Completeness and Doxastic Plurality for Topological Operators of Knowledge and Belief

Abstract. The first aim of this paper is to prove a topological completeness theorem for a weak version of Stalnaker's logic KB of knowledge and belief. The weak version of KB is characterized by the assumption that the axioms and rules of KB have to be satisfied with the exception of the axiom (NI) of negative introspection. The proof of a topological completeness theorem for weak KB is based on the fact that nuclei (as defined in the framework of point-free topology) give rise to a profusion of topological belief operators that are compatible with the familiar topological knowledge operator Int. With the help of nuclei, a canonical topological model for weak KB can be constructed. For this canonical model a truth lemma for the modal operators K and B holds such that a completeness theorem for weak KB can be proved in the familiar way.

The second aim of this paper is to show that the topological interpretation of knowledge as the interior operator Int comes along with a complete Heyting algebra of belief operators N° (inter-definable with nuclei N) that all fit the knowledge operator Int in the sense that the pairs (Int, N°) satisfy all axioms of weak KB. This amounts to a pluralistic relation between knowledge and belief: Knowledge does not fully determine belief, rather it designs a conceptual space for belief operators where different (competing) belief operators coexist that can be compared with each other. Thereby an intuitionistic calculus of belief operators related to a topological knowledge operator is set up.

Key words: Epistemic Logic, Topological Semantics, Heyting algebras, Nuclei, Doxastic Plurality, Weak KB logic of knowledge and belief.

1. INTRODUCTION. Understanding the relation between knowledge and belief is an issue of central importance in formal epistemology. Especially after the birth of knowledge-first epistemology, the question of what exactly distinguishes an item of knowledge and an item of belief and how one can be determined in terms of the other has become even more pertinent.

In the recent literature on the topological semantics of epistemological concepts such as knowledge and belief one may find two especially popular accounts. On the one hand, there is Stalnaker's combined logic KB of knowledge and belief that can be elegantly topologized as has been shown by the works of Baltag et al. (2017, 2019) and others. On the other hand, there is the work of Steinsvold and others that offers a formal account of belief and related epistemological concepts in a framework based on the notion of topological derivation (cf. Steinsvold (2006), Parikh et al. (2007), Bezhanishvili and van der Hoek (2014)). It is not quite clear, however, how these two accounts of a topological epistemology are related to each other. In this paper, it will be shown that both may be conceived as two special cases of a more general account based on the notion of (topological) nucleus. More precisely, Stalnaker's concept of belief can be characterized as one kind of nucleus, Steinsvold's concept of belief as another. To be specific, Stalnaker's account is characterized by a nucleus that can be described mathematically as the (unique) regular nucleus and Steinsvold' account is closely related to a nucleus that in this paper is called perfect nucleus. Beside these distinguished nuclei many others exist that give rise to their own concepts of belief.[1] Thus, to obtain a more comprehensive understanding of the relationship between knowledge and belief, it seems expedient to discuss the whole manifold of belief operators instead of restricting one's attention to the special operator N_S. In other words, for a given knowledge operator the doxastic plurality of correlated belief operators should be taken into account.

[1] In topological terms Stalnaker's operator N_S is defined as the concatenation IntClInt, where, as usual, the interior kernel operator of a topological space (X, OX) is denoted by Int and the topological closure is denoted by Cl. In this paper this (more or less standard) topological terminology is used throughout. In more detail it is explained in section 3.

In Stalnaker's KB system, the belief modality B turns out to be uniquely defined by the knowledge modality K. For this unique determination of B by K essentially the axiom of negative introspection (NI) is responsible. By abandoning (NI), as is done in this paper, a more complex and more flexible relation between the modalities K and B arises. In a nutshell, this relation may be described as a one-many-relation. The topological structure (X, OX) defines a family of belief operators N° that fit the knowledge operator Int in the sense that all pairs (Int, N°) satisfy the axioms of KB except (NI). The family of belief operators N° compatible with Int has the structure of a complete Heyting algebra. Stalnaker's belief operator N_S° turns out to be the top element of this Heyting algebra, its bottom element corresponds to Int that can be interpreted as the "ideal" or "optimal" belief operator that by definition cannot err.

For a given knowledge operator Int, the Heyting algebra of admissible belief operators can be conceived as an intuitionistic logic of belief operators: Different belief operators N° can be compared with each other according to their strengths and how far they deviate from the knowledge operator Int.

Traditionally, the relation between belief and knowledge has been conceptualized in a rather simple way: Either knowledge is defined as a special kind of belief, e.g., knowledge is "justified" true belief, or "correctly justified" true belief, or the like, as in many received accounts of knowledge, or, as in contemporary knowledge first accounts, knowledge is given conceptual priority and is used to define belief in a unique way. In a sense, this paper follows the knowledge-first approach but with a special twist. It is shown that for a given knowledge operator Int, there exists a pool of different admissible belief operators N° such that the pairs (Int, N°) all define well-behaved systems of epistemic logic satisfying the axioms of a weak KB system. Therefore, different cognitive agents who subscribe to the same knowledge operator Int may use different agent-specific belief operators that may be compared with each other according to their deviation from common knowledge. This amounts to a doxastic plurality of a multitude of belief operators based on a common root of one knowledge operator Int.

The organization of this paper is as follows: to set the stage, in section 2 we recall the axioms and rules of Stalnaker's KB logic of knowledge and belief. In section 3, we introduce the topological concepts that are necessary for defining a topological semantics of knowledge and belief. In section 4 we introduce the concept of (topological) nuclei that plays a central role for the definition of belief operators and their semantics. In section 5 nuclei are used to prove a topological completeness theorem for weak KB. The plurality of belief operators related to one topological knowledge operator is studied in more detail in section 6. In particular, we calculate (partially) the Heyting algebras of (consistent) belief operators for some important topological spaces. The structure of these algebras heavily depends on the topological structure of the underlying topological spaces. In section 7 we study the relation of the theory of nuclei and the theory of belief operators based on the topological notion of derived set operator (cf. Steinsvold et alii). It is shown that the dual t of the derived set operator d is "almost" a nucleus. More precisely, t can be characterized as pre-nucleus, i.e., a slight generalization of the concept of nucleus for the rather comprehensive class of T_D-spaces. Even more, for the very special class of DSO-spaces, the pre-nucleus t turns out to be an honest nucleus. With respect to the other way round, the nucleus defined by the perfect kernel of a set (already defined in section 6) can be shown to be the nucleus defined in natural way by the pre-nucleus defined by t. In sum, the framework of topological nuclei sheds new light not only on Stalnaker's account of belief and knowledge but also on the topological epistemology based on the topological concept of derived set. We conclude with some general remarks on the further elaboration of this nucleus-based approach formal epistemology in section 8.

2. STALNAKER'S LOGIC KB OF KNOWLEDGE AND BELIEF. First, for the sake of definiteness, let us recall the axioms and the inference rules of Stalnaker's system (cf. Stalnaker (2006), Baltag et al. (2017, 2019). For this purpose, we start with a standard unimodal epistemic language L_K with a countable set PROP of propositional letters, Boolean operators $\neg$, $\wedge$, and a modal operator K to be interpreted as knowledge. The formulas of L_K are defined as usual by the grammar

(2.1) $$\varphi ::= p \mid \neg p \mid \phi \wedge \psi \mid K\varphi \quad , \quad p \in PROP.$$

The abbreviations for the Boolean connectives $\vee$, $\rightarrow$, and $\leftrightarrow$ are standard.[2] Occasionally we use the abbreviations $\bot$ for $\varphi \wedge \neg\varphi$ and $\top$ for $= \varphi \vee \neg\varphi$.

Analogously to L_K, a bimodal epistemological language L_{KB} for modal operators K and B as an extension of L_K is defined. The grammar of L_{KB} is defined as usual:

(2.2) $$\varphi ::= p \mid \neg p \mid \phi \wedge \psi \mid K\varphi \mid B\varphi \quad , \quad p \in PROP.$$

Stalnaker's combined logic KB of knowledge and belief is defined as follows:

(2.3) Definition (Stalnaker's axioms and inference rules for modal operators K (knowledge) and B (belief)). A bimodal logic KB based on the bimodal language L_{KB} is a Stalnaker system iff it satisfies the following rules and axioms:

(CL)	All tautologies of classical propositional logic CL.	
(K)	$K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$	(Knowledge is additive).
(T)	$K\varphi \rightarrow \varphi$	(Knowledge implies truth).
(KK)	$K\varphi \rightarrow KK\varphi$	(Positive introspection for K).

[2] For a more detailed presentation of L_K, the reader may consult the recent papers of Baltag et al. (2017, 2019) and Aiello et al. (2003)).

(CB)	$B\varphi \rightarrow \neg B \neg \varphi$	(Consistency of belief).
(PI)	$B\varphi \rightarrow KB\varphi$	(Positive introspection of B).
(NI)	$\neg B\varphi \rightarrow K\neg B\varphi$	(Negative introspection of B).
(KB)	$K\varphi \rightarrow B\varphi$	(Knowledge implies belief).
(FB)	$B\varphi \rightarrow BK\varphi$	(Full belief).

Inference Rules:

(MP)	From φ and $\varphi \rightarrow \psi$, infer ψ.	(Modus Ponens).
(NEC)	From φ, infer $K\varphi$.	(Necessitation for K).♦

For the topological[3] approach to knowledge and belief, the axiom (NI) plays a special role as will be explained now. Let M = (X, OX, μ) be a topological model of L_K in the familiar sense (cf. Baltag et al. (2019)). Then for formulas Kφ we have μ(Kφ) = Int(μ(φ)). A semantics for L_{KB} is defined by setting for formulas Bφ the interpretation μ(Bφ) := IntClInt(μ(φ)). Then we can prove:

(2.4) Proposition. Let (X, OX) be any topological space. Under the semantics μ just given the topological model (X, OX, μ) validates all axioms and rules of Stalnaker's logic KB except the axiom of (NI) of negative introspection.

Proof. Check the definitions (2.3) and the standard properties of the topological operators Int and Cl (see section 3). Elementary examples of the real line (ℝ, Oℝ) and other familiar spaces show that (NI) fails to hold in general.[4] ♦

[3] The topological terminology used in this paper is fairly standard. Nevertheless, for the sake of definiteness, the topological concepts to be used will be explained in full detail in section 3 and 4.

[4] The topological reason for the failure of (NI) in general spaces may be informally described as the fact that (NI) requires that many clopen (= open and closed) subsets μ(Bφ) exist. General topological spaces, however, may lack sufficiently many clopen sets. For instance, connected spaces such as the Euclidean line (ℝ, Oℝ) have only Ø and ℝ as clopen subsets.

In Baltag et al. (2019) (and elsewhere) the following has been proved:

(2.5) Proposition (Baltag et al. (2019, Proposition 6). Under the semantics given above a topological space (X, OX) validates all axioms and rules of Stalnaker's system KB ((NI) included) iff it is an extremally disconnected space (ED space).♦

Thus, the first thing to note when we give up (NI) is that we gain greater generality in that the very special class of extremally disconnected spaces can be replaced by arbitrary topological spaces. On the other hand, we have to give up some "conceptual economy": The familiar proof that B is uniquely determined by K as $B \leftrightarrow \neg K \neg K$ is no longer available. Indeed, as will be shown in the following, the interpretation of the belief modality B is no longer uniquely determined by K.[5] Depending on the topological structure of models, there are many different possibilities for interpreting the belief modality B, not only the one that interprets belief as the "epistemic possibility of knowledge" as Stalnaker's KB-logic does. Giving up (NI) thereby amounts to obtaining a greater amount of conceptual flexibility with respect to B. This should be considered as a real bargain for a more comprehensive formal epistemology of knowledge and belief, or so I want to argue. A unique determination of belief by knowledge is not very plausible, or so I want to argue.

As a consequence, for the systems of knowledge and belief to be considered in this paper, the validity of (NI) will not be required. Rather, we will require only that our systems are weak Stalnaker systems in the following sense:

(2.6) Definition (Weak KB logic). A bimodal logic (with modal operators K and B) based on the bimodal language L_{KB} is a weak KB-logic iff it satisfies the following two conditions:

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[5] That is, the elegant equivalence of Stalnaker (2006) and Baltag et al. (2019) that $B \leftrightarrow \neg K \neg K$ is no longer valid. In contrast, weak KB turns out to be a truly bimodal extension of CL, i.e., B cannot uniquely be defined in terms of K.

(i) The modal operator B satisfies the Kripke axiom (K) and the axiom (4)*:

(K) $B(\varphi \rightarrow \psi) \rightarrow (B\varphi \rightarrow B\psi)$ (Belief is additive).

(4)* $BB\varphi \rightarrow B\varphi$.

(ii) All of Stalnaker's axioms and rules given in (2.3) except the axiom (NI) of negative introspection hold.♦

The axiom (4)* is the converse of the well-known axiom

(4) $B\varphi \rightarrow BB\varphi$

for the modal operator B that holds for the B-fragment of weak KB due to the axioms (PI) and (KB). That is, B is idempotent: $B = B^2$. More precisely, the B-fragment of weak KB is a special normal logic:[6]

(2.7) Corollary. The B-fragment of weak KB-logic is a KD4-logic:

(K) $B(\varphi \rightarrow \psi) \rightarrow (B(\varphi) \rightarrow B(\psi))$.

(D) $B\varphi \rightarrow \neg B \neg \varphi$.

(4) $B\varphi \rightarrow BB\varphi$.

Proof. By (2.6) the axioms (PI) and (KB) are valid. Thus, we obtain $B\varphi \rightarrow KB\varphi \rightarrow BB\varphi$ and therefore (4).The validity of (K) and (D) holds by definition.♦

(2.7) may be compared with the corresponding result for full KB logic according to which the B-fragment of full KB logic is a KD45 system (cf. Baltag et al. (2019, Proposition 4), Stalnaker (2006)).[7]

[6] For various equivalent definitions of a normal modal logic, see Chellas (1980, Theorem 4.3, p. 115).

[7] These authors characterize the B-fragment of KB logic only as a KD45 logic. Actually it is also a KD4*5 logic since for extremally disconnected spaces the belief operator ClInt clearly satisfies (4) and (4)* due to ClInt = ClIntClInt. Elementary examples based on the Euclidean line ($\mathbb{R}$, O$\mathbb{R}$) show that there are models of weak KB

(2.8) Proposition. Weak KB logic is (strictly) weaker than KB logic.

Proof. We have to show that the modal operator B of (full) KB logic satisfies the Kripke axiom (K) and (4)* of (2.6)(i). According to Stalnaker (2006) and Baltag et al. (2019) in (full) KB-logic one has B ↔ ¬K¬K ↔ K¬K¬K. As is easily checked, K¬K¬K is a normal operator, i.e., satisfies (K) and (4)*. Hence, as it should be, (full) KB logic is a weak KB logic. In order to show that weak KB is strictly weaker than KB, one has to find a formula that is valid for KB but not for weak KB. The formula $\neg K\neg K(\varphi \wedge \psi) = \neg K\neg K\varphi \wedge \neg K\neg K\psi$ will do.♦

In the following sections 3 and 4 a properly bimodal topological semantics for L_{KB} will be constructed that will be used to prove a topological completeness theorem for weak KB-logic and help elucidate the notion of doxastic plurality that is characteristic for weak KB. This semantics for weak KB is a bimodal extension of the familiar topological semantics of the unimodal topological semantics for K (cf. Baltag et al. (2019)). But first we have to recall the necessary rudiments of set-theoretical topology in more detail.

3. ON THE TOPOLOGY OF KNOWLEDGE OPERATORS.

In order to define a topological semantics for knowledge and belief operators, in this section we will recall the necessary rudiments of set-theoretical topology for topological epistemology. For a more detailed presentation, the reader may consult the recent works of Baltag et al. (2017, 2019).

First of all, recall the definition of a topological space:

(3.1) Definition. Let X be a set with power set PX. A topological space is an ordered pair (X, OX) with OX ⊆ PX that satisfies the following conditions:

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logic the B-fragments of which are not KD45 models (cf. Proposition (4.11)). Further, there are topological models of KB-(NI)+$(K)_B$+(4) that do not satisfy (4)*.

(i) $\emptyset$, X $\in$ OX.

(ii) OX is closed under finite set-theoretical intersections $\cap$ and arbitrary set-theoretical unions $\cup$.♦

The elements of OX are called the open sets of the topological space (X, OX). The set-theoretical complement A^C of an open set A is called a closed set. The set of closed subsets of (X, OX) is denoted by CX. The interior kernel operator Int and the closure operator Cl of (X, OX) are defined as usual: The interior kernel Int(A) of a set A $\in$ PX is the largest open set that is contained in A; the closure Cl(A) of A is the smallest closed set containing A. For details, see Willard (2004), Steen and Seebach Jr. (1982), or any other textbook on set-theoretical topology. The operators Int and Cl are well-known to satisfy the Kuratowski axioms:

<u>(3.2) Proposition (Kuratowski Axioms)</u>. Let (X, OX) be a topological space, A, B $\in$PX. The interior kernel operator Int and the closure operator Cl of (X, OX) satisfy the following (in)equalities

(i)	$Int(A \cap B) = Int(A) \cap Int(B)$.	$Cl(A \cup B) = Cl(A) \cup Cl(B)$.
(ii)	$Int(Int(A)) = Int(A)$.	$Cl(Cl(A)) = Cl(A)$.
(iii)	$Int(A) \subseteq A$.	$A \subseteq Cl(A)$.
(iv)	$Int(X) = X$.	$\emptyset = Cl(\emptyset)$.♦

These axioms are used in the following without explicit mention. Moreover, we will use freely the fact that the operators Int and Cl are inter-definable: $Int(A) = Cl(A^C)^C$ and $Cl(A) = Int(A^C)^C$. Further, it is expedient to conceive the operators Int and Cl as operators Int: PX—>PX and Cl: PX——>PX defined on PX. Hence, the concatenation of these operators makes perfect sense. Thus, the following definition makes sense: A subset A of X is called regular open iff Int(Cl(A)) = A. The set of regular open subsets of a topological space is denoted by O*X.

Dually, a subset A of X is called a regular closed set iff Cl(Int(A)) = A. In the following, concatenations of Int and Cl such as IntCl and IntClInt will play an important role.

The concept of a topological space (X, OX) is extremally general. For most applications it is expedient or even necessary to require that the topology satisfies axioms in addition to those generally required of topological spaces. One such collection of conditions is given by means of axioms called separations axioms. Some axioms of this kind that are important for the purposes of the present paper are the following ones:

(3.3) Definition (Separation Axioms for Topological Spaces). Let (X, OX) be a topological space.

(i) X is a T_0-space if, for every distinct a, b $\in$X there exists an open set A such that either a $\in$ A and b $\notin$ A, or b $\in$ A and a $\notin$ A.

(ii) X is a T_D-space if, for every a $\in$X, there exists an open set A such that a $\in$ A and that A–{a} is also open.

(iii) X is a T_1-space if, for every distinct a, b $\in$X there exist open sets A and B such that a $\in$ A and b $\in$B, such that b $\notin$ A, and a $\notin$ B.

(iv) X is a T_2-space if, for every distinct a, b $\in$X, there exist disjoint A and B containing a and b, respectively.♦

Each of these axioms is independent of the Kuratowski axioms of a topological space. In fact, there exist topological spaces which do not satisfy any of the separation axioms T_0 - T_2. More precisely, the following chain of implications hold:

(3.4) Proposition. The separation axioms T_i defined in (3.3) satisfy the following chain of implications: $T_2 \Rightarrow T_1 \Rightarrow T_D \Rightarrow T_0$. All implications of this chain are proper, i.e., they cannot be reversed.

Proof. Cf. Steen/Seebach Jr. (1978, p. 12) and Picado/Pultr (2012, p. 5).♦

The axioms T_2, T_1, and T_0 are classical. They are discussed (usually together with many other separation axioms) in full detail in most standard textbooks of topology (see also the particularly useful presentation in Steen and Seebach Jr. (1978)). The axiom T_D is rather new. It was first proposed in the 1960s by several authors for a variety of reasons (cf. Aull and Thron (1963)). For a modern presentation of many equivalent formulations of T_D see Picado and Pultr (2012, I.2). The axiom T_D will be especially useful for the calculation of the lattice NUC(OX) of belief operators for a wide class of topological spaces. Further, T_D turns out to be essential for dealing with nuclei and belief operators related to the derived set operator (cf. section 7 and Steinsvold (2006)).

For the definition of consistent belief operators, the concept of a dense subset of topological spaces will be important:

(3.5) Definition. Let (X, OX) be a topological space with interior operator Int and closure operator Cl, Y, Z $\in$ PX.

(i) Y is a dense subset of X iff Cl(Y) = X.

(ii) Z is a nowhere dense in X iff Int(Cl(Z)) = Ø.

(iii) A point $x \in X$ is isolated iff $\{x\} \in OX$.

(iv) A space (X, OX) is dense-in-itself iff it has no isolated points.♦

(3.6) Examples of dense and nowhere dense sets of topological spaces (X, OX).

(i) For the trivial coarse topology (X, {Ø, X}) every non-empty subset A $\in$ PX is dense and only Ø is nowhere dense. For the discrete topology (X, PX) only X is dense, and only Ø is nowhere dense.

(ii) Let ($\mathbb{R}$, O$\mathbb{R}$) be the real line endowed with the familiar Euclidean topology. Let F $\subseteq \mathbb{R}$ be a finite set. Then F is nowhere dense and the complement F^C of F is a dense open subset of ($\mathbb{R}$, O$\mathbb{R}$). More generally, the infinite set of integers $\mathbb{Z}$ is a nowhere dense subset of ($\mathbb{R}$, O$\mathbb{R}$).

(iii) The sets $\mathbb{Q}$ of rational numbers and $\mathbb{Q}^C$ of irrational numbers are disjoint dense subsets of ($\mathbb{R}$, O$\mathbb{R}$), i.e., $\mathbb{Q} \cap \mathbb{Q}^C = \varnothing$ and $Cl(\mathbb{Q}) = Cl(\mathbb{Q}^C) = \mathbb{R}$.

(iv) A more sophisticated example of a nowhere dense set is given by the Cantor dust D of the real line ($\mathbb{R}$, O$\mathbb{R}$) defined as follows: From the unit interval [0,1] of $\mathbb{R}$ remove the open middle interval (1/3, 2/3) obtaining the union of the closed intervals [0, 1/3] and [2/3, 1]. This set is denoted by D_1. From D_1 remove the open middle intervals (1/9, 2/9) and (7/9, 8/9) obtaining a set D_2 that consists of the four closed intervals [0, 1/9], [2/9, 1/3], [2/3, 7/9], and [8/9, 1]. And so on. Then the Cantor dust D is defined as the infinite intersection $D := \cap_{i \in \mathbb{N}} D_i$.

The Cantor dust is nowhere dense and perfect (= closed and having no isolated points) (cf. Steen and Seebach Jr. (1978, p. 57- 58)). Hence the complement D^C of the Cantor dust D is a dense open subset of ($\mathbb{R}$, O$\mathbb{R}$). In section 6 the complement D^C of the Cantor dust D will be used to define some interesting belief operators for the Euclidean space ($\mathbb{R}$, O$\mathbb{R}$).♦

After these preparations, topological models for the modal language L_K can be defined as usual (cf. section 2 and Baltag et al. (2019, 2.2.1)):

<u>(3.7) Definition</u>. Given a topological space (X, OX), a topo(logical) model for L_K is given by M = (X, OX, μ), μ a valuation function in the sense of McKinsey and Tarski. In particular, μ maps the propositional letters p $\in$ PROP onto elements of PX. The interior semantics for the model (X, OX, μ) is defined as usual. In particular, if a formula φ of L has the truth set $||\varphi||$, then the formula Kφ of L_K has the truth set $||K\varphi|| := Int(||\varphi||)$.♦

As will be shown in the next section, the topological structure of topological models (X, OX, μ) can be used not only to define a semantics for the knowledge modality K, but also for the belief modality B. For this purpose, it is necessary, however, to introduce some further topological apparatus, in particular the concept of (topological) nuclei of OX.[8] This will be done in the next section.

4. NUCLEI OF TOPOLOGICAL SPACES. In this section we introduce the concept of (topological) nuclei (cf. Johnstone (1982), Borceux (1994), Picado and Pultr (2012)). As said nuclei will be essential for the definition of belief operators B compatible with topological knowledge operators K.[9]

(4.1) Definition. Let (X, OX) be a topological space, and let A, D $\in$ OX. A map N: OX$\rightarrow$OX is called a nucleus of (X, OX) if it satisfies the following properties:

(i) $A \subseteq N(A)$. (Inflation)

(ii) $N(N(A)) = N(A)$. (Idempotence)

(iii) $N(A \cap D) = N(A) \cap N(D)$. (Distributivity)

The set of nuclei of a topological space (X, OX) is denoted by NUC(OX).♦

(4.2) Definition. A partial order ≤ on NUC(OX) is defined by the relation

$$N \leq N' \text{ iff } N(A) \subseteq N'(A), \text{ for all } A \in OX.$$♦

[8] The necessity of introducing nuclei for defining the semantics of belief operators distinguishes weak KB logic from original KB logic. Since for original KB systems the belief modality B can be defined in terms of K, in these systems the excursion into the theory of nuclei can be avoided, since the semantics of B can be defined in terms of the semantics of K.

[9] The literature on nuclei in point-free topology has reached a high level of technical sophistication. This paper does not aim to give a full-fledged introduction into the theory of nuclei. Instead, we intend to provide the basic definitions and facts so that the reader can understand that this theory has interesting applications regarding the modal theory of belief and knowledge. For a fuller account, the reader may consult Johnstone (1982), Borceux (1994), or Picado and Pultr (2012, 2021) and the extensive bibliographies mentioned there.

In the following, NUC(OX) is always endowed with this partial order. As is easily proved, the partial order ≤ renders NUC(OX) a complete lattice. Its bottom element 0 is the identity operator id_{OX}:OX→OX, and its top element 1:OX→OX is the trivial nucleus that maps every A∈ OX onto X.

Actually, much more is true about (NUC(OX), ≤). In order to express this in an appropriate way, one needs the following definition:

<u>(4.3) Definition (cf. Borceux (1994, Definition 1.3.1, Proposition 1.3.2(2)).</u> Let L be a complete lattice. For M ⊆ L denote the supremum of M by SUP(M). L is a complete Heyting algebra iff the following infinite distributive law holds: For all a ∈ L and M ⊆ L one has

$$a \wedge \mathrm{SUP}(M) = \mathrm{SUP}(a \wedge M).$$

For every complete Heyting algebra L a binary operation ⇒ is defined by

$$a \Rightarrow c := \mathrm{SUP}\{b \in L;\ a \wedge b \leq c\}$$

for a, c ∈L. The operation ⇒ is called the Heyting implication of L. ♦

The Heyting implication ⇒ has many interesting properties (cf. Borceux (1994, chapter 1.2), Johnstone (1982, I.1.10, p. 13), Picado/Pultr (2012, Appendix I, Section 7)) that have been studied by many authors.

The best-known examples of complete Heyting algebras are the lattices OX of open sets of topological spaces (X, OX) with A ⇒ D := Int(A^C ∪ D), A, D ∈ OX. For the purposes of this paper, however, another more general class of complete Heyting algebras plays an important role:

<u>(4.4) Proposition</u>. Let (X, OX) be a topological space. Then the lattice NUC(OX) of nuclei of OX is a complete Heyting algebra. The Heyting implication ⇒ of NUC(OX) is defined by

$$N \Rightarrow N'(D) := \mathrm{INF}\{N(E) \Rightarrow N'(E);\ E \geq D\} \qquad D, E \in OX.$$

Proof. (Johnstone (1982, II, 2.4, Lemma), Borceux (1994, Theorem 1.5.7)).♦

In the last decades, the investigation of NUC(OX) has turned out to be a fruitful pathway for studying topological problems of various kinds, particularly problems related to point-free topology (cf. Johnstone (1982), Borceux (1994), Picado and Pultr (2012)). In this paper, we conduct some modest steps to use the concept of nuclei to shed new light on the problems of modal systems that deal with the epistemological concepts of knowledge and belief. More precisely, we will deal with problems related to Stalnaker's KB logic of knowledge and belief and the theory of doxastic operators of Steinsvold and others. Before dealing with specific problems regarding this issue, it is expedient to give some examples of nuclei and to elucidate the structure of NUC(OX).

The following special class of nuclei will be the most important one for the purposes of this paper:

(4.5) Definition. A nucleus $N \in NUC(OX)$ is called a dense nucleus iff $N(\emptyset) = \emptyset$. The subset of dense nuclei of NUC(OX) is denoted by $NUC(OX)_d$.♦

(4.6) Examples (Johnstone (1982), Borceux (1994), Picado and Pultr (2012)). Let (X, OX) be a topological space and A, D ∈ OX. Denote the join and the Heyting implication of OX by ∪ and ⇒, respectively.

(i) The map k_A:OX—>OX defined by $k_A(D) := A \cup D$ is a nucleus. The nucleus k_A is called the closed nucleus defined by A. Clearly, for $A \neq \emptyset$ the nucleus k_A is not a dense nucleus.

(ii) The map $j_A: OX \rightarrow OX$ defined by $j_A(D) := (A \Rightarrow D)$ is a nucleus. The nucleus j_A is called the open nucleus defined by A. If A is a dense subset of X then j_A is a dense nucleus.[10]

(iii) The operator N_S: $OX \rightarrow OX$ defined by $N_S(D) :=$ IntCl((D)) is a nucleus. It is usually called the regular nucleus of OX.[11] Due to the Kuratowski axioms (3.2)(iii) and (3.2)(iv) for Int and Cl the nucleus N_S is a dense nucleus, i.e., $N_S(\emptyset) = Int(Cl(\emptyset)) = Int(\emptyset) = \emptyset$. Only for few topological spaces the nucleus N_S is an open nucleus, for most spaces (X, OX) there is no $A \in OX$, such that $N_S(D) = IntCl(D) = j_A(D) = A \Rightarrow D$.♦

Dense nuclei will play a central role in the following, as they define consistent belief operators. Now all ingredients are available to formulate a central definition of this paper:

(4.7) Definition. Let (X, OX) be the topological space of a topological model with interior kernel operator Int, let $N \in NUC(OX)$, and $i:OX \rightarrow PX$ the canonical inclusion. Denote the concatenation of $i:OX \rightarrow PX$, $N:OX \rightarrow OX$, and $Int:PX \rightarrow OX$ by N°. Then this operator N°: $PX \rightarrow PX$ is called the belief operator (related to Int and corresponding to the nucleus N).[12]♦

The natural next step is to show that (4.7) is a reasonable and fruitful definition that defines a family of well-behaved belief operators N° for a knowledge operator Int that enjoy all properties that one intuitively expects from "good" belief operators.

The task of justifying the predicate "*belief* operator" for N° is naturally divided into two subtasks:

(i) It has to be shown that (4.7) is formally adequate in the sense that the belief operators defined by (4.7) satisfy appropriate formal conditions of adequacy.

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[10] The reader should not be confused by this (established) terminology: every open subset A of X (as an element of OX) defines a closed nucleus and an open nucleus, namely, k_A and j_A, respectively.

[11] In this paper the regular nucleus IntCl is also called the Stalnaker nucleus and denoted by N_S, since it has played such a prominent role in the topological interpretation of Stalnaker's logic KB, cf. Baltag et al. (2017, 2019).

[12] Clearly, a nucleus N and its corresponding belief operator N° determine each other uniquely: N = iN°Int and N° = IntNi. Here, i is , of course, the canonical inclusion $OX \rightarrow PX$.

(ii) It has to be shown that sufficiently many philosophically interesting belief operators N° exist that fulfil the requirements of (4.7).

Epistemologists do not fully agree on what "good properties" for a belief operator are, of course, but the following qualities are rather uncontroversial candidates:

(4.8) Definition (Good belief operators). A good belief operator N° (related to a knowledge operator Int and corresponding to N) should satisfy the following conditions:

(i) A good belief operator N° may produce a false belief. Formally this means that there should exist a proposition $A \subseteq X$, such that $w \in N°(A)$ but $w \notin A$, i.e., a cognitive agent who uses N° believes that w is A-world, but actually w is not an A-world.

(ii) A good belief operator N° should be consistent, i.e., if the cognitive agent believes that w is an A-world, then he does not believe that w is not an A-world, i.e., $w \in N°(A)$ entails that $w \notin N°(A^C)$.

(iii) A good belief operator N° should be minimally compatible with its related knowledge operator Int, i.e., if it is known that w is an A-world, then it should be believed that w is an A-world, i.e., $w \in Int(A)$ entails $w \in N°(A)$. ♦

(4.9) Proposition. Let (X, OX) be a topological space and $N \in NUC(OX)_d$, $N \neq id_{OX}$. Then the belief operator N° corresponding to N is a good belief operator in the sense of (4.8).

Proof. We have to prove that N° satisfies the conditions (4.8)(i) – (iii).

(i): In order to show that there exists a proposition A such that N°(A) is possibly false one can argue as follows: By the definition of the partial order ≤ of nuclei (4.2), the smallest dense nucleus of OX is the identity id_{ox} with corresponding belief operator Int. Thus, according to the assumption $id_{OX} \neq N$, we may assume $id_{OX} < N$. Hence, there must be an $A \in OX \subseteq PX$ such that $A \subset N(A)$. By definition of N° and the fact that A is assumed to be open this entails that A is properly contained in N°(A), i.e., $A \subset N°(A)$. This is equivalent to $N°(A) \cap A^C \neq Ø$. In other words, there is a world $w \in N°(A) \cap A^C$. This means that an epistemic agent who uses N° believes that w is an A-world but actually w is not an A-world. In other words, the agent's belief is false. This proves (4.8)(i).

(ii) Due to the fact that N is dense and distributive with respect to $\cap$ ((4.1)(ii)) for all $A \in PX$ one has

$$Ø = N°(Ø) = i(N(Int(A \cap A^C))) = i(N(Int(A))) \cap i(N(Int(A^C))) = N°(A) \cap N°(A^C).$$

Hence $N°(A) \subseteq N°(A^C)^C$, i.e., the belief operator N° is consistent. This proves (4.8)(ii).

(iii) Since for all nuclei, by definition $Int(A) \subseteq N(Int(A))$ one obtains that $w \in Int(A)$ entails $w \in N°(A)$. This proves (4.8)(iii).

In sum, for all $N \in NUC(OX)_d$ with $N \neq id$, the corresponding belief operator N° is a good belief operator.♦

The condition (4.8)(i) is generally accepted as a necessary condition in order that an operator may be considered "as suitable for defining a doxastic logic" (cf. Bezhanishvili and van der Hoek (2014, p. 373) and Parikh et al. (2007, p. 329)). The conditions (4.8)(ii) and(iii) are also rather unanimously accepted among epistemologists. Thus, proposition (4.8) ensures that at least prima facie, nuclei may be considered as a promising source for a semantics of doxastic logic.

More systematically, one may require that good belief operators are those operators that define topological model of weak KB-logic. This can be carried out as follows. Let (X, OX) be a topological space and $N \in NUC(OX)_d$ a dense nucleus. Relying on a classical idea of McKinsey and Tarski we may use (X, OX, N) to define a valuation of weak L_{KB} by putting

- $\mu_N(p) \subseteq X$.
- $\mu_N(\neg\varphi) = X - \mu_N(\varphi)$.

- $\mu_N(\varphi \wedge \psi) = \mu_N(\varphi) \cap \mu_N(\psi)$.
- $\mu_N(\varphi \vee \psi) = \mu_N(\varphi) \cup \mu_N(\psi)$.
- $\mu_N(\varphi \rightarrow \psi) = \mu_N(\varphi)^C \cup \mu_N(\psi)$.
- $\mu_N(K\varphi) = \mathrm{Int}(\mu_N(\varphi))$.
- $\mu_N(B\varphi) = iN(\mu_N(K\varphi)) = iN\mathrm{Int}(\mu_N(\varphi))$.

Clearly, this definition is an extension of the classical definition of a valuation of the unimodal language L_K (cf. Aiello et al. (2003, p. 891)). Moreover, it should be noted that for the smallest dense nucleus $N = id_{OX}$ the last clause boils down the penultimate one, i.e., $\mu_N(B\varphi) = id_{OX}(\mu_N(K\varphi)) = \mu_N(K\varphi))$.

Now we can define the notion of a topological model of weak KB as follows.

(4.10) Definition. A topological model of weak KB is given by a quadruple M = (X, OX, N, μ_N), $N \in NUC(OX)_d$.♦

As usual, for topological models (X, OX, N, μ_N) the truth of a formula φ at a world $w \in X$ is inductively defined as follows:

- $M, w \vDash p$ iff $w \in \mu_N(p)$.
- $M, w \vDash \neg\varphi$ iff NOT($M, w \vDash \varphi$).
- $M, w \vDash \varphi \wedge \psi$ iff ($M, w \vDash \varphi$) AND ($M, w \vDash \psi$).
- $M, w \vDash \varphi \vee \psi$ iff ($M, w \vDash \varphi$) OR ($M, w \vDash \psi$).
- $M, w \vDash \varphi \rightarrow \psi$ iff NOT($M, w \vDash \varphi$) OR ($M, w \vDash \psi$)).
- $M, w \vDash K\varphi$ iff $\exists U(U \in OX(w \in U$ AND $\forall v \in U(M, v \vDash \varphi))$.
- $M, w \vDash B\varphi$ iff $\exists U(U \in OX(w \in N(U)$ AND $\forall v \in U(M, v \vDash K\varphi))$.

Then a formula φ is said to be true in the model $M = (X, OX, N, \mu_N)$ if $\mu_N(\varphi) = X$. A formula φ is said to be topologically valid if it is true in every topological model. Then we can easily prove:

(4.11) Proposition. The weak KB logic of knowledge and belief defined in (2.6) is sound with respect to the class of all topological models (X, OX, N, μ_N), i.e., all axioms and rules of weak KB hold for all topological models (X, OX, N, μ_N).

Proof. As is to be expected for proofs of soundness the proof is routine. Looking at the list of axioms and rules of weak KB-logic given in (2.6) the proof can be divided into three parts:

(i) axioms dealing only with the modal operator K;

(ii) axioms dealing only with the modal operator B;

(iii) axioms dealing with modal operators K and B (mixed axioms).

(i) The axioms of the first group are (K), (T) and (KK). The validity of (K), (T), and (KK) for topological models of weak KB is well-known because every model (X, OX, N, μ_N) of weak KB defines a model (X, OX, μ_N) for standard epistemic logic K (cf. Aiello et. al. (2003)). Moreover, modus ponens (MP) and necessitation (NE) hold for topological models.

(ii) The second group of axioms consists of the axioms that only deal with the modality B. This group contains the axioms

(CB) $B\varphi \rightarrow \neg B(\neg\varphi)$

(K) $B(\varphi \rightarrow \psi) \rightarrow (B\varphi \rightarrow B\psi)$

(4)* $BB\varphi \rightarrow B\varphi$

In order to show that (CB) holds in all models (X, OX, N, μ_N) of weak KB one has to show that $\mu_N(B\varphi) \subseteq \mu_N(B(\neg\varphi))^C$ for all φ. This can be seen as follows:

$$\mu_N(B\varphi) \subseteq \mu_N(B(\neg\varphi))^C \quad \text{iff} \quad \mu_N(B\varphi) \cap \mu_N(B(\neg\varphi))^{CC} = \emptyset$$

$$\text{iff} \quad iNInt(\mu_N(\varphi)) \cap iNInt(\mu_N(\neg\varphi)) = \emptyset \quad \text{iff} \quad iNInt(\mu(\varphi \wedge \neg\varphi)) = iN(\emptyset) = \emptyset.$$

since N is a dense nucleus.

The semantic interpretation of the modal operator B is defined as the concatenation of the three normal operators i, N, and Int and hence normal, i.e., (K) is satisfied. The validity of (4)* follows similarly from the definition of the belief operator: inIntiNint = iN^2Int = iNInt.

Clearly, the concatenation iNInt is idempotent, i.e., $(iNInt)^2$ = iNInt. Hence, (4)* is satisfied.

(iii) The third group of axioms comprises the axioms (PI), (KB), and (FB). The validity of these axioms can be read off directly from their definitions, the defining properties of nuclei, and the Kuratowski axioms.♦

For the proof of a topological completeness theorem for weak KB in section 5 we have to construct an adequate nucleus for the canonical topological model of weak KB that takes care of the belief modality B. For this construction, we show that the nuclei of OX give rise to appropriate sublocales of OX. We heavily rely on the detailed presentations of sublocales in Picado and Pultr (2012, Chapter III) and Johnstone (2002, Proposition 1.1.13, p. 481).

(4.12) Definition. Let H be a complete Heyting algebra. A subset $S \subseteq H$ is a sublocale of H iff S is closed under all meets of H and for every $s \in S$ and every $x \in H$, the Heyting implication $x \Rightarrow s \in S$.♦[13]

It may be observed that sublocales are defined for complete Heyting algebras in general. For our purposes we only need this concept for the special case of Heyting algebras OX arising from topological spaces (X, OX).

(4.13) Definition. A sublocale S is a dense sublocale of H iff S contains the bottom element 0 of H.♦

[13] Sublocales are also called $\Rightarrow$-ideals.

In the following we will only consider sublocales of Heyting algebras of the form OX for topological spaces (X, OX).

The smallest sublocale of OX is {X}. Clearly, {X} is a not a dense sublocale. Trivially, OX is the largest dense sublocale of OX. Probably the best-known non-trivial example of a dense sublocale of OX is the Boolean lattice of regular open sets $O^*X \subseteq OX$. Actually, O*X is a very special sublocale. According to Isbell's theorem, for every topological space (X, OX) O*X is the smallest dense sublocale of OX (see Theorem (6.5)).

Denote the set of sublocales of OX by SL(OX). Then SL(OX) is a complete lattice with respect to set-theoretical intersection $\cap$. Even more, with respect to $\cap$ the lattice SL(OX) is a complete co-Heyting algebra with the sublocale {X} as bottom element and the sublocale OX as its top element (cf. Picado and Pultr (2012, 3.2.1. Theorem, p.28)). Thus, we obtain:

(4.14) Proposition. For all $A \subseteq OX$ there is a (unique) smallest sublocale $S_A \in SL(OX)$ such that $A \subseteq S_A$, namely, the intersection of all sublocales that contain A.

Proof. The class of sublocales that contain A is not empty, since $A \subseteq OX$. Since SL(OX) is complete with respect to arbitrary set-theoretical intersections there is a smallest sublocale S_A that contains A, namely, the intersection of all sublocales that contain A.♦[14]

Given $A \subseteq OX$ a unique nucleus corresponding to the sublocale S_A may be constructed as follows. The inclusion i: $S_A \rightarrow OX$ has an adjoint map j: $OX \rightarrow S_A$. Then the concatenation ij:OX→OX is the desired nucleus. It may be called the nucleus generated by A.

[14] Actually there is an order-reversing bijection between nuclei and sublocales: A nucleus N:OX→OX is uniquely determined by its image i:N(OX)→OX. Indeed, $N(OX) \subseteq OX$ is a sublocale. Thus, a nucleus uniquely determines a sublocale. On the other hand, the inclusion map of a sublocale i:S→OX has an adjoint frame map j:OX→S such that the concatenation ij:OX→OX is a nucleus N (cf. Johnstone (2002, Proposition 1.1.3., p. 486), Picado and Pultr (2012, 5.3.2. Proposition. p. 32)). We don't need this result, however.

This correspondence will be important for constructing a canonical model for weak KB. More precisely, in the next section we will use this construction to define a canonical topological model (H, OH, N_B, μ_{NB}) which shows that weak KB logic is complete with respect to topological models (X, OX, N, μ_N).

<u>5. A TOPOLOGICAL COMPLETENESS THEOREM FOR WEAK KB.</u> In this section we will construct a canonical topological model (H, OH, N_B, μ_{NB}) for weak KB logic. This topological model will be used to prove a completeness theorem for weak KB. The proof follows closely the lines of the standard topological completeness proof of S4 epistemic logic for the knowledge modality K as carried out in Aiello et al. (2003). The only novelty is the construction of an appropriate dense nucleus $N_B \in NUC(OH)_d$ that is used to define the semantics of the belief modality B.

We start with the construction of a topological space (H, OH) for the canonical topological model of weak KB. Let φ be any well-formed formula of the bimodal extension L_{KB} of classical Boolean propositional logic. Call a set Γ of formulas L_{KB}-consistent if for no finite set $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma$ we have $KB \vdash \neg (\varphi_1 \wedge \ldots \wedge \varphi_n)$. A consistent set Γ is called maximally consistent if there is no consistent set of formulas properly containing Γ. Due to Lindenbaum's lemma (cf. Blackburn et al. (2001, Lemma 4.17, p. 197)) any consistent set of formulas can be extended to a maximal consistent one. It is well known that Γ is maximally consistent iff for any formula φ of L_{KB}, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$, but not both.

Now we can construct a topological space of maximally consistent sets of formulas for L_{KB} in a quite analogous way as this has been done for L_K (cf. Aiello et al. (2003)).

<u>(5.1) Proposition</u>. Define the canonical topological space (H, OH) for L_{KB} is as follows:

(i) H is the set of all maximally consistent sets Γ_{max} of formulas of L_{KB}.

(ii) For $\varphi \in L_{KB}$ let $[\varphi] := \{\Gamma_{max} \in H; \varphi \in \Gamma_{max}\}$. Define $S_K := \{[K\varphi]; \varphi \in L_{KB}\}$. Then OH is defined as the set of subsets of H generated by arbitrary unions of the S_K.

(H, OH) is a topological space, called the topological space of the canonical model of L_{KB}.

Proof. One has to show that S_K is a basis for a topology of H. This is carried out exactly in the same way as is done for the analogous assertion for L_K in Aiello et alii (2003, Lemma 3.2) by replacing L_K by L_{KB}.♦

Obviously, for the modal operator Int of (H, OH) a truth lemma can be proved in the same way as is done in Aiello et al. (2003) for the interior operator of the canonical topological space for S4.

Thus, the only missing ingredient for a full truth lemma of the bimodal logic of weak KB (and thereby a completeness theorem for weak KB) is the construction of an appropriate belief operator N° that satisfies a truth lemma for (H, OH). This will be carried out now. The key for this construction is the following observation:

(5.2) Lemma. For all formulas φ of L_{KB} one has $[B\varphi] = [BK\varphi] = [KB\varphi]$, i.e., the sets $[B\varphi]$ are basic open sets of the topological space (H, OH). Moreover, one has $[B\varphi] = [B^n\varphi]$ for all $n \geq 1$.

Proof. By the axioms (PI) and (T) one obtains that the formulas $B\varphi$ are logically equivalent to $KB\varphi$ and to $BK\varphi$. Hence, the sets $[B\varphi] = [KB\varphi]$ are basic open sets of (H, OH). Due to (N)* one has $[B\varphi] = [B^n\varphi]$ for all $n \geq 1$.♦

Now all ingredients are available for the definition of the canonical topological model of weak KB-logic. The construction of the underlying topological space is standard. The point is the definition of an appropriate nucleus that takes care of the modal operator B.

Since the lattice SL(OH) of sublocales of (H, OH) is closed with respect to arbitrary intersections by (4.14) the sublocale S(B) of the intersection of all sublocales that contain the elements of $\{[B\varphi]; \varphi \in L_{KB}\}$ exists. It is denoted S(B). As S(B) is a sublocale, for $i:S(B)\rightarrow OH$ an adjoint map $j:OH\rightarrow S(B)$ exists such that a nucleus $ij:OH\rightarrow S(B)\rightarrow OH$ is defined. This nucleus will be denoted by N_B. Clearly, $N_B \in NUC(OH)_d$, i.e., N_B is a dense nucleus, since by the axiom of consistency (CB) one has

$$[B(\bot)] \subseteq [B(\neg \bot]^C = [B(\top)]^C = [\top]^C = \emptyset.$$

The nucleus N_B defines a consistent belief operator $N_B°$. This $N_B°$ is used to define the canonical topological model of <u>weak</u> KB logic as follows:

<u>(5.3) Definition</u>. The canonical topological model of weak KB-logic is defined as (H, OH, N_B, μ_{NB}) with

(i) The elements of H are the maximally consistent sets of formulas Γ_{max} of L_{KB}.

(ii) The topology OH is generated by the basis of open sets $\{[K\varphi]; \varphi \in L_{KB}\}$.

(iii) The belief operator $N_B°$ is defined by the nucleus N_B generated by the sublocale S(BH) that is generated by the set of open subsets $\{[B\varphi]; \varphi \in L_{KB}\}$.

(iv) $\mu_{NB}(\varphi) := \{\Gamma_{max}; \Gamma_{max}$ is a maximally consistent set of formulas of L_{KB} with $\varphi \in \Gamma_{max}\}$.

This definition of the canonical model for weak KB is a straight-forward generalization of the analogous definition of the canonical model for K to be found in Aiello et al. (2003).[15] The

[15] According to Aiello et al. (2003, p. 896) the canonical topology of the canonical space is the intersection of the Kripke topology and the Stone topology. This entails that this space is compact and dense-in-itself.

definition of the belief operator $N_B°$ is based on the observation that the sets $\{[B\varphi]; \varphi \in L_{KB}\}$ are basic open sets of OH and $[B\varphi] = [B^n\varphi]$ for all $n \geq 1$. Therefore they define a dense sublocale of OH that corresponds to a dense nucleus. Since $N_B° \in NUC(OH)_d$, from (3.13) it follows that (Int, $N_B°$) satisfies the rules and axioms of a weak Stalnaker system, i.e., (H, OH, N_B, μ_{NB}) belongs to the class of sound models of weak KB-logic.♦

Now we prove a truth lemma for the canonical topological model (H, OH, N_B, μ_{NB}). This will be the essential ingredient for the proof of the desired completeness theorem:

(5.4) Theorem (Truth Lemma TL). Let (H, OH, N_B, μ) be the canonical topological model of weak KB. For all modal formulas φ of L_{KB} and all $w \in H$ one has: $w \vDash_{LKB} \varphi$ iff $w \in [\varphi]$.

Proof. Induction on the complexity of φ. The base case follows from the definition from (4.10). The case of the Booleans is also well known, see (Aiello et al. (2003, p. 895)). The interesting cases are the modal operators K and B. The proof for K is just a rehearsal of the well-known proof of the truth lemma TL for the unimodal case for K. Thus, it only remains to prove TL for B. Analogously to the proof for K, the proof of TL for B is divided into two parts:

(i) From truth to membership (If $w \vDash_{LKB} \varphi$ then $w \in [\varphi]$).

(ii) From membership to truth (If $w \in [\varphi]$ then $w \vDash_{LKB} \varphi$).

Proof of (i): Suppose $w \vDash_{LKB} B\varphi$. That means that there is a $U \in OH$ such that

$$w \in N(U) \text{ and } \forall v(\text{If } v \in U \text{ then } \forall v \in U(M, v \vDash K\varphi)).$$

Since we may assume that TL holds for K in OH this may be simplified to

$$w \vDash_{LKB} B\varphi \text{ iff there is a } K\psi \in OH \text{ such that } w \in BK\psi \text{ and} [K\psi] \subseteq [K\varphi].$$

Since (H, OH) is a topological model of S4 (with respect to K), due to the completeness of S4 the inclusion $[K\psi] \subseteq [K\varphi]$ entails the validity of $K\psi \rightarrow K\varphi$ in S4. By necessitation with respect to B in weak KB also $B(K\psi \rightarrow K\varphi)$ is valid in weak KB. Since B satisfies (K) in weak KB we have that $B(K\psi \rightarrow K\varphi) \rightarrow (BK\psi \rightarrow BK\varphi)$ is valid we obtain by MP that $(BK\psi \rightarrow BK\varphi)$ is valid in weak KB. Since weak KB is sound of (H, OH) we obtain that $[BK\psi] \subseteq [BK\varphi]$. By (5.2) we have $[BK\varphi] = [B\varphi]$. Thus, we eventually conclude $w \in [BK\psi] \subseteq [BK\varphi] = [B\varphi]$. In other words, "Truth entails membership".

<u>(2) From Membership to Truth</u>: Proof by induction on the complexity of formulas and reductio ad absurdum. Suppose $w \in [B\varphi]$. We can assume that the first part of TL "From truth to membership" has been proved for B. Suppose $w \not\models B\varphi$. Then by definition of $w \not\models B\varphi$ this is equivalent to

(i) $\text{NOT}(\exists U \in OH\ (w \in N(U)\ \&\ \forall v(\text{If } v \in U \text{ then } v \models K\varphi))$

Since we can assume that TL is already proved for K this can be simplified to

(i') $\text{NOT}(\exists U \in OH\ (w \in N(U)\ \&\ U \subseteq [K\varphi]))$

This is equivalent to

(ii) $\forall U \in OH\ (w \notin N(U) \text{ OR NOT } (U \subseteq [K\varphi]))$

In order to carry a reductio one has to find a U for which (ii) is false. Obviously, this is the case for $U = [K\varphi]$, since $[K\varphi] \subseteq [BK\varphi] = [B\varphi]$ by (5.2) and we have assumed that $w \in [B\varphi]$. Thus, the reductio ad absurdum has been carried out. Thereby the proof of the truth lemma for the modal operator K and the modal operator B is completed.♦

Now a completeness theorem for weak KB can be proved in the canonical way:

<u>(5.5) Completeness Theorem for weak KB</u>. For any consistent set of formulas Γ of L_{KB} one has

$$\text{If } \Gamma \models \varphi \text{ then } \Gamma \vdash_{wKB} \varphi.$$

Proof. Suppose that NOT($\Gamma \vdash_{wKB} \varphi$). For the proof of (5.5) we have to prove that this supposition entails NOT($\Gamma \vDash \varphi$). Then $\Gamma \cup \{\neg \varphi\}$ is consistent, and by a Lindenbaum Lemma it can be extended to a maximally consistent set $\Gamma_{max} \in H$ with $\{\neg \varphi\} \in \Gamma_{max}$, i.e., $\Gamma_{max} \in [\neg \varphi]$. According to the truth lemma (5.4) for (H, OH), this is equivalent to $\Gamma_{max} \vDash \neg\varphi$, whence NOT($\Gamma_{max} \vDash \varphi$). and we have constructed the required counter-model. ♦

In sum, the weak KB logic (2.6) of knowledge and belief is a normal, sound and complete bimodal extension of classical propositional logic CL defined by the two modalities K and B.[16] In the next section it will be shown that for a given knowledge operator Int a wealth of belief operators N° exists such that all the pairs (Int, N°) satisfy the rules and axioms of weak KB. For a given knowledge operator Int the pairs (Int, N°) are partially ordered by the partial order of NUC(OX) defined in (4.2) such that different belief operators N° can be compared with respect to the extent how much they deviate from knowledge Int.

6. ON THE DOXASTIC PLURALITY OF WEAK KB. In this section we deal with a peculiar feature of weak KB-logic, namely, its doxastic plurality. Doxastic plurality means that any topological knowledge operator Int always comes with a plurality of accompanying belief operators N° defined by the dense nuclei $N \in NUC(OX)_d$ such that all quadruples (X, OX, N, μ_N) are topological models of weak KB-logic. In other words, the complete lattice $NUC(OX)_d$ (which will be shown to be even a complete Heyting algebra in a moment) provides a framework for comparing weak KB logics (Int, N°), (Int, N'°) with respect to how much their belief operators N°, N'° deviate from knowledge Int. More precisely, $NUC(OX)_d$ sets up an

[16] If the axiom (NI) of negative introspection is assumed to be valid, the bimodal logic KB boils down to a unimodal logic defined by K since then the belief modality B can be uniquely defined in terms of K, namely B $\leftrightarrow$ = $\neg$K$\neg$K (cf. Footnote 4).

intuitionistic logic of competing belief operators related to one and the same knowledge operator Int.[17]

Up to now, we do not know much about the doxastic plurality of belief operators, since we do not know much about $NUC(OX)_d$. Given a topological space (X, OX) the only dense nuclei of this space that are known are the nucleus Int and the Stalnaker nucleus IntClInt. Thus, it is high time to overcome this shortage by providing other concrete examples of dense nuclei. This is the aim of this section. Moreover, to show that the new plurality of dense nuclei is really interesting for the logic of knowledge and belief, it has to be shown that the belief operators defined by these nuclei are conceptually appealing *as belief* operators. First of all, let us give some concrete examples of (dense) nuclei that are located between Int and IntClInt:

(6.1) Proposition (Macnab (1981, §6). Let (X, OX) be a topological space, $Y \subseteq X$, and $D \in OX$. Define a map $\mathcal{N}$:PX→NUC(OX) by $\mathcal{N}$(Y) by

$$\mathcal{N}(Y)(D) := Int(Y^C \cup D).$$

The nucleus $\mathcal{N}$(Y) is called the spatial nucleus defined by the subspace Y of X.

Proof. $\mathcal{N}$(Y) satisfies requirements (4.1)(i) – (iii) that define a nucleus:

(i): For $D \in OX$, one obtains $D \subseteq Int(Y^C \cup D)$. Hence, $D \subseteq \mathcal{N}(Y)(D)$.

(ii): Clearly, $\mathcal{N}(Y)(D) = Int(Y^C \cup D) \subseteq Int(Y^C \cup Int(Y^C \cup D))$. On the other hand, one calculates $Int(Y^C \cup Int(Y^C \cup D)) \subseteq Int(Y^C \cup (Y^C \cup D)) = Int(Y^C \cup D)$. Hence, $\mathcal{N}(Y)(\mathcal{N}(Y)(D)) = \mathcal{N}(Y)(D)$.

(iii): By the Kuratowki axiom (3.2) (i) one obtains $\mathcal{N}(Y)(D \cap D') = Int(Y^C \cup (D \cap D')) = Int(Y^C \cup D) \cap Int(Y^C \cup D') = \mathcal{N}(Y)(D) \cap \mathcal{N}(Y)(D')$.

Thus, for all subsets Y of X the map $\mathcal{N}(Y)$: OX→OX is a nucleus.♦

[17] This section is somewhat technical. Readers who the general assertion that a knowledge operator Int is always accompanied by many belief operators seems plausible may therefore skip this section on first reading.

(6.2) Proposition. If Y is dense in (X, OX) then $\mathcal{N}(Y)$ is a dense nucleus.

Proof. For Y dense in (X, OX) the Kuratowki axiom (3.2) (i) yields: $\mathcal{N}(Y)(\emptyset) = Int(Y^C) = Cl(Y^{CC})^C = Cl(Y)^C = X^C = \emptyset$.♦

(6.3) Corollary. If Y is dense in (X, OX) the belief operator $\mathcal{N}(Y)$ is a dense nucleus and the pair (Int, $\mathcal{N}(Y)°$) defines a weak KB-logic.

Proof. Since N(Y) is a dense nucleus, by (4.10) the pair of operators (Int, N(Y)°) defines a weak KB-logic.♦

In order to ensure that (6.3) actually yields different dense nuclei it is expedient to assume that the space (X, OX) is a T_D-space (cf. (3.4)).[18] Under this mild restriction one can prove:

(6.4) Proposition. For T_D-spaces (X, OX) the map $\mathcal{N}$:PX→NUC(OX) defined in (6.1) is an order-reversing monomorphism.

Proof. Suppose that Y and Y' are two distinct subsets of X and $\mathcal{N}(Y) = \mathcal{N}(Y')$. Suppose $x \in Y - Y'$. Since (X, OX) is a T_D-space (3.3)(ii), x has an open neighborhood D such that $D - \{x\}$ is open as well. Then we obtain $x \in Int(Y'^C \cup (D - \{x\}))$ but clearly $x \notin Int(Y^C \cup (D - \{x\}))$. This is a contradiction. Analogously, the assumption that there is an $x \in Y' - Y$ leads to a contradiction. Hence Y = Y', i.e., j is a monomorphism. $\mathcal{N}$ is order-reversing by definition.♦

Propositions (6.2) and (6.4) show that for "most" spaces (X, OX) many dense belief operators can be defined by dense subspaces Y of X that differ from Int and IntClInt, respectively. It is expedient to note, however, that for many familiar spaces there are important dense nuclei that

[18] T_D is a rather weak axiom satisfied by most topological spaces that "occur in nature". For instance, Euclidean spaces and, more generally, all T_2-spaces and all T_0-Alexandroff spaces are T_D-spaces.

cannot be characterized in this way. Rather, the belief operators defined by subspaces Y of X turn out to be only the most elementary class of belief operators. Indeed, the structure of NUC(OX) is much more complicated than the rather elementary structure of the classical Boolean algebra PX = {Y; Y $\subseteq$ X}.[19]

The fact that not all belief operators related to the knowledge operator Int arise from dense subsets should to be considered as an advantage of the concept of dense (= consistent) belief operators over dense subspaces. Or, the other way round, it has to be considered as a serious shortcoming of the concept of dense subspaces that there not enough of them. This may be explicated in some more detail as follows. As is well-known, for topological spaces (X, OX) the intersection of dense subspaces Y and Y' in general is not dense. A classic example is the real line (ℝ, Oℝ) for which the set of rational numbers ℚ and set of irrational numbers $\mathbb{Q}^C$ are both dense, but the intersection $\mathbb{Q} \cap \mathbb{Q}^C = \emptyset$ is clearly not dense. Hence, in general, a topological space (X, OX) does not have a unique smallest dense <u>subset</u>. But, due to Isbell's theorem (see (6.5)), there is a largest dense <u>nucleus</u>, namely, Stalnaker's nucleus N_S aka IntCl. More generally, the intersection $\cap Y_\lambda$ of arbitrarily many dense subspaces Y_λ of a topological space (X, OX) is usually far from being a dense subspace of X. Thus, the partial order of dense subspaces of a topological space (X, OX) (partially ordered by set-theoretical inclusion $\subseteq$) is a rather unwieldy structure. In sharp contrast, for dense nuclei of a topological space the situation is quite different. As will be proved in a moment, the set $NUC(OX)_d$ of dense nuclei is a (rather special) complete Heyting algebra.[20] Consequently, the class of belief operators related to a knowledge operator has a rather nice structure.

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[19] Already in Macnab (1981) it is proved that for T_D-spaces (X, OX) there is a Boolean isomorphism between PX and the Boolean algebra of <u>regular</u> elements of NUC(OX) (cf. Macnab (1981, Theorem (6.5)(5)). In contrast, even for the Euclidean line (ℝ, Oℝ) the full structure of NUC(Oℝ) is not fully known up to now (as far as I know).

[20] Roughly, the relation between dense subspaces and dense nuclei of a topological space (X, OX) may be compared with the relation between the field of rational numbers ℚ and the field of complex numbers ℂ with respect to their algebraic qualities. A very simple aspect of this issue concerns the solvability of polynomial equations. While there are enough complex numbers to solve all polynomial equations in a

By Proposition (4.3) we already know that the lattice NUC(OX) of nuclei of a space (X, OX) is a complete Heyting algebra. This result may be used to prove an analogous result for the set of dense nuclei $NUC(OX)_d$ by invoking a famous theorem of Isbell. Isbell's theorem asserts that every topological space (X, OX) has a greatest dense nucleus:

(6.5) Theorem (Isbell's Density Theorem). Let (X, OX) be any topological space. The dense nucleus N_S is the greatest element of $NUC(OX)_d$, i.e., for all dense nuclei N one has $N(D) \subseteq N_S(D)$, $D \in OX$.

Proof. The proof of this theorem goes well beyond the horizon of this paper. The reader is recommended to consult the excellent treatises Johnstone (1982, II. 2.4 Lemma, p. 50/51) or Picado and Pultr (2012, III, 8.3., p.40, also VI, 2.1, p. 101ff.)♦

Isbell's theorem is a very remarkable theorem, since it demonstrates that the dense nuclei of a topological space (X, OX) behave quite differently than the dense subspaces Y of X. More precisely, a space may have more dense nuclei N than dense subspaces (see propositions (6.7) and (6.8)): A pertinent example is the Euclidean line ($\mathbb{R}$, O$\mathbb{R}$) and its disjoint dense subsets $\mathbb{Q}$ and $\mathbb{Q}^C$ which entails that there is no largest dense subspace of ($\mathbb{R}$, O$\mathbb{R}$). Moreover, as will be shown in a moment, Stalnaker's nucleus N_S is not a spatial nucleus for ($\mathbb{R}$, O$\mathbb{R}$) (see (6.7)).
As has been pointed out by Johnstone and others, this difference between nuclei and subspaces may be considered as one of the great advantages of doing topology in the conceptual framework of "pointfree topology" - based on "nuclei", "(sub)locales", and related concepts - instead of traditional set-theoretical topology (cf. Johnstone 1991, p. 87-88). This paper is not the place to discuss this issue in any further depth, however. Just let us note the following elementary corollary of Isbell's density theorem:

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neat and elegant way, this does not hold for the more restricted domain of rational numbers $\mathbb{Q}$. It is quite difficult to say anything general about the solvability of polynomial equations in rational numbers.

(6.6) Corollary. The partially ordered family $NUC(OX)_d$ of dense nuclei is a complete Heyting algebra with bottom element id_{OX} and top element N_S = IntCl. These nuclei correspond to the dense belief operators Int and IntClInt, respectively.

Proof. $NUC(OX)_d$ is a subset of the complete Heyting algebra NUC(OX). By Isbell's theorem (6.5) the largest element of $NUC(OX)_d$ is the regular nucleus IntCl. Hence, $NUC(OX)_d$ is the downset $\downarrow N_S$ of nuclei N that are smaller than or equal to N_S, i.e., $\downarrow N_S := \{N; N \leq IntCl\}$. Thereby $NUC(OX)_d$ inherits canonically the structure of a complete Heyting algebra from NUC(OX) with bottom element id_{OX} and top element IntCl. By definition (4.2) of the partial order ≤ of NUC(OX) one calculates for $N \leq N_S$ that $N(\emptyset) \subseteq N_S(\emptyset) = Int(Cl(\emptyset)) = \emptyset$. Hence, the $N \in NUC(OX)_d$ are indeed dense nuclei.♦

Corollary (6.6) offers a neat intuitionist calculus of belief operators N° related to a given knowledge operator Int: Different belief operators N°, N°' can be compared with respect to riskiness. An operator N° is riskier than N°', i.e., more error-prone than N°', if and only if N°' ≤ N°. The least risky belief operator is, of course, the knowledge operator Int, since by definition $w \in Int(\mu(\varphi))$ always entails that $w \in \mu(\varphi)$, i.e., Int is factive. The riskiest belief operator is Stalnaker's belief operator $N°_S$, since by Isbell's theorem (6.5) $N \leq N_S$ for all $N \in NUC(OX)_d$. Hence, if one is guided by a cautionary principle in stating one's beliefs, it is advisable to base one's beliefs not on N_S but on a less risky operator N even if N_S may be considered as the operator that can be defined in the mathematically most elegant way.

In order to show that spatial nuclei $\mathcal{N}(Y)$ do not tell the whole story about nuclei of (X, OX) it is sufficient to give a prominent example of a nucleus N for which in general no generating subset Y exists:

(6.7) Proposition. For the Euclidean line ($\mathbb{R}$, O$\mathbb{R}$) the regular nucleus N_S is not a spatial nucleus, i.e., there is no subset Y of $\mathbb{R}$ such that $N_S = \mathcal{N}(Y)$.

Proof. Suppose the contrary, i.e., there is a $Y \subseteq \mathbb{R}$ such that $\mathcal{N}(Y)(D) = Int(Y^C \cup D) = IntCl(D)$ for all $D \in O\mathbb{R}$). Clearly, $Y \neq \emptyset$. Assume $x \in Y$. Take $D = \mathbb{R} - \{x\}$. The set D is open in ($\mathbb{R}$, $O\mathbb{R}$) since ($\mathbb{R}$, $O\mathbb{R}$) is a T_2-space (cf. (3.4)). Then, we get $\mathcal{N}(Y)(D) = Int(Y^C \cup D) = Int(D) = D$, but $N_S(D) = \mathbb{R}$. Hence, for ($\mathbb{R}$, $O\mathbb{R}$) the regular nucleus N_S is different from any spatial nucleus $\mathcal{N}(Y)$ of NUC($O\mathbb{R}$) whatsoever.♦

(6.8) Proposition. The join $\mathcal{N}(\mathbb{Q}) \vee \mathcal{N}(\mathbb{Q}^C)$ of the spatial nuclei $\mathcal{N}(\mathbb{Q})$ and $\mathcal{N}(\mathbb{Q}^C)$ is not a spatial nucleus.

Proof. Suppose $\mathcal{N}(\mathbb{Q}) \vee \mathcal{N}(\mathbb{Q}^C)$ is a spatial nucleus $\mathcal{N}(F)$ with $F \subseteq \mathbb{R}$. Then one obtains $\mathcal{N}(\mathbb{Q}) \leq \mathcal{N}(F)$ and $\mathcal{N}(\mathbb{Q}^C) \leq \mathcal{N}(F)$. By (6.4) this implies $F \subseteq \mathbb{Q}, \mathbb{Q}^C$ and therefore $F = \emptyset$. Hence $\mathcal{N}(\emptyset)(D) = Int(\mathbb{R} \cup D) = \mathbb{R}$ for all $D \in O\mathbb{R}$. This is a contradiction since $\mathcal{N}(\mathbb{Q})$ and $\mathcal{N}(\mathbb{Q}^C)$ are dense nuclei and therefore $\mathcal{N}(\mathbb{Q}) \vee \mathcal{N}(\mathbb{Q}^C)$ is also dense and at most as large as N_S. Hence $\mathcal{N}(\mathbb{Q}) \vee \mathcal{N}(\mathbb{Q}^C)$ cannot be a spatial nucleus.[21] ♦

Proposition (6.8) is a strong argument for the claim that there are not sufficiently many spatial nuclei for a satisfying theory of nuclei: The finite join N ∨ N' of two nuclei N and N' is a plausible and meaningful operation, if there is any such operation on these objects at all. If the domain of spatial nuclei is not closed under such an operation, this domain must be assessed as seriously incomplete. An appropriate strategy to overcome this deficit is to move from the domain of spatial nuclei to the domain NUC(OX) of all nuclei that may be considered as a kind of completion of the set of spatial nuclei.

[21] With some more effort it can be shown that there exist many spatial nuclei N_1, N_2 in $NUC(O\mathbb{R})_d$ such that $N_1 \vee N_2$ is non-spatial and different from $N_S = IntCl$.

For some spaces, however, N_S is a spatial nucleus. A simple case is provided by polar spaces introduced by Rumfitt to deal with the Sorites paradox in the framework of classical Boolean logic (cf. Rumfitt 2015).[22]

<u>(6.9) Definition</u>. Let X be a set and $\emptyset \neq P \subseteq X$ be a set of distinguished elements to be interpreted as prototypes, paradigmatic cases, or poles. Assume that for all $x \in X$ there is a non-empty set $m(x) \subseteq P$ of poles p. For all $x \in X$ and all $p \in P$ the sets m(x) are assumed to satisfy two requirements: (i) $\emptyset \neq m(x) \subseteq P$, and (ii) $m(p) = \{p\}$. These assumptions define a map X—m—>2^P in the obvious way. The map m is called a pole distribution and denoted by (X, m, P).♦

<u>(6.10) Proposition</u>. A pole distribution (X, m, P) defines a topology on X (cf. Rumfitt (2015), Mormann (2020)): For $A \subseteq X$ define the interior operator Int:PX→PX of the pole topology by $x \in Int(A) \Leftrightarrow (x \in A \text{ and } m(x) \subseteq A)$. Then the operator Int is a Kuratowski interior kernel operator and defines a topology OX. More precisely, (X, OX) turns out to be a (submaximal) Alexandroff space, i.e., arbitrary (not only finite) intersections of open sets are open.♦

More precisely, the topology of a polar spaces defined by (X, m, P) is calculated as follows:

<u>(6.11) Proposition (Topology of polar spaces)</u>. Let the pole distribution (X, m, P) define the polar space (X, OX). Then for $p \in P$ and $x \in X - P$ the following holds:

$Int(\{p\}) = \{p\}$, $Int(\{x\}) = \emptyset$, $Cl(\{x\}) = \{x\}$, $Cl(\{p\}) = \{x; p \in m(x)\}$

$IntCl\{p) = \{x; \{p\} = m(x)\}$, $\{x\} \cup m(x)$ is the smallest open set that contains x.

<u>Proof</u>. Just check the definitions. See Mormann (2022, Proposition 2.5).♦

[22] Rumfitt's polar spaces have been well known in topology. They may be characterized as submaximal Alexandroff spaces (cf. Bezhanishvili, Esakia, and Gabelaia (2004), Mormann (2022)).

From (6.11) one reads off that a polar space (X, OX) is a scattered T_D-space, i.e., X contains no non-empty dense-in-itself subsets (cf. Steen and Seebach Jr. (1982, p. 33)). Hence, we may apply a famous theorem of Simmons in order to obtain that NUC(OX) is Boolean:

<u>(6.12) Theorem</u>. (Simmons (1980), Picado and Pultr (2012)). Let (X, OX) be a scattered T_D-space. Then the map $\mathcal{N}$:PX→NUC(OX) defined in (6.1) is a Boolean isomorphism, i.e., all nuclei N are spatial, i.e., $N(D) = \mathcal{N}(Y)(D) = Int(Y^C \cup D)$, for some $Y \subseteq X$.♦

Clearly, by (6.11) a subset Y is dense in a polar space (X, m, P) iff $P \subseteq Y$. Hence, polar spaces are scattered T_D-spaces and we obtain:

<u>(6.13) Proposition</u>. Let (X, m, P) define a polar space (X, OX), $D \in OX$. Then $NUC(OX)_d = \{Y; P \subseteq Y \subseteq X\} = 2^{X-P}$. The bottom element Ø of 2^{X-P} corresponds to the largest dense subset of (X, OX), namely X, and is related to the nucleus id_{OX} by $\mathcal{N}(X)(D) = Int(X^C \cup D) = D$, and the top element 1 corresponds to the smallest dense subset of (X, OX), namely $P \subseteq X$ and is related to the Stalnaker nucleus N_S by $\mathcal{N}(P)(D) = Int(X\text{-}P \cup D) = IntClInt(D)$.♦

In sum, for the special case of polar spaces (X, OX) the family of consistent nuclei N, or, equivalently, the family of corresponding belief operators N°) related to Int has the structure of an atomic Boolean algebra. This entails, in particular, that for every nucleus N there exists a "complementary" nucleus N* such that $N \wedge N^* = id_{OX}$ and $N \vee N^* = IntCl$.

Moreover, propositions (6.9) and (6.7) show that the logics of belief (encapsulated in the complete Heyting algebras $NUC(OX)_d$ of polar spaces and "ordinary" topological spaces (like Euclidean spaces), respectively, strongly differ: For polar spaces the operator IntCl is spatial, i.e., induced by the subspace $P \subseteq X$ while for Euclidean spaces IntCl is not spatial.

As already explained in the previous section, $N_S°$ is the riskiest choice for a belief operator that is compatible with Int and still consistent. Certainly, $N_S°$ is an elegant choice for a belief operator that is available for all kinds of topological spaces whatsoever. Nevertheless, if one subscribes to a cautionary principle there is no reason to stick to the riskiest operator just for aesthetic reasons. For dense-in-themselves topological spaces there is a less risky alternative to N_S, namely, the perfect belief operator defined by the nucleus N_{PF}. This may be explained as follows.

The nucleus N_{PF} is a general operator in the sense that it relies on general features of the concept of topology and not on specific features of the underlying topological spaces. First recall that a subset $A \subseteq X$ is dense-in-itself in (X, OX) iff A has no isolated points (cf. (3.5) (iv)). Since the arbitrary union of dense-in-themselves subsets of X is dense-in-itself, the closure Cl(A) of a dense-in-itself set A is dense-in-itself, and the empty set Ø is clearly dense-in-itself (Kuratowski (1966)), for all closed subsets $A \subseteq X$ the largest dense-in-itself subset PF(A) of A is a well-defined concept. Clearly, for $A \in CX$ the set PF(A) is a closed and dense-it-itself set, i.e., a perfect set (cf. Steen/Seebach Jr. (1982, p.6)). Hence, PF(A) is usually called the perfect kernel of A (cf. Zarycki (1930), Oxtoby (1976)).

<u>(6.14) Proposition</u>. Let (X, OX) be a topological space, $A, D \in CX$. The perfect kernel PF(A) of A has the following properties:

(i) $PF(A) \subseteq A$ and PF(A) is closed.

(ii) If $A \subseteq D$ then $PF(A) \subseteq PF(D)$. (Monotony)

(iii) PF(PF(A)) = PF(A). (Idempotence)

(iv) $PF(A \cup D) = PF(A) \cup PF(D)$. (Distributivity with respect to $\cup$)[23].

[23] Zarycki (1930) erroneously claimed that PF is distributive with respect to $\cup$ for all subsets A, D of X, not only for closed ones. This error was observed by Vaidyanathaswatasmy (1947) and Oxtoby (1976). Oxtoby proved a more complex formula for all subsets A, D that yields (6.12)(iv) for closed sets. For our purposes it is sufficient that distributivity ((6.14)(iv)) holds for closed subsets of X. Simmons (1978, 1982) stated (without explicit proof) that (6.14) (iv) holds, i.e., that the operation PF is distributive with respect to $\cup$ for closed sets. He then went on to show that $PF(A^C)^C$ is a nucleus. Actually, Oxtoby proved his more general results on PF only for T_1-spaces. A closer inspection of his proof, however, reveals that for the distributivity of PF his proof works for all topological spaces.

Proof. The proofs of (i) – (iii) are obvious. A detailed proof of (iv) can be found in Oxtoby (1976).♦

If the space (X, OX) is dense-in-itself one has PF(X) = X. Then we can define a dense "perfect belief operator" N_{PF} as follows:

(6.15) Proposition. Let (X, OX) be topological space that is dense-in-itself, $A \in OX$. Define the operator N_{PF}: OX→OX by

$$N_{PF}(A) := PF(A^C)^C.$$

Then N_{PF} is a dense nucleus. N_{PF} is called the perfect nucleus of (X, OX).

Proof. We have to prove that N_{PF} satisfies the conditions (4.1)(i) – (iii) that define a nucleus.

(4.1) (i): Since the closure of a dense-in-itself subset of A^C is dense-in-itself and A^C is closed one clearly has that $PF(A^C) \subseteq A^C$. Hence $A = A^{CC} \subseteq PF(A^C)^C$.

By definition of PF the proofs for (ii) and (iii) are obvious. A detailed proof of a stronger and more general result than (6.14)(iv) can be found in Oxtoby (1976, section 2). Thus, N_{PF} is a nucleus.

If X is dense-in-itself one has X = PF(X) and N_{PF} is a dense nucleus since $N_{PF}(\emptyset) = PF(\emptyset^C)^C = (PF(X))^C = X^C = \emptyset$.♦

As explained before, the nucleus N_{PF} defines in a canonical way a belief operator denoted by $N_{PF}°$. By (4.8) we obtain:

(6.16) Theorem. Let (X, OX) be a dense-in-itself topological space. Then the pair (Int, $N_{PF}°$) of the interior operator Int and the perfect belief operator $N_{PF}°$ satisfies the rules and axioms of a weak KB system.♦

Examples of dense-in-themselves spaces abound. For instance, Euclidean spaces and other Polish spaces[24] are dense-in-themselves. Hence, (6.14) has wide applications. By Isbell's theorem the perfect nucleus N_{PF} is smaller than or equal to N_S, i.e., for all $A \in OX$ one has $N_{PF}(A) \subseteq N_S(A)$. For some spaces it can be shown that N_{PF} is indeed strictly smaller than N_S. Ignoring the mild restriction that the perfect nucleus N_{PF} is only defined for dense-in-itself spaces (X, OX) we may say that $N_{PF}°$ is another "general" belief operator (besides $N_S°$) in the sense that its definition does not depend on the specifics of the topological structure of (X, OX) as is the case, for instance, for spatial operators N(Y) defined by dense subsets $Y \subseteq X$. In other words, that forming beliefs on the basis of the perfect belief operator $N_{PF}°$ is, with respect to generality, on an equal footing as $N_S°$. It is natural to ask, whether $N_{PF}°$ and $N_S°$ are really different. The real line ($\mathbb{R}$, O$\mathbb{R}$) shows that $N_{PF}°$ is different from $N_S°$: Consider the Cantor dust D. As is well known, D is a perfect set and nowhere dense in $\mathbb{R}$, i.e., IntCl(D) = Int(D) = Ø. Hence D^C is open and one calculates for the belief operator $N_S°$ and $N_{PF}°$, respectively:

$$
\begin{aligned}
N_S°(D^C) &= \text{IntClInt}(D^C) = \text{IntCl}(D^C) = \mathbb{R}. \\
N_{PF}°(D^C) &= PF(D^{CC})^C = PF(D)^C = D^C.
\end{aligned} \tag{6.17}
$$

Hence, on ($\mathbb{R}$, O$\mathbb{R}$) the perfect belief operator $N_{PF}°$ is strictly smaller than Stalnaker's $N_S°$.♦

The Heyting algebra NUC(OX) defined by the underlying topological structure (X, OX) brings to the fore the doxastic plurality of weak KB-logic, i.e., the fact that there are many belief operators N° related to one given knowledge operator Int.

Concentrating on Stalnaker's $N_S°$ amounts to a considerable simplification. Following Stalnaker (2006), Baltag et al. (2019) rightly emphasize as an important feature of Stalnaker's

[24] A Polish space is a separable topological space that is homeomorphic to a complete metric space (cf. Jech (2002, Definition (4.12), p. 44)).

KB system that in KB the only admissible belief operator $N_S°$ can be defined in terms of the knowledge operator, namely, as $N_S°$= IntClInt.

According to the authors,

> this proposition constitutes one of the most important features of Stalnaker's combined system KB. This equivalence allows us to have a combined logic of knowledge and belief in which the only modality is K and the belief modality B is defined in terms of the former. We therefore obtain "...a more economical formulation of the combined belief-knowledge logic... (Baltag et al. (2019, p.221))

"Economy" is certainly an important feature of logical systems, but one may ask whether such an "economy" for a logic of knowledge and belief is actually desirable. It may be appropriate also to take into account that for the topological knowledge operator Int of an arbitrary topological space a numerous family of belief operators exists that are compatible with Int in the sense that all pairs (Int, N°) satisfy the axioms of weak KB logic. Acknowledging this fact adds a plausible dosage of epistemological pluralism and simultaneously maintain the "spirit" of Stalnaker's logic of knowledge and belief. It relates the operator of knowledge Int and the operators of belief N° in a more flexible manner than is done traditionally, when either N° is uniquely defined by Int, or Int is uniquely defined by N°.

We already have ensured that there exist many belief operators that satisfy the formal conditions that can reasonably be expected to hold for good belief operator. It remains to show that these operators are also philosophically plausible. In the following I'd like to argue that the class of novel belief operators introduced in this paper inherit their plausibility more or less directly from the philosophical plausibility of Stalnaker's operator classical belief operator $N_S°$ = ClInt that conceives (a strong version of) believing φ as not knowing that one does not know that φ (Stalnaker (2006, p. 195), Baltag et al. (2019, p. 220)). For extremally disconnected spaces (X, OX) this definition of full (or strong) belief – "Belief as possibility of knowledge" - is rendered formally as

(6.18) $B\varphi := \neg K \neg K\varphi$ or, in topological terminology N° := ClInt

As has been discussed in full detail this definition of belief has quite nice properties, namely, the pair (Int, ClInt) defines a topological model of (full) KB system iff the underlying space (X, OX) is extremally disconnected. For the more comprehensive class of all topological spaces, however, ClInt scores rather badly. As is easily calculated, already on (ℝ, Oℝ) the operator ClInt does not satisfy the axioms of (PI) of positive introspection, the axiom (CB) of consistency, nor the Kripke axiom of normality (cf. (2.6)(i)).[25] In other words, for general topological spaces (X, OX), ClInt is certainly not an acceptable belief operator.

There is a way out of this impasse. If one switches from ClInt to IntClInt the new operator preserves almost all plausible features of ClInt that qualified it as a nice normal belief operator. More precisely, IntClInt is a weak KB operator for all topological spaces, i.e., IntClInt satisfies all axioms of KB logic except the axiom (NI) of negative introspection. In other words, (Int, IntClInt) defines a weak KB system, but fails to be a (full) KB system, if (X, OX) is not an extremally disconnected space. Thus, IntClInt may be considered as a very well-behaved generalization of Stalnaker's original operator:

(i) it is conservative in the sense that it does not change anything for extremally disconnected spaces (X, OX), and

(ii) it minimally modifies the original ClInt where it is necessary.

Thus, the new general definition of belief IntClInt ("knowledge of the possibility of knowledge") faithfully preserves the spirit of Stalnaker's account of belief and renders it applicable to a much larger domain of topological universes than just extremally disconnected ones. Thus, everybody who considers Stalnaker's arguments that (full justified) belief ClInt as "conceptual possibility of knowledge" is philosophically convincing for belief in the case of

[25] Already Stalnaker (2006) pointed out that on general topological spaces the operator ClInt does not define a (reasonable) belief operator, since it is not a normal operator, i.e., does not satisfy (2.6)(i). (cf. Stalnaker (2006, p. 195)). Elementary instances for this fact are already obtained by the Euclidean line (ℝ, Oℝ): For A = (0, 1) and D = (1, 2) one calculates ClInt(A ∩ D) = Ø and ClInt(A) ∩ ClInt(D) = {1}. This contradicts the normality of ClInt.

extremally disconnected spaces, should accept IntClInt ("knowledge of possibility of knowledge") as a good formal explication of this concept in the more general case of arbitrary topological spaces.

Conceiving IntClInt as a good formal explication for belief also renders philosophically respectable other belief operators based on dense nuclei $N \in NUC(OX)_d$ as well. An essential ingredient for the proof of this thesis is Isbell's density theorem. Due to this theorem, one has for all nucleus-based dense belief operators N° = iNInt

(6.19) $$\text{Int}(A) \subseteq \text{iNInt}(A) \subseteq \text{IntClInt}(A).$$

Informally expressed this chain of inequalities asserts: a belief operator defined by any good (i.e., normal, consistent, …) nucleus-based belief operator iNInt is entailed by knowledge Int and is at least as strong as (and therefore entails) belief as defined by knowledge of possibility of knowledge IntClInt.

Nucleus-based belief operators iNInt take into account the specifics of the topological structure of the universes of possible worlds to strengthen the requirement of IntClInt. Topologically, the nucleus-based belief operators iNInt Int may be understood as approximations of Int in the sense that for all $A \in PX$ $\text{Int}(A) \subseteq \text{iNInt}(A)$ and iNInt(A) is extensionally close to Int(A), i.e., their set-theoretical difference $\text{iNInt}(A) \cap \text{Int}(A)^C$ is nowhere dense in (X, OX).

The simplest way of constructing an approximative knowledge operator in this sense is to ignore a small set of anomalies or exceptions that are not contained in Int(A) when a claim justified belief of approximative knowledge is made. Formally, this procedure is described by replacing the knowledge operator Int (applied to A) by the belief operator $\mathcal{N}(Y)°(A) = \text{Int}(Y^C \cup \text{Int}(A))$ for some appropriate set Y. It is important to note that the set Y^C of anomalies or exceptions has to be "small" in some appropriate sense. Not just any approximation of knowledge by Y^C is a reasonable approximation. The point is that Y^C has to be assumed as "small" or "negligeable" in some reasonable sense. Otherwise $\mathcal{N}(Y)°$ would not be dense A, i.d., the approximative belief based on Y would not be consistent. More precisely, one has to assume that $Int(Y^C) = Ø$, i.e., that Y is dense in (X, OX).

Accepting IntClInt as the "correct" generalization of the operator Clint originally defined for extremally disconnected spaces entails that all weak belief operators (i.e., belief operators that do not necessarily satisfy (NI) on general topological universes (X, OX)) may be considered as generalizations of the prototypical operator ClInt defined for extremally disconnected spaces and its logic S4.2. Thus, we may say that the nucleus-based approach of belief preserves the spirit of Stalnaker's approach, and, simultaneously, generalizes it.

The simplest way of asserting an approximative knowledge claim is by ignoring some small set of anomalies that are considered as irrelevant. Formally this is described by belief operators that are defined by spatial nuclei $\mathcal{N}(Y) \in NUC(OX)_d$, with Y a dense subspace of X. But depending on the topological structure, many other (non-spatial) methods of defining dense nuclei exists. As we have shown, already the formation of the finite supremum $N \vee N'$ of spatial nuclei N and N' may lead us beyond the realm of spatial nuclei. Thus, admitting only spatial nuclei for the definition of belief operators is rather inconvenient. Thus, one should give up the restriction to spatial nuclei and their belief operators and accept the larger domain $NUC(OX)_d$ of belief operators that can be defined by dense nuclei in general. The move from spatial nuclei to general nuclei is a kind of completion that is a procedure that takes place quite often in mathematics. An example in elementary algebra is the extension of the field of rational numbers $\mathbb{Q}$ to the field of complex numbers $\mathbb{C}$ in order to deal in a more comfortable way with the problems concerning the solution of polynomial equations.

In some sense, then, the nucleus-based theory of doxastic operators renders the concept of belief an open concept, since for many spaces the domain of nuclei $NUC(OX)_d$ is far from being completely understood up to now.

7. NUCLEI AND THE DERIVED SET OPERATOR. In this penultimate section we resume a topic that was mentioned already briefly in the introduction of this paper, namely, that (at least for certain topological universes of possible worlds) the derived set operator d of the topological structure may be used as a formal model of belief (cf. Steinsvold (2006), Parikh et al. (2007)). More precisely, in this section we want to show that the theory of topological nuclei not only sheds new light on a Stalnaker's account of knowledge and belief but also on the account of belief that is based on the notion of the derived set operator d. As it turns out both accounts of topological epistemology have interesting relations with the theory of nuclei. Let us start with the very definition of the derived set operator in topology:

(7.1) Definition (Steen and Seebach Jr. (1978, p. 5), Parikh et al. (2007, 11.2, p. 332/333)). Let (X, OX) be a topological space. A point $x \in X$ is called an accumulation (or limit) point of a set $A \subseteq X$ iff for each open neighborhood U of x we have $(U - \{x\}) \cap A \neq \emptyset$. The set of all accumulation points of A is denoted by d(A). The set d(A) is called the derived set of A and d is called the derived set operator.♦

As is well known, for each $A \subseteq X$ one has $Cl(A) = A \cup d(A)$, i.e., A is closed iff $d(A) \subseteq A$. In this section it is assumed throughout that (X, OX) is a T_D-space (cf. (3.3)). For this class of spaces one can prove:

(7.2) Proposition. If (X, OX) is a T_D-space and $S \in PX$, then $d(d(S)) \subseteq d(S)$, i.e., d(S) is closed (cf. Bezhanishvili and van der Hoek (2014, p. 373), van Benthem and Bezhanishvili (2007, p. 233, Definition 5.13)).

Proof. First, we should note that for a T_D-space for every open neighborhood U'(x) of x there is a possibly smaller open neighborhood of x such that $U(x) - \{x\}$is also open. By T_D we know that for any x there is an open neighborhood V(x) such that $V(x) - \{x\}$ is open. Hence, if U'(x) is any open neighborhood, then $U(x) := U'(x) \cap V(x)$ is an open neighborhood of x such that $U(x) - \{x\}$ is open. Now we want to show that d(S) is closed, i.e., $x \in d(S)$ entails $x \in d(d(S))$. By definition $x \in d(d(S))$ iff, for all open neighborhoods U(x), we have $U(x) \cap d(S) - \{x\} \neq Ø$. We may assume that $U(x) - \{x\}$ is open. For all $y \in d(S)$ one has that for all open neighborhood V(y) of y one has that $V(y) \cap S - \{y\} \neq Ø$. Clearly $(U(x) - \{x\}) \cap V(y)$ is an open neighborhood of y. Hence, $(U(x) - \{x\}) \cap V(y) \cap S - \{y\} \neq Ø$. Thus, U(x) is an open neighborhood of x such that $U(x) \cap S - \{x\} \neq Ø$, i.e., x is an accumulation point of S. That means $x \in d(S)$.♦

Now let $t(A) := d(A^C)^C$ be the dual operator of d, also called the co-derived operator of (X, OX). By (7.2), for all $A \subseteq X$ the set t(A) is open, since (X, OX) is T_D. We are going to show that t (restricted to OX) is "almost" a nucleus:

<u>(7.3) Proposition</u>. Let (X, OX) be a dense-in-itself T_D-space. The co-derived set operator $t:OX \to OX$ has the following properties for all $A, B \in OX$.

(i) $A \subseteq t(A)$.

(ii) If $A \subseteq B$ then $t(A) \subseteq t(B)$.

(iii) $t(A \cap B) = t(A) \cap t(B)$.

(iv) $t(A) \subseteq t(t(A))$.

(v) $t(Ø) = Ø$.

<u>Proof</u>. Check the definitions.♦

Thus, informally stated, the co-derived operator t is "almost" a nucleus. The only requirement that is missing for t being a nucleus is the inequality $t(t(A)) \subseteq (t(A)$. As is easily checked, in general this shortcoming cannot be eliminated. This can be seen as follows. To find an open A such that $t(t(A)) \neq t(A)$ is clearly equivalent to find a closed set S such that $d(d(S)) \neq d(S)$. A simple example of such a set is given by the following example (cf. Parikh et al. (2007, Example (6.21), p. 332)):

(7. 4) Example. Let S be the subset of real numbers $\mathbb{R}$ defined by

$$S := \{1/n + 1/(n + 1 + m)\} \cup \{1/n\} \cup \{0\}, \text{ for } n , m \geq 1.$$

The set S is obviously closed. Its set d(S) of accumulation points is $\{1/n\} \cup \{0\}$ and the set of accumulation points d(d(S)) of d(S) is $d(d(S)) = \{0\}$, and $d(d(d(S))) = Ø$. Hence $d(d(S)) \neq d(S)$ and equivalently $t(t(S^C)) \neq t(S^C)$.♦

In order to ensure that the co-derivative t is not only a pre-nucleus but even an honest nucleus one has to restrict the class of topological spaces considerably. Instead of dense-in-themselves T_D-spaces one has to specialize to DSO-spaces:

(7.5) Definition (Parikh et al. (2007, p. 334)). A topological space (X, OX) is called a DSO-space[26] if it is a dense-in-itself T_D-space such that d(A) is an open set for each $A \subseteq X$.

A simple example of a DSO-space is provided by the set of natural numbers $\mathbb{N}$ endowed with the finite-cofinite topology ($\mathbb{N}$, O$\mathbb{N}$). In this topology a subset U of $\mathbb{N}$ is open in ($\mathbb{N}$, O$\mathbb{N}$) iff its complement is finite or U = Ø.♦

For DSO-spaces we can prove:

(7.6) Theorem. Let (X, OX) be a DSO-space. Then the co-derived operator t is a dense nucleus.

Proof. By definition of DSO-spaces, for each $A \subseteq X$ the co-derivative d(A) is an open and closed subset of X. Since X is dense-in-it-self, d(A) is dense-in-itself as well. This means that d(A), as also being a closed set, is even a perfect set. Hence, by definition of being perfect

[26] DSO is an acronym for "Derived Sets are Open."

$d(d(A)) = d(A)$, i.e., for all i one has $d^i(A) = d(A)$. Trivially, the analogous equality holds for the co-derivative t, i.e., $t^i(A) = t(A)$. Thus, for DSO-spaces the pre-nucleus t is even an honest nucleus.♦

It should be emphasized that we have already met this nucleus – it is just the perfect nucleus N_{PF} defined in (6.15) by $N_{PF}(A) := PF(A^C)^C$, $PF(A^C)$ defined as the largest perfect (= closed and dense-in-itself) subset of A^C. Since $d = d^2$, $d(A^C)$ is perfect, and therefore a subset of $PF(A^C)$ as the largest perfect subset of A^C. On the other hand, $PF(A^C) = d(PF(A^C)$ is clearly a subset of $d(A) = d(d(A))$. Thus, $PF(A^C) = d(A^C)$ and $N_{PF}(A) = t(A) = d(A^C)^C$.

This relation between t and B_{PF} that exists for DSO-spaces can be generalized to an analogous relation for operators defined for the larger class of dense-in-itself T_D-spaces. One has to observe that for the series t, t^2, t^3 , … of pre-nuclei t^i a supremum $SUP(t^i)$ can be defined. This supremum turns out to be a nucleus that corresponds to the perfect nucleus B_{PF} (cf. (6.15)). The precise construction of $SUP(t^i)$ requires a more comprehensive investigation of the lattice of pre-nuclei of a topological space (X, OX). Extensive investigations in this area have been carried out by Simmons and others (cf. Simmons (1980)). Among other things it has been shown that this lattice of pre-nuclei is, analogously to the lattice NUC(OX) of nuclei, a complete Heyting algebra. We abstain from going into the details, since this would require the introduction of a considerable formal apparatus. Rather, we hope that already the special case of DSO-spaces may suffice to persuade the reader that the theory of nuclei is an appropriate general framework for doxastic operators that comprises not only Stalnaker's combined logics of knowledge and belief but also systems of doxastic operators based on the derived and the co-derived set operator, respectively (cf. Parikh et al. (2007)).

8. CONCLUDING REMARKS. This paper has two main results: First, a topological completeness theorem for Stalnaker's weak logic KB of knowledge and belief has been proved. Second, it has been shown that for weak KB-logic every knowledge operator Int is compatible with many different belief operators N° defined by dense nuclei N ∈ $NUC(OX)_d$. Thereby, for any given knowledge operator Int, a wealth of admissible belief operators N° exists such that all pairs (Int, N°) satisfy all axioms and rules of weak KB-logic.

This plurality of belief operators is an argument for doxastic tolerance: Different epistemic agents may rely on the same knowledge operator Int but subscribe to different belief operators $N_i°$ that all are compatible with Int in the sense that all pairs (Int, $N_i°$) satisfy the axioms of a weak KB-logic.

By subscribing to the axiom of strong negative introspection (NI), this doxastic plurality of different coexisting belief operators is eliminated in favor of one "dogmatically" imposed belief operator $N_S°$. This means, more precisely, that the intuitionistic Heyting algebra of belief operators encapsulated in $NUC(OX)_d$ boils down to the rather trivial Heyting algebra of two elements {Id, N_S}.

The existence of a unique riskiest consistent belief operator $N_S°$ for Int is a consequence of Isbell's density theorem. Mathematicians consider Isbell's theorem as a very important mathematical result. They have not been interested in any "philosophical" interpretation of it. Given the topological interpretation of Stalnaker's KB-logic by Baltag and others and the observation that Stalnaker's belief operator $N_S°$ is related to the regular nucleus IntCl that occurs in Isbell's theorem has the unexpected bonus that one can directly apply - without any extra conceptual effort - Isbell's theorem to obtain a non-trivial epistemological result, namely, the determination of the structure of set of dense belief operators as a complete Heyting algebra. Such short-circuits between mathematics and epistemology are rare. Usually much more philosophical efforts have to be invested to obtain interesting epistemological results. Here, almost everything has been done: On the mathematical side, Isbell's theorem is available, on the epistemological side, the topological interpretation of knowledge is a well-established theory.

Finally, a short remark on the relation between the strong (original) and the weak version of Stalnaker's logic KB of knowledge and belief. Formally, this relation can be described as follows:

The original (strong) version of KB (requiring the validity of (NI)) characterizes the relation between knowledge and belief by the very simple Heyting algebra $\mathbb{Z}_2$ = <Int, IntClInt> corresponding to the two extremal belief operators Int and IntClInt. In contrast, weak KB-logic conceptualizes the relation between knowledge and belief by the elements of a much larger complete Heyting algebra $NUC(OX)_d$ with bottom element Int and top element IntClInt. The structure of this algebra depends on the structure of the underlying topological space (X, OX). The structure of $NUC(OX)_d$ may vary considerably depending on (X, OX): For polar spaces (X, m, P), the Heyting algebra $NUC(OX)_d$ of nuclei has the simple structure of an atomic Boolean algebra $2^{|P|}$. In contrast, for the Euclidean line ($\mathbb{R}$, O$\mathbb{R}$), the structure of $NUC(O\mathbb{R})_d$ is, as far as I know, only partially known up to now.

From an epistemological point of view, the nucleus-based approach of this paper may be characterized as a "knowledge first" approach, since the belief-defining structure $NUC(OX)_d$ may be considered as "derived" from the underlying topological structure (X, OX) defined by the interior operator Int.

Thus, an important task for a comprehensive topological logic of knowledge and belief is the investigation of how the topological spaces (X, OX) underlying the topo-models of our epistemological logic determine the structure of the algebras $NUC(OX)_d$ of dense nuclei N that define the belief operators N° related to the knowledge operator Int as described by weak KB.

The results of this paper may be considered as some modest steps on the path towards a realization of this task.

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