and state-dependent utility theory." *J. Math. Econom.*, 29: 331–361.

Mongin, Philippe. 2017. "Bayesian Decision Theory and Stochastic Independence."

Mongin, Philippe, and Marcus Pivato. 2015. "Ranking multidimensional alternatives and uncertain prospects." *Journal of Economic Theory*, 157: 146–171.

Mongin, Philippe, and Marcus Pivato. 2016*a*. "Social evaluation under risk and uncertainty." In *The Oxford Handbook of Well-Being and Economic Policy.* , ed. Matthew D. Adler and Marc Fleurbaey, Chapter 24. Oxford.

Mongin, Philippe, and Marcus Pivato. 2016*b*. "Social Evaluation Under Risk and Uncertainty." In *Handbook of Well-being and Public Policy.* , ed. Matthew D. Adler and Marc Fleurbaey, Chapter 24, 711–745. Oxford University Press.

Nehring, Klaus. 2004. "The veil of public ignorance." *Journal of Economic Theory*, 119: 247–270.

Posner, Eric, and Glen Weyl. 2013. "FDA for financial innovation: Applying the insurable Interest doctrine to twenty-first-century financial markets." *Northwestern University Law Review*, 107(3): 1307.

Qu, Xiangyu. 2017. "Separate aggregation of beliefs and values under ambiguity." *Economic Theory*, 63(2): 503–519.

Risse, Mathias. 2003. "Bayesian group aggregation and two modes of aggregation." *Synthese*, 135: 347–377.

Savage, Leonard J. 1972. *The foundations of statistics.* New York:Dover Publications Inc.

Schmeidler, David. 1989. "Subjective probability and expected utility without additivity." *Econometrica*, 57(3): 571–587.

Segal, Uzi. 1992. "Additively separable representations on nonconvex sets." *J. Econom. Theory*, 56: 89–99.

Starr, Ross M. 1973. "Optimal production and allocation under uncertainty." *Quarterly Journal of Economics*, 87: 81–95.

Wakker, Peter. 1993. "Additive representations on rank-ordered sets. II. The topological approach." *J. Math. Econom.*, 22: 1–26.

Wakker, Peter P. 1989. *Additive representations of preferences.* Dordrecht:Kluwer.