Completions, Constructions, and Corollaries

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Abstract. According to Kant, pure intuition is an indispensable ingredient of mathematical proofs. Kant‘s thesis has been considered as obsolete since the advent of modern relational logic at the end of 19th century. Against this logicist orthodoxy Cassirer’s “critical idealism” insisted that formal logic alone could not make sense of the conceptual co-evolution of mathematical and scientific concepts. For Cassirer, idealizations, or, more precisely, idealizing completions, played a fundamental role in the formation of the mathematical and empirical concepts. The aim of this paper is to outline the basics of Cassirer’s idealizational account, and to point at some interesting similarities it has with Kant’s and Peirce’s philosophies of mathematics based on the key notions of pure intuition and theorematic reasoning, respectively.

0. Introduction. In his *paper A Renaissance of Empiricism in the Recent Philosophy of Mathematics?* (Lakatos 1978) Lakatos painted the history of Western epistemology with a broad brush as follows:

> “Classical epistemology has for two thousand years modeled its ideal of a theory ... on the conception of Euclidean geometry. The ideal theory is a deductive system with an indubitable truth-injection at the top (a finite conjunction of axioms) – so that truth, flowing down from the top through the safe truth-preserving channels of valid inferences, inundates the whole system.” (Lakatos 1978, 28)

From the Euclidean perspective, as Lakatos defined it, there is not much to say about proofs beyond the well known characterization that they are deductively valid arguments that necessarily lead from true premises to true conclusions. In the case of Euclidean geometry this means that the axioms of Euclidean geometry logically imply the theorems of Euclidean geometry. Today we take this assertion as a triviality. Philosophically, it might be less trivial as one think at first view. According to the founding father of modern epistemology – arguably Kant – the just mentioned “triviality” is not a triviality at all but a blatant falsehood. More precisely, in Kant we find the thesis that the axioms of Euclidean geometry do NOT logically imply the theorems of Euclidean geometry. This sounds a bit surprising, to say the least. Be this at it may, Kant insisted that we need something more than just pure logic, namely, pure intuition.

If this is true, then Kant does not belong to the tradition of Euclidean epistemology, as Lakatos defined it. Hence the question arises, whom else we can pick out as a good example of Lakatos's "Euclidean tradition"? A good choice would be Bertrand Russell who vigorously argued for the Anti-Kantian thesis:

> "The axioms of Euclidean geometry do logically imply the theorems of Euclidean geometry. More generally, proofs in mathematics must not contain any non-logical ingredients." (Russell 1903, §6)

Let's call this Russell's thesis. The first time Russell argued for it was in *The Principles of Mathematics* (Russell 1903). *The Principles* are heavily influenced by the logical and mathematical achievements of Peano, Cantor, and Frege. But Russell may be credited to have been the first professional philosopher who argued for this logicist thesis. If one accepts Russell's thesis the philosophy of mathematics on the one hand, and the philosophy of the empirical sciences become neatly separated: On the side of the empirical sciences one has a variety of procedures to obtain scientific knowledge ranging from deductive and inductive arguments to experiments of various kinds. On the other hand, mathematics has only one method of producing knowledge: proving theorems using the new relational logic. Not everybody subscribed to this neat apartheid model between philosophy of mathematics and philosophy of the empirical science. Among the dissenters one may mention (i) Peirce's Semiotic Pragmatism, (ii) Cassirer's Critical Idealism, (iii) Lakatos's Quasi-empiricism.

In the following I'll say nothing about Lakatos but will concentrate on Cassirer with some occasional glimpses to Peirce. I do not aim at elucidating the relation between Peirce's and Cassirer's philosophy in general, rather I'd like to concentrate on one pertinent issue, namely the role of intuition and symbolic constructions for mathematical knowledge. Both accounts may be characterized as attempts to do justice to Kant's philosophy of mathematics and at the same time to overcome the limitations of the traditional Kantian account of pure intuition in the realm of mathematical proofs. This meant in particular to withstand Russell's radical logicist stance according to which something like intuition is completely obsolete for modern mathematical and scientific knowledge.

The fundamental concept of Cassirer's philosophy of mathematics and science was the notion of <u>idealization</u>, or, more precisely, of <u>idealizing completion</u>. According to him, idealization played a fundamental role both in the formation of the concepts of empirical science as well as in the formation of mathematical concepts; idealizing completion was the common source of both mathematical and scientific concept formation. Thus, Cassirer occupied a rather peculiar position among the attempts to philosophically understand modern mathematics and its place among the other sciences: On the one hand, he was a vigorous supporter of the then new relational logic inaugurated by Frege, Peano, Russell, and others. On the other hand, his emphasis on the role of idealization in mathematics <u>and</u> the sciences may be interpreted as an attempt to revive something like Kant's pure intuition, or so I want to argue. The outline of my paper is as follows:

1. The Role of Intuition in Mathematics according to Kant
2. Russell's Logicist Expulsion of Intuition
3. Cassirer's Critical Idealism
4. Idealizations, Constructions and Corollaries
5. Concluding Remarks

<u>1. The Role of Intuition in Mathematics according to Kant</u>. The first issue we have to deal with is Kant's claim that the axioms of Euclidean geometry do not logically imply the theorems of Euclidean geometry. For instance, Kant contended that the theorem that the sum of the angles of a triangle is two right angles, is <u>not</u> implied the Euclidean axioms. First I'll give the textual evidence, then an explanation is offered why Kant made such a claim and why it is correct – even from our more modern point of view.

Kant's "anti-logical" thesis is expressed most clearly in the "Discipline of Pure Reason in Its Dogmatic Employment, where Kant contrasts philosophical with mathematical reasoning:

> "Philosophy confines itself to general concepts; mathematics can achieve nothing by concepts alone but hastens at once to intuition, in which it considers the concept in concreto, although still not empirically, but only in an intuition which it presents a priori, that is, which it has constructed, and in which whatever follows from the general conditions of the construction must hold, in general for the object of the concept thus constructed.
>
> Suppose a philosopher be given the concept of a triangle and he be left to find out, in his own way, what relation the sum of its angle bears to right angle. He has nothing but the concept of a figure enclosed by three straight lines, along with the concept of just as many angles. However long he meditates on these concepts, he will never produce anything new. He can analyze and clarify the concept of a straight line or of an angle or of the number three, but he can never arrive at any properties not already contained in these concepts. Now let the

> geometer take up this question. He at once begins by constructing a triangle. Since he knows that the sum of all the adjacent angles which can be constructed from a single point on a straight line, he prolongs one side of the triangle and obtains two adjacent angels which together equal two right angles. He then divides the external angle by drawing a line parallel to the opposite side of the triangle, and observes that he has thus obtained an external adjacent angle which is equal to an internal angle- and so on. In this fashion, through a chain of inferences guided throughout by intuition, he arrives at a solution of the problem that is simultaneously fully evident and general. In this fashion, through a chain of inferences guided throughout by intuition, he arrives at a solution of the problem that is simultaneously fully evident and general. (B743 – 745).

According to Kant, the only kind of logic available for the philosopher to analyze concepts was the traditional syllogistic logic. As already noted Peirce and Russell, syllogistic logic was not very helpful for proving theorems of geometry and other mathematical theories. Thus, Kant was right in claiming that the axioms of Euclidean geometry do not logically imply the theorems of Euclidean geometry. If we rely on syllogistic logic we need help from a non-logical source to carry out geometrical proofs. For Kant, this source was provided by pure intuition.

Among the experts on Kantian philosophy of mathematics there is no consensus about what exactly is to be understood by Kantian Pure Intuition. Here, I am not interested in doing Kant philology. Rather, I'd like to take Kant as a starting point. For me the important thing about "pure intuition" in a broad Kantian sense is that it renders mathematical proofs take place in some sort of ideal spatio-temporal scenarios, where some constructions are carried out according to certain rules that constitute the ideal domain in which this mathematical activity takes place. Something like this can already be found in Kant as is shown by the following quote from *the Critique of Pure Reason*:

> "I cannot represent to myself a line, however small, without drawing it in thought, that is gradually generating all its parts from a point. Only in this way can the intuition be obtained. ... Geometry together with its axioms, is based upon this successive synthesis of the productive imagination in the generation of figures." (B203 – 204).

This Kantian drawing of straight lines does not take place in real space-time. Rather, it referred to an ideal space-time. More precisely, it refers to an idealized Newtonian space-time. The constructions guided by pure intuition take place in an idealized space-time, where <u>ideal</u> points, <u>ideal</u> trajectories, <u>ideal</u> straight lines etc. existed, and where an <u>ideal</u> subject is able to draw perfect geometrical figures. The underlying theory of this ideal space is Newtonian mechanics.

Thus, in some sense, geometry presupposes Newtonian mechanics. Thus, for Kant, a "mixing" of physical and mathematical ideas was essential to the unity of his philosophy of mathematics. As we shall see similar features may be identified in Cassirer's and Peirce's accounts.

Summarizing, then, I propose to consider "pure intuition" as a faculty involved in checking proofs step by step to see that each rule has been correctly applied: in short: the intuition involved in "operating a calculus" (cf. Hintikka 1980). Kantian pure intuitions should be interpreted as having a strong operational or constructive component. Such a constructive version may help preserve a role for something like intuition even for modern mathematics. Cassirer took one step on this road by emphasizing the role of idealizing completions, Peirce took another one by pointing the importance of diagrammatic constructions. I do not contend that the thoughts of these authors are fully in line with Kant's original *Ansatz*. Rather, I'd like to show that Kant, Peirce, and Cassirer still have useful ideas to offer in the philosophical task of explicating the roles of idealization and conceptual constructions in the formation of mathematical concepts.

2. Russell's Logicist Expulsion of Intuition from Mathematics. Everything changed at the end of the 19th century when modern relational logic arrived on the stage. For Russell, whom we may take as a paragon of an anti-Kantian philosopher of mathematics, the date when this change happened can be determined quite precisely. In a letter to his friend Jourdain he wrote:

> "Until I got hold of Peano, it had never struck me that Symbolic Logic would be of any use for the Principles of Mathematics, because I knew the Boolean stuff and found it useless. Peano's EPSILON, together with the discovery that relations could be fitted into this system, led me to adopt symbolic logic." (From Proops (2006, 276))

"The Boolean stuff" Russell is mentioning here was Boole's *An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities* of 1854. Cum grano salis, then, we may identify "the Boolean stuff" with standard syllogistic logic. Russell considered this kind of logic rightly as rather useless for mathematics. It least, he was convinced that it was not able to do the job of deducing mathematical theorems from mathematical axioms. Thus, before he became acquainted with Peano's logic in 1900, Russell fully agreed with Kant in that "logic" is not of much use for mathematical proofs. Russell's argument for expelling Kantian intuition from mathematics was simply that pure intuition was no longer needed. Through the work of Peano, Cantor, and Frege a much more powerful logic became available that could do everything that in less fortunate times belonged to the ken of pure intuition.

"All mathematics, we may say – and in proof of our assertion we have the actual development of the subject – is deducible from the primitive propositions of formal logic: these being admitted, no further assumptions are required."

Russell's *The Principles of Mathematics* of 1903 is the source for a purely logicist conception of mathematical proofs. From Russell onwards, in the mainstream of philosophy of science mathematical proofs are conceptualized as purely logical derivations. Of course, intuition might continue to play a restricted role insofar as it may be considered as imprescindible to determine which axioms are true. But even from this last resort intuition was expelled, when axioms lost their status of indubitable truths and became mere conventions or implicit definitions. Thereby, the logicist philosophy of mathematics established a neat boundary between the realm of mathematics on the one hand, and realm of empirical science, on the other hand, since, obviously, deductive logic was not the only method to produce knowledge in the empirical sciences.

In one sense, however, Russell and Kant belonged to the same ilk. Both argued for a fixed and stable framework for doing mathematics: According to Kant, mathematics was based on some fixed pure intuition, for Russell it was based on some kind of equally fixed relational logic. Actually, things never stabilized in the way as Russell might have hoped, since the new relational logic never achieved the fixed and unique character that Russell expected.

In contrast to Kant's stable intuition and Russell's stable logic, Cassirer's idealizing completions were an unending conceptual process. The content and structure of this process was not to be determined by armchair philosophy once and for all, but unfolded in an unending process of the evolution of scientific conceptualizations.

3. Cassirer's Critical Idealism. Already in *Kant und die moderne Mathematik* (1907) Cassirer sketched an account that attempted to overcome the logicist separation between mathematics and the sciences by his neo-Kantian approach of critical idealism. He elaborated his approach in *Substance and Function* (1910) and later in the third volume of his opus magnum *The Philosophy of Symbolic Forms* (1929).

Cassirer was a keen partisan of the new relational logic. In *Kant und die moderne Mathematik* he enthusiastically welcomed Russell's *The Principles of Mathematics* as an important achievement for the philosophical understanding of modern mathematics. Nevertheless, he thought that Russell had not fully grasped the philosophical consequences of the new logic.

For reasons of space it is not possible developing here the essentials of Cassirer's philosophy of science account in an orderly manner. Only a very brief and condensed description of his main philosophical theses is given in the following lines.

According to Cassirer, philosophy of science is to be conceived as the theory of the formation of scientific concepts. Scientific concepts do not yield pictures of reality. Rather, concepts provide guidelines for the conceptualization of the world. The fundamental concepts of theoretical physics are blueprints for possible experiences. In this endeavor factual and theoretical components of scientific knowledge cannot be neatly separated. In a scientific theory "real" and "non-real" components are inextricably interwoven. Not a single concept is confronted with reality but a whole system of concepts. The unity of a concept is not to be found in a fixed group of properties, but in the rule, which lawfully represents the mere diversity as a sequence of elements. The meaning of a concept depends on the system of concepts in which it occurs. It is not completely determined by one single system, but rather by the continuous series of systems unfolding in the course of history. Scientific knowledge is a "fact in becoming" ("Werdefaktum"). Our experience is always conceptually structured, there is no non-conceptually structured "given". Rather, the "given" is an artifact of a bad metaphysics. Scientific knowledge does not cognize objects as ready-made entities. Rather, knowledge is organized objectually in the sense that in the continuous stream of experience invariant relations are fixated. The concepts of mathematics and the concepts of the empirical sciences are of the same kind. In the following, I'd like to concentrate on the last claim. As a start, it may be expedient to dwell upon it in some more detail quoting more fully from *Kant und die moderne Mathematik:*

> "What "critical idealism" seeks and what it must demand is a *logic of objective knowledge* (gegenständliche Erkenntnis). Only when we have understood that the <u>same foundational syntheses</u> (Grundsynthesen) on which logic and mathematics rest also govern the scientific construction of experiential knowledge, that they first make it possible for us to speak of a strict, lawful ordering among appearances and therewith of their objective meaning: only then the true justification of the principles is attained." (KMM, 44).

I'll refer to this thesis as the "sameness thesis". It lies at the heart of Cassirer's "critical idealist" philosophy of science. If one subscribes to the sameness thesis, the logicist separation between mathematics and science is not acceptable. According to critical idealism, philosophy of science had to concentrate neither on mathematics, as an ideal science, nor as an empirical science, but rather:

> "If one is allowed to express the relation between philosophy and science in a blunt and paradoxical way, one may say: The eye of philosophy must be directed neither on mathematics nor on physics; it is to be directed solely on the connection of the two realms." (KMM, 48)

Rather, he was looking for a common root from which both physics and mathematics sprang. This common root, he claimed, was the method of introducing ideal elements through which the idealizing character of scientific knowledge was established. In contrast to Russell, he did not attempt to neatly separate mathematics and the empirical sciences.

Today, when dealing with idealization in science, one implicitly assumes that idealization only concerns the empirical sciences. For instance, when one is discussing epistemological and ontological problems of idealization one is dealing with ideal gases, frictionless planes, ideal point masses etc. One rarely takes into account idealization within mathematics. Mathematics is thought to be already is on the ideal side, so to speak. Thus idealization is assumed to solely the empirical realm. According to Cassirer such a theory of idealization starts too late: for him, idealization has a role in mathematics and in the empirical sciences. A comprehensive theory of idealization had to take into account both mathematics and the empirical sciences.

Moreover, he insisted that one should not tackle this problem armed with "philosophical" presuppositions of what are the philosophically correct methods of idealization. The methods of idealization should be studied empirically, so to speak, no philosophical intuition will give us the answer. Rather, this has to be revealed by studying the history of science. Hence, philosophy of science has to pay attention to the ongoing evolution of science, it has to investigate and explicate the formation of scientific concepts in the real history of science.

In a nutshell, then, the sameness thesis contends that the "common foundational syntheses", on which mathematical knowledge and physical knowledge are based, are idealizing completions carried out by the introduction of "ideal elements". The point for Cassirer is that idealization is a common mark of all sciences *qua* sciences.

The primary role of idealization is to underwrite the constructive procedures used in mathematical argumentation, particularly in mathematical proofs. Idealizations aim to single out appropriate domains for doing mathematics in that they warrant that certain symbolic constructions and procedures can be carried out smoothly. In the elementary case of geometry this means, for instance, that certain points exist, more generally, that certain constructions are feasible. Less elementary, and very generally, the axiom of choice may be interpreted as an often indispensable idealizing assumption that guarantees the construction of choice functions, i.e., the possibility of picking out exactly one element of each set belonging to a given set of non-empty sets.

Idealizing completions, as it is to be understood in Cassirer's sense, intend to provide conceptual domains that offer comfortable and promising realms for a variety of symbolic constructions, transactions, and calculations. For instance, in an obvious sense, the domain of natural numbers is less apt to carry out less than elementary calculations than, say, the domain of real or complex numbers. The ideal character of a domain is not to be assessed by passively staring at its perfect and pure character but rather by the variety of possible symbolic actions for which it offers an expedient frame. Or, to put it the other way round, a domain is lacking ideal or conceptual completeness if we meet too many obstacles, exceptions, contradictions, and ad hoc assumptions in the course of our conceptual activities. The complete character of a conceptual domain is particularly evident in the case of geometry. In Here it shows in the possibility of carrying out a variety of geometrical constructions that ensure us of the existence of certain points, lines, and other geometrical entities. For Kant the warrant of the ideal completeness of the realm of geometry was pure intuition that ensured us that the ideal points, lines, and planes of geometry possessed the properties that rendered possible certain constructions. For Cassirer, idealization became a pluralist endeavor that evolved in the ongoing process of science in which the unity of pure thought was constituted. In both cases the ideal character of geometry showed itself in the richness of possible symbolic actions and transactions. In the next section we'll invoke some ideas from Peirce and Hintikka to further elaborate this active or even pragmatic feature of idealizing completion.

<ins>4. Idealizations, Constructions, and Corollaries.</ins> Cassirer's paradigmatic example of an idealizing completion in mathematics was the construction of Dedekind cuts. To understand

the guiding function of this example for the general theory of idealization let us briefly discuss an example of elementary geometry that shows how useful Dedekind completeness is for geometrical constructions. Moreover, this example clearly exhibits the resemblances between the Kant's pure intuition, Cassirer's idealizational and Peirce's diagrammatic thinking for mathematics and the empirical sciences.

Consider the problem of constructing in the Euclidean plane E an equilateral triangle with a given side AB of length 1. A "naive" construction proceeds as follows: Consider the circle C_A around A with radius of length 1 and the circle C_B around B with radius 1. Then the intersection of the two circles yields the third vertex X of the equilateral triangle ABX we were looking for. From a logicist point of view this "intuitive construction" is flawed. Assuming Euclid's original axioms the logicist will object that we do no know that the two circles C_A and C_B actually intersect. They may somehow avoid having a common point X, since one circle may slip through the other. This is more than a remote possibility. Indeed there are "unintended models" of Euclidean geometry showing that this indeed might happen. Consider the "rational plane" $\mathbf{Q}^2$ of ordered pairs of rational numbers (p, q) $\in \mathbf{Q}$ satisfies all geometrical axioms Euclid required, but for it the intersection point X does not exist. Assume A to have the coordinates (O, O) and B the coordinates (0, 1). Then X has the coordinates (1, √3). Therefore it does not belong to the rational plane $\mathbf{Q}^2$.

In order to ensure the existence of the intersection point X, one has to rely on a new axiom that does not appear in Euclid's *Elements*, namely Hilbert's axiom of continuity. As is well known the axiom of continuity is essentially equivalent to Dedekind's axiom that ensures the existence of sufficiently many "Dedekind cuts". In sum, the construction of the equilateral triangle can be carried out successfully only after we are operating on a completed plane, which makes sure that our constructions yield what we expect from them. In other words, the completion of the plane is a necessary presupposition to render possible "naive" constructions as that the vertex X as sketched above.

Completions of this kind are nothing special restricted to elementary geometry. Cassirer convincingly argued that idealizing completions are typical for all areas of mathematics (for some modern examples see Mormann 2008, forthcoming). For Kant some kind of ideal Newtonian space-time determined the variety of these constructions. In contrast to Kant, for the Neokantian Cassirer these conceptual frameworks no longer depend on some fixed ahistorical "pure intuitions", but emerge in the evolution of scientific knowledge itself. This gives Cassirer's philosophy of science a sort of Hegelian flavor (cf. Mormann 2008).

Designing conceptual frameworks or settings for doing mathematics is, however, certainly not the entire story to be told about the evolution of mathematics. The important point is to put these frameworks to work by formulating interesting problems and proving important theorems in them. Cassirer did not say too much about these more concrete aspects of the idealizational practice of mathematics. Here Peirce's philosophy of mathematics comes to the rescue, in particular the insight that Peirce self-confidently characterized as his "first real discovery":

> "My first real discovery about mathematical procedure was that there are two kinds of necessary reasoning, which I call the Corollarial and the Theorematic, because the corollaries affixed to the propositions of Euclid are usually arguments of one kind, which the more important theorems are of the other. The peculiarity of theorematic reasoning is that it considers something not implied at all in the conceptions so far gained, which neither the definition of the object of research nor anything yet known about could of themselves suggest, although they give room for it. Euclid for example, will add lines to his diagram which are not at all required or suggested by any previous proposition, and the conclusion that he reaches by this means says nothing about. I know that no considerable advance can be made in thought of any kind without theorematic reasoning." (Peirce 1976, vol. 4, 49)

For a detailed interpretation of the distinction between theorematic and corollarial reasoning and its significance see Hintikka's paper *C.S. Peirce's "First Real Discovery" and its Contemporary Relevance* (Hintikka 1980). For reasons of space I can give only some brief hints why this distinction can be used to maintain for diagrammatic or symbolic reasoning an indispensable role in mathematics that can withstand the logicist criticism Russell put forward more than a century ago. The first thing to note is that according to Peirce, theorematic reasoning, which in geometry may be characterized through the introduction of now points, lines, and other geometrical objects not present in the original formulation of a problem, is not restricted to geometry. Rather, theorematic reasoning pervades all of mathematics. As Hintikka points out what makes a deduction theorematic is not that it is based on some figures with some more less well defined properties but that we must take into account other individuals than those needed to state the premise of the argument (cf. Hintikka 1980, 306). The new individuals do not have to be visualized. They have to be mentioned and used in the argument. In contrast, an argument is corollarial in the sense of Peirce, if it is only necessary to imagine any case in which the premises are true in order to perceive immediately that the conclusion holds in that case (cf. Peirce (1976, 38)). With some good will, then, it seems appropriate to contend that corollarial reasoning is based on what Russell called "the Boolean stuff", i.e. elementary propositional logic and syllogistic logic. Theorematic deduction, on the other hand, is deduction in which it is necessary to carry out some sort of imaginary experiment in order to bring about some useful effects that may allow drawing further corollarial deductions that finally lead to the desired conclusion (ibidem).

Conceiving the distinction between theorematic and corollarial argumentation in this "logical" way (as Peirce and Hintikka do) it does not fall prey to Russell's logicist criticism. Russell had argued that there is no role for intuitions and figures in serious mathematical arguments after the advent of modern relation logic, because valid geometrical reasoning could be completely formalized. According to him, the reason why figures were thought of as indispensable was simply the incompleteness of earlier axiomatizations. This incompleteness made it necessary for mathematicians to go beyond their own explicit assumptions and to appeal to some sort of Kantian "pure intuition". Peirce, as one of the founding fathers of modern relational logic would be happy to subscribe to Russell's "complete formalization thesis". Nevertheless he would insist on the necessity to distinguish between different logical levels, to wit, corollarial and theorematic arguments. This distinction does not disappear even when geometrical arguments are "formalized". Moreover, as Hintikka has pointed out, if theorematic inference is characterized by the introduction of auxiliary individuals into the argument, one can consider the theorematic character of arguments as a gradual matter (cf. Hintikka 1980, 310).

In other words, one should not consider logic as a monolithic tool but allow for different degrees of complexity - in contrast to Russell's sweeping logicism that lumped together all of logic. Following the insights of Peirce and Cassirer, we obtain three different levels of "logical" reasoning in mathematics (and the sciences) ordered by its degree of complexity:

(1) Corollarial Reasoning

(2) Theorematic Reasoning

(3) Completional Reasoning

Mathematics has to do with all three levels. The most elementary level is the level of corollarial reasoning in the sense of Peirce, characterized logically by the employment of elementary propositional and syllogistic logics. On the second level one finds the realm of theorematic reasoning that has often been characterized as the realm of some kind of "Kantian" intuition. It is important, however, to conceive this kind of "pure intuition" not as a capacity of perceiving some kind of platonic reality but as the ability to carry out symbolic or ideal constructions of some kind or other. Logically, these constructions can be described as the introduction of new individuals and relations leading in particular to an increased level of quantificational complexity. Finally, on the highest level one finds the level of what may be called the level of completional or idealizing reasoning directed to the design of appropriate "settings" or frameworks in which successful diagrammatic or symbolic constructions in the sense of Peirce can be carried out. In other words, the axiom systems are proposals or blue prints how to produce useful constructions.

Idealizing completions offer the framework for theorematic constructions in the sense of Peirce. Frameworks are proposals whose "correctness" has to be assessed pragmatically. Hence Cassirer may be considered as subscribing to a "theoretical pragmatism" according to which

> "... the truth of concepts rests on the capacity "[to lead] to new and fruitful consequences. Its real justification is the effect, which it produces in the tendency toward progressive unification. Each hypothesis of knowledge has its justification merely with reference to this fundamental task..." (Substance and Function, chapter VII, 318ff).

Cassirer's theoretical pragmatism fits well with the implicit pragmatism uphold by working mathematicians. According to it they prefer settings in which theorems "one likes to be true" are actually true (see Mormann 2008). Similarly as theorematic reasoning has been blamed by a narrow logicist philosophy of mathematics as being based on vague intuitions that are of psychological interest only, the choice of "appropriate settings" has often been send off to the realm of subjective whims and matters of taste. The evolution of 20th century mathematics has shown that this assessment is hardly tenable. Constructing idealizing completions has become a routine activity in that there is an explicit theory that is dealing with these kinds of problems. Category theory offers a general framework in which mathematicians can discuss problems of appropriate settings in a way that goes beyond a subjectivist presentation of personal whims and preferences. In category theory problems of idealization and completion become explicit topics on the agenda of mathematics. Questions concerning the idealizational development of mathematical concepts are no longer restricted to informal "philosophical" considerations but obtain the status of well-defined mathematical problems.

5. Concluding Remarks. One of Cassirer's most fruitful philosophical insights in philosophy of mathematics was that idealizing completions such as Dedekind's were more than just mathematically interesting technical achievements. Rather, Dedekind's construction became a

prototype for the idealizational constructions that turned out to be essential for 20th century mathematics <u>and</u> for idealizational constructions in the empirical sciences as well. Evidence for this sweeping claim is not gathered by a priori considerations but by the "empirical observation" that "idealizations" and "completions" have become routine parts of the mathematician's daily work (cf. Mormann 2008). How these completed idealized frameworks organize the practice of mathematics may be studied by relying on the conceptual apparatus centering around the distinction between theorematic and corollarial reasoning introduced by Peirce, Hintikka, and others. In sum, this may offer an contribution Thereby to a naturalist approach of philosophy of mathematics in the sense that "real mathematics" is taken seriously, in contrast to traditional approaches that too closely stick to overly logical models of mathematics. Such a naturalist account would be quite in line with the general Neokantian attitude according to which philosophy has not the task of providing foundations for mathematics, the sciences or any other symbolic endeavors but rather to understand how they work, and to elucidate their ongoing evolution.

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