BELIEF MODALITIES DEFINED BY NUCLEI

Abstract. The aim of this paper is to show that the topological interpretation of knowledge as the interior kernel operator K of a topological space (X, OX) comes along with a partially ordered family of belief modalities B that fit K in the sense that the pairs (K, B) satisfy all axioms of Stalnaker's bimodal KB logic of knowledge and belief but the axiom of negative introspection (NI). The belief modalities B introduced in this paper are defined with the help of the dense nuclei of the Heyting algebra OX of the topological space (X, OX). The set $NUC(OX)_d$ of dense nuclei of OX is shown to have the structure of a complete Heyting algebra.[1] The elements of $NUC(OX)_d$ define a family of belief operators B such that the bimodal logics (K, B) satisfy all axioms of Stalnaker's KB system but (NI).

Key words: Epistemic Logic, Doxastic Logic, Topological Semantics, Heyting algebras, Nuclei, Mereology.

1. INTRODUCTION. Understanding the relation between knowledge and belief is an issue of central importance in formal epistemology. Especially after the birth of knowledge-first epistemology, the question of what exactly distinguishes an item of knowledge and an item of belief and how one can be determined in terms of the other has become even more pertinent. In the recent literature on the topological semantics of Stalnaker's KB system, only one belief operator of a whole family of plausible operators has been considered (cf. Baltag et al. 2013, 2017, 2019)), namely, the operator intclint (cl being the closure operator of (X, OX)). As will be explained in detail in section 3, this operator is defined by a very special nucleus (see (3.1)

[1] Many arguments concerning nuclei rely on the specific lattice-theoretical structures of Heyting algebras. Collections of useful formulas of the Heyting calculus can be found in Borceux (1994, chapter 1.2), Johnstone (1982, I, 1.10ff), and Picado and Pultr (2012, Appendix I, 7)). They are freely used in this paper.

Definition), namely, intclint is the largest dense nucleus of (X, OX). It should be emphasized, however, that this operator is not the only and arguably not the most plausible member of the family of belief operators B that are compatible with K.[2] Thus, to obtain a more comprehensive understanding of the relationship between knowledge and belief, it seems to be expedient to discuss the whole group of options instead of restricting one's attention only to rather special option.

In Stalnaker's KB system, the belief modality B turns out to be uniquely determined by the knowledge operator K. In this paper, a more complex and more flexible relation between knowledge K and belief B is proposed. In a nutshell, this relation may be described as follows. The topological structure (X, OX) defines a family of belief operators B that fit the knowledge operator defined by (X, OX) in a sense to be specified. The family of belief operators has the structure of a complete Heyting algebra $NUC(OX)_d$. Stalnaker's belief operator turns out to be just the top element of this Heyting algebra, its bottom element corresponds to the "ideal" or "optimal" belief operator K.

In other words, for a given knowledge operator K, the Heyting algebra of admissible belief operators can be conceived as a kind of (intuitionistic) logic of belief operators: Different belief operators can be compared with each other according to their strengths and how far they deviate from the knowledge operator K. In this vein, Stalnaker's belief operator B_S is only a special case among many other consistent belief operators B related to K. The unimodal systems of belief logics based on B, which can be derived from the full KB systems turn out to be KD4 logics in general.

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[2] For general topological spaces (X, OX) with a knowledge operator K (defined by the topological interior kernel operator int), a well-behaved belief operator is defined by intclint. This belief operator is the only one that is dealt with in Stalnaker (2006) and the various papers of Baltag et alii. For a very special class of topological spaces, namely, extremally disconnected spaces, the definition of this operator can be simplified to clint. As will become clear in the following sections, for general spaces, it is more natural to consider more general belief operators that have the form Bint, where B is a nucleus, i.e., an element of the complete Heyting algebra NUC(OX) of nuclei of OX.

The relation between topology and modal logic is often described as a relation between classes of modal logical systems (such as S4, S4.1, S4.2, S4.Z, etc.) and classes of topological spaces that can serve as models for these modal systems. The first example of this correspondence can be found in McKinsey and Tarski's trail-blazing paper of 1944, in which they proved such a correspondence between S4 and the class TOP of all topological spaces. More recent results in this area deal with the class WSC of weakly scattered spaces corresponding to S4.1, the class ED of extremally disconnected spaces corresponding to S4.2, the class of hereditarily extremally disconnected spaces (HED spaces) corresponding to S4.3, etc. Results of this kind have been obtained by Bezhanishvili and others (cf. (Bezhanishvili et al. (2004), (2015) Aiello et al. (2007)).

Usually, the relation between belief and knowledge is conceptualized in a rather direct way: either knowledge is defined as a special kind of belief (e.g., knowledge is "justified" true belief, or "correctly justified" true belief, or the like, as in many received accounts of knowledge), or knowledge—having conceptual priority—defines belief in a unique way. An example of the latte approach is Stalnaker's, who defines belief as the epistemic possibility of knowledge (cf. Stalnaker (2006), Baltag et al. (2017)). In a sense, this paper follows the knowledge-first approach but with a special twist. It is shown that for a given knowledge operator K, there exists a pool of different admissible belief operators B such that the pairs (K, B) all define well-behaved systems of epistemic logic satisfying the axioms of Stalnaker's KB system. Therefore, different cognitive agents who subscribe to the same knowledge operator K may use different agent-specific belief operators.

The present paper pursues the following strategy. We propose using the topological structure of the space (X, OX) (encapsulated in the Heyting algebra OX and unfolded by the Heyting algebra NUC(OX)) as a means to construct a family of appropriate belief operators. Every knowledge operator K (= int) comes with a class of appropriate belief operators B in the sense that the pairs (K, B) satisfy the axioms of Stalnaker's KB system, with the exception of the axiom of "negative introspection" (NI).

The organization of this paper is as follows: to set the stage, in the next section, we recall the axioms and rules of Stalnaker's KB logic of knowledge and belief. In section 3, we introduce some basic topological concepts that are necessary for the topological semantics of knowledge and belief. In section 4 we define the concept of a (topological) nucleus that plays a central role for the definition of belief operators. In section 5, the relation between nuclei and belief modalities is studied in detail. In section 6, we calculate the Heyting algebra of consistent belief operators for some important topological universes. We conclude with some general remarks on the further elaboration of this nucleus-based approach in section 7.

2. STALNAKER'S KB LOGIC OF KNOWLEDGE AND BELIEF

First, for the sake of definiteness, let us recall the axioms and the inference rules of Stalnaker's system (cf. Stalnaker (2006), Baltag et al. (2017, 2019):

(2.1) Definition (Stalnaker's axioms and inference rules for knowledge and belief).

(CL)	All tautologies of classical propositional logic.	
(K)	$K(\phi \rightarrow \psi) \rightarrow (K\phi \rightarrow K\psi)$	(Knowledge is additive).
(T)	$K\phi \rightarrow \phi$	(Knowledge implies truth).
(KK)	$K\phi \rightarrow KK\phi$	(Positive introspection for K).
(CB)	$B\phi \rightarrow \neg B \neg \phi$	(Consistency of belief).
(PI)	$B\phi \rightarrow KB\phi$	(Positive introspection of B).
(NI)	$\neg B\phi \rightarrow K\neg B\phi$	(Negative introspection of B).
(KB)	$K\phi \rightarrow B\phi$	(Knowledge implies belief).
(FB)	$B\phi \rightarrow BK\phi$	(Full belief).

Inference Rules:

(MP) From ϕ and $\phi \rightarrow \psi$, infer ψ. (Modus Ponens).

(NEC) From ϕ, infer $K\phi$. (Necessitation).♦

For the topological approach to knowledge and belief, the axiom (NI) plays a special role. It is easily shown that (NI) holds only for topological models of a very special kind, namely, models that are based on extremally disconnected spaces topological spaces. For the systems of knowledge and belief considered in this paper we will only require that they are weak Stalnaker systems in the following sense:

(2.2) Definition. A bimodal system based on the bimodal language L_{KB} is a weak Stalnaker system iff it satisfies all of Stalnaker's axioms and rules given in (2.1) for knowledge K and belief B but the axiom (NI) of negative introspection.♦

Now let us recall the basics of the interior semantics for modal epistemic logic as presented by Baltag, Bezhanishvili, Özgün, and Smets in various recent publications (cf. Baltag et al. (2013, 2015, 2016, 2019)). In the rest of this paper this semantics will be used throughout.

We start with a standard unimodal epistemic language L_K with a countable set PROP of propositional letters, Boolean operators $\neg$, $\wedge$, and a modal operator K to be interpreted as a knowledge operator. The formulas of L_K are defined as usual by the grammar

$$\varphi ::= p \mid \neg p \mid \phi\wedge\psi \mid K\varphi \quad , \quad p \in \text{PROP}.$$

The abbreviations for the Boolean connectives $\vee$, $\rightarrow$, and $\longleftrightarrow$ are standard. Analogously to L_K, a bimodal epistemological language L_{KB} for operators K and B is defined. For a more detailed presentation of topological semantics, the reader may consult the recent papers of Baltag et alii.

3. ON THE TOPOLOGY OF KNOWLEDGE AND BELIEF OPERATORS. Now let us recall the basics of the interior semantics for epistemic logic of knowledge and belief as presented by Baltag, Bezhanishvili, Özgün, and Smets (cf. Baltag et al. (2013, 2015, 2016, 2019)). This semantics will be used throughout the rest of this paper. First of all, recall the definition of a topological space:

(3.1) Definition. Let X be a set with power set PX. A topological space is an ordered pair (X, OX) with OX $\subseteq$ PX that satisfies the following conditions:

(i) Ø, X $\in$ OX.

(ii) OX is closed under finite set-theoretical intersections $\cap$ and arbitrary unions $\cup$.♦

The elements of OX are called the open sets of the topological space (X, OX). The set-theoretical complement **C**A of an open set A is called a closed set. The set of closed subsets of (X, OX) is denoted by CX. The interior kernel operator int and the closure operator cl of (X, OX) are defined as usual: The interior kernel int(A) of a set A $\in$ PX is the largest open set that is contained in A; the closure cl(A) of A is the smallest closed set containing A. For details, see Willard (2004), Steen and Seebach Jr. (1982), or any other textbook on set-theoretical topology). The operators int and cl are well-known to satisfy the Kuratowski axioms:

(3.2) Proposition (Kuratowski Axioms). Let (X, OX) be a topological space, A, B $\in$PX. The interior kernel operator int and the closure operator of (X, OX) satisfy the following (in)equalities

(i)	int(A $\cap$ B) = int(A) $\cap$ int(B).	cl(A $\cup$ B) = cl(A) $\cup$ cl(B).
(ii)	int(int(A)) = int(A).	cl(cl(A)) = cl(A).
(iii)	int(A) $\subseteq$ A.	A $\subseteq$ cl(A).
(iv)	int(X) = X.	Ø = cl(Ø).♦

In the following these axioms are used in the following without explicit mention. Moreover, we will use freely the fact that the operators int and cl are inter-definable:

$$\mathrm{int}(A) = \complement \mathrm{cl}(\complement A) \qquad \text{and} \qquad \mathrm{cl}(A) = \complement \mathrm{int}(\complement A)$$

Further, it should be pointed out that often it is expedient to conceive the operators int and cl as operators $PX \xrightarrow{\mathrm{int}} PX$ and $PX \xrightarrow{\mathrm{cl}} PX$ defined on PX. Hence, the concatenation of these operators makes perfect sense. In the following, concatenations such as intcl, intclint will play an important role.

The concept of a topological space (X, OX) is extremally general. For most applications it is expedient or even necessary to require that the topology must satisfy axioms in addition to those generally required of topological spaces. One such collection of conditions is given by means of axioms called separations axioms. Some very few axioms of this kind that are important for the purposes of the present paper are the following ones:

(3.3) Definition (Separation Axioms for Topological Spaces). Let (X, OX) be a topological space, a, b $\in$ X, and A, B $\in$ OX.

(i) X is a T_0-space iff there exists an A such that either a $\in$ A and b $\notin$ A, or b $\in$ A and a $\notin$ A.

(ii) X is a T_D-space iff every a $\in$ X has an open neighborhood A such that A–{a} is also open.

(iii) X is a T_1-space iff there exist A and B containing a and b respectively, such that b $\notin$ A, and a $\notin$ A.

(iv) X is a T_2-space iff there exist disjoint A and B containing a and b, respectively. ♦

Each of these axioms is independent of the axioms (3.1) of a topological space. In fact, there exist topological spaces which do not satisfy any of these separation axioms. More precisely, the following chain of implications hold:

(3.4) Proposition. The separation axioms T_i defined in (3.3)(i) – (iv) satisfy the following chain of implications:

$$T_2 \Rightarrow T_1 \Rightarrow T_D \Rightarrow T_0.$$

All implications of this chain are proper, i.e., they cannot be reversed. The axioms T_2, T_1, and T_0 may be called classical. They are discussed (usually together with many other separation axioms) in full detail in all standard textbooks of topology (see also the particularly useful presentation in Steen and Seebach Jr. (1978)). The axiom T_D is rather new. It was first proposed in the 1960s by several authors for a variety of reasons. For a modern presentation of T_D see Picado and Pultr (2012, I.2). The axiom T_D will be especially useful for the calculation of the family NUC(OX) of belief operators.

Dense subsets will be essential for the definition of consistent belief operators B:

(3.5) Definition. Let (X, OX) be a topological space with interior operator int and closure operator cl, let Y, Z $\in$ PX.

(i) Y is a dense subset of X iff cl(Y) = X.

(ii) Z is a nowhere dense in X iff int(cl(Z)) = Ø.♦

(3.6) Examples of dense and nowhere sets of topological spaces (X, OX).

$(i)_1$ For the trivial coarse topology (X, {Ø, X}) every non-empty subset A $\in$ PX is dense and only Ø is nowhere dense; $(i)_2$ For the discrete topology (X, PX) only X is dense, and only Ø is nowhere dense.

(ii) Let (**R**, O**R**) be the real line endowed with the familiar Euclidean topology. Let F $\subseteq$ **R** be a finite set. Then F is nowhere dense and the complement **C**F of F is a dense open subset of (**R**, O**R**). Analogously, the infinite set of integer **Z** is a nowhere dense subset of (**R**, O**R**).

(iii) The set $\mathbf{Q}$ of rational numbers and the set $\mathbf{CQ}$ of irrational numbers are disjoint dense subsets of ($\mathbf{R}$, O$\mathbf{R}$), i.e., $\mathbf{Q} \cap \mathbf{CQ} = \emptyset$ and cl($\mathbf{Q}$) = cl($\mathbf{CQ}$) = $\mathbf{R}$.

(iv) Let $\mathbf{Q} = \{q_1, q_2, \ldots\}$ be a linear ordering of $\mathbf{Q}$ and $U(q_i)$ the open neighborhood of q_i of diameter $1/2^{2i}$. Then the union U of the neighborhoods $U(q_i)$ is a proper dense open subset of ($\mathbf{R}$, O$\mathbf{R}$). Let λ be the Lebesgue measure of $\mathbf{R}$. As is well known, the set U is λ-measurable and has Lebesgue-measure $¼ < \lambda(U) < \sum 1/4^i = 1/(4-1) = 1/3$. Hence U is much smaller than $\mathbf{R}$. The complement $\mathbf{C}$U is closed and nowhere dense.

(v) A sophisticated example of a nowhere dense set is given by the Cantor set C of the real line ($\mathbf{R}$, O$\mathbf{R}$) defined as follows: From the unit interval [0,1] of $\mathbf{R}$ remove the middle open interval (1/3, 2/3) obtaining the union of the closed interval [0, 1/3] and [2/3, 1]. This set is denoted by C_1. From C_1 remove the open middle intervals (1/9, 2/9) and (7/9, 8/9) obtaining a set C_2 that consists of the four closed intervals [0, 1/9], [2/9, 1/3], [2/3, 7/9], and [8/9, 1]. And so on. Then the Cantor set is defined as the infinite intersection $C := \cap C_i$.

The Cantor set is nowhere dense and perfect (= closed and having no isolated points) (cf. Steen and Seebach Jr. (1978, p. 57- 58), Edgar (1990, 1.1), van Mill (2001, Example 1.5.4., p. 42ff)). Hence the complement $\mathbf{C}$C of the Cantor set C is a dense open subset of ($\mathbf{R}$, O$\mathbf{R}$). In section 6 the complement $\mathbf{C}$C of the Cantor set C will be used to define some interesting non-standard belief operators for the Euclidean space ($\mathbf{R}$, O$\mathbf{R}$). ♦

After these preparations, topological models for the modal languages L_K and L_{KB} and L_B can be defined as follows:

(3.7) Definition. Given a topological space (X, OX), we define a topo(logical) model for L_K as M = (X, OX, v), where P—v—> PX is a valuation function from the set of propositional letters P to PX.♦

Given a topological model (X, OX, v), the interior semantics for the language L_K is defined as usual. In particular, if a formula φ of L is interpreted as v(φ) = A ∈ PX, then the formula Kϕ of L_K is interpreted as v(Kϕ):= int(A). Usually, it is not necessary to explicitly mention the interpretation v of a model (X, OX, v). Hence, we use a set-theoretical denotation and write A, K(A), or int(A) instead of v(φ), Kv(φ), for A = v(φ), etc.
Given a topological model M = (X, OX, v), the knowledge operator K of M is always interpreted as the topological interior kernel operator int of (X, OX). Belief operators B will be defined by using the topological structure (X, OX) underlying M in various ways that will be explained in detail in the next sections.

4. NUCLEI OF TOPOLOGICAL SPACES (X, OX). In this section we introduce the concept of(topological) nuclei (cf. Johnstone (1982), Borceux (1994), Picado and Pultr (2012). Nuclei are the essential for the task of defining appropriate belief operators related to a topological knowledge operator K. The concept of a (topological) nucleus is basic for the rest of this paper. The literature on nuclei in (point-free) topology has reached a high level of technical sophistication. This paper does not aim to give a full-fledged introduction into the theory of nuclei. Instead, we intend to provide the basic definitions and facts so that the reader can understand that this theory has interesting applications regarding the modal theory of belief and knowledge. For a fuller account, the reader may consult Johnstone (1982), Borceux (1994), or Picado and Pultr (2012, 2021) and the extensive bibliographies mentioned there.

(4.1) Definition. Let (X, OX) be a topological space, and let $A \in OX$. A map $OX \xrightarrow{j} OX$ is called a nucleus of (X, OX) if it satisfies the following properties:

(i) $A \subseteq j(A)$. (Inflation)

(ii) $j(j(A)) \subseteq j(A)$. (Idempotence)

(iii) $j(A \cap B) = j(A) \cap j(B)$. (Distributivity)

The set of nuclei of a topological space (X, OX) is denoted by NUC(OX).♦

The set of nuclei NUC(OX) is partially ordered in a natural way by defining $j \leq j' := j(A) \subseteq j'(A)$ for all $A \in OX$. In the following, (NUC(OX), ≤) is always assumed to be endowed with this partial order. As is easily proved:

(4.2) Proposition. The partial order (NUC(OX), ≤) is a complete lattice.

Proof. Let $B_\lambda \in$ NUC(OX) be a family of nuclei. Then it is easily proved that the operator B defined by $B(D) := \mathrm{int}(\cap B_\lambda(D))$ is still a nucleus. The operator B is the pointwise infimum of the $B_\lambda(D)$ (cf. Borceux 1994, p. 34). Thus, (NUC(OX), ≤) is a complete inf-semilattice, and hence a complete lattice. Its bottom element 0 is the identity operator id, and its top element 1 is the (trivial) nucleus that maps every $D \in OX$ onto X.♦

Actually, more is true about (NUC(OX), ≤):

(4.3) Proposition. The lattice (NUC(OX), ≤) is a complete Heyting algebra. The Heyting implication ⇒ of NUC(OX) is defined by

$$j \Rightarrow k(D) := \mathrm{INF}\{j(E) \Rightarrow k(E);\ E \geq D\} \qquad (D, E \in OX)$$

Proof. (Johnstone 1982 (II, 2.4, Lemma), Borceux 1994 (Theorem 1.5.7).♦

In the last decades, the investigation of NUC(OX) has turned out to be a fruitful pathway for studying topological problems of various kinds, particularly problems related to point-free topology (cf. Johnstone (1982), Borceux (1994), Picado and Pultr (2012)). In this paper, we conduct some modest steps to use the concept of nuclei to shed new light on the problems of modal systems that deal with the epistemological concepts of knowledge and belief. More precisely, we will deal with problems related to Stalnaker's KB logic of knowledge and belief. Before answering the specific questions regarding this issue, it is expedient to give some examples of nuclei and to elucidate the structure of NUC(OX).

<u>(4.4) Examples (Johnstone 1982, Borceux 1994, Picado and Pultr 2012)</u>. Let (X, OX) be a topological space and A, B, D $\in$ OX. Denote the join and the Heyting implication of OX by $\cup$ and $\Rightarrow$, respectively.

(i) The identity map OX——id——>OX is a nucleus.

(ii) The map k_A defined by $k_A(D) := A \cup D$ is a nucleus. This nucleus is called the <u>closed</u> nucleus defined by A.

(iii) The map j_A defined by $j_A(D) = (A \Rightarrow D)$ is a nucleus. This nucleus is called the <u>open</u> nucleus defined by A.

(iv) Let int and cl denote the interior kernel operator and the closure operator of (X, OX), respectively. The operator OX——j**——>OX defined by j**(D):= intcl((D)) is a nucleus. It is called the regular nucleus of OX and is denoted by j**.♦

<u>(4.5) Definition</u>. A nucleus j $\in$ NUC(OX) is called a dense nucleus iff $j(\emptyset) = \emptyset$. The subset of dense nuclei of NUC(OX) is denoted by $\mathrm{NUC(OX)}_d$.♦

Dense nuclei will play a central role in the following, since they define consistent belief operators. Hence, it may be expedient to provide some explicit (counter)examples of this kind of nucleus:

(4.6) Examples of dense and not dense nuclei.

(i) The identity map $OX \xrightarrow{id} OX$ is a dense nucleus.

(ii) For $A \neq \emptyset$, the closed nucleus k_A is not dense since $k_A(\emptyset) = A \cup \emptyset = A$.

(iii) Recall that $A \in OX$ is dense in the topological sense iff $A^* := (A \Rightarrow \emptyset) = \emptyset$. The open nucleus j_A defined by $j_A(D) := A \Rightarrow D$ is a dense nucleus iff A is dense, since $j_A(\emptyset) = \mathrm{int}(A \Rightarrow \emptyset) = A^* = \emptyset$.

(iv) Applying the Kuratowski axiom (3.2)(iv) yields that the regular nucleus j^{**} is a dense nucleus: $j^{**}(\emptyset) = \mathrm{intcl}(\emptyset) = \emptyset$.

(v) The constant nucleus 1 that maps each element of OX onto X is a nucleus. Clearly, 1 is not dense since $1(\emptyset) = X$.♦

Now all ingredients are available to formulate the central definition of this paper:

(4.7) Definition. Let (X, OX) be the topological space of a topological model with interior kernel operator int, and let $j \in NUC(OX)$ be a nucleus $OX \xrightarrow{j} OX$ and $OX \xrightarrow{i} PX$ the canonical inclusion. Denote the concatenation $PX \xrightarrow{int} OX \xrightarrow{j} OX \xrightarrow{i} PX$ by B_j. For $A \in PX$, B_j is an operator $PX \xrightarrow{B_j} PX$ that maps A onto $B_j(A)$. The operator B_j is called a (nuclear) belief operator (related to K and defined by the nucleus j).♦

The rest of this paper is dedicated to the task of showing that (4.7) is a reasonable and fruitful definition that defines a family of well-behaved belief operators B_j for a knowledge operator K that enjoy all properties that one intuitively expects from "good" belief operators.

The task of justifying the predicate "belief operator" for B_j is naturally divided into two subtasks: First, it has to be shown that (4.7) is formally adequate in the sense that the belief operators defined by (4.7) satisfy appropriate formal conditions of adequacy. Second, it has to be shown that sufficiently many interesting belief operators B exist that fulfil the requirements of (4.7). Of course, there is no complete agreement of what "good properties for a belief modality" are, but the following properties are probably rather uncontroversial candidates:

<u>(4.8) Adequacy conditions for belief operators</u>. A good, intuitively plausible, belief operator B related to a knowledge operator K should satisfy the following conditions:

(i) A cognitive agent τ who relies on B may have a false belief. Formally, there should exist propositions $A \subseteq X$, such that $w \in B(A)$ but $w \notin A$, i.e., τ believes that w is an A-world, but w is not actually an A-world.

(ii) A good concept of belief should be consistent, i.e., if τ believes that w is an A-world, then τ does not believe that w is not an A-world, i.e., $w \in B(A)$ entails that $w \notin B(\complement A)$.

(iii) A good concept of belief should be compatible with the related concept of knowledge, i.e., if τ knows that w is an A-world, then τ believes A, i.e., $w \in K(A)$ entails $w \in B(A)$. ♦

<u>(4.9) Proposition</u>. Let (X, OX) be a topological space, $j \in NUC(OX)$, $j \neq id$. Then the dense belief operator $B_j :=$ jint is a good belief operator in the sense of (4.8).

<u>Proof</u>. (i) In order to prove that $B_j(A)$ may possibly be false one can argue as follows: By the definition of the partial order of nuclei (3.2), the smallest nucleus of OX is the identity id. Thus, according to our assumption id < j, there must be an open A such that $A \subset j(A)$ and $A \neq j(A)$. This is equivalent to $B_j(A) \cap \complement A \neq \emptyset$. In other words, there is a world $w \in B_j(A) \cap \complement A$. This

means that the epistemic agent who uses B_j believes w to be an A-world but in actuality, w is not an A-world. In other words, the agent's belief is false.

(ii) Due to the fact that the operator B_j is dense (by definition (4.5)) and distributive with respect to $\cap$ (by Definition (4.1)(ii)) one calculates $0 = B_j(A \cap \mathbf{C}A) = B_j(A) \cap B_j(\mathbf{C}A)$ iff $B(A) \subseteq \mathbf{C}B_j(\mathbf{C}A)$.

(iii) This holds due to the axiom (KB) of (2.1).♦

More systematically, one may require that for good belief operators, the pairs (K, B) should satisfy the rules and axioms of a weak Stalnaker system in the sense of (2.2). In the next section, we will prove that this is indeed the case for nuclear belief operators defined by (4.7). For the moment, we are content to prove that the belief fragment of the KB system defines a well-behaved logic of belief:

<u>(4.10) Proposition</u>. Let (X, OX) be a topological space with an interior kernel operator int, and $E, F \in OX$. Then, a dense nuclear belief operator $B_j: = j\text{int}$, $j \in NUC(OX)$, defines a KD4 logic of belief, i.e., the following axioms and rules are satisfied:

(i)	(K)	$B_j(E \to F) \to (B_j(E) \to B_j(F))$.	(Normality)
(ii)	(D)	$B_j(E) \to \mathbf{C}B_j(\mathbf{C}E)$.	(Consistency)
(iii)	(4)	$B_j(E) \to B_j(B_j(E))$.	(Idempotence)
(iv)	Inference Rules:		
(MP)	From E and $E \to F$, infer F.		(Modus ponens)
(NEC)	From E, infer $B_j(E)$.		(Necessitation)

<u>Proof.</u> (i) By definition (4.1), a nucleus j is distributive with respect to the intersection $\cap$. By Kuratowski (3.2)(i) the interior operator int is also distributive with respect to intersection $\cap$ as well. Hence the belief operator B_j as the concatenation of j and int is distributive as well.

Therefore, one has $B_j(E) \cap B_j(E \to F) = B_j(E \cap (E \to F)) = B_j(E \cap F) \subseteq B_j(F)$. Hence, we obtain $B_j(E) \cap B_j(E \to F) \subseteq B_j(F)$. This is set-theoretically equivalent to $B_j(E \to F) \subseteq B_j(E) \to B_j(F)$.

(ii) Using $\cap$–distributivity one calculates $\emptyset = B_j(E \cap \mathbf{C}E) = B_j(E) \cap B_j(\mathbf{C}E)$ iff $B(E) \subseteq \mathbf{C}B_j(\mathbf{C}E)$. In other words, axiom (D) holds for belief operator B_j.

(iii) To prove axiom 4, one observes that the belief operator B_j is idempotent: B_jB_j = jintjint = jjint = jint = B_j.

(iv) For (MP) there is nothing to show. The rule (NEC) of necessitation for B_j is valid since due to int(X) = X the rule of necessitation for K is valid. A fortiori one may infer from E that $B_j(E)$ holds, since $K(E) \subseteq B_j(E)$. Hence, the operators B_j define KD4 logics.♦

5. NUCLEAR BELIEF OPERATORS AND THE LOGIC OF KNOWLEDGE AND BELIEF.

After these preparations, we can state a main theorem of this paper. It describes, so to speak, the logic of belief operators B_j related to a knowledge topological operator K as an intuitionistic logic defined by the complete Heyting algebra $NUC(OX)_d$ of dense nuclei of (X, OX):

(5.1) Theorem. Let (X, OX) be a topological space, and let $j \in NUC(OX)_d$ be a dense nucleus. For the belief operator B_j := jint, the pairs (K, B_j) of modal operators K and B_j satisfy all axioms of a weak Stalnaker system, i.e., the system (K, B_j) satisfies all axioms and rules of (2.1) but the axiom of strong negative introspection (NI).

Proof. The axioms (K), (T), and (KK) involve only the interior kernel operator int and are classically known to hold for topological models of knowledge (cf. McKinsey and Tarski (1944)). Hence, it is sufficient to consider only axioms (CB), (PI), (KB) and (FB) that involve K and B_j.

(CB): Consistency of belief: Axiom (CB) is just axiom (D), which has already been proven in (3.7)(ii).

(PI): Positive introspection of B_j: The set $B_j(D)$ is, by the definition of B_j, open since j is a nucleus. Hence, $B_j(D) \subseteq int(B_j(D))$, i.e., (PI) is satisfied.

(KB): Knowledge implies belief: Clearly, $int(D) \subseteq jint(D) = B_j(D)$, since j as a nucleus is inflationary, i.e., id $\leq$ j.

(FB): Full belief: This holds by the definition of B_j and the (KK) principle of K.

In order to show that jint in general does not satisfy (NI) consider the following elementary example: Take (X, OX) = ($\mathbf{R}$, O$\mathbf{R}$). As is easily seen, for K = int and B = j**, the axiom (NI) of negative introspection is equivalent to $cl^{**}(D) \subseteq j^{**}int(D)$ for all $D \in P\mathbf{R}$. Clearly, for the open intervals $D = (a, b) \subset \mathbf{R}$ this is false, since $cl^{**}(D) = [a, b]$. Hence (NI) is not generally valid for the pair (K, B) = (int, j**int).[3]

This finishes the proof that for dense nuclei $j \in NUC(OX)$ the belief operators B_j:= jint satisy all axioms and rules of a weak Stalnaker KB system.♦[4]

To show that (5.1) is interesting for the logic of knowledge and belief, one must show that topological models (X, OX) do have a sufficient supply of interesting dense nuclei beyond j**. This is ensured by the following two propositions:

[3]Later we will show that also other pairs (K, B) of operators do no satisfy (NI).

[4] The soundness of KB being obvious, with respect to the completeness of KB one observes that the axioms of KB, as given in (2.1), are all Sahlquist formulas: K satisfies the axioms of KT4, B satisfies the axioms of KD4, and the connecting axioms $KB\varphi \longrightarrow B\varphi$, $B\varphi \longrightarrow KB\varphi$, and $K\varphi \longrightarrow B\varphi$ are clearly Sahlquist formulas (cf. Blackburn et al. (2010, Definition 3.51)). Hence, by the Sahlquist Completeness Theorem for (multi)modal languages one obtains that the logic KB is strongly complete with respect to the class of frames F_{KB} of KB (cf. Blackburn et al. (2010, Theorem 4.42)).

<u>(5.2) Proposition</u>. For (X, OX) let $Y \subseteq X$ be a subset of X. For $D \in OX$ the operator j_Y defined by $j_Y(D) := int(Y \rightarrow D)$ is a nucleus of OX. It is called a spatial nucleus defined by the subset Y of X.

<u>Proof</u>. One has to show that j_Y satisfies requirements (4.1)(i) – (iii):

(i): Clearly, $D \subseteq int(Y \rightarrow D) = int(\mathbf{C}Y \cup D)$. Hence, $D \subseteq j_Y(D)$.

(ii): By the Kuratowki axiom (3.2) (i) one obtains for the operator j_Y:

$j_Y(D \cap D') = int(\mathbf{C}Y \cup (D \cap D')) = int(\mathbf{C}Y \cup D) \cap int(\mathbf{C}Y \cup D') = j_Y(D) \cap j_Y(D')$.

(iii): Clearly, $j_Y(D) = int(\mathbf{C}Y \rightarrow D) \subseteq int(\mathbf{C}Y \rightarrow int(\mathbf{C}Y \rightarrow D))$. On the other hand,

$int(\mathbf{C}Y \rightarrow int(\mathbf{C}Y \rightarrow D)) \subseteq int(\mathbf{C}Y \rightarrow (\mathbf{C}Y \rightarrow D)) = int(\mathbf{C}Y \rightarrow D)$. Hence, $j_Y(j_Y(D)) = j_Y(D)$.

Thus, for all subsets Y of X the map OX—j_Y—>OX is a nucleus.♦

<u>(5.3) Corollary</u>. If Y is a dense subset of (X, OX), then the nucleus j_Y defined by (5.2) is a dense nucleus, i.e., $j_Y(\emptyset) = \emptyset$. For the belief $j_Y$int the pair (int, $j_Y$int) defines a weak Stalnaker system.

<u>Proof</u>. Let Y be a dense subset, i.e., cl(Y) = X. Then one calculates

$$j_Y(\emptyset) = int(\mathbf{C}Y \cup \emptyset) = int(\mathbf{C}Y) = \mathbf{C}cl\mathbf{C}\mathbf{C}(Y) = \mathbf{C}cl(Y) = \mathbf{C}X = \emptyset.♦$$

Following David Lewis according to whom the subsets of a set are its parts (cf. Lewis 1991), one may characterize the account of the present paper as a mereological epistemology, since the belief operators B_j are <u>mereologically</u> defined in the sense that there are determined by the parts Y of X (and the topology of (X, OX)). It is important to note that the characterization of the topological epistemology as mereological is incomplete, however. In the next section we will show that for many spaces (X, OX) there are important (dense) nuclei that cannot be characterized mereologically, i.e., they cannot be characterized with the help of subsets aka parts Y of X. Thus, the topological logic of belief operators is more complex than classical Boolean mereology.

In order to keep things as simple as possible, from now on it will be assumed that all topological spaces (X, OX) are T_D-spaces (cf. (3.4)). T_D is a rather weak separation axiom satisfied by most topological spaces that "occur in nature". For instance, Euclidean spaces and, more generally, all Hausdorff spaces and all T_0 -Alexandroff spaces are T_D-spaces. Thus, the restriction to T_D-spaces is a rather mild restriction.

(5.4) PROPOSITION. If (X, OX) is a T_D-space the map $PX \xrightarrow{\quad j \quad} NUC(OX)$ defined in (5.2) by $j_Y(D) := int(Y \rightarrow D)$ is a monomorphism.

Proof. Suppose that Y and Y' are two distinct subsets of X and $j_Y = j_{Y'}$. Suppose $x \in Y - Y'$. Since (X, OX) is a T_D-space x has an open neighborhood D such that $D - \{x\}$ is open as well. Then we obtain $x \in int(Y \cup (D - \{x\})$ but clearly $x \notin int(Y' - (D - \{x\})$. This is a contradiction. Analogously, the assumption that there is an $x \in Y' - Y$ leads to a contradiction. Hence Y = Y', i.e., j is a monomorphism.♦

It is well known that the requirement that (X, OX) is a T_D -space is necessary to guarantee that j is a monomorphism. (5.4) can be strengthened as follows:

(5.5) PROPOSITION. Let (X, OX) be a T_D-space. Denote the Boolean algebra of regular elements of NUC(OX) by NUC(OX)*.[5] Then the map $P \xrightarrow{\ j\ } NUC(OX)^*$ defined by $j(Y)(E) := int(Y \rightarrow E)$ ($= j_Y(E)$) is a Boolean isomorphism.

Proof. See Mcnab (1981, Theorem 6.5).♦

[5] Recall that for a Heyting algebra H with Heyting implication $\Rightarrow$ the regular elements are defined as elements b for which $b = b^{**} = (b \Rightarrow 0) \Rightarrow 0$.

As is well-known, for topological spaces (X, OX) the intersection of dense subspaces Y and Y' in general is not dense. A classic example is the real line (**R**, O**R**) for which the sets of rational numbers **Q** and of irrational numbers **CQ** are both dense, but the intersection **Q** $\cap$ **CQ** is clearly not dense. Hence, in general a topological space (X, OX) has no smallest dense subset Y. This entails that a topological space in general has no greatest dense nucleus j_Y such that $j \leq j_Y$ for all dense nuclei $j \in$ NUC(OX). In sharp contrast, for dense nuclei $j \in$ NUC(OX) the situation is quite different. A famous theorem of Isbell asserts that every topological space (X, OX) has a smallest dense nucleus:

(5.6) Theorem (Isbell's Density Theorem). Let (X, OX) be any topological space. Then, the dense nucleus j^{**} is the greatest element of NUC(OX)_d, i.e., for all dense nuclei j one has $j \leq j^{**}$, i.e., $j(D) \subseteq j^{**}(D)$, $D \in OX$.

Proof. (Johnstone (1982, II. 2.4 Lemma, p. 50/51), Picado and Pultr (2012, III, 8.3., p.40, also VI, 2.1, p. 101ff.) ♦

This is a very remarkable theorem, since it demonstrates that the dense nuclei of a topological space (X, OX) behave quite differently than the dense subspaces Y of X. More precisely, a space has more dense nuclei j than those that are defined by dense subspaces. A pertinent example is the Euclidean line (**R**, O**R**) and its disjoint dense subsets **Q** and **CQ**. As has been pointed out by Johnstone and others, this difference may be considered as one of the great advantages of doing topology in the conceptual framework of "pointfree topology" based on "(sub)locales", "nuclei", and their relatives instead of traditional set-theoretical topology (cf. Johnstone 1991, p. 87-88). This paper is not the place to discuss this issue in any further depth. Just let us note the following elementary corollary of Isbell's density theorem:

(5.7) Corollary. The family of dense nuclei $NUC(OX)_d$ is a complete Heyting algebra.

Proof. By definition, the set of dense nuclei $NUC(OX)_d$ is a subset of the complete Heyting algebra NUC(OX) of nuclei. By (5.6) the largest element of $NUC(OX)_d$ is the regular nucleus intcl. Hence, $NUC(OX)_d$ is the downset $\downarrow j^{**}$ of nuclei j, i.e., $\downarrow j^{**} := \{j;\ j \leq j^{**}\}$. Thereby $NUC(OX)_d$ inherits canonically the structure of a complete Heyting algebra from NUC(OX). ♦

The corollary (5.7) may be considered as a neat characterization of the intuitionist logic of belief operators B related to a given knowledge operator K.

6. AN ASSORTED CHOICE OF NUCLEI FOR EUCLIDEAN AND OTHER SPACES. In this section we will calculate some concrete nuclei of a variety of concrete topological spaces. For this purpose, it is useful to have available several equivalent ways of how nuclei can be characterized. All these equivalences are well known (cf. Johnstone 1982, Borceux 1994, Picado and Pultr 2012).

Let us start with the characterization of a nucleus j as a map $OX \xrightarrow{j} OX$ that is inflationary, idempotent, and distributive with respect to intersection (cf. (4.1)(i) – (iii))). First of all, let us note that due to (4.1) a nucleus j can be equivalently characterized by its image $j(OX) := \{j(a);\ a \in OX\}$. As is well known there is a canonical monomorphism $j(OX) \xrightarrow{i} OX$ (the "adjoint" of j) for the epimorphism $OX \xrightarrow{j'} j(OX)$ such that $ij' = id_{j(OX)}$, since j' is a frame map. Moreover, since j is idempotent the image j(OX) is just the fixed point set of j, i.e. $F_j := \{a;\ a = j(a) \text{ and } a \in OX\}$ of j (cf. Johnstone (2002, C1.1 481ff).). It is important to note that the join of two nuclei $j \vee l$ corresponds to the intersection of their images $j(OX) \cap l(OX)$ (or their fixsets $F_j \cap F_l$).

Now we will show that $j_{\mathbf{Q}}$ and $j_{\mathbf{CQ}}$ (and therefore $j_{\mathbf{Q}} \vee j_{\mathbf{CQ}}$) have fixed points that are not fixed points of j^{**}. Thus, $j_{\mathbf{Q}} \vee j_{\mathbf{CQ}} \neq j^{**}$.

Using different but equivalent characterizations of nuclei facilitates the calculation for Euclidean spaces and other spaces. In the first part of this section, we will deal with the Euclidean line and similar spaces. In the second part, which may be considered as complementary to the first one, we investigate a quite different type of topological spaces that exhibit a quite different behavior with respect to issues related to nuclei.

First, let us observe that the real line ($\mathbf{R}$, O$\mathbf{R}$) and similar spaces possess many dense nuclei. Since ($\mathbf{R}$, O$\mathbf{R}$) is a T_1-space, for all $x \in \mathbf{R}$ the singletons $\{x\}$ are closed and the complements $\mathbf{R} - \{x\}$ are dense and open. By (5.5) they give rise to different consistent belief operators B_x defined by $B_x(D) = j_{X-\{x\}}(D) = int(\{x\} \cup D)$.

Admittedly, the open subsets $X - \{x\}$ are not very interesting dense subspaces of T_1 -topo–logical spaces. They are very large and therefore almost trivial. At least they show that a large class of familiar topological spaces has plenty of different dense nuclei that may be used to define consistent belief operators.

For more specific examples of topological spaces (X, OX) one obtains more interesting examples of dense subsets (and thereby of consistent belief operators). These examples indicate that in general the structure of NUC(OX) seems to be difficult to calculate. Take, for instance, the Euclidean line ($\mathbf{R}$, O$\mathbf{R}$). For this space the set of rational numbers $\mathbf{Q}$ and the set of irrational numbers $\mathbf{CQ}$ are well known to be two dense, disjoint complementary subsets, i.e., $\mathbf{Q} \cap \mathbf{CQ}$

$= \emptyset$, $\mathbf{Q} \cup \mathbf{CQ} = \mathbf{R}$, and $cl(\mathbf{Q}) = cl(\mathbf{CQ}) = \mathbf{R}$. By (5.6) one obtains for the dense nuclei $j_{\mathbf{Q}}$ and $j_{\mathbf{CQ}}$ that $j_{\mathbf{Q}} \wedge j_{\mathbf{CQ}} = 0$ and $j_{\mathbf{Q}} \vee^* j_{\mathbf{CQ}} = 1$.[6]

Isbell's density theorem (5.6), however, yields a result that clearly shows that subspaces and nuclei of topological spaces may show a quite different behavior: Since both $j_{\mathbf{Q}}$ and $j_{\mathbf{CQ}}$ are dense, by Isbell's density theorem one has that $j_{\mathbf{Q}} \leq j^{**}$ and $j_{\mathbf{CQ}} \leq j^{**}$. Therefore, the join $j_{\mathbf{Q}} \vee j_{\mathbf{CQ}} \leq j^{**}$. On the other hand, one has $1 = j_{\mathbf{Q}} \vee^* j_{\mathbf{CQ}}$. That is, the join $j_{\mathbf{Q}} \vee j_{\mathbf{CQ}}$ of $j_{\mathbf{Q}}$ and $j_{\mathbf{CQ}}$ in $NUC(O\mathbf{R})_d$ is non-trivial and differs considerably from the trivial join $j_{\mathbf{Q}} \vee^* j_{\mathbf{CQ}} = j_{\mathbf{Q} \cap \mathbf{CQ}} = 1$ in $NUC(O\mathbf{R})^*$.

In the following it will be shown that it is well worth the effort to investigate the nuclei like $j_{\mathbf{Q}}$, $j_{\mathbf{CQ}}$, and j^{**} a bit further. First of all, let us show that the nucleus j^{**} for $(\mathbf{R}, O\mathbf{R})$ "has no points" in the following sense:

<u>(6.1) Proposition</u>. For the Euclidean line $(\mathbf{R}, O\mathbf{R})$ there is no subset Y of $\mathbf{R}$ such that $j^{**} = j_Y$.

<u>Proof</u>. Assume the contrary, i.e., there is a $Y \subseteq \mathbf{R}$ such that $j_Y(D) = int(\mathbf{C}Y \cup D) = intcl(D)$ for all $D \in O\mathbf{R}$. Clearly, $Y \neq \emptyset$. Assume the contrary, i.e., $Y = \emptyset$, and consider $D = (0,1) \in O\mathbf{R}$. Then one calculates $j_Y((0,1)) = int(\mathbf{C}\emptyset \cup (0,1)) = int(\mathbf{R}) = \mathbf{R}$, but $j^{**}((0,1)) = (0, 1)$. This is a contradiction. Hence, we may assume that $Y \neq \emptyset$. Assume $x \in Y$. Take $D = \mathbf{R} - \{x\}$. The set D is open in $(\mathbf{R}, O\mathbf{R})$ since $(\mathbf{R}, O\mathbf{R})$ is a T_2-space (cf. (3.4) Proposition). Then, we get $j_Y(D)$

[6] It should be observed that $\vee^*$ is the join of the Boolean algebra $NUC(O\mathbf{R})^*$ and <u>not</u> the join $\vee$ of the Heyting algebra NUC(OR)). Further, it should be noted that $(\mathbf{Q}, \mathbf{CQ})$ is not the only pair of disjoint dense ("equidense") subsets of $(\mathbf{R}, O\mathbf{R})$. Let $s > 0$ be a real number such that $\sqrt{s}$ is irrational $(\sqrt{s}\mathbf{Q}, \mathbf{C}\sqrt{s}\mathbf{Q})$ are other pairs of this kind.

$= \mathrm{int}(\mathbf{C}Y \cup D) = \mathrm{int}(D) = D$, but $j^{**}(D) = \mathbf{R}$. Hence, the regular nucleus j^{**} is different from any spatial nucleus of the type j_Y, Y being a subset of $\mathbf{R}$.♦

The "non-spatiality" of j^{**}, i.e., the fact that there is no subspace Y of $\mathbf{R}$ such that $j^{**} = j_Y$ is not thus exceptional as it may look like. There are other non-spatial operators as well. For instance, one obtains:

<u>(6.2) Proposition</u>. The join $j_{\mathbf{Q}} \vee j_{\mathbf{CQ}}$ of the spatial operators $j_{\mathbf{Q}}$ and $j_{\mathbf{CQ}}$ is non-spatial, i.e., it has no points.

<u>Proof</u>. A point $x \in \mathbf{R}$ is either rational or irrational, i.e., $x \in \mathbf{Q}$ or $x \in \mathbf{CQ}$. Define $D_x := \mathbf{R} - \{x\}$ and suppose $x \in \mathbf{Q}$. One obtains $j_{\mathbf{Q}}(D_x) = \mathrm{int}(\mathbf{CQ} \cup D_x) = \mathrm{int}(\mathbf{R} - \{x\}) = \mathbf{R} - \{x\}$. On the other hand, for $j_{\mathbf{CQ}}$ one calculates $j_{\mathbf{CQ}}(D_x) = \mathrm{int}(\mathbf{CCQ} \cup D_x) = (\mathrm{int}(\mathbf{R}) = \mathbf{R}$. Hence, for no $x \in \mathbf{R}$, D_x is invariant under $j_{\mathbf{CQ}}$ <u>and</u> $j_{\mathbf{CQ}}$, i.e., D_x is not invariant under $j_{\mathbf{Q}} \vee j_{\mathbf{CQ}}$. Analogously, one obtains that for $x \in \mathbf{CQ}$ the open sets $D_x := \mathbf{R} - \{x\}$ are not invariant under $j_{\mathbf{Q}} \vee j_{\mathbf{CQ}}$. In other words, the intersection of the fixed sets $F_{\mathbf{Q}}$ and $F_{\mathbf{CQ}}$ that characterizes $j_{\mathbf{Q}} \vee j_{\mathbf{CQ}}$ does not contain any point. Therefore $j_{\mathbf{Q}} \vee j_{\mathbf{CQ}}$ is (like j^{**}) a dense nucleus that is not induced by a (dense) subspace Y of $\mathbf{R}$ since it has no points.♦

According to Isbell's theorem j^{**} is the greatest dense nucleus. Hence, it is natural to ask whether $j_{\mathbf{Q}} \vee j_{\mathbf{CQ}} = j^{**}$ or $j_{\mathbf{Q}} \vee j_{\mathbf{CQ}} < j^{**}$. If $j_{\mathbf{Q}} \vee j_{\mathbf{CQ}} = j^{**}$ obtained, this would be equivalent to the fact that all $D \in O\mathbf{R}$ that are invariant under $j_{\mathbf{Q}}$ <u>and</u> $j_{\mathbf{CQ}}$, are invariant under j^{**} as well. The following example shows that this is not the case. Consequently, $j_{\mathbf{Q}} \vee j_{\mathbf{CQ}} < j^{**}$. More precisely, we show that the complement $D = \mathbf{C}C$ of the Cantor dust C is a non-regular open subset of $\mathbf{R}$

that is invariant under the nuclei $j_{\mathbf{Q}}$ and $j_{\mathbf{CQ}}$, but not invariant under j**. This entails that $j_{\mathbf{Q}} \vee j_{\mathbf{CQ}}$ is strictly smaller than j**.

In order to show that **C**C is invariant under $j_{\mathbf{Q}}$ <u>and</u> $j_{\mathbf{CQ}}$, the following topological lemma is needed:

<u>(6.3) Lemma</u>. Let C be the Cantor dust as defined in (3.6)(v). Then $C \cap \mathbf{Q}$ and $C \cap \mathbf{CQ}$ are dense in C i.e., for all $p \in C$. all open neighborhoods U(p) of p contain elements $q \in C \cap \mathbf{Q}$ different from p, and all open neighborhoods U(p) contain elements $r \in U(p) \cap \mathbf{CQ}$ different from p.

<u>Proof</u>. Assume $s \in C$. As is well known (cf. Steen and Seebach Jr. (1980, p. 57, Edgar (1990, Proposition 1.1.5, p. 4), s can be uniquely expressed to the base 3 without using the digit 1. If this representation is finite, then s is certainly rational, if the ternary representation is not finite, s may be irrational or rational. Now let $p \in C$ be represented by an infinite series $\sum a_k \, 1/3^k$. Then clearly p can be approximated arbitrarily closely by a finite sum $\sum^n a_k \, 1/3^k$ that is rational. Thus $C \cap \mathbf{Q}$ is dense in C. On the other hand, if p is any element of C it can be approximated by a finite rational p'. This p' can be approximated arbitrarily by an irrational $q \in C$, since C is a complete metrical space. Hence any neighborhood U(p') must contain irrational numbers q since C is uncountable. ♦

Now we can show that the regular nucleus j** and the join $j_{\mathbf{Q}} \vee j_{\mathbf{CQ}}$ are different by exhibiting elements of O**R** that are invariant under $j_{\mathbf{Q}} \vee j_{\mathbf{CQ}}$ but are not invariant under the regular nucleus j**:

(6.4) Proposition. Let ($\mathbf{R}$, O$\mathbf{R}$) be the Euclidean line and $\mathbf{C}$C the complement of the Cantor dust C as defined in (3.6)(v). Then $\mathbf{C}$C is invariant under $j_{\mathbf{Q}}$ and $j_{\mathbf{CQ}}$, i.e., $j_{\mathbf{Q}}$ ($\mathbf{C}$C) = $j_{\mathbf{CQ}}$($\mathbf{C}$C) = $\mathbf{C}$C, but $\mathbf{C}$C is not invariant under j**, since j**($\mathbf{C}$C) = $\mathbf{R}$.

Proof. First, let us show that C is not invariant under j**. As is well known, the Cantor dust C is nowhere dense and closed in $\mathbf{R}$. Hence, $\mathbf{C}$C is open and one calculates

$$j^{**}(\mathbf{C}C) = \mathrm{int}(\mathrm{cl}(\mathbf{C}C)) = \mathrm{int}\mathbf{C}\mathrm{int}\mathbf{C}(\mathbf{C}C)) = \mathrm{int}(\mathbf{C}\mathrm{int}(C)) = \mathrm{int}(\mathbf{C}(\emptyset)) = \mathrm{int}(\mathbf{R}) = \mathbf{R}.$$

Hence, $\mathbf{C}$C is not invariant under j**. In order to prove that $\mathbf{C}$C is invariant under $j_{\mathbf{Q}}$ and $j_{\mathbf{CQ}}$ one proceeds as follows. First, consider $j_{\mathbf{CQ}}$(C) = int($\mathbf{Q} \cup \mathbf{C}$C). Assume x $\in$ (($\mathbf{Q} \cup \mathbf{C}$C) $\cap$ $\mathbf{CC}$C) = ($\mathbf{Q} \cap$ C). We show that x $\notin$ int($\mathbf{Q} \cup \mathbf{C}$C). Suppose x $\in$ int($\mathbf{Q} \cup \mathbf{C}$C) and x $\in$ $\mathbf{Q}$. Thus, there exist an open neighborhood U(x) $\subseteq$ $\mathbf{Q} \cup \mathbf{C}$C. By Lemma (6.3) there is a y in U(x) such that y $\in$ $\mathbf{CQ}$ and y $\in$ C. This is a contradiction. Hence x $\notin$ int($\mathbf{Q} \cup \mathbf{C}$C). In other words, $j_{\mathbf{CQ}}$($\mathbf{C}$C) = int($\mathbf{Q} \cup \mathbf{C}$C) = $\mathbf{C}$C, i.e., $\mathbf{C}$C is invariant under $j_{\mathbf{CQ}}$. Now consider $j_{\mathbf{Q}}$(C) = int($\mathbf{CQ}$ $\cup$ $\mathbf{C}$C). Analogously, by (6.3) one shows that $\mathbf{C}$C is invariant under $j_{\mathbf{Q}}$ as well. Hence, $\mathbf{C}$C is invariant under under $j_{\mathbf{Q}} \vee j_{\mathbf{CQ}}$. Thus, the nuclei j** and $j_{\mathbf{Q}} \vee j_{\mathbf{CQ}}$ are different. More precisely, due to Isbell's theorem one has $j_{\mathbf{Q}} \vee j_{\mathbf{CQ}} < j^{**}$.♦

These calculations carried out so far should be sufficient to convince the reader that the structure of NUC(OX)$_d$, i.e., the structure of the family of epistemological logics (K, B) definable with the aid of the topology of spaces such as the Euclidean line ($\mathbf{R}$, O$\mathbf{R}$) (and similar spaces), is far from trivial.

One reason for the complexity of NUC(O$\mathbf{R}$) (and NUC(O$\mathbf{R}$)$_d$) seems to be that these spaces are <u>resolvable</u> in the sense that they possess disjoint dense subsets A and B that are complements of each other, i.e., $A \cap B = \emptyset$, $A \cup B = X$ and $cl(A) = cl(B) = X$.[7]

More directly this vague conjecture is confirmed by the examples to be discussed in the rest of this section, namely by spaces that are irresolvable in the sense that they do not possess disjoint dense subsets. At least for some of these spaces (X, OX) the Heyting algebras NUC(OX) seem to be more easily calculable than is the case for apparently elementary spaces such as ($\mathbf{R}$, O$\mathbf{R}$).

The task of finding topological spaces (X, OX) whose Heyting algebra NUC(OX) (and NUC(OX)$_d$) can be calculated more easily, has been treated in the literature (cf. Simmons (1980), Ávila, Bezhanishvili, Morandi, and Zaldívar (2021)). For the philosophical intentions of this paper, it is expedient to deal with spaces that are philosophically or logically relevant that fulfil this requirement. Fortunately, the recent literature contains some appropriate examples, for instance, Rumfitt's polar spaces (cf. Rumfitt 2015, Bobzien 2015)[8]. The following examples indeed confirm the conjecture that the Heyting algebras NUC(OX) of nuclei of irresolvable spaces (X, OX) are more easily calculable. A very simple case is provided by polar spaces introduced by Rumfitt to deal with the Sorites paradox in the framework of classical Boolean logic (cf. Rumfitt 2015).[9]

[7] For some recent results that point in this direction, see Baboolal, Picado, Pillay, Pultr (2019).

[8] Polar spaces were introduced by Rumfitt to deal with the logic of vague concepts (cf. Rumfitt (2015)). In topology, polar spaces and related classes of spaces have been discussed for some time (although not under this name, of course) (cf. van Benthem, Bezhanishvili (2007), Bezhanishvili, Mines, and Morandi (2003), Bezhanishvili, Esakia, and Gabelaia (2004), and Gabelaia (2001)). The main example of a polar space discussed by Rumfitt (2015) is the so-called color circle: Given a set X of color experiences, a subset P of X is selected the elements of which are to be interpreted as prototypical or paradigmatic elements of X. For instance, the elements of P are to be taken as a "typically" blue object or a "typically" red object etc. The elements of P are called poles of the color space X. The selection of poles is assumed to define a pole distribution in the sense of (5.8).

[9] Rumfitt's polar spaces have been well known in topology. They may be characterized as submaximal Alexandroff spaces (cf. Bezhanishvili, Esakia, and Gabelaia (2004), Mormann (2021)).

(6.5) Definition. Let X be a set and $P \subseteq X$ be a (non-empty) set of distinguished elements to be interpreted as prototypes, paradigmatic cases, or poles. Assume that for all $x \in X$ there is a non-empty set $m(x) \subseteq P$ of poles p. For all $x \in X$ and all $p \in P$ the sets m(x) are assumed to satisfy two requirements: (i) $Ø \neq m(x) \subseteq P$, and (ii) $m(p) = \{p\}$. These assumptions define a map $X \xrightarrow{m} 2^P$ in the obvious way. The map m is called a pole distribution and denoted by (X, m, P).♦

(6.6) Proposition. A pole distribution (X, m, P) defines a topology on X (cf. Rumfitt 2015, Mormann 2020): For $A \subseteq X$ define the operator $PX \xrightarrow{int} PX$ by $x \in int(A) \Leftrightarrow (x \in A$ and $m(x) \subseteq A)$. Then the operator int is a Kuratowski interior kernel operator and defines a topology OX on X. More precisely, (X, OX) turns out to be a (submaximal) Alexandroff space.♦

More precisely, the topology of a polar spaces defined by (X, m, P) is calculated as follows:

(6.7) Example (Topology of Polar spaces). Let (X, m, P) define the polar space (X, OX). Then for $p \in P$ and $x \in P - X$ the following holds:

$int(p) = \{p\}$, $\quad int(x) = Ø$, $\quad cl(x) = \{x\}$, $\quad cl(p) = \{x; p \in m(x)\}$

$j^{**}(p) = intcl(p) = \{x; \{p\} = m(x)\}$, $\{x\} \cup m(x)$ is the smallest open set that contains x.♦

From (6.7) one reads off that a polar space (X, OX) is a scattered T_D-space (i.e., X contains no non-empty dense-in-itself subsets (cf. Steen and Seebach Jr. (1982, p. 33)). Hence, we may apply a famous theorem of Simmons according to which NUC(OX) is Boolean:

(6.8) Theorem. (Simmons (1980), Picado and Pultr (2012)). Let (X, OX) be a scattered T_D-space. Then $PX \xrightarrow{j} NUC(OX)$ is a Boolean isomorphism, i.e., all nuclei j are spatial, i.e., $j(D) = j_Y(D) := int(Y \rightarrow D)$, for some $Y \subseteq X$.♦

Clearly, by (6.7) a subset $Y \subseteq X$ is dense in a polar space (X, m, P) iff $P \subseteq Y$. Hence, polar spaces are scattered T_D-spaces and we obtain:

<u>(6.9) Proposition</u>. Let (X, m, P) define a polar space (X, OX). Then $NUC(OX)_d = 2^{X-P}$. The bottom element 0 of 2^{X-P} corresponds to the largest dense subset of (X, OX), namely X, related to the nucleus id, and the top element 1 corresponds to the smallest dense subset of (X, OX), namely P. Clearly, P is related to the regular nucleus $j^{**} = j_P$.♦

In sum, for the special case of polar spaces (X, OX) one obtains that the family of consistent belief operators B related to the knowledge operator K has the structure of a complete Boolean algebra. This entails, in particular, that for every belief B there exists a "complementary" belief operator B* such that $B \wedge B^* = int$ and $B \vee B^* = intclint$.

Proposition (6.9) may further be used to show that the logics of belief and knowledge of polar spaces and "ordinary" topological spaces (like Euclidean spaces) strongly differ:

<u>(6.10) Proposition</u>. Let (X, OX) be a dense-in-itself T_1-space. Then, the nucleus j** is not open, i.e., there is no $A \in OX$ such that $j^{**}(D) \neq A \Rightarrow D$ for all D.

<u>Proof</u>. Assume the contrary, i.e., there is $A \in DOX$ such that $D^{**} = A \Rightarrow D$ for all $D \in OX$. Consider $D_x := X - \{x\}$ for $x \in X$. Clearly, D_x is dense, i.e., $D_x^{**} = X$. Hence, $A \subseteq D_x$ for all $x \in X$ according to Borceux (1.2.3 (1)). Thus, $A \subseteq \cap D_x = \emptyset$. This is a contradiction since Ø is not dense. Hence, the regular nucleus j** is not open.♦

In sum, the family of belief operators related to a topological knowledge operator K may be used to exhibit the specific structures of universes of possible worlds that one is using. This capacity of distinguishing different possibilities is lost when attention is paid exclusively on Stalnaker's operator intclint. Admittedly, things become simpler. Following Stalnaker (2006), Baltag et alii emphasize as an important feature of Stalnaker's KB system that only one belief operator B_S (= intcl), i.e., the belief operator can be defined in terms of the knowledge operator. According to them,

> this proposition constitutes one of the most important features of Stalnaker's combined system KB. This equivalence allows us to have a combined logic of knowledge and belief in which the only modality is K and the belief modality B is defined in terms of the former. We therefore obtain "...a more economical formulation of the combined belief-knowledge logic... . (Stalnaker (2006, p. 179), Baltag et al. (2019, p.221))

"Economy" is certainly an important feature of logical systems, but one may ask whether such an "economy" for a logic of knowledge and belief is actually desirable. It may be more plausible that for a given knowledge operator K a family of belief operators B exists that are compatible with K in the sense that all pairs (K, B) satisfy the axioms of weak Stalnaker systems, i.e., all of Stalnaker's original axioms but (NI). In other words, the account of this paper may be said to maintain the "spirit" of Stalnaker's logic of knowledge and belief, and, at the same time, adds to it a plausible dosage of epistemological pluralism that relates K and B in a more flexible manner.

Formally, the account presented in this paper may be considered as a neat generalization of Stalnaker's approach: Instead of the Boolean lattice of two elements {0, 1} corresponding, respectively, to the knowledge operator K - to be interpreted as "ideal belief " -, and to Stalnaker's belief operator B - to be interpreted as the "most error-prone", but still consistent belief operator - we have a Heyting algebra $NUC(OX)_d$ of belief operators the bottom element of which corresponds to knowledge K (= int) and the top element of which corresponds to Stalnaker's belief operator B (= intclint):

(6.10) Theorem. Let (X, OX) be a T_D-space and $NUC(OX)_d$ its Heyting algebra of dense nuclei. Then $(NUC(OX)_d, \leq)$ is a complete Heyting algebra that can be interpreted as an (intuitionist) logic of belief operators B related to the topological knowledge operator K of the space (X, OX).

7. CONCLUDING REMARKS. The main result of this paper has been the theorem that any one topological operator of knowledge K is compatible with many different belief operators B_j defined by dense nuclei $j \in NUC(OX)_d$. Thereby, for any given knowledge operator K, depending on the underlying topological structure (X, OX) of the models, a wealth of “admissible” or “fitting” belief operators B related to K can be defined such that all pairs (K, B) satisfy all axioms and rules of Stalnaker’s KB system but (NI). This plurality may be interpreted as a formal argument for doxastic tolerance: two epistemic agents τ_1 and τ_2 may rely on the same knowledge operator K but subscribe to different nuclear belief operators B_1 and B_2 that are compatible with K in the sense that both (K, B_1) and (K, B_2) satisfy Stalnaker’s axioms. By subscribing to the strong axiom of negative introspection (NI), the possibility of a pluralism of different coexisting belief operators compatible with a given knowledge operator is eliminated in favor of one “dogmatic” system that allows only one acceptable belief operator. The topological approach to knowledge and belief presented in this paper is an essentially pluralistic “knowledge first” approach in the sense that for one knowledge operator K, a complete Heyting algebra $NUC(OX)_d$ of belief operators B is constructed such that the pairs (K, B) of modal operators satisfy all of Stalnaker’s axioms (but (NI) plus some other adequacy conditions for knowledge and belief.

In sum, instead of Stalnaker’s KB system that only takes into account two epistemological operators of knowledge K (= int) and belief B (= intcl) the topological account presented in this paper conceptualizes the relation between knowledge and belief as a complete Heying algebra $NUC(OX)_d$. The structure of this Heyting algebra of belief operators depends on the structure of (X, OX) that underlies the topological models of the concepts of knowledge and belief. For polar spaces (X, m, P) $NUC(OX)_d$ has a rather simple structure, for others such as Euclidean spaces the structure of $NUC(OX)_d$ seems to be rather complicated.

Thus, an essential task for a topological logic of knowledge and belief is the investigation of how the structure of the topological spaces (X, OX) that underlie the topo-models of our epistemological logic determines the structure of the Heyting algebra $NUC(OX)_d$ of belief operators B related to K.

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