Weak Arithmetics and Kripke Models[1]

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Abstract

In the first section of this paper we show that $i\Pi_1 \equiv W\neg\neg l\Pi_1$. In the second section of the paper, we show that for equivalence of forcing and satisfaction of Π_m-formulas in a linear Kripke model deciding Δ_0-formulas, it is necessary and sufficient that the model be Σ_m-elementary. This implies that if a linear Kripke model forces PEM_{prenex}, then it forces PEM. We also show that, for each $n \geqslant 1$, $i\Phi_n$ does not prove $\mathcal{H}(I\Pi_n)$. Here, Φ_n's are Burr's fragments of HA.

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0. Preliminaries

We fix the language $L = \{+, \cdot, <, 0, 1\}$. The principle PEM (some of whose restrictions will appear below) of Excluded Middle is $\forall \overline{x}(\varphi(\overline{x}) \vee \neg\varphi(\overline{x}))$.

Heything arithmetic HA and its fragments $(PA^-)^i$, iop, lop, $i\Delta_0$, $i\Sigma_n$ and $i\Pi_n$, $n \geq 1$, are the intuitionistic counterparts of first order Peano Arithmetic PA and its fragments PA^-, Iop, Lop, $I\Delta_0$, $I\Sigma_n$ and $I\Pi_n$. More generally for any set Γ of formulas we will use notations such as $i\Gamma$ and $l\Gamma$ in the same manner. $\neg\Gamma$ is the class of formulas of the form $\neg\varphi$ with $\varphi \in \Gamma$.

By $W\neg\neg LNP$, we mean the scheme $\forall \overline{y}\neg\neg(\exists x\varphi(x, \overline{y}) \rightarrow \exists x(\varphi(x, \overline{y}) \wedge \forall z < x \neg\varphi(z, \overline{y})))$.

We use the usual terminology about Kripke structures as in [TD]. Here we mention two facts about Kripke models. The proofs are straightforward (see [AM]).

Fact 0.1 Suppose α is a node of a Kripke model and φ is an L_α-sentence:

1) $\alpha \Vdash \varphi$ iff $\beta \Vdash \varphi$ for each $\beta \geq \alpha$.

[1] This version corrects an error in the journal version.

2) $\alpha \Vdash \neg\varphi$ iff $\beta \nVdash \varphi$ for each $\beta \geq \alpha$.

3) $\alpha \Vdash \neg\neg\varphi$ iff for each $\beta \geq \alpha$ there exists $\gamma \geq \beta$ such that $\gamma \Vdash \varphi$.

Fact 0.2 Suppose $\mathcal{K} \Vdash (PA^-)^i$ (resp. $\mathcal{K} \Vdash i\Delta_0$) and $\varphi \in \exists_1$ (resp. $\varphi \in \Sigma_1$). Then for each $\alpha \in K$, we have:

$$\alpha \Vdash \varphi \Leftrightarrow M_\alpha \vDash \varphi.$$

If $\psi \in \forall_1$ (resp. $\psi \in \Pi_1$) then:

$$\alpha \Vdash \psi \Leftrightarrow \forall\beta \geq \alpha \; M_\beta \vDash \psi.$$

Therefore, a $\forall_1$ (resp. Π_1)-formula is forced at a node α of a Kripke model of $(PA^-)^i$ (resp. $i\Delta_0$) if and only if it is satisfied in the union of the worlds in any path above α.

1. $i\Pi_1$ and its Kripke models

It was observed in [MM, Sec. 6] that, the second proof in [TD, p.131] for $HA \vdash W\neg\neg LNP$ actually proves the following:

Fact 1.1 If a fragment $i\Gamma$ of HA is m-closed under the negative translation and $I\Gamma \vdash L\Gamma$, then for any formula $\varphi(x, \overline{y}) \in \Gamma$, $i\Gamma \vdash \forall\overline{y}\neg\neg(\exists x\varphi(x, \overline{y}) \to \exists x(\varphi(x, \overline{y}) \wedge \forall z < x\neg\varphi(z, \overline{y})))$.

As a corollary, it was proved that $iop \equiv W\neg\neg lop$ where $W\neg\neg lop$ is the intuitionistic theory axiomatized by $(PA^-)^i$ plus $W\neg\neg LNP$ on open formulas. Here we prove a similar result for $i\Pi_1$.

Note that by the above fact $i\Pi_1 \vdash W\neg\neg l\Pi_1$. Also, using $i\Pi_1 \equiv i\neg\Pi_1$, see [W2, Cor. 6], we have $i\Pi_1 \vdash W\neg\neg l\neg\Pi_1$ where $W\neg\neg l\neg\Pi_1$ is the intuitionistic theory axiomatized by $i\Delta_0$ plus $W\neg\neg LNP$ on $\neg\Pi_1$ formulas.

Proposition 1.2 $W\neg\neg l\neg\Pi_1 \vdash i\Pi_1$.

Proof Assume $\mathcal{K} \Vdash W\neg\neg l\neg\Pi_1$. Let $\alpha \in \mathcal{K}$ does not force $I_x\varphi(x, \overline{y})$, for some Π_1-formula φ. Therefore, by the above facts, there will exist a node $\gamma \geqslant \alpha$ with $a, \overline{b} \in M_\gamma$ ($\overline{b}$ of the same arity as $\overline{y}$), such that

(i) $\gamma \Vdash \varphi(0, \overline{b}) \wedge \neg\varphi(a, \overline{b})$,

(ii) $\gamma \Vdash \forall x(\varphi(x, \overline{b}) \to \varphi(x+1, \overline{b}))$.

By $\mathcal{K} \Vdash W\neg\neg l\neg\Pi_1$, we get $\gamma \Vdash \neg\neg\exists x(\neg\varphi(x, \overline{b}) \wedge \forall z < x\varphi(z, \overline{b}))$. Therefore, for some $\delta \geq \gamma$ and some (necessarily nonzero) $d \in M_\delta$, $\delta \Vdash \neg\varphi(d, \overline{b}) \wedge \forall z < d\varphi(z, \overline{b})$. This is a contradiction to the fact that γ (and therefore, δ) forces $\forall x(\varphi(x, \overline{b}) \to \varphi(x+1, \overline{b}))$. $\Box$

Proposition 1.3 $W\neg\neg l\Pi_1 \vdash i\neg\Pi_1$.

Proof Let α be a node of a Kripke model $\mathcal{K} \Vdash W\neg\neg l\Pi_1$, $\varphi(x, \overline{y})$ negation of a Π_1-formula, and $\overline{a} \in M_\alpha$ of the same arity as $\overline{y}$. To prove $\alpha \Vdash I_x\varphi(x, \overline{a})$, assume without loss of generality that $\alpha \Vdash \varphi(0,\overline{a})$. It is enough to show that for every $\beta \geq \alpha$, there exists $\delta \geq \beta$ such that, $M_\delta \Vdash I_x\varphi(x,\overline{a})$, since $\neg\neg I_x\varphi(x,\overline{a}) \vdash I_x\varphi(x,\overline{a})$. Fix $\beta \geq \alpha$. If $\beta \Vdash \forall x\varphi(x,\overline{a})$, then we may take $\delta = \beta$. Otherwise, by $\beta \Vdash W\neg\neg l\Pi_1$, there will exist $\gamma \geq \beta$ such that, for some non-zero $d \in M_\gamma$, $\gamma \Vdash \neg\varphi(d,\overline{a}) \wedge \forall z < d\varphi(z,\overline{a})$. Clearly, such a node δ has the desired property. □

Corollary 1.4 $i\Pi_1 \equiv W\neg\neg l\Pi_1 \equiv W\neg\neg l\neg\Pi_1$.

2. Forcing and truth

For a class Γ of formulas and a Kripke structure $\mathcal{K}$, $\Vdash\Leftrightarrow_{\mathcal{K},\Gamma}\models$ (or just $\Vdash\Leftrightarrow_\Gamma\models$ if $\mathcal{K}$ is understood) means that for any node α of $\mathcal{K}$, formula $\varphi(\overline{x},\overline{y}) \in \Gamma$ and $\overline{a} \in M_\alpha$, we have $\alpha \Vdash \varphi(\overline{x},\overline{a})$ if and only if $M_\alpha \models \varphi(\overline{x},\overline{a})$.

Lemma 2.1 For any Kripke structure $\mathcal{K}$ and any $m \geq 0$, we have:

(i) If $\Vdash\Leftrightarrow_{\Pi_m}\models$, then $\Vdash\Leftrightarrow_{\Sigma_{m+1}}\models$.

(ii) If $\Vdash\Leftrightarrow_{\Sigma_m}\models$ and $\mathcal{K}$ is a Σ_m-elementary-extension model, then $\mathcal{K} \Vdash PEM_{\Sigma_m}$.

(iii) If $\mathcal{K} \Vdash PEM_{\Sigma_m}$ is linear, then $\Vdash\Leftrightarrow_{\Pi_m}\models$.

Proof (i) and (ii) are straightforward.

(iii) Clearly for any $\mathcal{K} \Vdash PEM_{\Delta_0}$, we have $\Vdash\Rightarrow_{\text{Prenex}}\models$. Conversely, assume $\mathcal{K} \Vdash PEM_{\Sigma_{m+1}}$ is linear, α is a node of $\mathcal{K}$, $\psi(\overline{x},\overline{a}) \in \Delta_0$ and $\alpha \nVdash \forall x_{m+1}\exists x_m \cdots Qx_1\psi(\overline{x},\overline{a})$, where $Q \in \{\forall,\exists\}$. Using $PEM_{\Sigma_{m+1}}$, it suffices to show $\alpha \Vdash \neg\neg\exists x_{m+1}\forall x_m \cdots Q^*x_1\neg\psi(\overline{x},\overline{a})$, where Q^* is the quantifier dual to Q. If not, there would exist $\beta \geq \alpha$ such that $\beta \Vdash \neg\exists x_{m+1}\forall x_m \cdots Q^*x_1\neg\psi(\overline{x},\overline{a})$ and so by PEM_{Σ_m}, $\beta \Vdash \forall x_{m+1}\exists x_m \cdots Qx_1\psi(\overline{x},\overline{a})$. By $\alpha \nVdash \forall x_{m+1}\exists x_m \cdots Qx_1\psi(\overline{x},\overline{a})$, there exists $\gamma \geq \alpha$ and $c \in M_\gamma$ such that $\gamma \nVdash \exists x_m \cdots Qx_1\psi(\overline{x},\overline{a})[x_{m+1}/c]$ and so by PEM_{Σ_m} again, $\gamma \Vdash \neg\exists x_m \cdots Qx_1\psi(\overline{x},\overline{a})[x_{m+1}/c]$.

But then $\delta = \mathrm{m}ax\{\beta,\gamma\}$ leads to a contradiction. □

Corollary 2.2 Let $\mathcal{K} \Vdash PEM_{\Delta_0}$ be linear. Then the following are equivalent:

(i) $\Vdash\Leftrightarrow_{\Pi_m}\models$.

(ii) $\mathcal{K}$ is a Σ_m-elementary-extension Kripke model.

(iii) $\mathcal{K} \Vdash PEM_{\Sigma_m}$.

It is known that in intuitionistic predicate logic, unlike its classical counterpart, the prenex-normal form theorem does not hold. This is also the case for intuitionistic arithmetic. Indeed, it was proved, by Visser and Wehmeier, that $iPNF$ is Π_2-conservative over $i\Pi_2$, were $iPNF$ is the intuitionistic theory axiomatized by $(PA^-)^i$ plus the induction scheme restricted to prenex formulas, see [W2, Thm. 3]. However, we have the following:

Corollary 2.3 If $\mathcal{K} \Vdash PEM_{\text{prenex}}$ is linear, then $\mathcal{K} \Vdash PEM$.

For a set T of sentences, T^i denotes its intuitionistical closure. In [Bus], the intuitionistic theory of the class of T-normal Kripke structures is denoted $\mathcal{H}(T)$. Buss axiomatized $\mathcal{H}(T)$ by the universal closures of all formulas of the form $(\neg\theta)^{\varphi}$, where θ is semipositive (i.e. each implicational subformula of θ has an atomic antecedent) and $T \vdash_c \neg\theta$. It was proved in [M, Cor. 1.2] that, $T^i \in \text{range}(\mathcal{H})$ iff $T^i = \mathcal{H}(T)$. As a corollary, no fragment of HA extending $i\Pi_1$ belongs to the range of $\mathcal{H}$.

Burr's fragments Φ_n of HA are defined as follows, see [Bur2, Sec. 7b]:

(i) $\Phi_0 = \Delta_0$,

(ii) $\Phi_1 = \Sigma_1$,

(iii) For $n \geqslant 2$, Φ_n consists of all formulas $\forall x(B \to \exists y C)$, where $B \in \Phi_{n-1}$ and $C \in \Phi_{n-2}$.

Burr showed that these fragments can be considered as normal forms for the formulas of intuitionistic arithmetic. More precisely, he proved:

(i) $I\Pi_n = I\Phi_n$ for $n \geqslant 0$,

(ii) $\bigcup_{n\in\omega} \Phi_n$ =Form(L) (modulo equivalence in $i\Delta_0$),

(iii) $I\Pi_n$ and $i\Phi_n$ prove the same Π_2-formulas for $n \geqslant 0$.

The following was proved by T. Polacik, see [P, lemma 1]:

Fact 2.4 Fix $n \geq 0$. Let $\mathcal{K} \Vdash PEM_{\Delta_0}$ be an Σ_n-elementary extension Kripke model. Then, for each $\alpha \in \mathcal{K}$ and each $\varphi \in \Phi_n$ we have: $\alpha \Vdash \varphi$ if and only if $M_\alpha \models \varphi$.

Proposition 2.5 For each $n \geq 1$, we have $\mathcal{H}(I\Pi_n) \nsubseteq i\Phi_n$.

Proof We construct a Kripke structure by putting a model of $I\Pi_n$ above a Σ_n-elementary substructure of it which is not a model of $I\Pi_n$, see [HP, P. 222-223] for the existence of such substructures. Using the above fact, it is easy to see that this Kripke model forces $i\Phi_n$. So we get a non-$I\Pi_n$-normal Kripke model of $i\Phi_n$. On the other hand, as it was observed in [AM] (in the proof of 2.1 (iv)), any theory of the form $\mathcal{H}(T)$ is closed under Friedman's translation and so by [W1], each finite Kripke model of it is $\mathcal{H}(T)^c$-normal. So, by [M, lemma 1.2], it must be T-normal. □

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