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Gettier Cases can neither be Known nor Consistently Believed

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Abstract. The aim of this paper is to give an epistemological characterization of Gettier cases from the perspective of a topological account of knowledge and belief. Relying on a topological semantics of Stalnaker's logic KB of knowledge and belief the following will be shown: For most topological models of KB Gettier cases occur in a natural way, i.e., most models of KB contain sets of possible worlds that can be interpreted as Gettier cases where certain true justified beliefs obtain that cannot be considered as knowledge. Topologically, Gettier cases are characterized as nowhere dense sets. This entails that Gettier cases are "epistemically invisible", i.e., they can neither be known nor consistently believed. The proof that Gettier cases cannot be known is elementary, the proof that they cannot be consistently believed relies on a highly non-trivial theorem of modern point-free topology, namely, Isbell's density theorem. This theorem renders it possible to prove this result for all belief operators B compatible with a given knowledge operator K.

Keywords. Gettier cases; Moore cases; Topological epistemology; Interior semantics; Nuclei; Stalnaker's KB-logic; Epistemic Invisibility; Isbell's Density Theorem.

1. Introduction. Understanding the relation between knowledge and belief is an issue of central importance in epistemology. A meanwhile classical problem for the relation between knowledge and belief has been put on the agenda of epistemology by Edmund Gettier, when he asked whether knowledge could be defined as justified true belief or not (Gettier 1963), and put forward several examples that cast serious doubt on an affirmative answer to his question

(Borges, de Almeida, Klein (2017), Machery (2017), Nagel (2013)). Nevertheless, there are philosophers who maintain that the classical account of knowledge as true justified belief is essentially sound notwithstanding Gettier cases (Sellars 1975). The topological account of knowledge and belief to be presented in this paper may help to clarify this issue.

A mild preliminary semi-formalization may be useful to explain how we will proceed in the following. Let X be a set of possible worlds. A subset $A \subseteq X$ is called a proposition. The negation of A is defined as the set-theoretical complement $\complement A$ of A. For a proposition A the subsets $K(A) \subseteq X$ and $B(A) \subseteq X$ are to be interpreted as the propositions "It is known that A" and "It is believed that A", respectively.

A proposition A is <u>true</u> in a world w iff $w \in A$; otherwise, A is false in w. The empty set Ø is to be interpreted as the contradictory proposition that is false at every world w. The set X is the trivial tautological proposition that is true at every world. A proposition A is <u>known</u> in a world w iff $w \in K(A)$. Analogously, a proposition A is <u>believed</u> in w iff $w \in B(A)$. A Gettier proposition G(A) is defined as the proposition such that $G(A) := A \;\&\; B(A) \;\&\; \complement K(A) \neq \emptyset$. A world w is a Gettier world iff $w \in G(A)$. Such a world is an A-world, it is believed that A is a A-world, but nevertheless it is not known that A is an A-world.

The aim of this paper is to characterize Gettier cases G(A) topologically and episte–mologically. On the basis of a topological semantics of Stalnaker's logic KB of knowledge and we will prove the following:

(i) Whether Gettier cases G(A) exist, depends on the topological structure of the models of KB. For most topological universes Gettier cases exist. A special class of topological models, however, can be shown to have no Gettier cases. Moreover, a slight change of the knowledge operator K results in an elimination of Gettier cases.

(ii) Gettier cases are epistemologically invisible for Stalnaker's logic of knowledge and belief. This means, Gettier cases G(A) can neither be known nor consistently believed, since K(G(A)) = Ø and B(G(A)) = Ø.

(iii) Topologically, Gettier cases can be characterized as nowhere dense sets of (X, OX), i.e., intcl(G(A)) = Ø for all Gettier cases G(A).

The proof that K(G(A)) = Ø is elementary.[1] In contrast, to prove that B(G(A)) = Ø for all consistent belief operators B compatible with K requires some more effort, since there are many such operators. In order to show that Gettier cases G(A) are invisible for all consistent belief operators B one has to invoke a highly non-trivial theorem of modern point-free topology, namely, Isbell's density theorem (cf. Johnstone (1982, 2.4., Lemma, p. 50-51), Picado, Pultr, and Tozzi (2004, 2.13, Proposition, p. 50), Picado and Pultr (2012, Ch. III, 8.3., Proposition, p. 40)). More precisely, one has to rely on the logico-mathematical achievement, long ago proved by McKinsey and Tarski (1944) that the modal logic S4 can be identified with the logic of topological spaces. In other words, S4 is sound and complete with respect to the class of models defined as topological spaces (X, OX). Isbell's theorem implies that all consistent belief operators are smaller than or equal to Stalnaker's belief operator B_S. This entails that it is sufficient to show that Gettier cases are epistemically invisible with respect to B_S.

The existing literature on the topological semantics of Stalnaker's KB systems (cf. Baltag et al. 2013, 2017, 2019)), only takes the operator B_S = intclint (cl = the closure operator of (X, OX)) into account. This operator, is, however, not the only one and, arguably, not the most plausible one of a large family of belief operators B that are compatible with K.[2] To obtain a

[1] It is on a similar technical level as Fine's recent results concerning the epistemic invisibility of "Rumsfield ignorance" and "higher-order ignorance" (cf. Fine (2017)).

[2] For general topological spaces (X, OX) for which the interior kernel operator int is interpreted as knowledge operator, a well-behaved belief operator is defined by intclint. This belief operator is the only one that is dealt

more comprehensive understanding of the relationship between knowledge and belief, it seems expedient to consider the whole set of belief operators B compatible with K (cf. Stalnaker (2006), Baltag et al. (2013, 2017, 2019).

In Stalnaker's KB logic, the belief operator B_S is uniquely determined by the knowledge operator K. This is due to the fact that the "axiom of negative introspection" (NI)for the belief operator is assumed. This axiom is, to put it mildly, debatable. It claims that if one does not believe A, one knows that one does not believe A. In this paper, a more flexible relation between knowledge K and belief B is proposed. In a nutshell, this relation may be described as follows. A topological structure (X, OX), i.e., a knowledge operator K defines a family of belief operators B that fit the knowledge operator K in the sense that the pairs (K, B) of modal operators satisfy the axioms of a Stalnaker logic (except the axiom (NI) of negative introspection). The family of belief operators B related to K can be shown to have the structure of a complete Heyting algebra. Stalnaker's belief operator B_S turns out to be the top element of this Heyting algebra, its bottom element is the knowledge operator K that may be considered as the "ideal" or "optimal" belief operator, while B_S is the most "risky" belief operator in the sense that it maximally deviates from knowledge K without becoming inconsistent. Different belief operators compatible with the same K can be compared with each other according to their strengths and how far they deviate from K.

Usually, the relation between belief and knowledge is conceptualized in a rather direct way: either knowledge is defined as a special kind of belief (e.g., knowledge is "justified" true belief, or "correctly justified" true belief, or the like, as in many received accounts of knowledge), or, in some version of "knowledge first" epistemology, knowledge is given conceptual priority and defines belief in a unique way. An example of the latter approach is Stalnaker's, who

with in Stalnaker (2006) and the various papers of Baltag et alii. For a very special class of topological spaces, namely, extremally disconnected spaces, the definition of this operator can be simplified to clint.

defines belief as the epistemic possibility of knowledge (cf. Stalnaker (2006), Baltag et al. (2017)). The theorem of Stalnaker's logic of knowledge and belief the operator B can be defined uniquely in terms of the operator K depends on the assumption that the contentious axiom (NI) of "negative introspection" is assumed to be valid.

As will be shown in this paper, if (NI) is abandoned, the unique definability of belief by knowledge is no longer valid. Rather, a profusion of different belief operators all of which are compatible with the same knowledge operator K appears on the scene.

This plurality is to be considered as an advantage for a good logic of knowledge and belief even it renders the theory more complicated. It may be characterized still as a "knowledge-first" approach – but with a special twist. It is shown that for a given knowledge operator K, there exists a pool of different admissible belief operators B such that all the pairs (K, B) define well-behaved systems of epistemic logic satisfying the axioms of Stalnaker's KB system. Therefore, different cognitive agents who subscribe to the same knowledge operator K may use different agent-specific belief operators B.

The "epistemic invisibility" of Moore and Gettier cases holds for all (consistent) belief operators B compatible with K. This claim is easily stated but not so easily proved. Rather, the proof relies on a theorem of Isbell the proof of which requires a considerable technical apparatus from modern point-free topology (Johnstone (1982), Picado and Pultr (2012)).

Stalnaker's approach (Stalnaker 2006) does not contribute anything to the solution of this problem, since he requires the axiom (NI) of negative introspection as valid thereby forcing a 1-1-relation between the operators K and B. Abandoning (NI) results in a novel pluralist relation between knowledge and belief. For the a more detailed investigation of this relation modern point-free topology offers useful conceptual devices, or so I want to argue.

Due to the trail-blazing result of McKinsey and Tarski (1944) the modal logic S4 is sound and complete with respect to the class of topological spaces. That is to say, every theorem of S4 is a theorem of topology and vice versa.

The construction of more and more ingenuous and contrived examples of Gettier cases by a flourishing "Gettier industry" in the last 50 years suggests that Gettier situations are somehow not exceptional situations, they occur only under extraordinary, especially lucky (or unlucky) circumstances (Zagzebski 2015). The topological approach of Gettier cases reflects this feature in a novel way by characterizing Gettier cases G(A) topologically as nowhere dense subsets of (X, OX). This entails that for the KB-concepts of knowledge and belief Gettier cases G(A) are epistemically invisible in the sense that one cannot know and not even consistently that a world w is a Gettier situation, i.e., $w \in G(A)$. This holds, since $K(G(A)) = Ø$ and $B(G(A)) = Ø$.

The outline of the paper is as follows. In the next section we recall the basics of Stalnaker's logic of knowledge and belief. Section 3 recalls the basics of the topological semantics of Stalnaker's logic elaborated by Baltag et al. in several recent publications (cf. Baltag et al. (2013, 2017, 2019) bvb m). Section 4 characterizes Gettier cases as topologically nowhere dense subsets of topological spaces. In section 5 is it shown that a given topological knowledge operator K is accompanied by a profusion of belief operators B, such that all the pairs (K, B) satisfy the rules and axioms of a weak Stalnaker system. In Section 6 the main theorems are proved, namely, that Gettier cases are epistemically invisible for topological knowledge and consistent belief operators. We close with some remarks suggesting some possible roads for future research on the relation of knowledge and belief in section 7.

2. Stalnaker's Logic of Knowledge and Belief. Now let us recall the basics of the grammar and syntax of the bimodal logic KB of knowledge and belief put forward by Stalnaker (2006). In recent years Baltag, Bezhanishvili, Özgün, and Smets in various recent publications proposed a topological semantics for KB (cf. Baltag et al. (2013, 2015, 2016, 2019)). This semantics will be also be the basis of the used in this paper. The main novelty of this paper is the introduction of a new semantics for the belief operators B of KB, since, in contrast to Stalnaker and Baltag et al. belief operators B are no longer uniquely definable in terms of K. The main ingredient for the more flexible semantics of B is the concept of a nucleus, introduced in the 1980s in modern point-free topology (cf. Johnstone (1982), Picado and Pultr (2012)).

We start with a standard unimodal epistemic language L_K with a countable set PROP of propositional letters, Boolean operators $\neg$, $\wedge$, and a modal operator K to be interpreted as a knowledge operator. The formulas of L_K are defined as usual by the grammar

$$\varphi ::= p \mid \neg p \mid \phi \wedge \psi \mid K\varphi \quad , \qquad p \in \text{PROP}.$$

The abbreviations for the Boolean connectives $\vee$, $\rightarrow$, and $\longleftrightarrow$ are standard. Then, analogously to L_K, a bimodal epistemological language L_{KB} for operators K and B is defined. For a more detailed presentation of topological semantics, the reader may consult the recent papers of Baltag et alii mentioned above.

First, for the sake of definiteness, let us recall the axioms and the inference rules of Stalnaker's KB-systems (cf. Stalnaker (2006), Baltag et al. (2017, 2019): The language of the KB-systems is an extension of classical (Boolean) propositional language by two modal operators K(knowledge) and B(belief) that have to fulfil the following rules and axioms:

<u>(2.1) Definition (Stalnaker's axioms and inference rules for knowledge and belief)</u>.

(CL)	All tautologies of classical propositional logic.	
(K)	$K(\phi \rightarrow \psi) \rightarrow (K\phi \rightarrow K\psi)$	(Knowledge is additive).
(T)	$K\phi \rightarrow \phi$	(Knowledge implies truth).
(KK)	$K\phi \rightarrow KK\phi$	(Positive introspection for K).
(CB)	$B\phi \rightarrow \neg B \neg \phi$	(Consistency of belief).

(PI)	$B\phi \rightarrow KB\phi$	(Positive introspection of B).
(NI)	$\neg B\phi \rightarrow K\neg B\phi$	(Negative introspection of B).
(KB)	$K\phi \rightarrow B\phi$	(Knowledge implies belief).
(FB)	$B\phi \rightarrow BK\phi$	(Full belief).

Inference Rules:

(MP)	From ϕ and $\phi \rightarrow \psi$, infer ψ.	(Modus Ponens).
(NEC)	From ϕ, infer $K\phi$.	(Necessitation).♦

For the topological approach to knowledge and belief, the axiom (NI) plays a special role. It is easily shown that (NI) holds only for topological models of a very special kind, namely, models that are based on extremally disconnected spaces. For the systems of knowledge and belief considered in this paper we will only require that they are weak Stalnaker systems in the following sense:

(2.2) Definition. A bimodal system based on the bimodal language L_{KB} is a weak Stalnaker system iff it satisfies all of Stalnaker's axioms and rules given in (2.1) except possibly the axiom (NI) of negative introspection.♦

There are various reasons for abandoning (NI): first, (NI) is a rather implausible requirement for belief. Second, from a topological perspective, the axiom (NI) is very restrictive. Only a severely restricted class of topological models satisfies (NI), namely, extremally disconnected spaces (ED-spaces). Most spaces that occur "in nature", do not belong to this class. For in–stance, the familiar Euclidean spaces and their derivatives are far from being extremally disconnected. Finally, the axiom (NI) of negative introspection leads to a 1-1-relation between knowledge K and belief B. This is, as I will argue in the following, an implausible and too simplistic understanding of the complex relation between knowledge and belief. Rather, for a given a topological knowledge operator K, it can be shown that many belief operators B are compatible with K in the sense that the pairs of operators (K, B) satisfy all axioms of a weak Stalnaker system.

3. Topological Semantics of Knowledge and Belief Operators. In a topological setting some propositions are necessarily unknown due to the fact that for a non-trivial, i.e., not discrete topology, necessarily some subsets A ≠ Ø of worlds exist that have an empty interior, i.e., int(A) = Ø. Otherwise, the topology is discrete, i.e., int(x) = {x} for all x ∈ X what would amount to the trivial assumption of omniscience.

Now let us recall the basics of the interior semantics for epistemic logic of knowledge and belief as presented by Baltag, Bezhanishvili, Özgün, and Smets (cf. Baltag et al. (2013, 2015, 2016, 2019)). This semantics will be used throughout the rest of this paper. First of all, recall the definition of a topological space:

(3.1) Definition. Let X be a set with power set PX. A topological space is an ordered pair (X, OX) with OX ⊆ PX. The subset OX has to satisfy the following conditions:

(i) Ø, X ∈ OX.

(ii) OX is closed under finite set-theoretical intersections ∩ and arbitrary unions ∪. ♦

The elements of OX are called the open sets of the topological space (X, OX). The set-theoretical complement **C**A of an open set A is called a closed set. The set of closed subsets of (X, OX) is denoted by CX. The interior kernel operator int and the closure operator cl are defined as usual: The interior kernel int(A) of a set A ∈ PX is the largest open set that is contained in A; the closure cl(A) of A is the smallest closed set containing A. For details, see

Willard (2004), Steen and Seebach Jr. (1982), or any other textbook on set-theoretical topology. The operators int and cl are well-known to satisfy the Kuratowski axioms:

<u>(3.2) Proposition (Kuratowski Axioms)</u>. Let (X, OX) be a topological space, A, B $\in$ PX. The interior kernel operator int and the closure operator cl of (X, OX) satisfy the following (in)equalities

(i)	$\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap \mathrm{int}(B)$.	$\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.
(ii)	$\mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A)$.	$\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.
(iii)	$\mathrm{int}(A) \subseteq A$.	$A \subseteq \mathrm{cl}(A)$.
(iv)	$\mathrm{int}(X) = X$.	$\mathrm{cl}(\emptyset) = \emptyset$.♦

The operators int and cl are inter-definable, i.e., int = **C**cl**C** and cl= **C**int**C**. In the following these and some other well-known formulas will be used without explicit mention.

It is expedient to conceive the operators int and cl as operators $PX \xrightarrow{\mathrm{int}} PX$ and $PX \xrightarrow{\mathrm{cl}} PX$ operating on on PX. Then, clearly, OX and CX appear as images of int and cl, respectively. That is, OX =int(PX) = {int(A); A $\in$ PX}, and CX = cl(PX) = {cl(A); A $\in$ PX}. Hence, the concatenation of these operators makes perfect sense. Moreover, for later use we observe that the interior kernel operator int factors as a concatenation of an epimorphism (by a slight abuse of denotation denoted also by int) and a monomorphism i: $PX \xrightarrow{\mathrm{int}} OX \xrightarrow{i} PX$. In the following, concatenations such as intcl, clint and intclint will play an important role.

Dense and nowhere dense subsets of (X, OX) will be essential for discussion of consistent belief operators and investigating the concept of epistemological invisibility:

(3.3) Definition. Let (X, OX) be a topological space with interior operator int and closure operator cl. A subset Y ∈ PX is dense iff cl(Y) = X, Y is nowhere dense iff intcl(Y) = Ø. Clearly, if Y is nowhere dense, the complement **C**Y is dense.♦

(3.4) Examples. (i) For the trivial coarse topology (X, {Ø, X}) every non-empty subset A ∈ PX is dense and only Ø is nowhere dense.

(ii) For the discrete topology (X, PX) only X is dense, and only Ø is nowhere dense.

(iii) Let (**R**, O**R**) be the real line endowed with the familiar Euclidean topology. Let F ⊆ **R** be a finite set. Then F is nowhere dense and the complement **C**F of F is a dense open subset of (**R**, O**R**). Also the infinite set of integers **Z** is a nowhere dense subset of (**R**, O**R**). The set **Q** of rational numbers and the set **CQ** of irrational numbers are disjoint dense subsets of (**R**, O**R**), i.e., **Q** ∩ **CQ** = Ø and cl(**Q**) = cl(**CQ**) = **R**.♦

Perhaps not so well known is the following formula that will be used in the proofs of the main theorems in section 5:

(3.5) Lemma.[3] Let (X, OX) be a topological space with interior kernel operator int and closure operator cl, A, B ∈ PX. If A or B ∈ OX, then

$$\mathrm{intcl}(A \mathbin{\&} B) = \mathrm{intcl}(A) \mathbin{\&} \mathrm{intcl}(B).♦$$

After these preparations, we can tackle the task of defining a topological semantics for the bimodal language L_{KB} defined as an extension of propositional logic by the modal operators K and B. L_{KB} is considered as an extension of the unimodal language L_K by an unimodal

[3] It is well known, of course, that the operator intcl distributes over & if A and B are open. The point of (3.5) is that only one of these factors has to be open in order to ensure that intcl distributes. This fact will be used to prove that Gettier cases G(A) are nowhere dense (cf. (5.4)). An explicit proof of the dual statement of (3.5) (intcl is replaced by clint, and & by ∨, respectively) can be found in Kuratowski and Mostowski (1976, I.8. (18), p. 31).♦

operator B. Consequently, the topological models of L_{KB} will be extensions of the topological models of L_K. The topo-models of L_K are standard in the pertinent literature (cf. Aiello et al. (2003, 2006), Baltag et al. (2019). For convenience of the reader let us briefly recall the basics of this approach:

(3.6) Definition. Let (X, OX) be a topological space and FORM(L_K) the set of well-formed formulas of L_K. A valuation v of the formulas of L_K into (X, OX) is a map that is recursively defined as follows:

$v(p) \subseteq X$, $v(\neg p) = \mathbf{C}v(p)$, $v(\varphi\wedge\psi) = v(\varphi) \wedge v(\psi)$, $v(K\varphi) = int(v(\varphi))$

Other operators such as $\vee$, $\rightarrow$, $\longleftrightarrow$ etc. are defined in terms of $\neg$, $\wedge$ and K as usual and their valuations are defined correspondingly.♦

Analogously to K, the semantics for belief operators B is defined with the help of operators $PX \xrightarrow{B} PX$ that satisfy appropriate structural requirements and can be used to define belief operators that fulfil the axioms of Stalnaker's KB. The following preparatory definition is essential for this task:

(3.7) Definition (Johnstone (1982, 2.2., p. 48f), Borceux (1994, 15.1, Definition, p. 29)). Let (X, OX) be a topological space. An operator $OX \xrightarrow{j} OX$ is called a nucleus of OX iff it satisfies the following requirements

(N1) $A \subseteq j(A)$.

(N2) $j(j(A)) \subseteq j(A)$.

(N3) $j(A \,\&\, B) = j(A) \,\&\, j(B)$.♦

For the definition of consistent belief operators, a special class of nuclei is needed:

(3.8) Definition (Johnstone (1982, 2.4(c), p. 50)). A nucleus $OX \xrightarrow{j} OX$ is called a dense nucleus iff $j(\emptyset) = \emptyset$. ♦

Now the central concept of a topological belief operator B can be defined as follows:

(3.9) Definition. Let (X, OX, v) be a topological model of the language L_K, and let j be a dense nucleus $OX \xrightarrow{j} OX$ of OX. Define a belief operator B as the concatenation $PX \xrightarrow{int} OX \xrightarrow{j} OX \xrightarrow{i} PX$. Then a topological model (X, OX, B, v) of L_{KB} is defined by extending the valuation v defined by formulas of L_K to formulas of L_{KB} by $v(B\varphi) := B(v(\varphi))$. In order to simplify notation, the valuation v of topo-models (X, OX, B, v) will not be explicitly mentioned if not needed. In this way, $v(\varphi)$, $\neg v(\varphi)$, $v(\varphi \wedge \psi)$, $B(v(\varphi))$ etc. are denoted set-theoretically as A, $\complement A$, $A \cap D$, B(A) etc. assuming that $v(\varphi) = A$, $v(\psi) = D$ etc. ♦

The operators B and j uniquely determine each other.[4] Explicitly, one easily observes that $B = i \bullet j \bullet int$ and $j = int \bullet B \bullet i$. Thus, the theory of belief operators B that are compatible with a topological knowledge operator K could be formulated as a theory of operators $OX \xrightarrow{j} OX$ that satisfy the just mentioned requirements (1) – (4). This is more than a formal nicety. The conceptual advantages to replace B by j are considerable: While belief operators B may be considered as ad hoc and contrived inventions of philosophers, the theory of operators j satisfying (1) – (3) has turned out in the last forty years or so as a very valuable and versatile conceptual tool of (point-free) topology (Borceux (1994), Johnstone (1982, 2002), Picado and Pultr (2012)). There is no need to go into the technical details of these theories here, the only point we want to make is that that the theory of nuclei entails (via Isbell's theorem)

[4] All belief operators that will be considered in the following are defined with the aid of nuclei j. Usually, it is not necessary, however, to mention nuclei explicitly.

that for any topological universe (X, OX) whatsoever the partial order of consistent belief operators has a maximal element, namely, Stalnaker's operator B_S.[5]

<u>(3.10) Proposition</u>. Let (X, OX, B) a topological model of L_{KB}. Let K be the knowledge operator of (X, OX) and B be a belief operator defined by a dense nucleus OX—j—>OX. Then the pair of operators (K, B) satisfies all axioms and rules of a weak Stalnaker system.

<u>Proof</u>. First, the rules and axioms that only deal with the knowledge operator K are clearly satisfied for (X, OX, B).

Next, by definition of the belief operator B as a concatenation B = int • j • i the operator B satisfies the axioms (PI) of positive introspection, the axiom (KB) that knowledge implies belief, and the axiom (FB) of full belief.

It remains to show that B satisfies the axiom (CB) of consistency. For the proof we essentially use that the nucleus j defining B is assumed to be dense. By definition B is distributive with respect to &. Then density of B and the definition of the valuation v entail that

$$\varnothing = B(\varnothing) \quad \text{iff} \quad B(\varphi \wedge \neg \varphi) = B(\varphi) \cap B(\neg\varphi) = \varnothing \quad \text{iff} \quad v(B(\varphi)) \subseteq \mathbf{C}B(\mathbf{C}v(\varphi)). \blacklozenge$$

<u>4. On the Topology of Gettier Cases</u>. In the last 50 years a profusion of ingenious examples has been produced that describe Gettier situations informally (cf. Borges, de Almeida, Klein (2017), Machery (2017)). As already explained in the Introduction in the framework of a topological epistemology Gettier cases appear as (not trivial) propositions G(A) := A & B(A) & CK(A). Then, for a topological epistemology the following questions arise naturally:

[5] Indeed, in mathematics, the operators j are called (dense) nuclei of the Heyting algebra OX of the topological space (X, OX) (Johnstone 1982, Picado and Pultr 2012). This term has been chosen, of course, without any reference to the concept of "belief operators". The general format of the nuclei j of a Heyting algebra OX is given in (Johnstone 2002, Lemma 1.1.17, p. 485).

(1) For which models (X, OX, B) of KB-logic do Gettier cases G(A) := A & B(A) & NOT(K(A)) exist and for which they do not exist?

(2) Can cognitive agents, who use K, know Gettier cases G(A) in the sense that there are worlds w with w $\in$ K(G(A))?

(3) Can cognitive agents, who use B, consistently believe Gettier cases G(A) in the sense that there are worlds w $\in$ B(G(A))?

(4) How can Gettier cases G(A) be characterized topologically?

The answer of the first question is as follows: For most topological models (X, OX, B) of KB-logic Gettier cases can be constructed in a direct and straightforward way. There are, however, topological spaces for which no Gettier cases exist, i.e., for all A $\in$ PX one has G(A) = A & B(A) & NOT(K(A)) = Ø.

The answers to questions (2) and (3) are, perhaps surprisingly, negative. Gettier cases G(A) are epistemically invisible, i.e., cognitive agents who rely on K and B in their epistemic practice on as the operators K and B can neither know nor even consistently believe that a world w be a G(A) world, since K(G(A)) = Ø and B(G(A)) = Ø.

Question (4) can be answered as follows: Gettier cases G(A) are nowhere dense, i.e., intcl(G(A) = Ø. Intuitively, this means that Gettier cases are "exceptional" or "thin" sets.

For every (non-discrete) topological space (X, OX), some propositions A $\in$ PX cannot be known, i.e., one has A $\neq$ Ø but int(A) = Ø. Those propositions may be said to be epistemically invisible. An analogous question may be asked for the bimodal logic KB of knowledge and belief. For instance, given a Gettier case G(A) in a topo-model of KB one may ask whether one can know or believe G(A) or not.

(4.1) Definition. Let (X, OX) be a model of L_{KB}. A set A $\in$ PX defines a Gettier case G(A) for (X, OX, B) iff G(A):= A & B(A) & NOT(K(A)) $\neq$ Ø. Analogously, the set A defines a Moore case M(A) in (X, OX) iff M(A) := A & NOT(K(A)) $\neq$ Ø. ♦

In more detail, G(A) is to be read as follows. A is assumed to be a set of possible worlds. A world w $\in$ G(A) is to be interpreted as a situation in which A obtains ("w is an A-world"), w $\in$ B(A) is to be interpreted as the fact that the cognitive agent who uses B as his doxastic operator believes that w in an A-world, and w $\in$ NOT(K(A)) means that nevertheless it is not known that w is an A-world. The following elementary example should suffice to convince the reader that Gettier cases can be easily constructed for many topological models of KB-logic:

(4.2) Gettier cases in the Euclidean plane. Let ($\mathbf{R}^2$, O$\mathbf{R}^2$) be the Euclidean plane with the standard Euclidean topology. ($\mathbf{R}^2$, O$\mathbf{R}^2$) defines a topo-model of KB by taking the interior kernel operator int of ($\mathbf{R}^2$, O$\mathbf{R}^2$) as knowledge operator K and B the Stalnaker belief operator= intclint. A Gettier case is defined as follows: Denote the rational numbers by $\mathbf{Q} \subseteq \mathbf{R}$. Let $\mathbf{R} \xrightarrow{i} \mathbf{R}^2$ be the embedding of the real numbers $\mathbf{R}$ into the plane defined by i(r) := (r, 0), r $\in$ $\mathbf{R}$. Let A $\subseteq$ $\mathbf{R}^2$ be the subset defined by A := $\mathbf{R}^2$ – i($\mathbf{Q}$). Let B_S be the Stalnaker belief operator B_S := intclint. One calculates B(A) = intclint(A) = $\mathbf{R}^2$ and int(A) = $\mathbf{R}^2$ – i($\mathbf{R}$). Thus, the set G(A) defined by G(A): = B(A) $\cap$ A $\cap$ $\mathbf{C}$K(A) = i($\mathbf{Q}$) $\neq$ Ø is a Gettier case. Nevertheless, one clearly has K(G(A)) = K(i($\mathbf{Q}$)) = Ø. That is, whether a world w exemplifies the Gettier case G(A), i.e., whether or not w $\in$ G(A), cannot be known according to the conditions of KB-logic since int(G(A) = Ø. Analogously one obtains B(G(A)) = B(i($\mathbf{Q}$)) = Ø. In other words, the Gettier case G(A) is epistemically invisible for K and B. Later, we will prove that for all topo-models (X, OX, B) all Gettier cases are epistemically invisible. ♦

The fact that familiar topological spaces such as the Euclidean plane give rise to topo-models of KB that possess Gettier cases should not be taken as evidence that this is the case for all topo-models. Indeed, the following recipe shows that topo-models of KB-logic exist that are free of Gettier cases:

<u>(4.3) Theorem</u>. A topo-model (X, OX, B_S) of KB has no Gettier cases if and only if (X, OX) is a nodec space (aka α-space), i.e., iff int(A) = A $\cap$ intclint(A) for all A $\subseteq$ X.

<u>Proof</u>. First, assume that (X, OX) has no Gettier cases with respect to intclint, i.e., G(A) = A $\cap$ intclint(A) $\cap$ **C**int(A) = Ø for all A $\subseteq$ X. This implies A $\cap$ intclint(A) $\subseteq$ int(A). The reverse inclusion int(A) $\subseteq$ A $\cap$ intclint(A) holds by the Kuratowski axiom of int. Thus, (X, OX) is an α-space.

Assume that (X, OX) is an α-space, i.e., int(A) = A $\cap$ intclint(A). Then clearly any G(A) = A $\cap$ intclint(A) $\cap$ **C**(A $\cap$ intclint(A)) = Ø. In other words, (X, OX) has no Gettier cases.♦

That Every topological space (X, OX) with interior kernel operator int gives rise to an α-space (X, O_αX) defined by int_α (A) := A $\cap$ intclint(A) that is free of Gettier cases. The move from (X, OX) to (X, O_αX) amounts only to a topologically small change of the defining topologies in the sense that for all A the set-theoretical difference of int(A) and int_α(A) is nowhere dense with respect to intcl, i.e., intcl(int_α(A) $\cap$ **C**int(A)) = Ø.

The logic of α-topological spaces (X, O_αX) is well known. Among other properties, these spaces are spaces, for which all nowhere dense subsets A of (X, O_αX) are closed. Further, those spaces are models for the so-called Zeman logic S4.Zem (cf. Bezhanishvili, Esakia, and Gabelaia (2005), Zeman (1969)). This logic is an extension of S4 that is characterized by the modal axiom

(4.4) $$S4.Zem = S4 + intclint(A) \ \& \ A \longrightarrow int(A).$$

In sum, from a topological point of view, the JTB-theory of knowledge and belief is not simply false, rather, it can be described as a theory of a limited range, namely, the realm of α-spaces as its models. Further, the construction of α-topological spaces $(X, O_\alpha X)$ reveals that α-spaces $(X, O_\alpha X)$ are not just a special class of topological spaces (X, OX). Rather, every topological space (X, OX) is accompanied, so to speak, by its own α-space $(X, O_\alpha X)$. Both spaces are related to each other in a natural functorial way. This means, among other things, that repeating the α-construction yields nothing new, since $(X, O_{\alpha\alpha}X) = (X, O_\alpha X)$. Being free of Gettier cases is a stable quality of topological models of KB. If one considers the thesis that knowledge is true justified belief as a claim of a "JTB-theory" then Gettier cases can be considered as a kind of Kuhnian anomalies for that theory that refute the theory in rare cases.

Example (4.2) illustrates this by a toy example: Let T_A be the "theory" that all points (p, q) of the plane $\mathbf{R}^2$ are A-points. A cognitive agent can come to know this assertion for all points (p, q) that do not lie on the line $\mathbf{R}$, i.e., (p, q) with $q \neq 0$. For this realm, T_A is true can be known to be true. Nevertheless, T is true as well beyond this "safe" area, namely, for some points of the line $\mathbf{R} = \{(p, 0)\}$ that satisfy $p \notin \mathbf{Q}$. Due the fact that every open neighborhood of such a (p, 0) includes points that do not belong to A, one cannot know that $(p, 0) \in A$. Nevertheless, it seems a justified belief that A is the whole plane. Indeed, one obtains $B(A) = intclint(A) = \mathbf{R}^2$. Then $(p, 0) \in A$ with $p \notin \mathbf{Q}$ is a justified true belief that cannot be recognized as knowledge, since there is no open neighborhood U(p, 0) of (p, 0) in $\mathbf{R}^2$ that can be considered as evidence that (p, 0) is an A-point. ♦

5. On the Plurality of Belief Operators. The topological approach to epistemology is, in a rather obvious sense, a "knowledge first" approach. Defining knowledge as the interior kernel operator int of a topological space (X, OX) is a direct and quite plausible way to bring topology to work in the epistemological realm. In comparison with knowledge, it is much less clear how to handle the concept of belief in a topological framework. In this section I'd like to demonstrate that the concept of belief is underdetermined by the axioms and rules of weak Stalnaker systems. More precisely, I'd like to show that for any topologically defined knowledge operator K there is a profusion of different belief operators B such that (K, B) satisfies all axioms and rules of a weak Stalnaker system.

The belief operators B related to K are partially ordered in a natural way. The smallest belief operator is the operator K (= int), the largest belief operator is Stalnaker's operator intclint. The larger B, the riskier it is to use B for stating one's beliefs, since they may considerably deviate from K. Nevertheless, as long as one's belief operator is not greater than B_S one can assume that it is still consistent.

Clearly, the underdetermination of the logic of knowledge and belief by the axioms of a weak Stalnaker systems renders the logic of knowledge and belief more complex but also more interesting and more realistic. There is no convincing reason to assume knowledge uniquely determines belief.

Fortunately, for the purposes of this paper, the multiplicity of belief operators is not an insuperable obstacle to prove the epistemological invisibility of Gettier cases, since the invisibility for intclint entails the invisibility for all other (consistent) belief operators. This is be proved by Isbell's theorem.

In this paper I'd like to deal with the issue of the plurality of belief operator only in barest outline. We only point out that for given knowledge operator K a lot of compatible belief operators B exists. This is explicitly shown by the following proposition:

(5.1) Proposition. Let (X, OX) be a topological space with interior kernel operator int. For a dense subset $F \subseteq X$ define the operator $PX \xrightarrow{B_F} PX$ by

$$B_F(A) := \mathrm{int}(F \longrightarrow \mathrm{int}(A))$$

Then the pair (K, B_F) of operators satisfies all rules and axioms of a weak Stalnaker system.

Proof. In order to show that the pairs of operators (K, B_F) of operators define weak Stalnaker systems, it only remains to show that B_F satisfies the axioms of consistency (CB), positive introspection (PI), knowledge implies belief (KB), and full belief (FB). All these assertions easily follow from the definition of B_F and elementary calculations using the Kuratowki axioms.♦

Now we show by an elementary example that the recipe (5.1) yields many different compatible belief operators B that are all compatible with a knowledge operator K - Stalnaker's operator B_S is only one among many.

(5.2) Examples. Let $(\mathbf{R}, O\mathbf{R})$ be the real line, $p \in \mathbf{R}$, let $\mathbf{Q}$ denote the subset of rational numbers, $A \in P\mathbf{R}$. Define operators B_p and $B_{\mathbf{Q}}$ by

(i) $B_p(A) := \mathrm{int}(\mathbf{R} - \{p\} \longrightarrow \mathrm{int}(A))$.

(ii) $B_{\mathbf{Q}}(A) := \mathrm{int}(\mathbf{Q} \longrightarrow \mathrm{int}(A))$.

(iii) $B_{C\mathbf{Q}}(A) := \mathrm{int}(C\mathbf{Q} \longrightarrow \mathrm{int}(A))$.

(iv) $B_S(A) := \mathrm{intclint}(A)$.

Then (i) – (iv) define different belief operators all compatible with the knowledge operator int.[6]♦

[6] It should be noted that the examples (i) – (iv) in no way describe all belief operators compatible with the standard knowledge operator of $((\mathbf{R}, O\mathbf{R})$.

The set of compatible belief operators B of K has a very complex structure that up to now is not fully known even for "elementary" spaces such as ($\mathbf{R}$, O$\mathbf{R}$). This paper is not the appropriate place to deal with this issue. Rather, we are content to show by some examples that there is a plurality of belief operators that accompany a given knowledge operator. A comprehensive topological logic of knowledge and belief should take into account the whole spectrum of belief operators B compatible with a given knowledge operator K. It cannot settle for the investigation of only one belief operator that has been chosen for one reason or other. On the other hand, even for apparently "simple" spaces like the real line ($\mathbf{R}$, O$\mathbf{R}$) a complete determination of its belief operators is a formidable task. Fortunately, for the questions that are dealt with in this paper, a full knowledge of the set of belief operators is not necessary. Some general information is enough. The first essential observation is that the set of belief operators is endowed with a natural partial order:

(5.3) Definition. Let (X, OX) be a topological space, B_1, B_2 belief operators compatible with the interior operator K of (X, OX). A partial order $\leq$ between belief operators is defined by

(5.3) $$B_1 \leq B_2 := B_1(A) \subseteq B_2(A) \text{ for all } A \in PX.$$

As is easily seen the unique minimal element of the set of belief operators is the identical map $PX \xrightarrow{id} PX$. ♦

Next one needs to show that the partially ordered class of consistent belief operators has a unique greatest element, namely, Stalnaker's operator B_S. That is to say: For every consistent belief operator B, one has $B(A) \subseteq B_S(A)$ for all $A \subseteq X$. Formulated with the concept of nuclei, Isbell's theorem asserts that every topological space (X, OX) has a largest dense nucleus OX—J—>OX, namely, the regular nucleus j = intcl.[7]

(5.4) Isbell's Density Theorem. Every complete Heyting algebra (X, OX)[8] has a largest dense nucleus OX——j——>OX, namely, the nucleus j defined as j(A) := intcl(A), A $\in$ OX. In other words, for all dense nuclei OX——j ——>OX and for all A one has B'(A) $\subseteq$ intcl(A).♦

The proof of Isbell's theorem goes well beyond the horizon of this paper. The reader may consult any of the standard sources of point-free topology such as Johnstone (1982, 2002), Picado, Pultr, and Tozzi (2006) or Picado and Pultr (2012). For our purposes we need only for the following immediate consequence of Isbell's theorem:

(5.5) Corollary. For all topological spaces (X, OX), all consistent belief operators B are smaller than or equal to Stalnaker's belief operator B_S := intclint, i.e., for all A $\in$ PX one has B(A) $\subseteq$ intclint(A).

Proof. By definition there is 1-1-correspondence between belief operators and dense nuclei (cf. (3.9)).♦

As will be shown in detail in the next section, (5.5) is enough to prove that Gettier and Moore cases G(A) and M(A) are epistemically invisible for epistemic operators of all models of KB-logics.

[7] This fact is in stark contrast with the analogous assertion in set-theoretical topology, according to which every topological space has a smallest dense subspace. This As assertion is blatantly false. The disjoint dense subsets **Q** and **CQ** of rational numbers and irrational numbers of **R** is a classical counterexample.

[8] Actually, Isbell's theorem is more general. It does not only hold for complete Heyting algebras of the form OX but for all complete Heyting algebras.

6. Epistemic Invisibility of Moore and Gettier cases: Proofs of the main theorems. Now everything is prepared to prove the epistemic invisibility for Moore and Gettier cases with respect to knowledge and belief operators which satisfy the laws of Stalnaker's KB-systems. For the convenience of the reader, we explicitly recall once again the general setting. Let (X, OX) be any topological model of KB-logic of knowledge operator K and belief operator B. These operators are assumed to satisfy the axioms of a weak Stalnaker system. For $A \in PX$ a Moore case is defined as $M(A) := A \cap \mathbf{C}(K(A)) \neq \emptyset$ and a Gettier case as $G(A) := A \cap B(A) \cap \mathbf{C}(K(A)) \neq \emptyset$, respectively. M(A) and G(A) are said to be epistemically invisible in (X, OX) if and only if $K(M(A)) = K(G(A)) = \emptyset$ and $B(M(A)) = B(G(A)) = \emptyset$.

The proof that Gettier cases are epistemologically invisible with respect to the knowledge operator K is elementary. It can be carried out by some elementary calculations with S4-formulas. In contrast, the proofs of epistemic invisibility with respect to belief operators B depends on Isbell's theorem that is to be characterized as a highly non-trivial theorem of point-free topology. Isbell's theorem ensures that Stalnaker's belief operator B_S = intclint is the largest consistent belief operators B compatible with the knowledge operator K.

(6.1) Theorem. Let (X, OX) be a topological model of KB-logic, M(A) and G(A) a Moore case and a Gettier case, respectively. Then M(A) and G(A) cannot be known, i.e., $K(M(A)) = \emptyset$ and $K(G(A)) = \emptyset$.

Proof. $K(M(A)) = K(A \,\&\, \mathbf{C}(K(A)) = K(A) \,\&\, K(\mathbf{C}(K(A)))$

$$= K(K(A)) \,\&\, K(\mathbf{C}(K(A))) = K(K(A) \,\&\, \mathbf{C}(K(A))) = K(\emptyset) = \emptyset.$$

Since $G(A) \subseteq M(A)$ and K is monotone one obtains $K(G(A)) \subseteq K(M(A)) = \emptyset$. Hence, Moore cases M(A) and Gettier cases G(A) are invisible for the topological knowledge operator K.♦

To show the epistemic invisibility of Moore and Gettier cases with respect to all consistent belief operators B requires some more work. First of all, we show that Moore and Gettier sentences cannot be believed by Stalnaker's belief operator B_S:

<u>(6.2) Theorem</u>. Moore cases M(A) and Gettier cases G(A) are epistemically invisible with respect to Stalnaker's belief operator B_S = intclint, i.e., B_S (M(A)) = Ø and B_S(G(A)) = Ø.

<u>Proof</u>. B_S(M(A)) = intclint(M(A)) = intclint(A & **C**(K(A))

= intclint(A) & intclint(**C**(int(A)) = intclint(A) & **C**(clintclint(A))

= intclint(A) & **C**(clint(A) $\subseteq$ clint(A) & **C**(clint(A)) = Ø.

Hence, Moore cases M(A) are epistemologically invisible for B_S. Since G(A) $\subseteq$ M(A) and B_S is monotone, one obtains that *a fortiori* Gettier cases G(A) are epistemologically invisible for B_S, i.e., B_S(G(A))) = Ø.♦

<u>(6.3) Corollary</u>. Moore cases M(A) and Gettier cases G(A) are epistemically invisible with respect to all consistent belief operators B, i.e., B(M(A)) = Ø and B(G(A)) = Ø.

<u>Proof.</u> By Isbell's theorem one has B(M(A)) $\subseteq$ B_S(M(A)) = Ø. Belief operators B are monotone, hence from G(A) $\subseteq$ M(A) one obtains that B(G(A)) $\subseteq$ B(M(A)) = Ø.♦

In sum, with respect to epistemological invisibility Moore cases and Gettier cases behave the same way. Nevertheless, they may behave topologically quite differently. To prove this assertion, we use again the technical lemma (3.5):

<u>(6.4) Theorem</u>. Gettier cases G(A) are nowhere dense in (X, OX), i.e., intcl(G(A)) = Ø. In contrast, Moore cases may be topologically dense in (X, OX).

Proof. Let G(A) be a Gettier case. By the axiom of positive introspection (PI) (2.1) and the facticity of K the set B(A) is open. Hence, we can apply (3.5) and obtain

$$\mathrm{intcl}(A \,\&\, B(A) \,\&\, \mathbf{C}(K(A)) \subseteq \mathrm{intcl}(B(A) \,\&\, \mathbf{C}(K(A))$$

$$= \quad \mathrm{intclint}(B(A)) \,\&\, \mathrm{intcl}(\mathbf{C}(K(A)) = \mathrm{intcl}(\mathrm{int}(A)) \,\&\, \mathrm{intcl}(\mathbf{C}(K(A)))$$

$$= \quad \mathrm{intcl}(\mathrm{int}(A) \,\&\, \mathbf{C}\mathrm{clintint}(A)) \subseteq \mathrm{clint}(A) \cap \mathbf{C}\mathrm{clint}(A) = \emptyset. \blacklozenge$$

The following example shows that Moore cases may be dense in (X, OX):

(6.5) Example of a dense Moore case. Let (**R**, O**R**) be the real line with Euclidean topology, and **Q** the rational numbers $\mathbf{Q} \subseteq \mathbf{R}$. Then $M(\mathbf{Q}) := \mathbf{Q} \,\&\, \mathbf{C}(\mathrm{int}(\mathbf{Q})) = \mathbf{Q}$, and one obtains: $\mathrm{intcl}(M(\mathbf{Q})) = \mathrm{intcl}(\mathbf{Q}) = \mathbf{R}$, i.e., the Moore case M(**Q**) is dense in (**R**, O**R**).♦

Finally, let us mention that for KB even the difference between knowledge K and (not necessarily true) justified belief B is epistemically invisible, even nowhere dense:

(6.6) Proposition. Let D(A) := B(A) & NOT(K(A)) be the set-theoretical difference between justified (consistent) belief B(A) and knowledge K(A). Then intcl(D(A)) = Ø.

Proof. Using (3.5) and Isbell's theorem one calculates:

$$\mathrm{intcl}(B(A) \,\&\, \mathbf{C}(K(A)) \;=\; \mathrm{intcl}(B(A)) \,\&\, \mathrm{intcl}(\mathbf{C}(K(A)))$$

$$= \quad \mathrm{intclint}(A) \,\&\, \mathbf{C}(\mathrm{clint}(A)) \subseteq \mathrm{clint}(A) \,\&\, \mathbf{C}(\mathrm{clint}(A)) = \emptyset. \blacklozenge$$

In other words, for the KB-logic justified belief B(A) (even if it is not true) and proper knowledge K(A) are topologically very close to each other. Their difference is topologically "negligible" aka nowhere dense. Thus, from the perspective of KB-logic there is only a small difference between knowledge and justified belief.

7. Concluding Remarks. This paper evidences that topological epistemology can handle the issue of Gettier cases in a flexible way. On the one hand, it shows that Gettier cases can be constructed in a natural and systematic way. On the other hand, it shows that Gettier cases can be avoided systematically by choosing an appropriate topological design for the underlying topological spaces that are used for the semantics of KB-logic. More precisely, for any topological model (X, OX, B) one may find a Gettier-free model (X, $O_\alpha X$, B_S) that minimally deviates from the original model (X, OX, B). Third, it explains the special epistemic status of Gettier cases, namely, that Gettier cases are described as contrived situations that depend on rare combinations of lucky and unlucky circumstances. They use to be met with a certain incredulity (cf. Nagel (2013), Machery (2017)). In the framework of topological epistemology, Gettier cases cannot only be not known and not even be consistently believed according to the standards of KB-logic. Nevertheless, they can be predicted on general topological grounds.[9]
Topology offers various ways to construct models free of Gettier cases. Depending on the structure of conceptual spaces, Gettier cases G(A) as counterexamples to JTB may be predicted to occur, or, complementarily, they can be avoided if one is prepared to accept certain requirements for the conceptual spaces that underly the models. The occurrence of Gettier cases is a question of the topological design of the models that one favors. As Zagzebski emphasized, absolute precision is only one criterion for a good explanation of knowledge. Other criteria such as simplicity, conciseness, theoretical illumination as well are important.[10] Hence, Gettier cases do not automatically refute the JTB-theory (cf. Zagzebski 2017, 188).

[9] Hence, the topological semantics of KB-systems presented in this paper supports Williamson's thesis according to whom "there are natural formal models that provide robust evidence against JTB" (cf. Williamson 2015, 139) – at least partially. Being more flexible than Williamson's relational account, topological epistemology also has a place for models of KB that confirm the classical JTB-account.
[10] For some remarks on a general "design theory" of conceptual spaces see Douven and Gärdenfors (2019) and Douven (2019).

In a nutshell, whether Gettier cases occur or not, depends on the underlying topological structure of a model. Moreover, the move from (X, OX) to (X, $O_\alpha X$) allows to eliminate all Gettier cases by only a slightly changing the topological structure. Thus, the topological approach of the logic of knowledge and belief shows that the occurrence of Gettier cases is a matter of the topological design of underlying models used. Gettier cases generated by ever more ingenuous thought-experiments do not automatically refute the traditional JTB-conception of knowledge. Formal examples, in particular, topological examples may be considered a complementary source for arguments that JTB cannot be the last word in the discussion of the relation between knowledge and belief. Whether one should go thus far as Sellars is a different question:

> The explication of knowledge as "justified true belief", though it involves many pitfalls to which attention has been called in recent years, remains the orthodox or classical account and is, I believe, essentially sound. (Sellars 1975, p. 99)

Sellars did not explain what he meant by “essentially sound.” The topological account gives a precise explication of the intuitive assessment of the “essential soundness” of the classical JTB-account of knowledge. Topologically, Gettier cases are characterized as nowhere dense subsets of a topological universe, i.e., they are “rare” or “exceptional” in a quite precise sense. Moreover, they can easily be eliminated by only changing slightly the underlying topological structure. With a lot of good will, these facts may be interpreted as evidence that the JTB explication of knowledge is “essentially sound.”

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