A Representational Reconstruction of Carnap's Quasianalysis

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1. Introduction

According to general wisdom, quasianalysis belongs to the large family of Carnap's ingenious, but finally failed contributions to epistemology and philosophy of science. In this paper I want to show that this is not the case. Rather, Carnapian quasianalysis is to be considered as a promising theory of a representational constitution of scientific objects. That is to say, I intend to embed Carnap's approach of quasianalytical constitution in the framework of a general theory of meaningful representation (cf. Mundy (1986)).

The outline of this paper is as follows: In section 2 I recall the basics of the quasianalytical approach, taking into consideration not only the well-known account in "Der Logische Aufbau der Welt" (*Aufbau*) but also a rather unknown first version of quasianalysis ("Quasizerlegung") which Carnap developed in an unpublished manuscript written in 1923. This paper deserves attention not only for philosophico-historical reasons, rather it contains quite a lot of interesting features of the quasianalytical approach which do not appear in the *Aufbau* account. In section 3 I reformulate Carnap's account of quasianalysis in the framework of a representational theory of similarity measurement. This allows us to consider the theory of quasianalysis as a special case of a general theory of structural representation. In section 4 it is shown how Goodman's objections against the feasibility of any quasianalytical account may be defused in the new framework. As an application of representational quasianalysis, in section 5 I sketch how Quine's thesis of empirical underdetermination of theories may be elucidated in the framework of a representational quasianalysis.

2. Carnap's Quasianalysis of 1923

Carnap distinguishes that there are two essentially different ways of describing a set of elements. The first way is to say what are the properties or parts every element has. This method he calls the method of *individual* description. The second way is to tell what are the relations between the elements. It might be called the method of *relational* description. The relational description has the advantage of being an *internal* description, it does not go beyond the set it intends to describe: the elements of the set

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in question are *not* decomposed in parts that (usually) do not belong to that set. Rather, the relational description characterizes them by appropriate subsets of Cartesian products of the basic set itself. On the other hand, the method of relational description has the drawback of being rather clumsy. Thus it would be desirable to have a method which transforms a relational description in a handier individual one but keeping the virtue of being an immanent description thereby joining the advantages of both. This is the method of quasianalysis. The term "quasianalysis" ("Quasizerlegung") appears for the first time in an unpublished manuscript of 1923 which has the programmatic title "Quasi-Analysis - A Method to order non-homogenous sets by means of the theory of relations". It is a purely formal theory that might be considered as a generalization of the well-known Russell-Whitehead theory of equivalence classes. Carnap describes the task of quasianalysis as follows:

> Suppose there is given a set of elements, and for each element the specification to which it is similar. We aim at a description of the set which only uses this information but ascribes to these elements quasicomponents or quasiproperties in such a way that it is possible to deal with each element separately using only the quasiproperties, without reference to other elements. Carnap (1923, 4)

The ascription of quasiproperties is not arbitrary, of course, but should obey four basic conditions (cf. Carnap (1923, 4-5):

(C1) If two elements are similar they coincide in at least one quasiproperty.
(C2) If two elements are not similar they do not coincide in any quasiproperty.
(C3) If two elements a and b are similar to exactly the same elements, i.e., if they have the same similarity neigborhood, they have the same quasiproperties.
(C4) There is no quasiproperty which can be removed such that the conditions (C1) - (C3) are still satisfied.

As Carnap observes, these axioms are consistent and independent of each other. In the *Aufbau* only the conditions (C1) and (C2) appear. It is useful to have formal definitions of the concepts used in the following. A similarity structure, denoted by (S,~), is to be a set S endowed with a similarity relation. A similarity relation is a reflexive and symmetric, but not necessarily transitive, relation ~S x S. The similarity neighborhood of $x \in S$ is denoted by $co(x) := \{y \mid x \sim y\}$. Most often our examples are *finite* similarity structures, i.e., the underlying set S is a finite set. Then it can be conveniently represented by a finite numbered graph such that two elements define an edge if and only if they are similar as displayed in the following example (cf. Carnap (1923, 5):

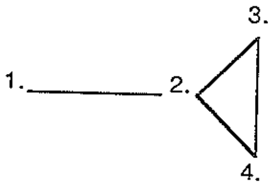

Figure 2.1

This graph is to be interpreted as a similarity structure with underlying set S = {1,2,3,4} where 1 and 2 are similar, 2,3,4 are similar to each other, and no other pairs of (different) elements are similar to each other. A quasianalysis of a finite similarity structure (S,~) can be succinctly described by a list (cf. Goodman 1954, Ch. VI): as above denote the elements of S by natural numbers 1,2, ..., n. The quasiproperties are denoted by a,b,c,... Then a quasianalysis of (2.1) can be given as a list of the following kind: {1.a, 2.ab, 3.bc, 4.bc} to be read in the obvious way, to wit, 1 has the quasiproperty a, 2 has the quasiproperties a and b etc. Sometimes it's convenient to combine lists and graphs in the following way:

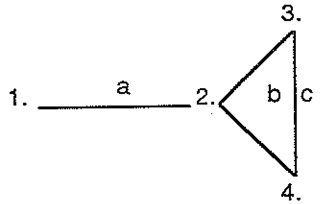

Figure 2.2

For later purposes let us mention one famous example, which may be called "Goodman's triangle":

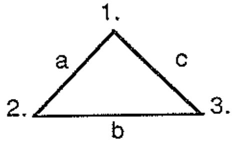

Figure 2.3

It is a quasianalysis in the sense of the simplified definition of the *Aufbau*, i.e., it satisfies (C1) and (C2). However, it is not a quasianalysis according to the original definition since it does not satisfy (C3).

3. Quasianalysis (QA) in the framework of a representational theory of similarity measurement

Now I embark on the task of reformulating QA in the framework of a representational theory of similarity measurement. This will enable us to exploit some interesting analogies of QA with the representational theory of measurement. The starting point is the following representational reformulation of quasianalysis:

(3.1) Definition.

(i) A *weak* quasianalysis of (S,~) is a map f: $S \rightarrow 2^Q$ which satisfies the following properties:

(1) $s \sim s' \Rightarrow f(s) \cap f(s') \neq \emptyset$

(2) $f(s) \cap f(s') \neq \emptyset \Rightarrow s \sim s'$

(ii) A *strong* quasianalysis is a weak quasianalysis which satisfies the following two further conditions:

(3) co (x) = co (y) $\Rightarrow$ f (x) = f (y)

(4) No elements of Q can be removed, unless the resulting f does not satisfy at least one of the conditions (1) - (3).

In (3.1) I apparently have introduced the set Q of quasiproperties as independent of the set S. As the reader will remember, according to Carnap one of the main virtues of quasianalysis is that it does allow us to consider the quasiproperties as derived, i.e., set theoretically constructed entities. Quasianalysis in the sense of (3.1) can be considered as an *immanent* description in the following way:

(3.2) Lemma

Let f: $S \to 2^Q$ be a (weak or strong) quasianalysis. Denote the power set of S by Po(S). Define f*: $S \to 2^{Po(S)}$ by $f^*(s) := \{q^* | q \in f(s)\}$. Then f* is a (weak or strong) quasianalysis.

(3.2 gives rise to the equivalence relation of *extensional equivalence*: two quasianalysis f: $S \to 2^Q$ and f':$S \to 2^{Q'}$ are *extensionally equivalent* iff f* = f'*. In the following I'll consider quasianalysis only "up to extensional equivalence". This means, I essentially work with "immanent" quasianalysis in Carnap's sense. In the *Aufbau* Carnap introduced the distinction between quasianalysis of the first and the second kind which can be rendered precise as follows:

(3.3) Definition

A (weak, strong) quasianalysis of the *first kind* is a (weak, strong) quasianalysis f: $S \to 2^Q$ for which f*(q) for each $q \in Q$ satisfies the following two requirements:

(i) (x) (y) $(x,y \in f^*(q) \Rightarrow x \sim y)$
(ii) (x) $(x \notin f^*(q) \Rightarrow \exists y\ (y \in f^*(q)$ and $x \not\sim y)$

If these conditions are not satisfied f is said to be of the *second kind*. A subset of S satisfying (i) and (ii) is a called a *similarity circle*. The set of similarity circles is denoted by SC(S).

Stated informally the condition of (3.3) requires that an element x which is similar to all y having the quasiproperty q also has the quasiproperty q, or, in Carnap's own terms it requires that the extension of each quasiproperty is a *similarity circle* (see Carnap (1928, § 70 f). As can easily be verified, Goodman's triangle is a weak quasianalysis of the second kind.

We may consider the set f(s) of quasiproperties of s as a *model* of s, i.e., a quasianalysis is a kind of *theoretical representation*: the elements s of the similarity structure are represented by their models f(s), and the similarity relation "~" is represented by the set theoretical relation of intersection. Of course, this representation is not arbitrary but has to satisfy certain conditions of adequacy, to wit, the conditions (3.1) (1) - (4).

Considering the quasianalysis of a similarity structure as a representation immediately leads us to ask the following questions: Does a *Representation Theorem* hold, i.e., given a similarity structure (S,~), is there a quasianalytical representation f: $S \to 2^Q$? This question was already positively answered by Carnap in 1923.

More interesting is whether an *Uniqueness Theorem* holds: having established the existence of a quasianalytical representation for all similarity structures, the natural question arises whether it is "essentially" unique? Carnap knew very well that weak quasianalysis usually are *not* unique. All the authors who criticized the quasianalytical approach only treated weak quasianalysis, and usually they considered its non-uniqueness as a fatal blow. As far as I know nobody has ever treated the uniqueness question for quasianalysis which satisfy something like (C1) - (C4) except Brockhaus (1963). To show that in general a strong quasianalysis of the first kind is not unique it suffices to give a counter-example. The smallest I've been able to find is the following one:

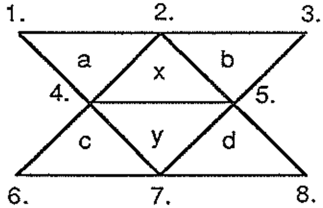

Figure 3.4

As one easily verifies this similarity structure has *two* essentially different quasianalytical representations satisfying (C1) - (C4): one has the quasiproperties a,b,c,d, and x, while the other has a,b,c,d, and y. A quasianalysis with fewer properties does not exist.

Thus, in general, even for strong quasianalysis of the first kind, not to mention weak quasianalysis, a *Uniqueness Theorem* does *not* hold. Does this show that the quasianalytical approach is doomed to fail, as many authors maintained? I don't think so. One way out is to switch to a quasianalysis of the second kind thereby eventually reaching uniqueness. This path is beset with certain difficulties which I cannot discuss in this paper. Another more promising rout is trying to find a special class C of similarity structures such that all members of C have a *unique* quasianalysis. In this paper I deal only with the second path starting with a theorem which we owe to Brockhaus (1963).

(3.5) Theorem

A similarity structure $(S,\sim)$ has a unique quasianalysis $f: S \rightarrow 2^Q$ of the first kind iff $(S,\sim)$ has the following property: there is a set $SC(S,2) \subseteq SC(S)$ satisfying the following requirements:

(i) $\cup SC(S,2) = S$

(ii) for $T_i \in SC(S,2)$ there are two (not necessarily different) elements $x_i, y_i \in T_i$ such that $co(x_i) \cap co(y_i) = T_i$.

The proof is lengthy but elementary, i.e., it does not use any new concepts or methods that would not have been available to Carnap in 1923. Counter-examples show that the condition that f is of the first kind cannot be removed.

The content of this theorem can be formulated as the statement that a strong quasianalysis of the first kind is unique iff each of its quasiproperties is *generated extensionally* by at most two elements. To get a feeling for this condition let us make the following remarks:

(1) If (S,~) is a transitive similarity structure then, obviously its quasiproperties, i.e., its equivalence classes, all are generated by *one* x_q. However, the reverse is not true: there are a lot of similarity structures (S,~) which satisfy this condition but are not equivalence structures. The smallest example is given by the following similarity structure:

1.________2.________3.

(2) All examples of similarity structures encountered in the literature satisfy the condition of (3.5). Hence they have a unique strong quasianalysis of the first kind.

(3) In the counter-example (3.4) either the quasiproperty x or y belongs to each strong quasianalysis, and x and y have exactly *three* generators.

(3.5) gives the motivation to characterize similarity structures according to how many generators are needed for their strong quasianalysis:

(3.6) Definition.

A similarity structure (S,~) is *of the nth-order* iff there is a set $SC(S,n) \subseteq SC(S)$ satisfying the following requirements:

(i) $\cup SC(S,n) = S$

(ii) for $\in T_i$ SC(S,n) there are $x_{i1}, \ldots x_{in} \in T_i$ such that

$$co(x_{i1}) \cap \ldots \cap co(x_{in}) = T_i$$

After this preparatory definition we are able to succinctly express the main result of this section as follows:

(3.7.) Theorem.

A similarity structure (S,~) has a unique strong quasianalysis of the first kind if and only if it is of the first or second order.

Summarizing we may say that due to (3.7) similarity structures of the second kind indeed provide a "natural" realm where the quasianalytical approach works even if we rely on the most severe requirement of uniqueness. This realm is strictly larger than the class of transitive similarity structures which can be considered as the genuine field of the Russell-Whitehead method of equivalence classes. Thus, Carnapian quasianalysis actually is a working generalization of the latter.

4. Criticism of Criticisms of Quasianalysis

Almost all the authors who have dealt with the formal aspects of Carnap's quasianalysis have followed Goodman's criticism launched against this approach in Goodman (1954). The only exceptions known to me are Brockhaus (1963), Moulines (1991) and Proust (1984). In our representational reconstruction of quasianalysis the basic line of Goodman's criticism can be reconstructed as follows: given a similarity structure (S,~), one singles out a certain distribution of properties $f_G: S \to 2^P$ as the "real" one. The only conditions imposed on f_G are (C1) and (C2). f_G might be considered as God's distribution chosen by Him for some reason we mortals don't know.

Then, according to Goodman, the task of quasianalysis is to reconstruct f_G from relational information only, i.e., only from the information contained in the extensional list of the similarity relation "~". Of course, this is in general not possible because for weak quasianalysis a *Uniqueness Theorem* does not hold. The simplest and most famous example is Goodman's triangle mentioned above.

We would be better off if we'd required God's property distribution to satisfy not only (C1) and (C2) but to be a strong quasianalysis of the first kind. This plot would allow us to get rid of Goodman's triangle (and a lot of other "counter-examples") since it does not satisfy (C3) and is not of the first kind. However, Goodman and his followers might try harder confronting us with similarity structures of the third or higher order which definitively do not possess a unique strong quasianalysis (even of the first kind). In this case, it would be mere luck if our property distribution f coincided with f_G. A first objection to this strategy of disavowing the quasianalytical approach could claim that similarity structures of higher kind ($n \geq 3$) are too complicated so that they might be found in nature. This contention is supported by the fact that till now in the literature no similarity structure of the third or of higher order has been discussed.

But I think we can do better: let us grant that there might be "natural" similarity structures of higher order. Even then the thesis that Carnap's quasianalytical approach is doomed to fail is drawn much too hastily. It is only justified as long as we accept that the main goal of quasianalysis is to reconstruct a pre-given property representation f_G. When we challenge this premiss the perspective of the quasianalytical approach doesn't look that bleak anymore. This objection has been put forward by Proust (cf. Proust (1984). According to her, Goodman realistically misunderstands the very intentions of the quasianalytical approach: "Goodman's objections ... reestablish in spite of him the fiction of an omniscient God capable of controlling through originary intuition, that is, without construction, what the constitution derives from its extensional data." (Proust (1984, 299) In a less picturesque language this just amounts to challenge the legitimacy of a "real" property distribution f_G: $S \rightarrow 2^P$ as the one and only guiding star. Instead one should take seriously the quasianalytical perspective: if we have no other means of constituting properties than through the quasianalysis of our elementary experiences, it might very well happen that these experimental data are *not* sufficient to single out a uniquely determined "objective" property distribution. This amounts to admitting the possibility of the empirical underdetermination of a quasianalytical representation of a similarity structure as a *theory* of that structure, or so I want to argue in the next section.

5. Application: The Thesis of Underdetermination in the Framework of Quasianalysis

The thesis of the empirical underdetermination of theories maintains that there are incompatible theories which are empirically equivalent. Usually the underdetermination thesis has been studied in the standard approach which considers theories as sets of sentences. Without arguing for it, I propose a structural approach which considers quasianalysis of similarity structures as prototypes of empirical theories. This claim is in line with Carnap's contention, put forward in the *Aufbau*, that we might conceive the world as a huge similarity structure (cf. *Aufbau* § 27). More precisely, this can be spelt out as follows: the theory's domain of data is a similarity structure (S,~). A map f: $S \rightarrow 2^Q$ (not necessarily a quasianalysis) is a kind of *theory* of this structure in the following sense: The "theory-map" f represents the data s by conceptual models f(s) which are bundles of quasiproperties, relations of data are represented by relations of their models. For example, suppose that for $s, s' \in S$ we have $f(s) \cap f(s') \neq \emptyset$. This is to say that the theory f claims the following *observation categorical* to be true: "Whenever x = s and y = s' then x and y are similar to each other". Whether all universal sentences of this type are true depends on whether f is a structure preserving map, i.e., whether it satisfies ($s \sim s' \Leftrightarrow f(s) \cap f(s') \neq \emptyset$), i.e., (C1) and (C2). Considering a theory f as adequate iff all the observation categoricals implied by it are true, we get that a quasianalysis of the world's similarity structure can be considered as an empirically adequate theory of that world. Now the underdetermination thesis claims that a theory "is bound to have empirically equivalent alternatives which, if we were to discover them, we would see no way of reconciling by reconstrual of predicates". (Quine 1975. 327). Empirically equivalent alternatives should be "equally good", i.e., their theoretical virtues such as simplicity, economy etc. should be roughly the same (cf. Bergström (1993, 335). In the quasianalytical approach this is captured by the requirements (C3) and (C4). Quine's "reconstrual of predicates" can be reconstructed as follows: Assume we have two quasianalysis $f: S \rightarrow 2^Q$ and $f': S \rightarrow 2^{Q'}$ of the same similarity structure $(S,\sim)$. Then the quasianalytical counterpart of a reconstrual of predicates is an appropriate map $g: 2^Q \ 2^{Q'}$ which makes the following diagram commutative:

$$\begin{tikzcd} & 2Q \arrow[dd, "g"] \\ S \arrow[ur, "f"] \arrow[dr, "f'"'] & \\ & 2Q' \end{tikzcd}$$

Figure 5.1

Looking at the example (3.4) it is easy to see that there is no map from the set Q = (a,b,c,d,x) to the set Q' = {a,b,c,d,y} of rival quasiproperties which renders (5.1) commutative. This means the quasianalytical systems based on Q and Q', respectively, are incompatible. Now, depending on the contingent structure of the world we are ready to prove or to disprove the underdetermination thesis:

If the world $(S,\sim)$ happens to be a similarity structure of the first or second order the underdetermination thesis is wrong: according to (3.5) there is one and only one empirically adequate quasianalytical theory of the world.

On the other hand, if the world happens to be a similarity structure of higher order (at least of order three) the underdetermination thesis is true. At least this is the case as long as we don't find other theoretical virtues which allow us to establish a ranking between different strong quasianalysis of the first kind.

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