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THE PARADOX OF THE BAYESIAN EXPERTS

1 INTRODUCTION

Suppose that a group of experts are asked to express their preference rankings on a set of uncertain prospects and that all of them satisfy the economist's standard requisite of Bayesian rationality. Suppose also that there is another individual who attempts to summarize the experts' preference judgments into a single ranking. What conditions should the observer's ranking normatively be expected to satisfy ? A natural requirement to impose is that it be Paretian, i.e., it should respect unanimously expressed preferences over pairs of prospects. Another condition which appears to be desirable is that the observer's and the experts' rankings should conform to one and the same decision theory, i.e., the observer himself should be Bayesian. The next question is then, are these seemingly compelling normative assumptions compatible with each other ?

As a specific application, think of an insurer who considers selling a new insurance policy and consults a panel of experts before deciding which specification of the insurance policy, if any, should be marketed. Suppose that the insurer knows little or nothing about how to elicit the experts' subjective probabilities. The only way in which he could take advantage of the panel's expertise seems to be this: he will require the experts to state their preferences between the logically possible specifications, and then aggregate these data to define his own ranking. If one further assumes that the experts are Bayesian, the question naturally arises of how the insurer could conform to the double consistency requirement just explained. Notice that this question makes perfectly good sense even if one has assumed that the insurer is not familiar with Bayesian elicitation methods. Writers in the tradition of de Finetti (1974–75) have emphasized that to conform to the Bayesian axioms is tantamount to being "coherent" in one's betting behaviour, regardless of whether or not one knows the probability calculus.

Several writers in the field of collective choice or decision-making have investigated aggregative problems that are formally similar to the Bayesian experts problem. Their nearly unexceptional conclusion is that logical difficulties will result from the double imposition of Bayesianism and Paretianism on the observer's preference. In an earlier paper [Mongin, 1995], we provided an up-to-date analysis of these difficulties, using the axiom system which enjoys the highest theoretical status among Bayesians, i.e., Savage's [1972]. Essentially, the imposition of relatively weak Paretian conditions, such as Pareto Indifference or Weak Pareto, leads to impossibility results in a quasi-Arrovian style, i.e., to dictatorial conclusions, whereas the imposition of the Strong Pareto condition involves a sheer logical impossibility, unless the individuals have identical probabilities or utilities. In each case, the inference depends on "technical" assumptions the role and relevance of which should carefully be ascertained; unexpected possibility results emerge when they are relaxed. Our Savagean conclusions encompass most of the more partial or elementary variants of the Bayesian experts paradox that have been discussed thusfar. We refer the reader to this earlier paper for references and comparisons.

The logical difficulties of "consistent Bayesian aggregation" led some writers to relinquish the first consistency condition — Paretianism — while others abandonned the second — Bayesianism. Either way out of the paradox involves a *diminutio capitis*. We have just suggested that the two requirements seemed equally natural. There are indeed serious arguments in favour of each, which makes the choice of a weaker version of consistency very awkward. In the present paper, we shall explore an altogether different potential solution to the paradox of the Bayesian experts. It consists in retaining the double consistency condition while varying the chosen notion of "Bayesianism". The impossibility results derived from applying Savage's axioms suggest that one should take a fresh look at them. A natural candidate to play the culprit's part is the sure-thing principle; but it is not our intention here to weaken it. If only for the purpose of theoretical experimentation, we want to remain within the confines of Bayesianism. There are also significant and well-recognized problems connected with the use of Savage's divisibility axiom. The present paper will take notice of them, but its primary target is to investigate the role of those axioms which ensure that the utility value of consequences is independent of what state of the world occurs.

Several writers in the Bayesian tradition, such as Karni [1985] and Drèze [1987], have repeatedly emphasized that state-independence is an inappropriate assumption to make in general. The standard example to support this claim involves the partition of states into the events "the agent lives" and "the agent dies". Insurance economics is replete with examples of a less dramatic sort in which the assumption of state-independence appears to be indefensible, both normatively and factually. On the constructive side, Drèze, Karni and others have devised axiom systems which deliver *state-dependent* subjective expected utility representations. They provide the generalization of Bayesianism that we want to put to the test. The general question of this paper is then, does state-dependent utility theory offer a solution to the paradox of the Bayesian experts ?

For reasons of tractability rather than of substance, most of the work on state-dependent utility does not employ Savage's framework but the alternative, highly accessible framework introduced by Anscombe and Aumann [1963]. We shall follow the existing literature and rephrase both the paradox and its tentative solutions accordingly. As is well-known, Anscombe and Aumann's (AA) approach to uncertainty involves a loss of generality with respect to Savage's, in that it assumes a lottery structure on the consequence set. On the other hand, it makes it possible to dispense with Savage's divisibility axiom, and thus to deal with finite state sets – a welcome extension of subjective expected utility (SEU) theory. As far as technical derivations are concerned, the analysis of "consistent Bayesian aggregation" la Savage depended on the measure-theoretic properties of his construction, in particular the nonatomicity of his derived subjective probability. The reader should expect the present analysis to revolve around the convexity properties of the *consequence set*, as conveniently assumed by AA.

The paper is organized as follows. Section 2 presents the definitions and axioms from SEU theory that will be used throughout. We shall briefly contrast AA's initial system – which is state-independent – with two later state-dependent variants. It is easy enough to axiomatize a *completely state-dependent* system of SEU. The well-known difficulty with this construction is that it leaves the individual's subjective probability indeterminate. Most of the work by Drèze, Karni and others has actually consisted in defining systems of *intermediary strength*, which allow for a state-dependent utility valuation of consequences but preserve the determination (if perhaps not the uniqueness) of the individual's subjective probability. Among the variants of AA's construction, only these intermediary systems can be claimed by Bayesianism. To accept complete state-dependence is really to take off the edge of the doctrine; this appears to be a well-recognized point. We have selected here an influential intermediary system first introduced by Karni, Schmeidler and Vind [1983].

Section 3 restates the initial paradox by applying AA's own state-independent system. As suggested, the results exactly parallel those reached in the Savage case but are easier to derive. Section 4 makes a start with an easy possibility result: the paradox disappears in the pure state-dependent generalization of AA's system. Then, we proceed to reexamine the paradox in the light of the relevant intermediary system of Karni, Schmeidler and Vind (KSV). The general conclusion of section 4 is that impossibility results can be derived in the KSV framework too, but the required technical assumptions are even stronger than those put to use in the state-independent case. This is why we choose to impose these assumptions only on a subset of the state space, and accordingly obtain only *local* analogues of our earlier dictatorial or logical impossibility theorems. Section 5 discusses a two-individual illustration of our impossibility results and compares them with those of Schervisch, Seidenfeld and Kadane (1991), who have also investigated a state-dependent version of the Bayesian experts problem. Section 6 elaborates on the implications of the present results for the theory of collective decision-making. The proofs of all formal statements are in Appendix A.

2 DEFINITIONS AND AXIOMS FROM SUBJECTIVE EXPECTED UTILITY THEORY

As in Anscombe and Aumann [1963], we assume that there is a finite set S of states of the world, to be denoted by $s = 1, \cdots, T$, and there is a set X of final outcomes, to be denoted by $A, B, C, \cdots$ Throughout, we require that there are at least two distinct states and two distinct final outcomes. (A stronger cardinality restriction will be introduced in section 3.) The consequence set is $\mathcal{R} = \Delta_F(X)$, i.e., the set of all simple probabilities on X. The set of uncertain prospects, or (to use Savage's word) acts, is the set H of all functions $S \to \mathcal{R}$, to be denoted by

$f, g, h, \cdots$ Then, $f(s)$ is a simple probability on X; denote by $f(s, A)$ the value it gives to $A \in X$. Since the state set is finite, it is often convenient to denote acts as vectors:

$$f = [R_1, \cdots, R_s, \cdots, R_T],$$

where R_s stands for $f(s)$. (Then, we rewrite $f(s, A)$ as $R_s(A)$.) Finally, consider the set $\mathcal{R}^* = \Delta_F(H)$, i.e., the set of all simple probabilities on H. A typical element of $\mathcal{R}^*$ may be written as:

$$(\lambda_1 f^{\mathbf{1}}, \cdots, \lambda_k f^k) = (\lambda_1[R_1^{\mathbf{1}}, \cdots, R_T^{\mathbf{1}}], \cdots, \lambda_k[R_1^k, \cdots, R_T^k]),$$

where $\lambda = (\lambda_1, \cdots, \lambda_k)$ is a probability vector and indexes $1, \cdots, k$ refer to particular acts in H.[1] Now, following many writers in AA theory, we can identify this element of $\mathcal{R}^*$ with the following, altogether different mathematical object:

$$\lambda_1 f^{\mathbf{1}} + \cdots + \lambda_k f^k \in H.$$

As is well-known, to identify these two mathematical entities with each other is equivalent to assuming AA's "reversal of order" axiom. The resulting simplification has a price, because it then becomes impossible to discuss the extension of AA's approach to "moral hazard", as promoted by Drèze [1987].

Granting the identification $\mathcal{R}^* \simeq H$, H becomes the decision-maker's choice set. The preference relation $\succeq$ on H can then be subjected to all or part of the following axioms.

AXIOM 1 (VNM axiom) $\succeq$ satisfies the von Neumann-Morgenstern axioms.

Any axiomatic version of VNM theory will do (see [Fishburn, 1982], for details).

AXIOM 2 (Nontriviality) There are two outcomes A^*, A_* such that

$$[A^*, \cdots, A^*] \succ [A_*, \cdots, A_*].$$

The axiom below relies on the derived concept of a conditional preference. For any $s \in S$, define $\succeq_s$ ("the preference conditional on s") by[2]:

$$f \succeq_s g \text{ iff } [\forall f', g' \in H : f'_{-s} = g'_{-s}, f(s) = f'(s), g(s) = g'(s)]\ f' \succeq g'.$$

Define a state s to be null if its conditional preference $\succeq_s$ is trivial, i.e., $\succeq_s = H \times H$.

[1] We shall adopt the convention of using square brackets for uncertain prospects and curved ones for risky prospects (i.e., prospects with preassigned probabilities).

[2] We use the game-theoretic notation f'_{-s}, g'_{-s} to refer to the subvectors obtained from f', g' by deleting their s-component.

AXIOM 3 (State-Independence) For all non-null states s, t, and all constant $f, g \in H$, $f \succeq_s g$ iff $f \succeq_t g$.

Under state-independence it becomes meaningful to identify the constant act $[R, \cdots, R]$ with its value $R \in \mathcal{R}$, so that $\succeq$ induces a preference relation on $\mathcal{R}$. In this context Axiom 2 states that there are $A^*, A_* \in X$ such that $A^* \succ A_*$. Another version of Axiom 2 will also be used at a later stage in this paper:

AXIOM 2$'$. There is a non-null state s.

PROPOSITION 1 ([Anscombe and Aumann, 1963]). *If $\succeq$ satisfies Axioms 1, 2 and 3, there exist a nonconstant VNM function u on $\mathcal{R}$ and a probability $p = (p_1, \cdots, p_T)$ on S such that for all $f, g \in H$:*

$$(*) \qquad f \succeq g \ \text{ iff } \ \sum_{s\in S} p_s u(f(s)) \ \geq \ \sum_{s \in S} p_s u(g(s)).$$

State s is null iff $p_s = 0$. Any other pair (u', p') that satisfies the same properties as (u, p) is such that $p' = p$ and u' is a positive affine transform of u.

PROPOSITION 2 ([Fishburn, 1970]). *If $\succeq$ satisfies Axiom 1, there exist VNM functions $u_1, \cdots, u_T$ on $\mathcal{R}$ such that for all $f, g \in H$:*

$$(**) \qquad f \succeq g \ \text{ iff } \ \sum_{s\in S} u_s(f(s)) \ \geq \sum_{s \in S} u_s(g(s)).$$

The function u_s represents the conditional preference $\succeq_s$, and is constant iff s is null. Any other T-tuple $(u'_1, \cdots, u'_T)$ satisfying the same properties as $(u_1, \cdots, u_T)$ is a positive affine transform of the latter vector.

By a VNM function u on $\mathcal{R}$ we mean a function which has the following, mixture-preserving property: for any $\lambda \in [0, 1]$ and any R, R',

$$u((\lambda R, (1-\lambda)R')) \ = \ \lambda u(R) + (1-\lambda) u(R'),$$

or equivalently (since we are considering here only simple probabilities R, R'), a function which has the expected-utility form:

$$u(R) \ = \ \sum_{A \in X} R(A) u(A).$$

The fact that the utility representations defined on the *consequence set*, i.e., u in Proposition 1 and the u_i in Proposition 2, are VNM is a characteristic feature of the Anscombe-Aumann approach as a whole. Conceptually, this feature is irrelevant

to the aim of the construction; technically, it is an ingenious device – actually, comparable with Savage's $(P6)$ — to facilitate the derivation of the SEU formula.[3]
OBSERVATION. If $\succeq$ satisfies not only Axiom 1, but also Axiom 2, there are VNM functions $v_1, \cdots, v_T$ on $\mathcal{R}$ and a probability $p = (p_1, \cdots, p_T)$ on S such that for all $f, g, \in H$:

$$(***) \qquad f \succeq g \ \text{ iff } \ \sum_{s \in S} p_s v_s(f(s)) \ \geq \ \sum_{s \in S} p_s v_s(g(s)).$$

Any $(T+1)$-tuple $(q, v'_1, \cdots, v'_T)$ such that q is a probability on S and:

$$p_s = 0 \ \Leftrightarrow \ q_s = 0,$$
$$v'_s = \frac{p_s}{q_s} v_s \ \text{ if } \ q_s \neq 0, \ v'_s \ \text{ arbitrary otherwise},$$

can be substituted for $(p, v_1, \cdots, v_T)$ in (***).

Notation. We define $U(f) = \sum_{s \in S} p_s u(f(s))$ and $V(f) = \sum_{s \in S} p_s v_s(f(s))$ when the suitable assumptions hold. As a rule, U, V, W will refer to representations of preferences over acts, and u, v, w to representations of preferences over consequences.

The comparison between Proposition 1, Proposition 2, and the ensuing well-known observation, brings out the classic difficulty of state-dependent utility theory. The system consisting of only Axioms 1 and 2 is not rich enough to determine the decision-maker's subjective probability. To add Axiom 3 makes it possible to uniquely determine p if one selects a state-independent representation u on $\mathcal{R}$, as do AA in their seminal article. However, Axiom 3 is too restrictive; it amounts to excluding the relevant complication of state-dependent preferences. Hence a dilemma of determination and relevance; see, among others, [Fishburn, 1970; Drèze, 1987; Karni, 1985; Karni, 1993; Schervish *et al.*, 1990].[4]

Various methods have been put forward to escape from the dilemma just suggested. Most (but not all) of them consist in assuming Axioms 1 and 2, and then introducing further axioms to determine the state-dependent functions $v_1, \cdots, v_T$ that underlie the uninformative representations $u_1, \cdots, u_T$ of Proposition 2. Once

[3]We made another assumption on the consequence set $\mathcal{R}$ which — by contrast to the VNM assumption — is dispensable within the AA approach. To save notation, we assumed a state-independent consequence set $\mathcal{R}$. Some expositions of state-dependent utility theory, such as Fishburn's [1970], and actually the original paper by Karni, Schmeidler and Vind [1983], adopt a more general framework in which not only the evaluations but also the availability of consequences vary from one state to another. As far as we can judge, the results of the present paper can be extended unproblematically to this more general framework. On the issue of state-dependent consequences in expected utility theory, see also [Hammond, 1996].

[4]The last paper usefully emphasizes that Anscombe and Aumann's choice of a state-independent u on $\mathcal{R}$ is to some extent question-begging. Even when Axiom 3 holds, it is trivially possible to replace (*) with infinitely many equivalent *state-dependent* representations, each of which corresponds to one particular subjective probability.

the vector $(v_1, \cdots, v_T)$ is known (up to a positive affine transformation, PAT),[5] it becomes possible to write $u_s = p_s v_s$, where the p_s are well-determined (and ideally unique) probability values. We shall not attempt at covering all the variants of this axiomatization strategy. A representative system will be enough for the purpose of this paper.

As in Karni, Schmeidler and Vind [1983], we introduce an auxiliary binary relation $\tilde{\succeq}$. It is meant to describe the preference that the decision-maker would express between acts *if his subjective probability were some given* $q = (q_1, \cdots, q_T)$. KSV's strategy is to infer the agent's actual state-dependent utilities from the (supposedly meaningful and even observable) hypothetical preference $\tilde{\succeq}$, and then determine his actual, unknown subjective probability p by using this utility-relative information.

Formally, fix a probability $q = (q_1, \cdots, q_T)$ on S with $q_s > 0$ for all s and associate with each $f \in H$ an *hypothetical act* f' defined as follows: f' is on S, and for each s, $f'(s) = f(s)q_s$, that is to say, $f'(s)$ is that function on X which satisfies $f'(s, A) = f(s, A)q_s$ for all $A \in X$. Note that $f'(s)$ is *not* a probability on X, unlike $f(s)$, but f' can be viewed as a probability on $S \times X$ unlike f. (The computation is obvious.) Given the positivity assumption made on q, the set H' of all hypothetical acts is clearly in a one-to-one relationship to H; this makes the notation f, f' unambiguous. The element f' describes the effect of compounding the given probability q on S with each of the lotteries that f assigns to $s = 1, \cdots, T$. Define the *hypothetical preference* ($\tilde{\succeq}$) to be a preference relation on the set of hypothetical acts. This formal construct is meant to capture the modification in the individual's preferences "if his subjective probability were q".[6]

It is consistent to impose the same decision-theoretic constraints on both $\tilde{\succeq}$ and $\succeq$, i.e., to subject hypothetical preference to the VNM axioms. Beyond this, some coordinating condition should relate $\succeq$ to $\tilde{\succeq}$. In effect, KSV impose the ("consistency") axiom that conditional preferences $\tilde{\succeq}_s$ and $\succeq_s$ are the same whenever s is non-null for $\succeq$. The point of this axiom is to ensure that hypothetical preference data deliver usable information on the individuals' state-dependent utilities.[7]

AXIOM 4 (Hypothetical Preference). For all $s \in S$ that are non-null with respect to $\succeq$, and for all $f, g \in H$,

$$f \succeq_s g \Leftrightarrow f' \tilde{\succeq}_s g',$$

where f' and g' are the elements in H' associated with f and g respectively.

[5] Formally, two vectors $(v'_1, \cdots, v'_T)$ and $(v_1, \cdots, v_T)$ are identical up to a PAT if there are a number $\mu > 0$ and a vector $(\nu_1, \cdots, \nu_T)$ such that $(v'_1, \cdots, v'_T) = \mu(v_1, \cdots, v_T) + (\nu_1, \cdots, \nu_T)$.

[6] Notice carefully that although hypothetical acts carry preassigned probabilities with them, they do not reduce to VNM lotteries. States of the world matter in the construction of hypothetical acts.

[7] KSV's exposition is considerably more complex, due to their detailed analysis of null states. Wakker [1987], and Schervish, Seidenfeld and Kadane [1990], provide alternative restatements; we do not follow them here.

PROPOSITION 3 ([Karni *et al.*, 1983]). *Assume that* $\succeq$ *satisfies Axioms 1 and* $2'$. *Take any probability* q *on* S *with* $q_s > 0$ *for all* $s \in S$. *Assume that the induced hypothetical preference* $\tilde{\succeq}$ *also satisfies Axioms 1 and* $2'$, *and that* $\succeq$ *and* $\tilde{\succeq}$ *jointly satisfy Axiom 4. Then, there are VNM functions* $v_1, \cdots, v_T$ *on* $\mathcal{R}$ *and a probability* $p = (p_1, \cdots, p_T)$ *on* S *such that for all* $f, g \in H$*:*

(i) $f \succeq g$ *iff* $\sum_{s \in S} p_s v_s(f(s)) \geq \sum_{s \in S} p_s v_s(g(s))$

(ii) $f' \tilde{\succeq} g'$ *iff* $\sum_{s \in S} q_s v_s(f(s)) \geq \sum_{s \in S} q_s v_s(g(s))$.

If s *is non-null for* $\succeq$, *then* $p_s > 0$. *Any other* $(T+1)$*-tuple* $(v'_1, \cdots, v'_T, p')$ *that satisfies conditions (i) and (ii) is such that the vector* $(v'_1, \cdots, v'_T)$ *is a PAT of* $(v_1, \cdots, v_T)$ *and* $p'_s/p'_t = p_s/p_t$ *for all* s, t *non-null for* $\succeq$.

Notice that the theorem does not entirely determine the agent's probability on null states: if s is null for $\succeq$ and not for $\tilde{\succeq}$, then comparison of (i) and (ii) leads to the conclusion that $p_s = 0$; if s is null for *both* $\succeq$ and $\tilde{\succeq}$, nothing can be said about p_s (but necessarily, v_s = constant).

Although the conclusions of the theorem are stated atemporally, they might be interpreted in terms of the two step-experiment mentioned at the outset. Irrespective of whether Axiom 4 provides a satisfactory formal rendering, the experiment *itself* raises a conceptual problem: the agent might well attach no sense to the expression of his preferences conditionally on the use of a subjective probability which is not his own.[8]

At least, the KSV procedure has a significant negative argument to recommend itself: to the best of our knowledge, existing alternatives *either* entail only a partial solution to the indeterminacy-of-probability issue, *or* involve the same operational difficulties as the KSV procedure, *or* imply an even more radical departure from standard Bayesian assumptions. Karni and Schmeidler's [1993] state-dependent variant of Savage's axiomatization exemplifies the first problem. In the AA framework, Karni's [1993] assumption of *given* transformations between the $v_s(\cdot)$ illustrates the second problem, while Drèze's [1987] use of a "moral hazard" assumption illustrates the third. For all its shortcomings, KSV's article is a serious representative of the work done in the field of state-dependent utility theory. This is sufficient to make it relevant to a paper which is primarily concerned with theoretical experimentation.

3 IMPOSSIBILITY RESULTS IN THE STATE-INDEPENDENT CASE

The present section will first introduce a multi-individual extension of the AA approach broadly speaking, and then restrict attention to the state-independent case

[8][Drèze, 1987] expresses his critique of the KSV approach differently. He claims that it relies on information obtained from *verbal* behaviour, which he says is unreliable in principle and should be ignored. In essence, Drèze disqualifies KSV's contribution on the grounds that they do not follow the methodology of revealed preference theory. The critical point in the text does not depend on one's adhering to a revealed preference methodology.

with a view of deriving the AA variant of the Bayesian experts paradox.

Let us assume that there are individuals, to be represented by indexes $i = 1, \cdots, n$, and an observer, to be represented by index $i = 0$, who express their subjective probabilities indirectly, i.e., by stating their preferences $\succeq^i$ over uncertain prospects. Throughout, we shall require $\succeq^i$ to satisfy some subset of the axioms of section 2, for all $i = 0, 1, \cdots, n$. This requirement reflects the assumption that both the individual experts and the observer are Bayesian; it encompasses one of the two consistency conditions discussed in the introduction. The remaining, Paretian condition can be made precise in terms of one of the following standard requirements: for all $f, g \in H$,

$$f \sim^i g,\ i = 1, \cdots, n \ \Rightarrow\ f \sim^0 g. \qquad \text{(C)}$$
$$f \succeq^i g,\ i = 1, \cdots, n \ \Rightarrow\ f \succeq^0 g. \qquad (\text{C}_1)$$
$$f \succ^i g,\ i = 1, \cdots, n \ \Rightarrow\ f \succ^0 g. \qquad (\text{C}_2)$$
$$f \succeq^i g,\ i = 1, \cdots, n \ \text{and}\ \exists j \in \{1, \cdots, n\} : f \succ^j g \ \Rightarrow\ f \succ^0 g. \qquad (\text{C}_3)$$

In social choice theory, these are the conditions of Pareto-Indifference, Pareto-Weak Preference, Weak Pareto, Strict Pareto, respectively. We also introduce the Strong Pareto condition:

$$(\text{C}^+) \ = \ (\text{C}) \,\&\, (\text{C}_3).$$

Obviously, $(\text{C}_1) \Rightarrow (\text{C})$ and $(\text{C}_3) \Rightarrow (\text{C}_2)$. Given the rich structure of the consequence set in the AA approach, more can be said on the logical relations between the Pareto conditions. It will shortly be seen that under a minor restriction on preferences, (C_1), hence (C) are implied by any other condition. Let us introduce the following restriction of Minimum Agreement on Acts:

$$(\text{MAA}) \qquad \exists f_*, f_{**} \in H,\ \forall i = 1, \cdots, n,\ f_* \succ^i f_{**}.$$

Notice the difference with the requirement of Minimum Agreement on Consequences used in Mongin [1995, Section 3]. In the present context the latter would state that:

$$(\text{MAC}) \qquad \exists R_*, R_{**} \in \mathcal{R},\ \forall i = 1, \cdots, n,\ R_* \succ R_{**}.$$

In a pure state-independent context such as that of the earlier article, (MAC) provided an appropriate notion of minimum agreement among the individuals. We want a weaker condition here since it should also be applicable to the state-dependent context of the following sections.

For $i = 0, 1, \cdots, n$ denote by U^i the SEU representation of $\succeq^i$ when this relation satisfies all of the assumptions of Proposition 1, and V^i the more general additive representation of $\succeq^i$ that satisfies the unique assumption of Proposition 2. Then, $U^i(f) = \sum_{s \in S} p^i_s u^i(f(s))$ and $V^i(f) = \sum_{s \in S} u^i_s(f(s))$. For any vector-valued function $(\varphi_1, \cdots, \varphi_k)$, denote its range (i.e., set of values) by $Rge(\varphi_1, \cdots, \varphi_k)$. A basic consequence of imposing *any* of the AA systems of section 2 on the observer's and individual preferences $\succeq^0, \succeq^1, \cdots, \succeq^n$ is that the vector of corresponding utility representations has a convex range. Lemmas 4, 5 and 6 spell out this fact and its important consequences in terms of the V^i representations. The same results obviously apply to the U^i since they are restricted forms of the V^i.

LEMMA 4. *If $\succeq^0, \succeq^1, \cdots, \succeq^n$ satisfy the assumption of Proposition 2 (= Axiom 1), then $Rge(V^0, V^1, \cdots, V^n)$ is convex.*

De Meyer and Mongin [1995] have investigated the aggregative properties of a real function φ_0 which is related to given real functions $\varphi_1, \cdots, \varphi_n$ by unanimity conditions analogous to $(\mathrm{C}), (\mathrm{C}_1), \cdots$ and by the assumption that $(\varphi_0, \varphi_1, \cdots, \varphi_n)$ has convex range. These aggregative results are applicable here because of Lemma 4 and will be used throughout the paper. Here is the first application:[9]

LEMMA 5. *If $\succeq^0, \succeq^1, \cdots, \succeq^n$ satisfy Axiom 1, then (C) holds if and only if there are real numbers $a_1, \cdots, a_n, b$ such that $V^0 = \sum_{i=1}^n a_i V^i + b$. (C_1) [resp. (C^+)] holds if and only if this equation is satisfied for some choice of non-negative [resp. positive] numbers $a_1, \cdots, a_n$.*

Another consequence of Lemma 4 is the following tightening of the logical implications between unanimity conditions:

LEMMA 6. *If $\succeq^0, \succeq^1, \cdots, \succeq^n$ satisfy the assumptions of Proposition 1, and if (MAA) holds, then*

$$(\mathrm{C}_2) \ \Rightarrow (\mathrm{C}_1) \ \textit{and} \ (\mathrm{C}_3) \ \Leftrightarrow (\mathrm{C}^+).$$

Thus, the list of conditions becomes simplified under (MAA). Returning now to the conclusion of Lemma 5, we know that it can be applied to the state-independent representations U^i. Hence, it seems as if this lemma delivered an aggregative rule of the familiar sort — what social choice theorists call generalized utilitarianism (e.g., [d'Aspremont, 1985]). A simple algebraic argument adapted from Mongin [1995, Section 4]) will demonstrate that this is *not* the case in general. Impossibility results lurk behind the apparently well-behaved affine decomposition $U^0 = \sum_{i=1}^n a_i U^i + b$. Dictatorial rules will emerge from the analysis of the weaker unanimity conditions $(\mathrm{C}), (\mathrm{C}_1), (\mathrm{C}_2)$, while sheer logical impossibility will result from imposing the stronger conditions (C_3) or (C^+).

Given a preference profile $\succeq_0, \succeq_1, \cdots, \succeq_n$ satisfying the assumptions of Proposition 1, hence representable by

$$U^0(\cdot) \ = \ \sum_s p_s^0 u^0(\cdot), \ U^1(\cdot) = \sum_s p_s^1 u^1(\cdot), \cdots, U^n(\cdot) = \sum_s p_s^n u^n(\cdot),$$

[9]Lemma 5 is an encompassing version of a famous social aggregation theorem first stated by Harsanyi [1955].

we say that i is a *probability dictator* if $p^0 = p^i$, that i is a *utility dictator* if $u^0 = u^i$ (up to a PAT), and that i is an *overall dictator* if he is both a probability and a utility dictator. We define i to be an *inverse* utility dictator or an *inverse* overall dictator by changing the clause that $u^0 = u^i$ into $u^0 = -u^i$. We shall also say that *probability agreement* prevails if $p^1 = \cdots = p^n$ and that *pairwise utility dependence* (p.u.d.) prevails if for all $i, j \geq 1, u^i = u^j$ (up to an affine transformation of any sign). Probability agreement and p.u.d. are two degenerate cases of individual profiles; in general, both probabilities and utilities should be expected to vary from one individual to another.

How to capture individual diversity in the language of formal choice theories is a difficult problem. As in Coulhon and Mongin [1989], or Mongin [1995], we shall use the convenient shortcut of defining diversity in terms of algebraic independence. Recall that a set of elements $\{\varphi_1, \cdots, \varphi_k\}$ of a vector space is *affinely independent* if for any set of real numbers $a_1, \cdots, a_k, b$,

$$a_1\varphi_1 + \cdots + a_k\varphi_k + b = 0 \;\Rightarrow\; a_1 = \cdots = a_k = b = 0.$$

This concept, rather than the weaker one of linear independence, provides the relevant notion of algebraic independence in the case of utility functions. Plainly, affine and linear independence become equivalent in the case of probabilities. A relevant fact to report here is that a set of VNM functions $u^1, \cdots, u^n$ is affinely independent if and only if these functions are "separated" from each other by suitable lotteries. This equivalence can be immediately extended to AA representations:

LEMMA 7. *Suppose that $u^1, \cdots, u^n$ are VNM utility functions on $\mathcal{R}$. They are affinely independent if and only if for every $i = 1, \cdots, n$, there are $R^i_*, R^i_{**} \in \mathcal{R}$ such that:*

$$u^i(R^i_*) > u^i(R^i_{**}) \text{ and } u^j(R^i_*) = u^j(R^i_{**}) \text{ for } j \neq i.$$

Similarly, the $V^1, \cdots, V^n$ derived in Proposition 2 are affinely independent if and only if for every $i = 1, \cdots, n$, there are f^i_*, f^i_{**} such that:

$$V^i(f^i_*) > V^i(f^i_{**}) \text{ and } V^j(f^i_*) = V^j(f^i_{**}).$$

If affine independence assumptions formalize individual diversity in an obvious sense, it is also the case that in a VNM context, they imply some form of minimum agreement between individuals. This rather curious consequence deserves emphasis here since it means that in some algebraic contexts (MAA) and (MAC) are given for free:[10]

LEMMA 8. *Suppose that $u^1, \cdots, u^n$ are affinely independent VNM functions. Then, (MAC) holds. Similarly, if the $V^1, \cdots, V^n$ of Proposition 2 are affinely independent, (MAA) holds.*

[10] Compare with the related statements in [Weymark, 1993, Proposition 3] and [Mongin, 1995, Corollary 4.3].

We are now in a position to state the two impossibility theorems which formalize the paradox of the Bayesian experts. In part (*) of both Propositions 9 and 10 we introduce a linear independence restriction on individual probabilities. To ensure that this restriction applies, we shall assume in part (*) that the state space S has cardinality at least n.

PROPOSITION 9. *Assume that* $\succeq^0, \succeq^1, \cdots, \succeq^n$ *satisfy Anscombe and Aumann's axioms of state-independent utility, i.e., the assumptions of Proposition 1. Denoting by* $p^1, \cdots, p^n$ *the probabilities and by* $u^1, \cdots, u^n$ *the utility functions on consequences provided by Proposition 1, assume that either:*

(*) $p^1, \cdots, p^n$ *are linearly independent,*
or
(**) $u^1, \cdots, u^n$ *are affinely independent.*

Then, if (C) holds, there is either a utility or an inverse utility dictator in case (*), and there is a probability dictator in case (**). There is an overall or an inverse overall dictator when both (*) and (**) apply. If either (C_1) *or* (C_2) *holds, the same results follow, except that there is always a utility dictator in case (*).*

When there is an overall dictator, all of the unanimity conditions are obviously satisfied, so that we could have stated part of Proposition 4 in terms of "if and only if" conditions.[11] This observation also implies that the problem of Consistent Bayesian Aggregation does not involve any *logical* impossibility in the case of conditions (C), (C_1) and (C_2). The stronger conditions (C_3) and (C^+) lead to altogether different conclusions.

PROPOSITION 10. *The assumptions are as in Proposition 4. Then, if* (C_3) *or* (C^+) *holds, case (*) implies that pairwise utility dependence prevails and that there is a utility dictator; case (**) implies that probability agreement prevails and that there is a probability dictator.*

Notice that in both Proposition 9 and 10, (MAA) is an inference, not an assumption. A modest strengthening of the first part of Proposition 5 would follow from assuming (MAC). Then, *positive* p.u.d. prevails (i.e., all individual utilities are identical up to a positive scale factor).

Proposition 10 can be restated as follows: under the assumptions of Proposition 9, (MAA) and (C_3), if either the n probabilities are linearly independent and (at least) two utility functions are affinely independent, or the n utility functions are affinely independent and (at least) two probabilities are distinct, then there is no solution to the Bayesian experts problem. This wording makes it clear that under appropriate distinctiveness restrictions, (C_3) is a *logical* impossibility; given these restrictions, *even dictatorship* fails to deliver a solution.

[11] Note also that inverse utility dictatorship is impossible when (C_2) and (MAA) hold. Utility dictatorship and inverse utility dictatorship can coexist with each other under the weaker assumption (C_1), as the following shows: take $n = 2, U_0 = U_1$ and $U_2 = -U_1$.

A word of comparison with the Savagean formulation of the paradox is in order. The main technical step in Mongin [1995] was to derive a version of Lemma 5. Since Savage does not assume anything on the consequence set, this had to be done by a special construction based on his divisibility-of-events axiom ($P6$). Once the affine decomposition of Lemma 5 is obtained, the algebra of impossibility results follows similar paths in the Savage and the Anscombe-Aumann variants.[12]

4 THE STATE-DEPENDENT CASE

Suppose that we just impose Axiom 1 on the preference relations $\succeq^0, \succeq^1, \cdots, \succeq^n$. This is the pure state-dependent case, as characterized by Proposition 2; each $\succeq^i$ is represented by $V^i(f) = \sum_{s \in S} u_s^i(f(s))$. It is easy to check that nontrivial solutions to the aggregation problem now exist, whatever individual preferences might be. To see that, take any profile $\succeq^1, \cdots, \succeq^n$ that satisfies Axiom 1 and consider the added preference relation $\succeq^0$ defined by means of the following representation:

$$(+) \qquad V^0(f) = \sum_{i=1}^{n} a_i (\sum_{s \in S} u_s^i(f(s)),$$

where $a_i > 0$ for all i. Obviously, $\succeq^0$ satisfies the whole list of Pareto conditions (C), $\cdots$, (C^+). It is also clear that $\succeq^0$ satisfies Axiom 1 (since a sum of VNM functions is also VNM, and Axiom 1 does not require anything beyond that property). A little more explicitly, (+) can be rearranged as:

$$(++) \qquad V^0(f) = \sum_{s \in S} u_s^0(f(s)),$$

by defining $u_s^0 = \sum a_i u_s^i$ for all $s \in S$. This rewriting makes it plain that $\succeq^0$ and the $\succeq^i$ obey the same (weak) decision theory.

Hence, in the pure state-dependent case, the paradox of the Bayesian experts vanishes. This mathematically trivial resolution can strike one as conceptually relevant only if one regards Axiom 1 as a sufficient foundation for Bayesianism. We have already suggested that this is *not* a sensible position to take. Without some restriction on the many subjective probabilities that are compatible with state-dependent utilities, Bayesianism vanishes at the same time as the paradox.

Before we proceed to axiomatic systems of intermediary strength, we should complete the analysis of the first paragraph. What is not so trivial as the "resolution" just sketched is the fact that equation (+) delivers a *necessary* solution to

[12] A two-person version of Propositions 4 and 5 was obtained by [Seidenfeld *et al.*, 1989], using an expected utility framework in the style of Anscombe and Aumann. Schervish, Seidenfeld and Kadane [1991, Theorem 2] state this result more formally. We defer comparison to section 5.

the aggregation problem. This fact follows from Lemma 2 above, when conditions (C), (C_1) or (C^+) hold, and from Lemmas 5 and 6 when (C_2) and (C_3) hold (assuming (MAA)). Let us take stock of the characterization just obtained:

PROPOSITION 11. *Assume that $\succeq^0, \succeq^1, \cdots, \succeq^n$ satisfy the unique assumption of Proposition 2, i.e., axiom 1, and that $V^i(f) = \sum_{s\in S} u_s^i(f(s)), i = 0, 1, \cdots, n$ are the state-dependent representations derived in Proposition 2. Then, (C) holds if and only if there are real numbers $a_1, \cdots, a_n, b$ such that:*

$$\forall s \in S, \quad u_s^0 = \sum_{i=1}^{n} a_i u_s^i + b.$$

Similarly, (C_1) [or (C_2), if one assumes (MAA)] holds if and only if there are $a_i \geq 0, i = 1, \cdots, n$, and b such that this equation holds; and assuming (MAA), (C_3) holds if and only if there are $a_i > 0, i = 1, \cdots, n$, and b such that the equation holds.

The remainder of this section investigates a multi-agent application of the KSV approach. We shall assume that hypothetical probabilities q are used to determine the observer's and individuals' state-dependent utilities, following the procedure implicitly described in Proposition 3. More precisely, each of $i = 0, 1, \cdots, n$ is endowed with a preference relation $\succeq^i$, as well as an hypothetical preference relation $\tilde{\succeq}^i$, to be thought of here as i's preference over acts conditionally on some given, strictly positive q^i. We know from section 2 that if $\succeq^i$ and $\tilde{\succeq}^i$ conform to axioms 1, 2′ and 4, for $i = 0, 1, \cdots, n$, there are VNM functions $v_1^i, \cdots, v_T^i$ on $\mathcal{R}$ and subjective probabilities p^i on S such that:

(i) $f \succeq^i g$ iff $\sum_{s\in S} p_s^i v_s^i(f(s)) \geq \sum_{s\in S} p_s^i v_s^i(g(s))$
(ii) $f' \tilde{\succeq}^i g'$ iff $\sum_{s\in S} q_s^i v_s^i(f'(s)) \geq \sum_{s\in S} q_s^i v_s^i(g'(s))$.

These equivalences and the accompanying uniqueness properties will lead to the negative results below. We shall make full use of the flexibility implied by the KSV approach, and take the auxiliary probabilities q^i to be sometimes identical, sometimes different from one individual to another. The upshot of this analysis is that if there is sufficient diversity among the individuals' state-dependent utility functions, a variant of the earlier probability dictatorship and probability agreement theorems holds. Correspondingly, a variant of the earlier utility dictatorship and dependence theorem holds, but as will be explained, the symmetry between probability and utility breaks down in the state-dependent case.

To state these negative results, some further terminology is required. For any $S' \subset S, S' \neq \phi$, we shall say that i is a *probability dictator for* S' if either $p^0(S') = p^i(S') = 0$, or $p^0(S') \neq 0 \neq p^i(S')$ and for all $s \in S'$,

$$p^0(s|S') = p^i(s|S'),$$

and that *probability agreement prevails on* S' if for all $i, j = 1, \cdots, n$, either $p^i(S') = p^j(S') = 0$, or $p^i(S') \neq 0 \neq p^j(S')$ and for all $s \in S'$, $p^i(s|S') =$

$p^j(s|S')$. Similarly, we shall say that i is a *utility dictator on* S' if for all $s \in S', v_s^0 = v_s^i$ (up to a PAT (which might depend on the particular s); and that *pairwise utility dependence* (p.u.d.) *prevails on* S' if for all $s \in S'$ and for all $i, j = 1, \cdots, n, v_s^i = \pm v_s^j$, up to PATs (which might depend on s).

The exposition of impossibility results in this section does not follow the order of last section. We first analyze the probabilistic variant of paradox, and then move to its variant in terms of utility functions.

PROPOSITION 12. *Assume that* $\succeq^0, \cdots, \succeq^n$ *and (for some common q)* $\tilde{\succeq}^0, \cdots, \tilde{\succeq}^n$ *satisfy Axioms 1, 2′ and 4. Denote by* $p^1, \cdots, p^n$ *the individuals' subjective probabilities, and by* $v_1^1, \cdots, v_T^1, \cdots, v_1^n, \cdots, v_T^n$ *the individuals' state-dependent utilities, which are provided by Proposition 3. Assume that (C) applies to both sets of preferences. Then, if* S' *is some nonempty subset of* S *such that for all* $s \in S', v_s^1, \cdots, v_s^n$ *are affinely independent, there is a probability dictator on* S'.

If (C) is replaced by either (C_1), or (C_2) together with (MAA), the same results hold; if (C) is replaced with (C_3) and (MAA), probability agreement prevails, and there is a probability dictator, on S'.

As a particular application of Proposition 12, take S' to be the whole subset of those states which are non-null for at least one $i = 1, \cdots, n$. Then, depending on the Pareto conditions, either the dictator imposes his *absolute* probability, or *absolute* probability agreement prevails, exactly as in the state-independent case. In order to obtain this conclusion, one should resort to the strong assumption that for every relevant state s, the $v_s^1, \cdots, v_s^n$ are affinely independent. As explained in section 3, the significance of this assumption can be appreciated using its equivalent reformulation: for every relevant state, and every individual i, there are lotteries R_*^i, R_{**}^i that "separate" v_s^i from the others' utilities v_s^j. One would hesitate to impose such a strong assumption uniformly across states. To take an example in the style of Savage's, suppose that s' is good weather and s'' bad weather, and that individuals i and j have the following preferences: when s' prevails, i — the adventurous vacationer — prefers rockclimbing to canooing and is indifferent between going to a picnic or taking a swim, while j – the quiet vacationer – is indifferent between the first two lotteries but strictly prefers one of the last two to the other; when s'' prevails, both i and j are indifferent between the four lotteries. Or, to take an economic example, suppose that final outcomes are money amounts and that in some states, widely different amounts are available, whereas in others, only trivial increments around a given money amount are.[13] The "separation" property might well be satisfied in the former case but fail in the latter (since this case might be formalized in terms of linear, hence identical utility functions for money). This discussion suggests that the case in which S' is maximal might be irrelevant. It explains why we chose to emphasize local (i.e., event-relative) properties as in Proposition 12.

[13] Admittedly, this example does not quite fit in the formalism of this paper since it involves not only state-dependent utilities but also state-dependent consequences.

The next proposition deals with a utility variant of the paradox. It is concerned with the special case of an admixture of state-dependence and state-independence. To deal with this case appropriately, we determine the KSV procedure beyond what was done by these authors. Suppose that there is a subset S' of states — all of which we take to be non-null — having the following property: conditional preferences on constant acts do not vary across states in S, whereas they vary across any two $s \in S, t \notin S'$. Thus, as far as S' is concerned, event-, rather than state-dependence, prevails. Restricting attention to acts taking some fixed value on each $t \notin S'$, it can be seen that the standard Anscombe–Aumann theorem (Proposition 1) applies. Thus, using the AA representation, we have a probability π on S'. The assumptions underlying the KSV procedure in Proposition 3 do not ensure that the conditional of the derived probability p on S' will coincide with π. Since π can be revealed by standard betting techniques, it seems natural to require that the two probabilities be equal. The way of obtaining this result while applying the KSV procedure is to impose that the conditional of the hypothetical probability q on S' be equal to the (independently revealed) π.

Formally (in the notation of section 2):

ASSUMPTION 13. Suppose that there is $S' \subset S, |S'| \geq 2$ such that every $s \in S'$ is non-null, and for every pair of constant acts $f, g \in H$:

$$\forall s, t \in S', f \succeq_s g \text{ iff } f \succeq_t g.$$

Then, we require q in the KSV system to satisfy:

$$\forall s \in S', \frac{q(s)}{q(S')} = \pi(s),$$

where π is the probability on S' derived by applying the assumptions of Proposition 1 to the restriction of $\succeq$ to those acts in H which take some fixed set of values on $S \setminus S'$.

Now, we are ready for the last variant of the paradox.

PROPOSITION 14. *Assume that* $\succeq^0, \cdots, \succeq^n$ *and* $\tilde{\succeq}^0, \cdots, \tilde{\succeq}^n$ *obey the KSV system, i.e., they satisfy Axioms 1, 2' and 4. Suppose that there is* $S' \subset S, |S'| \geq 2$ *such that:*

for all $i = 0, 1, \cdots, n,$ *every* $s \in S'$ *is non-null*; $(*)$

for all $i = 0, 1, \cdots, n,$ *all* $s, t \in S'$, $(**)$
and all pairs of constant acts $f, g \in H, f \succeq^i_s$ *iff* $f \succeq^i_t g.$

Denote by $p^0, \cdots, p^n$ *the subjective probabilities given by Proposition 3 for some set of hypothetical probabilities satisfying Assumption 1, and suppose that:*

$(***)$ $(p^1_s)_{s \in S'}, \cdots, (p^n_s)_{s \in S'}$ *are linearly independent vectors.*

Then, if (C) applies to both sets of preferences, there is a utility dictator on S'.

If (C) is replaced with either (C_1) or (C_2), the same result holds. If (C_3) is used instead, positive pairwise utility dependence prevails, and there is a utility dictator, on S'.[14]

Supposing that for all i and s, s is non-null, Proposition 14 can be applied to $S' = S$ as a particular case, but this would be an uninteresting application. The conclusions would just repeat the utility-relative impossibility results already obtained in Propositions 9 and 10. The point of Proposition 14 is to extend these results slightly by emphasizing event-relative properties. Condition (**) is a limited state-independence assumption. It is compatible with a generally state-dependent framework. Notice also that it does not involve any uniformity from one i to another beyond the mere fact that the preferences of each just depend on S' (rather than on the particular state in S'). Going back to one of the previous examples, take $S' = \{$excellent weather, fair weather$\}$. It can no doubt happen that the two vacationers' and the observer's preferences are non-trivial and uniform across S', as required by (*) and (**), respectively. As far as condition (***) is concerned, it is perhaps no more problematic here than it was in the state-independent framework. Take again the vacationers' example: for (***) to be met, it is enough that they entertain different probabilities of the weather turning fair or excellent. Notice however that there is a rough trade-off in plausibility between (**) and (***): the smaller S' is, the more plausible is (**) but the *less* plausible (***).

5 THE TWO-INDIVIDUAL CASE

In the two-individual case, the impossibility conclusions of sections 3 and 4 can be sharpened, as the following corollaries show.

COROLLARY 15. *Assume that $\succsim^0, \succsim^1, \succsim^2$ satisfy Anscombe and Aumann's axioms of state-independent utility. Assume that $p^1 \neq p^2$ and that u^1, u^2 are not identical up to an affine transformation. Then, (C) holds if and only if there is an overall or an inverse overall dictator; (C_1) or (C_2) holds if and only if there is an overall dictator; and it is impossible for either (C_3) or (C^+) to hold.*

COROLLARY 16. *Assume that $\succsim^0, \succsim^1, \succsim^2$ and $\tilde{\succsim}^0, \tilde{\succsim}^1, \tilde{\succsim}^2$ satisfy Karni, Schmeidler and Vind's system of state-dependent utility as restated in section 2. If (C), (C_1), or (MAA) and (C_2) hold, then for each state $s \in S$, either $p^0(s) = p^1(s)$ or $p^0(s) = p^2(s)$ or $v^0(s, \cdot) = v^1(s, \cdot) = v^2(s, \cdot)$ up to affine transformations. If (MAA) and (C_3) hold, for each state $s \in S$, either $p^0(s) = p^1(s) = p^2(s)$ or $v^0(s, \cdot) = v^1(s, \cdot) = v^2(s, \cdot)$ up to affine transformations.*

These two corollaries are closely related to the results of Schervish, Seifenfeld and Kadane [1991, Theorems 2 and 4], who formalize a version of the Bayesian

[14]We included Axiom 2$'$ among the assumptions just for clarity, since condition (*) makes it redundant.

experts paradox in the two-individual case by assuming an AA framework of first state-independent, and then state-dependent theory. In the state-dependent case, they use a special variant of the KSV procedure.[15] Like the main theorem of their paper, our Corollary 2 comes close to predicting that under relevant assumptions, for each state s, either probability agreement or utility-dependence prevails on $\{s\}$. The difference between this wording and their formal statement appears to come mostly from the complication of the null states in the variant they adopt.

It should be clear that the analysis of "consistent Bayesian aggregation" cannot be pursued just in the two-individual case. The conclusion corresponding to Corollary 15 loses its elegant simplicity when $n \geq 3$. The mild requirements that p^1, p^2 should be distinct and that u^1, u^2 should not be essentially identical or opposite functions become the more technical, less interpretable restrictions that $p^1, \cdots, p^n$ are linearly independent, and that $u^1, \cdots, u^n$ are affinely independent. Earlier examples demonstrate that in the state-independent framework, nontrivial solutions to the aggregation problem emerge in the absence of suitable independence assumptions.[16] As far as the state-dependent framework is concerned, Corollary 16 appears to derive a quasi-impossibility theorem without making technical restrictions. Again, the simplicity of this conclusion disappears when $n \geq 3$. We shall give a three-individual example to illustrate how easily nontrivial solutions to the aggregation problem might emerge from the state-dependent case, when algebraic independence restrictions are omitted.

Take $X = \{x_1, x_2, x_3\}$, so that $\Delta(X)$ is S_3^1, i.e., the unit simplex of $\mathbb{R}^3$. Denote the elements of $\mathcal{R} = \Delta(X)$ as $R = (R_1, R_2, R_3)$. In the notation used throughout, (C) implies that for all $R \in \mathcal{R}$:

$$(i) \qquad \sum_s p_s^0 v_s^0(R) \ = \ \sum_i a_i \sum_s p_s^i v_s^i(R)$$

and (considering now hypothetical instead of actual preferences):

$$(ii) \qquad \sum_s q_s^0 v_s^0(R) \ = \ \sum_i b^i \sum_s q_s^i v_s^i(R).$$

It is easy to find specific values such that the KSV assumptions and (C) hold, but for some $S' \subset S$, neither probability dictatorship nor any form of utility dictatorship holds. Take:

[15] See their other paper [Schervish *et al.*, 1990] for a statement of this variant.

[16] See [Goodman, 1988] and [Mongin, 1995, Example 4]. We have belatedly heard of Goodman's contribution to the n-person analysis. Thanks are due to Teddy Seidenfeld for bringing this and other references to our attention.

$$p^1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \; p^2 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \; p^3 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}, \; p^0 = q = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix},$$

$$\begin{aligned} u^1_{s_1} &= R_1, u^1_{s_2} = R_3, u^1_{s_3} = 2R_2 + 3R_3, \\ u^2_{s_1} &= R_1 + 3R_2, u^2_{s_2} = R_1, u^2_{s_3} = R_2, \\ u^3_{s_1} &= R_2, u^3_{s_2} = R_3 + 2R_1, u^3_{s_3} = R_3, \\ u^0_{s_1} &= 2R_1 + 4R_2, u^0_{s_2} = 2R_3 + 3R_1, u^0_{s_3} = 3R_2 + 4R_3. \end{aligned}$$

The vectors p^0, p^1, p^2, p^3 can be thought of as KSV probabilities. In the particular instance, the common hypothetical probability q is taken to be equal to the observer's. To see that the above data agree with (C), notice that equations (i) and (ii) hold with:

$$a_1 = \frac{4}{3}, a_2 = 2, a_3 = \frac{8}{3}, b_1 = b_2 = b_3 = 1.$$

In contradistinction to the impossibility result stated in Proposition 12, probability dictatorship does not hold on $S' = S$. This fact can be traced to the failure of only one assumption in Proposition 12 — i.e., affine independence. Indeed, the individuals' state-dependent utilities are linearly dependent in each state.

Notice that no form of utility dictatorship prevails either. This frustrates the hope of extending the impossibility conclusion of Corollary 16 without adding suitable technical assumptions.

We close the discussion of Corollary 16 by noting that it is a consequence of Proposition 12 *alone*. That is to say, the probability-relative impossibility result implies a restriction on the observer's utility whenever there are only two individuals. This convenient property is lost in the general case.

6 FINAL COMMENTS: THE *EX POST* SOLUTION TO THE PARADOX.

The present paper has offered a comprehensive treatment of the paradox of the Bayesian experts within the framework of the Anscombe-Aumann approach, first by assuming complete state-independence of utility, second by considering the opposite case of complete state-dependence, and third by applying the "intermediary" system of Karni, Schmeidler and Vind [1983] in which utility is state-dependent but the subjective probability is shown to be unique. Propositions 4 and 5 state the paradox in its pure form. They are the AA counterparts of the impossibility theorems recently proved within a state-independent, Savagean framework [Mongin, 1995, Propositions 5 and 7]. Proposition 11 states an easy, but unimpressive possibility result for the pure state-dependent case. By assuming the more informative KSV framework, Propositions 12 and 14 reinstate the paradox, although in a significantly different variant from the initial one.

One might perhaps have conjectured that the paradox would reappear in essentially its original form, once the state-dependence assumption is compounded with a procedure to determine subjective probabilities uniquely. This conjecture fails. The state-by-state analysis uncovers novel and curious situations: combining the assumptions of Propositions 12 and 14 on a sufficiently large state set, one might end up with juxtaposing probability dictatorship or agreement on some events with utility dictatorship or dependence on other events, and a nondescript state of affairs elsewhere. More generally, if "consistent Bayesian aggregation" leads to any paradoxical consequences in the state-dependent framework, these are bound to be state- or at least event-relative. This is the most obvious difference between the negative conclusions delivered by Propositions 9 and 10, on the one hand, and Propositions 12 and 14, on the other.

For more than two experts, technical conditions must be employed to derive dictatorship or uniformity conclusions. We argued that these conditions would be too stringent if they were to apply to each and every subset of the state set. This is why we selected *local* formulations of both our technical conditions and impossibility results. Accordingly, it might be argued that the latter are not really impossibility results, i.e., that the paradox of the Bayesian experts has not been reproduced in the state-dependent framework of this paper. This would be an exaggerate conclusion. It would be tantamount to abstracting from the important differences between a completely unconstrained aggregative rule (Proposition 11) and a relatively constrained one (Propositions 12 and 14). The correct interpretation probably lies half-way between the initial expectation that any sophisticated theory of state-dependent utility would reinstate the original paradox, and the extreme view now under discussion.

Given that we cannot conclude that state-dependent utility theory is the way of escape from the logical difficulties of collective Bayesianism, a more radical alternative must be sought. Within the province of decision theory at large, it remains to investigate suitable relaxations of the sure-thing principle. Within the confines of the present paper, which is restricted to Bayesianism, the remaining logical possibilities are to relax either Paretian consistency or Bayesian consistency. The former solution is illustrated in the field of welfare economics by those writers who reject the *ex ante* formulation of the Pareto principle (i.e., the version which was investigated in this paper), while retaining an *ex post* (i.e., consequence-relative) version.[17] By contrast, the latter solution consists in denying that the aggregate should inherit the individuals' method of decision.

We should like to indicate a (highly qualified) preference for the former over the latter direction of analysis. Even more clearly than some earlier and formally similar cases in welfare economics, the Bayesian experts problem implies that Bayesian consistency should be taken seriously. In the present setting the aggregate does not refer to a collective entity, but to some person acting as an observer.

[17] A leading exponent is Hammond [1982; 1983]. Among the recent applications of the *ex post* point of view, see in particular Zhou's [1996] axiomatization of Bayesian utilitarianism.

To go back to the example of section 1, the aggregate represents the insurer who attempts to summarize the experts' opinions. Given the nature of the observer in this problem situation, it seems natural to subject his preferences to the same choice-theoretic constraints as those prevailing on the (other) individuals' preferences.[18]

Here is a scenario which is compatible with the relaxation of Paretian consistency, whereas Bayesianism is preserved throughout. If, *contrary to our initial assumption*, the insurer understands Bayesian elicitation methods, he will be able to estimate the experts' underlying probability and utility functions. In the state-dependent case, we should then assume that he *himself*, rather than the experimenter, applies the KSV procedure. Once in possession of individual probability and utility data, he will process them *separately* to construct a summary probability and a summary vector of state-dependent utilities. These two items can then be combined unproblematically in the way prescribed by SEU theory. The Pareto principle will be used in the construction of the summary utility vector, but not necessarily in the construction of the summary probability. When applied to the individuals' utilities in each state, it functions as an *ex post* unanimity principle.

The previous paragraph shows that there is one (actually, well-known) way of keeping Bayesian consistency intact while preserving some form of Paretianism. We have just rephrased in terms of our decision-theoretic example the aggregation procedure which has long been recommended by the *ex post* school of welfare economics. In doing so, we have emphasized that there are definitive cognitive assumptions underlying the *ex post* approach — a point which is rarely mentioned in welfare theory. Before arguing that the *ex post* method is a feasible solution to the paradox of the Bayesian experts, one should check whether these assumptions apply. A definitive advantage of the *ex ante* method examined throughout this paper is that it does not require much knowledge on the observer's part.

To claim that the *ex post* approach provides not only a feasible, but a good resolution, a closer examination of the Pareto principle is needed. Implicitly, the defence of the principle trades on a distinction between *factual* and *normative* considerations. The essence of Paretianism is to proclaim that individuals are sovereign in normative matters; this means that their judgments in these matters should never be scrutinized or criticized, but taken for granted. "Normative" here can be diversely understood by reference to values, objectives, or even tastes, as in the "consumer sovereignty" doctrine. These interpretations would correspond to particular statements and defences of the Pareto principle. It is not our task here to list and compare them. The crucial point is that the individuals' sovereignty can be, and has been, argued for in the context of various notions of "normative" judgments, while *there is no concept of factual judgment for which this principle makes sense*. Factual judgements *should* be scrutinized and criticized. Theoretically at

[18]There is a modelling alternative which would make it even clearer that the observer here is just another individual. One could possibly endow him with *two* binary relations, one of which would represent his preferences *qua* ordinary person, the other his preferences *qua* observer. The insurer would then include his own private opinions among those which he tries to amalgamate.

least, they are susceptible of ascertainable truth values; they can be justified or dismissed by logic and evidence. So the Pareto conditions can only hold of a special class of unanimous judgments. Once all this is made clear, it seems as if the *ex post* variant is automatically warranted, and the *ex ante* variant automatically rejected. As it were, the former reaps all the benefits of the normative versus factual distinction.

Let us first clarify the negative part of the argument. Assuming a standard Bayesian framework, comparisons of prospects normally depend on *both* how the agent assesses the values of consequences and how he estimates the likelihood of events. Stochastic dominance, in which only the values of consequences matter, is an exceptional case. Thus, the scope of the *ex ante* Pareto principle exceeds the province of normative judgments. This, in itself, would not make it invalid, just dubious. What makes it invalid is that the excess content of the *ex ante* principle — its encroachment upon the province of factual judgments — leads to spurious recommendations. Here is one: under any state-independent variant of SEU theory, whenever all individuals agree on the strict ranking of two particular consequences, the principle implies that unanimous *probability* judgments should be respected, regardless of the evidence available to each individual. This is a spurious recommendation: evidence should matter to the observer. It can be shown that when conditioning partitions differ from one individual to another, a Bayesian observer who knows what these partitions are will sometimes violate the probabilistic form of the Pareto principle.[19]

Now, consider the positive argument in favour of the *ex post* Pareto principle. It says that the latter is justified because it involves only normative judgments. But in real life, judgments about consequences are infected with factual considerations. A hole in the ozone layer strikes one as an undesirable consequence because of certain scientific facts and laws. To own a large fortune becomes less desirable, or might even become absolutely undesirable, to somebody who knows that he will die tomorrow; and so on. All this suggests that the *ex post* principle could in turn fall a prey to the argument against the *ex ante* principle. By itself, the normative versus factual distinction does not provide the former with a sufficient foundation.[20]

From the above discussion we might conclude that the factual versus normative distinction cuts both ways, and that the foundations of *ex post* reasoning are shaky. But they are at least solider than the foundations of *ex ante* reasoning, which — this paper has attempted to demonstrate — appears to be simply flawed. And in the absence of a third alternative,[21] the *ex post* solution has at least the advantage of providing a feasible way out of the conundrum of collective Bayesianism.

[19]We are indebted here to Ed Green and David Schmeidler. Probabilistic unanimity is discussed at grater length in Mongin [1997].

[20]For a similar argument, see [Broome, 1990].

[21]A paper by Levi [1990] appears to sketch a third alternative by defining restrictions on the *ex ante* principle. This is an interesting avenue to explore.

APPENDIX

A PROOFS

Proof [of Lemma 4] Proposition 2 implies that for $i = 0, 1, \cdots, n$, V^i preserves convex combinations of acts. Hence the vector $(V^0, V^1, \cdots, V^n)$ has convex range. ■

Proof [of Lemma 5] See [De Meyer and Mongin, 1995, Proposition 1]. ■

Proof [of Lemma 6] See [De Meyer and Mongin, 1995, Propositions 1 and 2]. ■

Proof [of Lemma 7] The former conclusion is proved in Coulhon and Mongin [Coulhon and Mongin, 1989]. In view of Axiom 1, the latter is an immediate application of the former. ■

Proof [of Lemma 8] Take $R^1_*, R^1_{**}, \cdots, R^n_*, R^n_{**}$ as in the statement of Lemma 7 and construct the following elements of $\mathcal{R}$:

$$P_* \;=\; (\frac{1}{n}R^1_*, \cdots, \frac{1}{n}R^n_*) \text{ and } P_{**} \;=\; (\frac{1}{n}R^1_{**}, \cdots, \frac{1}{n}R^n_{**}).$$

Then, from the mixture-preserving property of $u^1, \cdots, u^n$:

$$u^i(P_*) \;>\; u^i(P_{**}),\; i = 1, \cdots, n,$$

so that (MAC) holds. The case of affinely independent $V_1, \cdots, V_n$ can be dealt with similarly. ■

Proof [of Proposition 9] Suppose that (C) and (*) hold. Lemma 5 implies that there are $a_1, \cdots, a_n, b$ such that

$$(A1) \qquad U^0 \;=\; \sum_{i=1}^{n} a_i U^i + b.$$

One of the a_i must be nonzero because of AA's nontriviality assumption (Axiom 2). We may select any $\overline{R} \in \mathcal{R}$ and put $u^0(\overline{R}) = u^1(\overline{R}) = \cdots = u^n(\overline{R}) = 0$; there is no assumption of substance in this normalization. Let us now consider the following class of $f \in H$: there are $s \in S$ and $R \in \mathcal{R}$, such that

$$\begin{cases} f(s) & = \; R \\ f(t) & = \; \overline{R} \text{ for all } t \neq s. \end{cases}$$

Applying AA's representation theorem in equation (1), we get:

$$\forall f \in H, \quad \sum_{s\in S} p^0_s u^0(f(s)) \;=\; \sum_{i=1}^{n} a_i \sum_{s\in S} p^i_s u^i(f(s)).$$

When we restrict attention to the class of acts just defined, this becomes:

$$\forall s \in S, \forall R \in \mathcal{R}, \quad p^0_s u^0(R) \;=\; \sum_{i=1}^{n} a_i p^i_s u^i(R).$$

From now on in the proof, we shall use functional notation. The last equation becomes:

$$(A2) \qquad p^0 u^0 \;=\; \sum_{i=1}^{n} a_i p^i u^i,$$

where the functions on the right- and the left-hand sides are defined on $S \times \mathcal{R}$.

Given that in the state-independent case, constant acts may be identified with consequences, equation (A1) also implies that:

$$(A3) \qquad u^0 \;=\; \sum_{i=1}^{n} a_i u^i.$$

Replacing (A3) into (A2) we get:

$$(A4) \qquad \sum_{i=1}^{n} a_i u^i [p^0 - p^i] \;=\; 0.$$

If the $p^0 - p^1, \cdots, p^0 - p^n$ were linearly independent, one would have that:

$$a_i u^i \;=\; 0 \text{ for all } i = 1, \cdots, n,$$

which is impossible since Axiom 2 implies that the u^i are nonconstant and one a_i must be nonzero. Hence, there is $j \in \{1, \cdots, n\}$ such that for some $b_1, \cdots, b_{j-1}, b_{j+1}, \cdots, b_n$:

$$(A5) \qquad p^0 - p^j \;=\; \sum_{i \neq j} b_i (p^0 - p^i).$$

Now, $\sum_{i \neq j} b_i \neq 1$ in view of (*). (Assume that $\sum_{i \neq j} b_i = 1$; then (A5) leads to the absurd equation $p^j = \sum_{i \neq j} b_i p^i$.) We can rewrite (A5) as:

$$p^0 \;=\; (1 - \sum_{i \neq j} b_i)^{-1} p^j \;-\; \sum_{i \neq j} (1 - \sum_{i \neq j} b_i)^{-1} b_i p^i,$$

which provides a linear decomposition of p^0 in terms of $p^1, \cdots, p^n$. Changing the notation, we have just derived:

$$(A6) \qquad p^0 \;=\; \sum_{i=1}^{n} c_i p^i, \text{ for some } c_1, \cdots, c_n \text{ such that } \sum c_i = 1,$$

so that at least one c_i is positive.

Now, replacing (A6) into (A2) leads to:

$$\sum_{i=1}^{n} (c_i u^0 - a_i u^i) p^i \;=\; 0.$$

Using (*) again, we conclude that for all $i = 1, \cdots, n$,

$$(A7) \qquad c_i u^0 \;=\; a_i u^i.$$

One of the a_i must be non-zero, and for any $i = 1, \cdots, n, a_i \neq 0$ if and only if $c_i \neq 0$ (because $u^0, u^1, \cdots, u^n$ are nonconstant). Hence, there is a utility or inverse utility dictator, as was required to show.

Consider the effect of assuming (C_1) instead of (C), while still assuming that (*) holds. The argument just made remains available, since (C_1) trivially implies (C). But there is now a sign restriction on the a_i (Lemma 5). This restriction, together with the fact that one c_i must be positive, implies that there is a utility dictator.

To deal with (C_2) in case (*) we first note that the latter property implies that:

$$(***) \qquad U^1, \cdots, U^n \text{ are affinely independent.}$$

(Suppose not, and consider the special class of acts at the beginning of the proof; then, for some $j \in \{1, \cdots, n\}$, there are coefficients $d_i, i \neq j$, such that $p^j u^j = \sum_{i \neq j} d_i p^i u^i$, a contradiction.) Then, Lemma 8 says that (MAA) holds, and from Lemma 6 the results reached for (C_1) apply to (C_2).

When (C) and (**) hold, equations (A1) to (A4) remain unchanged. Then, the affine independence property of the u^i implies that for all i,

$$(A8) \qquad a_i(p^0 - p^i) \;=\; 0,$$

whence we conclude that there is a probability dictator. This conclusion still holds under either (C_1) or (C_2). The latter case is dealt with by noting that (**) also implies (***). ■

Proof [of Proposition 10] Using Lemmas 6 and 8 as in the previous proof, we see that under either (*) or (**), (C_3) becomes equivalent to (C^+). Hence, from Lemma 5 there is an affine decomposition:

$$U^0 \;=\; \sum_{i=1}^{n} a_i U^i + b$$

with positive a_i for $i \;=\; 1, \cdots, n$. Now, assuming that (*) is the case, we can reproduce the reasoning of the previous proof and conclude, as in (A7) above, that for all $i = 1, \cdots, n$,

$$c_i u^0 \;=\; a_i u^i.$$

Since all of the a_i are positive, and $a_i \neq 0$ if and only if $c_i \neq 0$, we conclude that for $i = 1, \cdots, n$, there are $\alpha_i \neq 0$ such that $u^0 = \alpha_i u^i$, and pairwise utility dependence prevails among $u_1, \cdots, u_n$. Remember that one of the c_i must be positive; this implies that there is a utility dictator. To analyze case (**), we revert to equation (A8) in the proof above:

$$a_i(p^0 - p^i) \;=\; 0,$$

and now conclude that probability agreement prevails and that there is a probability dictator. ■

Proof [of Proposition 12] Throughout the proof, we write $v^i(s, R)$ instead of $v^i_s(R)$ and $p^i(s)$ instead of p^i_s, and assume that for some $R_* \in \mathcal{R}$,

$$v^i(s, R_*) \;=\; 0 \text{ for all } s \in S \text{ and all } i = 0, 1, \cdots, n.$$

(This normalization is permitted by the uniqueness part of Proposition 3.)

From Proposition 3 we know that if (C) holds, $\succeq^i$ is represented by:

$$V^i(f) \;=\; \sum_{s \in S} p^i(s) v^i(s, f(s)),\; i = 0, 1, \cdots, n.$$

Lemma 2 and the chosen normalization imply that:

$$V^0(f) \;=\; \sum_{i=1}^{n} a_i V^i(f)$$

for some $a_1, \cdots, a_n$. By identifying the two expressions for $V^0(f)$, and restricting them to those acts which have values R on s and R_* elsewhere, we conclude that:

$$(A9) \qquad p^0(s) v^0(s, R) \;=\; \sum_{i=1}^{n} a_i p^i(s) v^i(s, R) \quad \forall s \in S, \forall R \in \mathcal{R}$$

Repeating the argument for the auxiliary preferences $\tilde{\succeq}^0, \cdots, \tilde{\succeq}^n$ and their functional representations leads to:

$$q(s)v^0(s,R) \;=\; \sum_{i=1}^{n} b_i q(s) v^i(s,R) \quad \forall s \in S, \forall R \in \mathcal{R},$$

for some $b_1, \cdots, b_n$. Since $q(s) > 0$ for all s,

$$(A10) \qquad v^0(s,R) \;=\; \sum_{i=1}^{n} b_i v^i(s,R) \quad \forall s \in S, \forall R \in \mathcal{R}.$$

Replacing (A10) into (A9) we have that:

$$(A11) \qquad \sum_{i=1}^{n} [b_i p^0(s) - a_i p^i(s)] v^i(s,R) = 0 \quad \forall s \in S, \forall R \in \mathcal{R}.$$

Now, consider S' as in the first part of the Proposition. For any fixed $s \in S'$, since the $v^i(s,\cdot)$ are linearly independent, the equation in R:

$$(A12) \qquad \sum_{i=1}^{n} [b_i p^0(s) - a_i p^i(s)] v^i(s,\cdot) \;=\; 0$$

implies that:

$$(A13) \qquad b_i p^0(s) \;=\; a_i p^i(s), \quad i = 1, \cdots, n.$$

Consider the sets of indexes:

$$I \;=\; \{i = 1, \cdots, n |\; a_i \neq 0\} \text{ and } J \;=\; \{i = 1, \cdots, n |\; b_i \neq 0\}.$$

From Axiom 2$'$, as applied to $\succeq_0$ and $\tilde{\succeq}_0$ respectively, we know that $I \neq \phi \neq J$. Suppose that $I \cap J = \phi$. Then, (A13) implies that:

$$p^0(s) = 0, \text{ and for at least one } i \in I, p^i(s) = 0.$$

If we repeat the reasoning for $s' \in S, s \neq s'$, we find that $p^i(s') = 0$ for the same i. Hence, in the case in which $I \cap J = \phi$, there is i such that $p^0(S') = p^i(S') = 0$, a case of probability dictatorship.

Now, consider the case in which $I \cap J \neq \phi$. There is i such that $a_i \neq 0 \neq b_i$, and:

$$p^0(s) = a_i b_i^{-1} p^i(s), \quad \forall s \in S'.$$

Either $p^i(S') = 0 = p^0(S')$, or $p^i(S') \neq 0$ and

$$\frac{p^0(s)}{p^0(S')} = \frac{p^i(s)}{p^i(S')}, \forall s \in S',$$

which again shows that probability dictatorship prevails. The analysis of the other conditions than (C) makes use of Lemmas 6 and 7, as in the corresponding parts of the proofs of Propositions 10 and 11. Details are left for the reader. ■

Proof [of Proposition 14] We first spell out the implications of Assumption 13 for each KSV representation taken individually. Axiom 1 can be applied to the restriction of $\succeq^i$ to the set $H_{s'}$ of acts having some fixed set of values outside S', and because of (*) and (**), Axiom 2 and a version of Axiom 3 hold for this preference relation (which we also denote by $\succeq^i$). From Proposition 1 there is a state-independent function w^i on $\mathcal{R}$ and a probability π^i on S' such that:

$$\forall f, g \in H_{S'},\ f \succeq^i g \text{ iff } \sum_{s \in S'} \pi_s^i w^i(f(s)) \geq \sum_{s \in S'} \pi_s^i w^i(g(s)).$$

Now, the conclusions of Proposition 3 also apply to the restricted preference. Using Assumption 1, the $(|S'|+1)$-tuple $(w^i, \cdots, w^i, \pi^i)$ is seen to satisfy conditions (i) and (ii) in Proposition 3, as applied to acts in $H_{S'}$, so that by the uniqueness part of this proposition:

$$(v_s^i)_{s \in S'} = (w^i, \cdots, w^i) \text{ up to a PAT},$$

and:

$$\pi^i(s) = \frac{p^i(s)}{p^i(S')}, \forall s \in S'.$$

Hence, for $i = 0, 1, \cdots, n$, we may replace the initial vector of KSV representations relative to $H_{S'}$ by $(w^i, \cdots, w^i)$, and use (C) and condition (***) to prove impossibility results as if state-independence prevailed. The reader is referred to the relevant parts of the proofs of Propositions 9 and 10. ■

Proof [of Corollary 15] Immediate from Propositions 9 and 10. ■

Proof [of Corollary 16] If $v^1(s, \cdot), v^2(s, \cdot)$ are affinely independent, Proposition 12 implies that either $p^0(s) = p^1(s)$ or $p^0(s) = p^2(s)$ whenever (C), (C_1), or (MAA) and (C_2) hold, a conclusion which is strengthened into $p^0(s) = p^1(s) = p^2(s)$ whenever (MAA) and (C_3) holds. If $v^1(s, \cdot), v^2(s, \cdot)$ are affinely dependent, the conclusion that $v^0(s) = v^1(s) = v^2(s)$, up to relevant affine transformations, follows from inspecting equation (A2) in the proof of Proposition 12. ■

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