Mathematical Aspects of Similarity and Quasianalysis - Order, Topology, and Sheaves

Abstract. The concept of similarity has had a rather mixed reputation in philosophy and the sciences. On the one hand, philosophers such as Goodman and Quine emphasized the „logically repugnant“ and „insidious“ character of similarity that allegedly rendered it inaccessible for a proper logical analysis. On the other hand, a philosopher such as Carnap assigned a central role to similarity in his constitutional theory. Moreover, the importance and perhaps even indispensibility of the concept of similarity for many empirical sciences can hardly be denied. The aim of this paper is to show that Quine’s and Goodman’s harsh verdicts about this notion are mistaken. The concept of similarity is susceptible to a precise logico-mathematical analysis through which its place in the conceptual landscape of modern mathematical theories such as order theory, topology, and graph theory becomes visible. Thereby it can be shown that a quasi-analysis of a similarity structure S can be conceived of as a sheaf (etale space) over S.

Key Words: Similarity, Order, Topology, Sheaf theory, Quasi-analysis.

1. Introduction. The notion of similarity has had a rather mixed reputation in 20th century philosophy and science. According to Quine

> we cannot easily imagine a more familiar or fundamental notion than this, or a notion more ubiquitous in its application. On this score it is like the notions of logic: like identity, negation, alternation and the rest. And yet, strangely, there is something logically repugnant about it (Quine (1969, 117)).

Nelson Goodman had an even more critical, positively hostile attitude toward the concept of similarity. According to him, similarity “is a pretender, an impostor, a quack.” (Goodman (1972, 437 – 447)). On the other hand, for Rudolf Carnap th concept of similarity played an essential role in the constitutional theory put forward in his *opus magnum The Logical Construction of the World* (Carnap 1928 (1961), henceforth *Aufbau*). Indeed, the main target for Goodman’s harsh remarks about similarity was Carnap’s *Aufbau*. Goodman wanted to show that Carnap’s similarity-based approach was doomed to fail from the outset (cf. Goodman (1951, chapter V)).

For several decades Goodman’s criticism of the *Aufbau* was almost unanimously accepted as definitive. It became an unquestionable truth that Carnap’s similarity-based constitution of the world as sketched in the *Aufbau* could not succeed. Goodman seemed to have established once and for all that

Carnap's attempt of building up the world from a set of elementary experiences endowed with a similarity relation was not feasible. Goodman was seconded by Quine, who for some more general reasons, not directly related to problems with Carnap's *Aufbau*, asserted that the concept of similarity was "alien to logic and set theory" (Quine (1969, 121)). The aim of this paper is to show that Quine and Goodman are wrong. As I want to show the concept of similarity is not "alien to logic and set theory". On the contrary. I argue that the notion of similarity can be submitted to a rich mathematical analysis that can help to a fuller philosophical and logical understanding of this notion. Basically, such an analysis may be traced back to Carnap's *Aufbau*. This is not to say that a comprehensive elucidation of the concept of similarity could get along with the restricted formal means that Carnap had at his disposal, or, at least, was prepared to deploy. Neither, it is necessary to follow Quine's implicit suggestion that "logic and set theory" are the appropriate tools for coming to terms with the concept of similarity. Rather, I contend that the appropriate frame to deal with similarity and related concepts is modern structural mathematics – more precisely, theories such as topology, order theory, and graph theory. In other words, I propose to submit the concept of similarity to a precise and fruitful mathematical analysis. This would be in line with the important role that similarity plays in a variety of disciplines such as such as psychology, linguistics, cognitive science, and many others. In all these sciences the concept of similarity plays a central role and is considered as anything but a "quack". For every scientifically minded philosopher it should be a matter of concern if philosophy on the one hand, and the sciences on the other, come to such opposite assessments. History of science and philosophy should have taught us that it was seldom fruitful and expedient for philosophy to blame science for its allegedly low conceptual level and to preen herself of her high conceptual standards that render it obligatory to expel logically doubtful concepts such as similarity from the realm of respectable notions.

As more recent work in cognitive psychology and related sciences evidence show (cf. Tversky (1977), Gärdenfors (2000) and others) the dismissal of the concept of similarity by philosophers has been too hasty.[1] Also in philosophy Goodman's verdict on this notion has begun to be put into doubt since some time (cf. Proust (1989), Mormann (1994, 2009)).

The outline of the paper is as follows: In section 2 the basic definitions of similarity relations, order relations are introduced. In section 3 we show that similarity structures (S, ~) may be endowed in a natural way with a topological structure, to wit, the so called order topology (or Alexandrov topology). The basic topological properties of similarity structures are studied, and some important classes of similarity structures with particularly nice topological properties are defined. In section 4 an example of a similarity structure (S, ~) is constructed whose Alexandrov space is not sober. In section 5 continuous maps between similarity structures are defined and some of their basic properties are studied. As a particularly interesting class of continuous maps between similarity structures continuous quasi-analyses are singled out. More precisely, a quasi-analysis of a similarity structure S is shown to define a sheaf (etale space) over S. We close with some more detailed explanations of how the

[1] For a nice survey of the history of „Similiarity after Goodman" see the survey of Decock and Douven (2011).

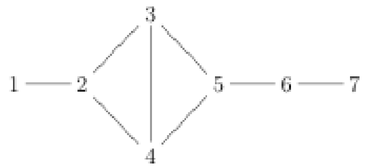
account of similarity and quasi-analysis put forward in this paper is related to Carnap's original account of these issues as presented in his early manuscript *Quasizerlegung* (Carnap 1923).[2]

2. Similarity and Order. To set the stage for the following sections, in this section we recall the necessary rudiments of the theories of similarity structures and order theory, that will be used in the rest of this paper.

(2.1) Definition. A similarity structure is a relational structure (S, ~), S being a set and ~ a reflexive and symmetric, but not necessarily transitive, relation defined on S. For x, y ∈ S the fact that (x, y) ∈ ~ is denoted by x ~ y, and x and y are said to be similar to each other. If there is no danger of confusion (S, ~) is simply denoted by S.♦

Similarity structures (S, ~) may be conceived of as simple undirected graphs: The elements of S correspond to the vertices of the graph, and two different elements x, y ∈ S are similar to each other if and only if they are the vertices of an edge:

(2.2)



This graph represents a similarity structure (S, ~) with underlying set S = {1, 2, 3, 4, 5, 6, 7} such that each i ∈ S is assumed to be similar to itself, i = 1, 2, …, 7. Moreover, the element 1 is similar to 2, the element 2 is similiar to 1, 3, and 4, the element 3 is similar to 2, 4, and 5, the element 4 is similar to the elements 2, 3, and 5, the element 5 is similar to the elements 3, 4, and 6, the element 6 is similar to 5 and 7, and 6 and 7 are similar; no other similarity relations between the elements of S obtain.

The most important concept for a detailed study of similarity structures is the concept of similarity neighborhood:

(2.3) Definition. Let (S, ~) be a similarity structure. The similarity neighborhood co(x) of x (x ∈ S) is defined as co(x) := {y; x ~ y}.♦

(2.4) Example. The elements of the similarity structure (2.2) have the following similarity neighborhoods:

[2] This paper only deals with the simplest concept of similarity, namely, similarity conceived of as a binary relation ~ that is assumed to be reflexive and symmetric, i.e., for all objects a, b of the domain under consideration one has a ~ a, and that a ~ b entails b ~ a. This is not a severe restriction: As has been shown in Mormann (2009) a binary similarity relation suffices to define a variety of more complex notions of similarity - for instance ternary relations ("a is more similar to b than to c") and even higher ones such as ("a is more similar to b than c is to d") which behave quite reasonable in so far as they satisfy most of the adequacy conditions discussed in Williamson (1988).

co(1) = {1, 2}   co(2) = {1, 2, 3, 4}   co(3) = {2, 3, 4, 5}
co(4) = {2, 3, 4, 5}   co(5) = {2, 4, 5}   co(6) = {5, 6, 7 }   co(7) = {6, 7}.♦

Since the relation ~ is reflexive and symmetric, co(x) contains always at least one element, namely, x itself. Further, one always has x ∈ co(y) ⇔ y ∈ co(x). In general, a similarity neighborhood co(x) is not an equivalence class, however, since ~ need not be transitive, i.e., one may have x ~ y and y ~ z, but not x ~ z. An element x ∈ S with co(x) = {x} may be called isolated. A similarity structure (S, ~) with co(x) = {x} for all x ∈ S is called the trivial discrete similarity structure on S. Another type of trivial similarity structures are those structures (S, ~) with co(x) = S for all x ∈ S. In an obvious intuitive sense all interesting similarity structures on S are somewhere in-between these two trivial similarity structures.
Virtually all structures defined for similarity structures to be considered in this paper arise from the basic structure of similarity neighborhood co(x) (2.3). Perhaps the simplest one that (at least implicitly) already occurs in Carnap (1923) is the following equivalence relation:

(2.5) Definition. Let (S, ~) be a similarity strudcture with similarity neighborhood co. Then co defines an equivalence relation $=_{co}$ on (S, ~) by x $=_{co}$ y := co(x) = co(y). The equivalence class defined by x ∈ S with respect to $=_{co}$ is denoted by [x]. The set of equivalence classes is denoted by S*, i.e., S* := {[x]; x ∈ S}. Then ~ defines a similarity relation ~* on S* by [x] ~* [y] := x ~ y, x ∈ [x], y ∈ [y].♦

Elements x and y with co(x) = co(y) cannot be distinguished structurally. Hence one may contend that „nice“ similarity structures (S, ~) should satisfy the following axiom:

(2.6) Definition (Axiom of Similarity Neighborhood Identity (SNI). A similarity structure (S, ~) satisfies the axiom (SNI) if for all x, y ∈ S co(x) = co(y) entails x = y.♦

This axiom can already be found in Carnap (1923) where it is called *Verwandtschaftsgleichheit.* From a structuralist point of view, elements whose similarity neighborhoods coincide should be considered as essentially equivalent, since they cannot be distinguished structurally (see *Aufbau*). By definition for every similarity structure (S, ~) the similarity structure (S*, ~*) satisfies (SNI). Hence one may assume that all similarity structures to be considered satisfy (SNI). If (S, ~) does not satisfy (SNI) it may be replaced by the quotient structure (S*, ~*). Thereby one obtains in a canonical way a similarity structure that does satisfy (SIN). As an example of how the transition from (S, ~) to (S*, ~*) looks like consider (2.2). As is easily checked, for (2.2) one has co(3) = co(4) while for all other elements x, y ∈ S their similarity neighborhoods co(x) and co(y) differ. Hence for S* we obtain the following linear graph:

(2.7)   [1]———[2]———[3]———[5]———[6]———[7]

The equivalence relation $=_{co}$ already appears in Carnap (1923). Nevertheless, most scholars who dealt with Carnap's quasi-analytical account of similarity (in particular Quine and Goodman) ignored it. For the following, more important than $=_{co}$ will turn out a partial order relation ≤ defined by the relation co as well. This relation is new, i.e., it does appear in Carnap (1923).
Recall that a quasi-order on S is a binary relation ≤ that is reflexive and transitive. A quasi-order ≤ is a partial order if it is also anti-symmetric, i.e., from x ≤ y and y ≤ x one can conclude x = y. Then one can prove the following elementary lemma:

(2.8) Lemma. Let (S, ~) be similarity structure with similarity neighborhood co. On S a canonical quasi-order ≤ is defined by x ≤ y := co(x) $\subseteq$ co(y). This quasi-order is a partial order if and only if (S, ~) satisfies (SNI). ♦

Later the axiom (SNI) will reappear in topological terms. For the moment let us concentrate on its order-theoretical aspects. First note that by moving from a similarity structure (S, ~) to its order structure (S, ≤) one may lose information, i.e., it may happen that two different similarity structures (S, ~) and (S*, ~*) may define isomorphic order structures (S, ≤) and (S*, ≤*).

(2.9) Example. Let **Z** be the integers … -2, -1, 0, 1, 2, … endowed with the similarity relation ~ defined for m, n $\in$ **Z** by n ~ m := $|m - n| \leq 1$

-2 -1 0 1 2

… —•——•——•——•——•— …

Intuitively, n and m are similar to each other if they are equal or are direct neighbors in the familiar order of the real line **R**. With respect to this similarity relation there are no non-trivial inclusion relations between the similarity neighborhoods co(n), i.e., one has co(n) $\subseteq$ co(m) if and only if for m = n. Hence the order structure (**Z**, ≤) related to the similarity structure (**Z**, ~) is the trivial order structure i.e., the same order structure as the one that is correlated to the trivial discrete similarity structure (**Z**, =):

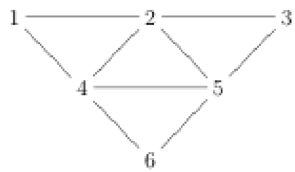
(2.10) -2 -1 0 1 2

… • • • • • …

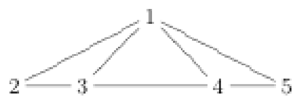
Thus, although the similarity structures (**Z**, ~) and (**Z, =)** are distinct as similarity structures, they have the same order structure (**Z**, ≤). This shows that several different similarity structures may have the same order structures.
Clearly, trivial order structures such as (2.8) are not very interesting from an order-theoretical perspective. To single out a class of similarity structures (S, ~) with interesting order structures the following definition is useful:

(2.11) Definition. Let (S, ~) be a similarity structure with order structure (S, ≤) and n ≥ 1. The depth of (S, ~) is at least n - 1 if and only if there is a strictly increasing chain $s_1 < s_2 < \ldots < s_n$ in (S, ≤). The depth d(S) of (S, ~) is n if and only if there is no strictly increasing chain of length n + 1.
If there are strictly increasing chains of arbitrary length the depth of (S, ~) is said to be infinite. ♦

As is easily checked the depth d(S) of the structures (2.5) and (2.6) is 0. An example of a similarity structure of depth 1, already discussed by Goodman and later by many others is the following one:
(2.12)



As is easily calculated, (2.12) contains chains of length 2, namely, 1 < 2, 3 < 2, 1 < 4, 6 < 4, 3 < 5, and 6 < 5, and clearly no chain of length 3. Hence d(S) =1. To conclude for the moment this discussion about depth of similarity structures, let us finally consider a similarity structure of depth 2:
(2.13)



Since (2.13) contains two chains of length 3, namely, 2 < 3 < 1 and 5 < 4 < 1, but none of length 4, it has the depth 2. Later it will be shown that there are similarity structures (S, ~) with d(S) = n for any n ∈ **N**. with some more effort one can even prove that there are similarity structures of infinite depth d(S). In section 5 a natural description of depth in terms of the property distributions of a quasi-analysis will be given. The depth d(S) of a similarity structure (S, ~) may be considered as a measure of its complexity, that is to say, *ceteris paribus* a similarity structure (S, ~) is the more complex the greater its depth d(S).

3. Topology of Similarity Structures. The second conceptual perspective from which we will study similarity structures (S, ~) is the topological perspective. In order to set the stage for the following sections we succinctly recall the basic definitions of topology and give some examples and counter-examples.

(3.1) Definition. Let X be a set and PX its powerset. A topology on X is a subset OX ⊆ PX satisfying the following requirements:

(i) Ø, X ∈ OX.

(ii) Arbitrary unions $\cup A_i$ and binary intersections $B_1 \cap B_2$ of elements A_i, B_1, B_2 ∈ OX are in OX.

The elements of OX are called open sets, the set-theoretical complements of open sets are called closed sets. The set of closed sets of X is denoted by CX. Topological spaces are denoted by (X, OX) or simply by X.♦

As is well-known the set of open subsets OX of a topological space (X, OX) endowed with the partial order induced by set-theoretical inclusion $\subseteq$ has the structure of a complete Heyting algebra (cf. for example Borceux (1994), Vickers (1989)). Dually, the set of closed sets CX has the structure of a complete Co-Heyting algebra.

In a conceptually different, but strictly equivalent way topological structures may be characterized by topological closure operators. This perspective was developed mainly by Kuratowski and his collaborators in the 1920s (cf. Kuratowski and Mostowski (1976)):

<u>(3.2) Definition</u>. A topological closure operator cl on a set X is an operator PX—cl—>PX satisfying the following requirements for all A, B $\in$ PX (cf. Kuratowski and Mostowski 1976, 27):

(i) $cl(A \cup B) = cl(A) \cup cl(B)$. (ii) $cl(cl(A)) = cl(A)$.

(iii) $A \subseteq cl(A)$. (iv) $cl(\varnothing) = \varnothing$.

A subset A $\in$ PX is called closed if and only if A = cl(A). The set of closed sets of (X, cl) is denoted by CX. The set-theoretical complements of elements of CX are called the open sets of (X, cl) and are denoted by OX.♦

The definitions (3.1) and (3.2) are strictly equivalent. Assuming (3.2) one easily shows that due to (3.2)(i)-(iv) OX is a topological structure in the sense of (3.1). In the other direction, a topological space (X, OX) in the sense of (3.1) gives rise to a topological closure operator cl by defining $cl(A) := \cap\{B; A \subseteq B, B \in CX\}$. As one easily checks this operator satisfies the Kuratowski axioms. Hence, (3.1) and (3.2) are strictly equivalent.

On every set X there exist two extreme topological structures:

(3.3) $O_0X := \{\varnothing, X\}$ $O_1X := PX$

O_0X is called the indiscrete topology of X, and O_1X is called the discrete topology. The indiscrete topology O_0X is defined by the closure operator $cl_0(A) = X$, for $\varnothing \neq A$, and the closure operator cl_1 of O_1X is the identity, i.e., $cl_1(A) = A$, for all A $\in$ PX. For later use, let us mention some more interesting examples of topologies:

<u>(3.4) Important Examples of Topological Spaces</u>.

(i) (Metrical Topology). Let (X, d) be a metrical space with distance function d: X x X ——>**R**. The metrical topology OX defined by d is the smallest topology that contains arbitrary unions of subsets {y; d(x, y) < r, x ∈ X}, 0 < r ∈ **R**. The sets {y; d(x, y) < r, x ∈ X} are called open balls of radius r and center x.

(ii) (Co-finite Topology). Let **N** be the natural numbers 0, 1, 2, … . Let O**N** be the set of all subsets U ⊆ **N** with finite complement **C**U or U = Ø. Thereby on **N** a topology O**N** is defined. This topology is called the cofinite topology on **N**.

(iii) (The completed natural numbers **N** as a topological Space). Denote by **N** the natural numbers and assume ω to be an object not in **N**. Define O(**N** ∪ {ω})) as the set consisting of Ø and all U containing ω and such that **N** – U is finite. Then(**N** ∪ {ω}, O(**N** ∪ {ω})) is a topological space.

Topological structures arising from metrical structures are probably the best known examples of topological spaces. The standard example of a metrical topology is, of course, the topology of Euclidean space. The topological spaces (3.4)(ii) and (3.4)(iii) are well-known examples of "pathological" topological spaces, they are often use to produce counter-examples for disproving plausible conjectures that arise from too narrow an orientation along standard topological spaces.♦

Topological structures OX on a set X are partially ordered by set-theoretical inclusion. For every topology OX one has $O_0X \subseteq OX \subseteq O_1X$. Much more is true: Since an arbitrary intersection $\cap O_iX$ of topologies O_iX on X is again a topology the set of topologies on X can be rendered a complete lattice with respect to ⊆. (cf. Davey and Priestley (1989)).♦

Topological structures are very general. In order to obtain interesting results it is expedient to introduce more specific axioms that single out specific classes of topological spaces. In the book of Steen and Seeberg (1978) hundreds of different types of topological spaces with sometimes rather exotic properties are discussed in detail. For the purposes of this paper it suffices to have a look on some of them defined by some classical separation axioms (cf. Steen and Seeberg (1978, 11)):

(3.5) Definition (Separation Axioms). Let (X, OX) be a topological space.

(i) X is a T_0 – space if and only if for all distinct a, b ∈ X there exists an open set A ∈ OX such that either a ∈ A and b ∉ A, or b ∈ A and a ∉ A.

(ii) X is a T_1 – space if and only if for all distinct a, b ∈ X there exist A, B ∈ OX containing a and b respectively, such that such b ∉ A and a ∉ B. Equivalently, X is a T_1-space if and only if {a} = cl(a) for every a ∈ X.

(iii) X is a T_2 – space (or Hausdorff space) if and only if there disjoint A, B $\in$ OX containing a and b respectively.♦[3]

Traditional topology has concentrated on topological spaces that satisfy rather strong separation axioms such as the Hausdorff axiom T_2 and even stronger ones (cf. Steen and Seebach Jr. (1978)). In contrast, the topology of similarity structures points in a different direction. For the topological aspects of similarity structures the higher separation axioms turn out to uninteresting, since they most similarity structure (S, ~) do not satisfy them. Rather, appropriate separation axioms for similarity structures are located somewhere between T_0 and T_1. Before we tackle this task, first we have to define appropiate topological structures (S, OS) for similarity structures (S, ~). This is done as follows:

(3.6) Definition. Let (S, ~) be a similarity structure with partial order structure (S, ≤). The set A $\subseteq$ S is called an upper set if and only if x $\in$ A and x ≤ y entails y $\in$ A for all x, y $\in$ S. Then a topological space (S, OS) is defined by the stipulating that A $\subseteq$ S is open if and only if it is up-closed with respect to ≤, i.e., if x $\in$ A and x ≤ y entail that y $\in$ A:

$$(x)(y)\ (x \in A\ \&\ x \leq y \Rightarrow y \in A).$$♦

As is easily checked arbitrary unions and arbitrary (!) intersections of upper sets are upper sets. Hence the upper sets of (S, ≤) define a topology on S. This topology is called the order topology or the Alexandrov topology on S.[4] In the following sections we study this topological structure (S, OS) of similarity structures (S, ~). As will become clear, Alexandrov topologies of similarity structures (S, ~) behave rather differently than most of the topologies we are accostumed to, such as the topologies of Euclidean spaces and their generalizations. For instance, most "higher" separation axiom T_i ($i \geq 2$) are uninteresting for Alexandrov spaces.
Before these and related topics can be discussed in reasonable detail, let us make some elementary observations concerning the special features of Alexandrov topologies that distinguish them from other topologies:

(3.7) Lemma. Let (S, ≤) be a partial order with Alexandrov topology (S, OS). For every x $\in$ S a distinguished open set and a distinguished closed set can be defined as follows:

(i) The set $\uparrow x := \{y; x \leq y\}$ is the smallest open set of OS that contains x.
(ii) Dually, the set $\downarrow x := \{y; y \leq x\}$ is the closure cl(x) of the singleton {x}

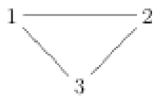
[3] T_0, T_1, and T_2, are only the first members of a longer series of ever stronger separation axioms, to be continued by T_3, T_4, ..., and so on. Later, in this „classical" series, quite a few „intermediate" axioms such as $T_{1\ 1/2}$, $T_{2\ 1/2}$, have been inserted. (cf. Steen and Seeberg (1978, 11ff)).
[4] The Alexandrov topology on (S, ≤) may not be the only interesting topology for order structures (S, ≤). It is only the simplest one. Arguably, the Scott topology or the Larson topology are more useful topological structures (cf. Vickers (1989), Picado and Pultr (2012)).

and the smallest closed subset of CS that contains x.

Proof. (i) holds by definition of the order topology. (ii) In order to prove that ↓x = cl(x) we first show that the complement S - ↓x is open in OS. By definition an open set A of OS is upper closed, i.e. ↑A = A. Assume z ∈ S -↓x. This entails that NOT(z ≤ x). A fortiori for any w ≥ z one has NOT(w ≤ x). In other words, ↑(S - ↓x) = (S - ↓x) is open and therefore ↓x is closed in (S, OS).
Since cl(x) is the smallest closed set containing x one has cl(x) ⊆ ↓x. In order to prove equality, suppose that ↓x ⊆ cl(x) does not hold. Then there is a y ≤ x with y ∈ S - cl(x). Since this set is open we would get x ∈↑y ⊆ S - cl(x) and therefore x ∈ S - cl(x). This is absurd. Hence cl(x) = ↓x.♦

Now let us consider in more detail how the Alexandrov topologies (S, OS) of similarity structures (S, ~) behave with respect to the traditional separation axioms as defined in (3.3). The following example shows that in general similarity structures do not even the satisfy axiom T_0.

(3.8) Example ("Goodman's Triangle"). Consider the similarity structure (S, ~) represented by the following graph:



Since co(1) = co(2) = co(3) = {1, 2, 3} the order topology OS on S is the trivial indiscrete topology, i.e., OS = {{1, 2, 3}, Ø}. Hence, the topological space (S, OS) resulting from Goodman's triangle does not satisfy T_0.♦

A remedy to overcome this deficiency is to require that "nice" similarity structures satisfy the axiom (SNI) discussed in the previous section:

(3.9) Lemma. The Alexandrov topology (S, OS) of a similarity structure (S, ~) satisfies T_0 if and only if (S, ~) satisfies (SNI).

Proof. (i) Assume (SNI) to hold, and let x and y be distinct elements of S. One has to show that there is an open set U such that either x ∈ U and y ∉ U or y ∈ U and x ∉ U. Suppose that such a set U does not exist. This entails that for the open sets ↑x and ↑y one has y ∈ ↑x AND x ∈↑y. This entails co(x) ⊆ co(y) AND co(y) ⊆ co(x), and therefore co(x) = co(y). By (SNI) we get x = y. This is a contradiction. Hence (S, OS) is T_0.
(ii) On the other hand, assume T_0 to hold for (S, OS) and suppose that (SNI) does not hold for (S, ~). Then there are distinct x and y with co(x) = co(y). Since for all z ∈ S the set ↑z is the smallest open neighborhood that contains z, due to T_0 we may assume that x ∈ ↑x and y ∉ ↑x. Hence co(x) ≠ co(y). This is a contradiction to our original assumption. Hence (SNI) holds for (S,).♦

The topological structures collected in (3.5) provide only a tiny selection of the many topological structures which have been studied in detail since the beginnings of topology as a proper mathematical theory since its inception in the early decades of the 20th century. Despite far-reaching structural differences they have something in common, namely, they all arise from certain specific features of the underlying structured sets: (3.4)(i) from the metrical structure, (3.4)(ii) from the essential distinction between finite and infinite cardinalities of sets, and (3.4)(iii) from specific topological features of real intervals. In other words, topology may be conceived of as a general perspective or point of view from which quite different mathematical structures may be studied in a way that often reveals novel aspects that would have remained invisible otherwise. As I want to show in the following, this also holds for similarity structures. More precisely, I want to argue that the topological perspective provides a useful framework for the investigation of similarity structures (S, ~).♦

As is well-known, Hausdorff spaces, in particular metrical spaces such as Euclidean spaces, satisfy T_1 (see (3.4)(ii)). Hence, satisfying T_1 in no way is an "exotic" topological property for a topological space. Nevertheless, the following example suggests that it is not reasonable to expect that similarity structures in general satisfy T_1.

(3.10) 1————2————3

Although the similarity structure (3.10) is a reasonable similarity structure in every respect, one obtains co(1) = {1, 2}, co(2) = {1, 2, 3}, co(3) = {2,3}. Thus the basic open sets are ↑1 = {1, 2}, ↑2 = {2}, ↑3 = {2, 3}. This entails that there are no open sets U and V that separate 1 and 2, i.e., open sets which satisfy $1 \in U$ and $2 \notin U$, and $1 \notin V$ and $2 \in V$. Analogously, 2 and 3 cannot be separated. Hence, (3.10) does not satisfy the axiom T_1. Even worse, T_1-similarity structures turn out to be in general topologically uninteresting:

<u>(3.11) Proposition</u>. Let (S, ~) be a similarity structure. If (S, OS) satisfies T_1, then the Alexandrov topology OS is the trivial discrete topology O_1X = PX.

<u>Proof</u>. Assume that (S, OS) is T_1. Then every singleton {x} is closed, i.e., cl(x) = {x}. Since OS is Alexandrov one has cl(x) = ↓x := {y; x ≤ y}. Hence every x is a minimal and a maximal element of the partial order ≤. In other words, ≤ is trivial. Therefore all points of S are isolated, i.e., open and closed subsets of S.♦

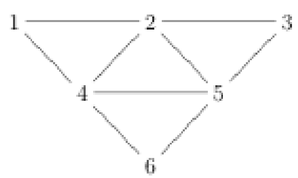
From (3.11) one immediately infers that a fortiori the Hausdorff separation axiom T_2 is quite useless for the elucidation of the topological aspects of similarity structures. In brief, the results are somewhat disappointing. None of the classical separations axioms does cut any ice for the topological aspects of similarity structures. This does not mean, however, that topology has nothing to offer to the conceptual analysis of similarity and related concepts. It should only be considered as evidence that the topological aspects of similarity, if there are any, are not classical. In the rest of this section I want to show that non-classical topology scores better with respect to this task. Indeed, there are certain non-classical separation axioms that are helpful for classifying different types of similarity structures according to their topological features.

All of classical general topology restricts its attention on spaces for which at least the Hausdorff separation axiom T_2 is valid. Spaces that do satisfy this axiom were considered as mere curiosities. This situation has changed. In the contemporary literature one may find dozens of „lower separation axioms“ somewhere between T_0 and T_2. It goes without saying, that it is impossible to deal with a a rep sample all of them. Let us consider only one example. Recall that a topological space (X, OX) is $T_{1/2}$-space if for each $x \in X$ the singleton {x} is either closed or open (cf. Dunham (1977, Theorem 2.5, p. 161). Then one easily checks that the similarity structure (3.10) defines a $T_{1/2}$-space: The singletons {1} and {3} are closed while {2} is open. The same obtains for generalizations of (3.10) such as

(3.12) 1———2—— …. ——n (n $\in$ **N)**.

also give rise to $T_{1/2}$ spaces that fail to be T_1-spaces. This has nothing to do with the fact that these similarity structures are “linear”, as is shown by the similarity structure already considered in (2.12): (cf. Goodman (1951), Mormann (1994, 2009), Leitgeb (2008):

(3.13)



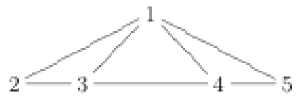
One checks the singletons {2}, {4}, and {5} are open, while the singletons {1}, {3}, and {6} are closed. Hence also (3.13) also yields a $T_{1/2}$ space that is not T_1. There have been defined a variety of other weak separation axioms such as $T_{1/4}$ and $T_{3/4}$ somewhere inbetween T_0 and T_1 but we will not pursue this line of thought further in this paper. We are content to state the following elementary result:

<u>(3.14). Proposition</u>. For similarity structures (S, ~) depth and topology are related as follows:

(i) d(S) = 0 if and only if (S, OS) is T_1.
(ii) d(S) = 1 if and only if (S, OS) is $T_{1/2}$.

Let us assume from now on that the topologies of all similarity structures are T_0. If a similarity structure (S, ~) does not satisfy T_0, it may be replaced by the similarity structure (S*, ~*) as defined in section 2. Then (S*, ~*) does satisfy T_0. In the light of the examples of similarity structures discussed so far one might conjecture that the axiom $T_{1/2}$ might be a generally valid axiom for similarity structures. The following structure, already considered in section 2 (2.13), shows that this is not the case:

(3.15)



In the similarity structure (3.15) the singleton {1} is open, the singletons {2} and {5} are closed but not open, while the singletons {3} and {4} are neither closed nor open. Hence (3.15) is not $T_{1/2}$. Thus, also $T_{1/2}$ does not help much in the task of singling out out an interesting class of topologically interesting similarity structures. One has to look elsewhere if one wants to find topological axioms (other than T_0) that are satisfied by all "reasonable" smilarity structures. Indeed, there are some although they may not be considered as classical ones. Rather, they emerged in the context of modern point-free topology especially designed for the needs of logic and information sciences (cf. Johnstone (1982), Vickers (1989), Picado and Pultr (2012)).
In the rest of this section two of these axioms shall be studied in some detail, namely, the so called T_D-axiom and the axiom SOB of sobriety. These axioms are stronger than T_0 but they do not entail T_1 nor are entailed by T_1. Moreover, they are independent of each other.

(3.16) Definition. The topological space (X, OX) satisfies the axiom T_D if and only if for each $x \in X$ there is an open neighborhood $U(x) \in OX$ such that $U(x) – \{x\} \in OX$.♦

Clearly, the axiom T_1 entails T_D: If (X, OX) is T_1 every open neighborhood U of x yields an open set U – {x} since cl(x) = {x}. The example (3.4)(iii) shows that T_D does not entail T_1. The relations between T_D and T_0 are as follows:

(3.17) Proposition. A T_D space (X, OX) satisfies T_0.

Proof. Assume $x, y \in X$, $x \neq y$, and $x \in U(x)$ with U(x) – {x} open. Then either $y \in U(x) – \{x\}$ or $y \notin U(x) – \{x\}$ and therefore $y \notin U(x)$.♦

Example (3.4)(iii) shows that in general T_0 does not entail T_D. The point ω has no open neighborhood U such that U – {ω} is open. Nevertheless (X, OX) does satisfy T_0. But (X, OX) is not sober since there is a irreducible element of CX that does not have the form cl(x) for a point $x \in X$, namely, X. By definition the only closed sets of X are Ø, the finite subsets of **N**, and X itself.

(3.18). Proposition: If (S, ~) satisfies (SNI) the Alexandrov topology (S, OS) satisfies T_D.
Prooof. It suffices to show that ↑x and ↑x – {x} are open for all $x \in S$. Assume $y \in$ ↑x and $y \in$ ↑x – {x}, i.e., $x \neq y$. The set ↑x – {x} is open if and only if ↑y ⊆ ↑x – {x}, i.e., if for all z with $y \leq z$ one has $z \in$ ↑x – {x}. From $x \leq y \leq z$ one gets $x \leq z$. Hence $z \in$ ↑x – {x} if and only if $x \neq z$. By reductio assume x = z.

Then one obtains from $x \leq y$ and $y \leq z$ and $z = x$ that $y = x$ by (SNI). This is a contradiction to the assumption $x \neq y$.♦

In sum, we have obtained a neat topological characterization of those similarity structures (S, ~) that satisfy Carnap's structuralist requirement that "nice" similarity structures should satisfy the axiom (SNI) of "similarity neighborhood identity", namely, that their Alexandrov spaces (S, OS) have to satisfy the separation axiom T_D. Nevertheless, this characterization may be appear somewhat disappointing in-sofar as for Alexandrov spaces the local closedness axiom T_D is nothing but the weakest topological axiom T_0 in disguise.
In the next section we will study another "non-classical" topological axiom of similarity structures (S, ~) that takes into account the concept of depth. This axiom, the so-called axiom of sobriety, is in general not equivalent with T_D - there are similarity structures that do NOT satisfy it, although they satisfy T_0 (and T_D).

4. Sober Similarity Structures**.** For the definition of the concept of sobriety some preparatory definitions are needed**.**

(4.1) Definition . Let (X, OX) be a topological space. (i) An open set P is irreducible if and only if for all $A, B \in OX$ with $A \cap B \subseteq P$ one has $A \subseteq P$ or $B \subseteq P$. (ii) Dually, a closed set $Q \in CX$ is irreducible if and only if for all $F, G \in CX$ with $Q \subseteq F \cup G$ one has $Q \subseteq F$ or $Q \subseteq G$.♦

(4.2) Examples of Irreducible Sets.
(i) Let (E, OE) be the Euclidean space, endowed with its standard metrical topology OE. Then the open sets $E - \{x\}$, $x \in E$ are irreducible. For open sets U such that $U \cap V \subseteq E - \{x\}$ either $U \subseteq E - \{x\}$ or $V \subseteq E - \{x\}$, since otherwise $x \in U \cap V$ and hence $U \cap V$ would not be a subset of $E - \{x\}$. Moreover, it can be easily shown that for (E, OE) there are no other irreducible elements of OE.

(ii) More generally, for any topological space (X, OX) the open sets $X - cl(\{x\})$ are irreducible (cf. Picado and Pultr (2012)).

(iii) For closed sets irreducibility can be expressed more simply than by (4.1)(ii) by stipulating that a closed set Q is irreducible if it is not the proper union of two closed subsets F and G. Clearly, for $x \in X$ the closures cl(x) of singletons {x} are irreducible. Further, if X is a T_1-space then the singletons {x} are irreducible.♦

Further, it can be easily shown that for Hausdorff spaces (such as the Euclidean space (E, OE) the irreducible elements $X - cl(x)$ of OX, and cl(x) of CX, respectively of irreducible sets are the only irreducible elements. That is to say that for Hausdorff spaces there is 1-1 correspondence between points of X and irreducible elements of OX and CX, respectively. This is not the case for all topological spaces. Take, for instance (E, O_oE), i.e., the set of points of Euclidean space endowed with the trivial indiscrete topology. Then clearly, there is no such 1-1 correspondence between points of E and irreducible elements of O_oE or C_oE. Hence, the following definition makes sense:

(4.3) Definition. A topological space (X, OX) is sober if and only if there is a 1-1-correspondence between the points $x \in X$ and the irreducible elements of X – cl(x) of OX, or, equivalently, between the points $x \in X$ and the irreducible elements cl(x) of CX.♦

As we want to show now this is an interesting definition since the Alexandrov spaces (S, OS) of many similarity structures (S, ~) considered so far are sober. As is well-known, sober spaces are T_0: Assume a, b $\in$ S and a ≠ b. Suppose a $\in$ cl(b) and b $\in$ cl(a). Then cl(a) $\subseteq$ cl(b) and cl(b) $\subseteq$ cl(a) and therefore cl(a) = cl(b). Since X has been assumed to be sober, this entails a = b. This is a contradiction. Hence one has a $\notin$ cl(b) or b $\notin$ cl(a). Assume a $\notin$ cl(b). Then a $\in$ S – cl(b) $\in$ OS, but b $\notin$ X – cl(b). In other words, S satisfies T_0. On the other hand, it is well known that the cofinite topology of **N** is T_0 but not sober. Hence, in general, sobriety and T_0 are not equivalent. The following example shows that sober spaces need not satisfy $T_{1/2}$:

(4.4) Example. The similarity structure (3.15) is sober but not does not satisfy $T_{1/2}$. The closures cl(x) of the elements of (3.15)are calculated as:

↓1 = {1, 2, 3, 4, 5}	↓2 = {2}	↓3 = {2, 3}
↓4 = {4, 5}	↓5 = {5}	

Clearly, these are the only irreducible closed subsets of (3.15) and each cl(x) is the closure of a unique generating point. Hence, (3.15) is sober.♦

By proposition (3.17) T_D entails T_0 but not viceversa. Hence one may wonder whether every T_D-similarity structure (S, ~) leads to a sober space (S, OS). The answer to this question is No (cf. Picado and Pultr (2012), 6.4.4. Proposition, p. 22)). Instead of simply relying on this general result, it is more elucidating to construct a concrete example of a similarity structure (S, ~) with an Alexandrov space (S, OS) that is T_D but not sober.

The first step is the construction of similarity structures (S, ~) of arbitrary finite depth n (n $\in$ **N)**. Let A = {0, 1, …, n} and A* := {1*, …, n*} be two disjoint sets of cardinality n+1 and n, respectively. Define a similarity relation ~ on (A $\cup$ A*, ~) by the following stipulations:

(4.5)

co(0) := A

co(1) := A $\cup$ {1*}   co(1*) := {1,2,3, …, n, 1*}

co(2) := A $\cup$ {1*, 2*}   co(2*) := {2,3, …, n, 2*}

...   …

co(n) := A $\cup$ {1*, 2*, ..., n*} ,   co(n*) := {n, n*}, In other words, the subgraph A of $A \cup A^*$ is a complete graph of n+1 elements in which each vertex is joined to each of the others by exacly one edge; the subgraph A^* of $A \cup A^*$ is the null graph of n elements with no edges. The elements of the subgraphs A and A^* respectively are joined as indicated in (4.5). It is easily checked that distinct elements $a, b \in A \cup A^*$ have distinct similarity neighborhoods co(a) and co(b). More precisely we obtain

$$co(0) \subset co(1) \dots \subset co(n)$$

while the similarity neighborhoods co(k*) of vertices of A^* in $A \cup A^*$ are incomparable, i.e., there are no nontrivial inclusion relations between the co(k*). Hence the Alexandrov space $(A \cup A^*, O(A \cup A^*))$ of $(A \cup A^*, \sim)$ is T_0. Moreover, the partial order $(A \cup A^*, \leq)$ contains the strictly increasing chain $0 < 1 < 2 < \dots < n$ and does not contain any strictly longer increasing chain. Hence $(A \cup A^*, \sim)$ is a T_0 Noetherian similarity structure of depth n.

A similar construction can be used to construct an infinite non-sober similarity relation (S, ~) by replacing the finite sets A and A^* by countably infinite sets. Consider the disjoint sets

(4.6) $\quad \mathbf{N} := \{0, 1, 2, 3, 4, ..., n, ...\} \quad , \quad \mathbf{N^*} := \{1^*, 2^*, 3^*, ..., n^*, ...\}.$

On $\mathbf{N} \cup \mathbf{N^*}$ a similarity relation ~ is defined by the following recipe: Assume **N** to be endowed with the trivial similarity relation according to which all $n \in \mathbf{N}$ are similar to each other, and **N*** endowed with the discrete similarity structure according to which $n^* \sim m^*$ if and only if $n^* = m^*$. The similarity relation ~ between elements of N and of N* is defined as follows:

(4.7)

$$\begin{array}{ll} co(1) := \mathbf{N} \cup \{1^*\} & co(1^*) := \{1,2,3, \dots, n, \dots , 1^*\} \\ co(2) := \mathbf{N} \cup \{1^*, 2^*\} & co(2^*) := \{2,3, \dots, n, \dots , 2^*\} \\ \dots & \dots \\ co(k) := \mathbf{N} \cup \{1^*, 2^*, ..., k^*\} & co(k^*) := \{k, k+1, \dots, n, \dots , k^*\}, \\ \dots & \dots \end{array}$$

For $n \in \mathbf{N}$ one obtains $\uparrow n := \{x; co(n) \leq co(x)\} = \{n, n+1, n+2, \dots \}$ and $\uparrow k^* = \{k^*\} \cup \{k, \{k, k+1, \dots\}$. The order structure $(\mathbf{N} \cup \mathbf{N}^*, \leq)$ of the similarity structure $(\mathbf{N} \cup \mathbf{N}^*, \sim)$ has an infinite strictly increasing monotone chain $0 < 1 < 2 < \dots < n < \dots$, i.e., $(\mathbf{N} \cup \mathbf{N}^*, \leq)$ is not Noether. In order to prove that (4.7) is not sober it is shown that there is no 1-1 correspondence between the points of $\mathbf{N} \cup \mathbf{N}^*$ and the irreducible elements of $O(\mathbf{N} \cup \mathbf{N}^*)$. More precisely, it is shown that the empty set Ø as an open subsets of $\mathbf{N} \cup \mathbf{N}^*$ is irreducible but there is no point $u \in \mathbf{N} \cup \mathbf{N}^*$ with $cl(u) = \mathbf{N} \cup \mathbf{N}^*$. This entails that Ø is an irreducible subset $\mathbf{N} \cup \mathbf{N}^*$ that does not have the form $\mathbf{N} \cup \mathbf{N}^* - cl(u)$. Hence, $\mathbf{N} \cup \mathbf{N}^*$ is not sober.

In order to show that Ø is irreducible one has to prove that for all open U, V with U ∩ V = Ø one has U = Ø or V = Ø. Since (**N** ∪ **N***, O(**N** ∪ **N***)) is Alexandrov, it is sufficient to prove this assertion for the special open sets ↑a and ↑b. One may distinguish three different cases: (i) ↑n ∩ ↑k = Ø, (ii) ↑n ∩ ↑k* = Ø, and (iii) ↑n* ∩ ↑k* = Ø. From (4.7) one obtains ↑n = {n, n+1, n +2, …} and ↑k* = {k*} ∪ {k, k+1, k+2, …}. The intersection of these sets, be they of type (i), (ii), or (iii), is always non-empty. Hence Ø is irreducible.

On the other hand, for all points u ∈ **N** ∪ **N*** one either has cl(u) = {0, 1, …, n} ∪ {1*, 2*, …, n*} for u = n, or cl(x) = {k*} for u = k*. Hence, it is impossible for any u ∈ **N** ∪ **N*** that Ø = **N** ∪ **N*** - cl(u). Hence there is no 1-1 correspondence between the points u ∈ **N** ∪ **N*** and the irreducible elements of O(**N** ∪ **N***). In other words, the Alexandrov space (**N** ∪ **N***), O(**N** ∪ **N***)) of (**N** ∪ **N***, ~) is not sober.

Using the fact that the Alexandrov space (S, OS) of an order structure (S, ≤) is sober if and only if (S, ≤) satisfies SNI and is Noether (cf. Picado and Pultr (2012) we obtain in particular that all finite similarity structures (S, ~) are sober and T_D, while similarity structures with infinitely many elements are sober if and only if they don't contain proper chains of elements with increasing depths.

Thus, when starting the Carnapian project of an *Aufbau* (construction) of the world from the basic level of a similarity structure (S, ~) of *elementary experiences* one has to make some assumptions concerning the complexity of the world that is to be constituted. The simplest proposal would be to conceptualize the world as a similarity structure S of a certain finite depth d(S). As transpires from proposition (3.14) small values of depths such as d(S) = 0, or 1 appear to be conceptually rather implausible. Choosing a fixed finite value d(S) = n in advance, is difficult to justify, the most plausible choice would be to subscribe to a general Noetherian framework: the basic level of the world is to be conceptualized as a similarity structure (S, ~) that is Noether, i.e., (S, ≤) does not contain proper chains of infinite depths. This does not exclude the possibility that the set S of elementary experiences is of infinite cardinality, of course. Moreover, it does not even exclude that there elements of arbitrarily large depth. In the next section it will be explained that this requirement can be formulated in a natural way as a condition on their quasi-analytical property distributions.

Before we come to this, let us conclude this section with some remarks on the concept of sobriety. Traditionally, topological spaces are defined as relational structures constituted by a set X of points, and a certain set-theoretical structure OX on X, consisting of subsets U of X that are to be interpreted as the "open neighborhoods" of points x ∈ U. Thereby in certain cases interesting relations can be established between genuinely topological elements such as points x ∈ X on the one hand, and lattice-theoretical objects, such as open sets U ∈ OX, on the other. For instance, if (X, OX) is a Hausdorff space there is a 1-1-correspondence between points x ∈ X and maximal elements X – {x} of the Heyting algebra OX, or between between points x ∈ X and atoms {x} of the co-Heyting algebra CX. Thus one may ask the general question whether (set-theoretical) points are really necessary to do topology. Perhaps, under certain appropriate circumstances, it may be sufficient to deal with the lattice OX (or CX)? This is indeed the case, and the "appropriate circumstances" can be precisely characterized, namely, X has to be sober. To put it bluntly: A sober topological space (X, OX) is completely characterized by its Heyting algebra (OX, ≤) of open sets. The method to prove this result may be traced back to the proof of Stone's trail-blazing representation theorem for Boolean algebras in the 1930s but is much simpler.

Let us start with the following definition:

<u>(4.8) Definition</u>. Let L be a complete lattice and F a proper subset of L. F is a filter on L if it satisfies the following three requirements:

(i) $0 \notin F$.

(ii) F = is F is upper closed, that is for all a, b $\in$ L a $\in$ F and a ≤ b entails that b $\in$ F.

(iii) F is closed with respect to finite meets, i.e., a, b $\in$ F entails that $a \wedge b \in F$.

F is a completely prime filter if moreover it satisfies the further condition that for all K with $K \subseteq L$ and $\sup(K) \in F \Rightarrow \exists a\,(a \in K$ and $a \in F)$. This requirement is often expressed by saying that F is "inaccessible by arbitrary suprema": if sup(K) is a element of F, then there must be an element $a \in K$ that already belongs to F.♦

<u>(4.9) Example.</u> Let (X, OX) be a topological space, $a \in X$. Then the set N(a) := {U; $a \in U \in OX$} of open neighborhoods of a is a completely prime filter on OX.♦

After these preparations we can precisely formulate the above assertion that a sober topological space (X, OX) is completely characterized by its lattice OX of open subsets. This entails, in particular, that the point set X can be reconstructed by purely lattice-theoretical means from OX:

<u>(4.10) Theorem (cf. Picado and Pultr (2012, 1.4. Corollary)</u>. Let L be a complete lattice isomorphic to the Heyting algebra OX of a sober space (X, OX). The (X, OX) is homeomorphic to a topological space (Y, OY) defined as follows:

Y = {F; F completely prime filter in L}   OY = {a*; $a \in L$ with a* = {F; $a \in F$}}

Informally, the points of the topological space (Y, OY) are the completely prime filters of F, and the elements of OY are the sets a* of completely prime filters F containing a.♦

<u>5. Continuous Maps and Quasianalyses</u>. Sometimes topology is simply characterized as the study of topological spaces. Stricly speaking, this tells us only half of the story. At least as important is the study of structure-preserving maps between topological spaces. This is a general feature of modern structural mathematics. The investigation of mathematical structures goes hand in hand with the investigation of structure-preserving maps between those structures. Although there may be several

different kinds of structure-preserving maps, in the case of topological spaces the most important kind of structure-preserving maps are certains continuous maps between spaces. In an analogous manner

<u>(5.1) Definition.</u> Let (S, ~) and (T, ~) be similarity structures. A structure-preserving map (S, ~) ——f——>(T, ~) is a set-theoretical map S ——f——>T that is structure preserving in the sense that for all x, y ∈ S x ~ y entails that f(x) ~ f(y). If there is no danger of confusion, structure-preserving maps between similarity structures (S, ~) and (T, ~) are simply denoted as S ——f——>T. ♦

The concatenation of similarity-preserving maps S——f——>T and T ——g——>U of similarity structures S, T, and U is again a structure-preserving map S ——g•f——>U. Further, the identity map S ——id——>S is structure-preserving and concatenation is associative. Hence, similarity structures and structure-preserving maps define a "local mathematical universe", i.e., a category in the sense of Eilenberg and Mac Lane (cf. Awodey 2010, Picado and Pultr (2012, Appendix II)) that may be denoted by SIM.
In an analogous way one may define structure-preserving maps between order structures (X, ≤). Given order structures (A, ≤) and (B, ≤) a set-theoretical map A——h——>B is structure-preserving with respect to the order structure if and only if for all c, d ∈ A the relation c ≤ d entails h(c) ≤ h(d). Analogously one obtains that order structures and order-preserving maps define a category ORD.
Now the natural question arises how these categories are related. By (2.6) every similarity structure (S, ~) defines in a canonical way an order structure (S, ≤). Hence one may ask: Given a structure-preserving map (S, ~) ——f——>(T, ~) does f induce a structure-preserving map (S, ≤) ——f——>(T, ≤) between the corresponding order structures? As elementary examples show, in general the answer to this question is no, there are similarity-preserving maps that are not order preserving. More interestingly, similarity maps that do preserve order structures can be characterized in topological terms. To understand how this works, first recall the following basic definition:

<u>(5.2) Definition</u>. Let (X, OX) and (Y, OY) topological spaces. A set-theoretial map X——f——>Y is continuous (with respect to the topologies OX and OY) if and only if for all U ∈ OY one has $f^{-1}(U)$ ∈ OX. ♦

It is routine to check that also topological spaces and continuous maps define a category TOP in the sense of mathematical category theory. The following proposition points to an interesting relation between the category ORD of order structures and the category TOP of topological spaces:

<u>(5.3) Proposition</u>. Let (X, ≤) ——f——>(Y, ≤) be an order-preserving map between the order structures (X, ≤) and (Y, ≤). Then f defines a continuous map between the corresponding Alexandrov topological spaces (X, OX) ——f——>(Y, OY).

Proof. Since the topological structures considered are Alexandrov topologies it suffices to prove for all $x \in X$ the sets $f^{-1}(\uparrow f(x))$ are open in X. Assume $a \in f^{-1}(\uparrow f(x))$ and $a \leq b$ with $a, b \in X$. In order to show that $f^{-1}(\uparrow f(x))$ is open one has to show that $b \in f^{-1}(\uparrow f(x))$. Now $b \in f^{-1}(\uparrow f(x))$ iff $f(b) \in \uparrow f(x))$ iff $f(x) \leq f(b)$. This holds true due to the assumption that $a \leq b$ and f is order-preserving. Hence, f is continuous. ♦

Informally, an order-preserving map $(S, \leq) \xrightarrow{f} (T, \leq)$ between order structures (S, ~) and (T, ~) defines a continuous map $(S, OS) \xrightarrow{f} (T, OT)$ between the Alexandrov spaces (S, OS) and (T, OT). In category-theoretical terms this amounts to the assertion that the correspondence between order structures (S, ≤) and topological spaces (S, OS) and their respective structure-preserving maps gives defines a functor (cf. Awodey 2006).

After these general considerations we are prepared to deal with the issue of the topological aspects of quasi-analysis of similarity structures. For this purpose, it is expedient to consider the following special class of similarity structures that will play a crucial role for quasi-analyzing general similarity structures:

(5.4) Definition. Let Q be an non-empty set. Let P*Q the set of non-empty subsets of Q. Then a similarity relation on P*Q is defined by $A \sim B := A \cap B \neq \emptyset$. In the following P*Q is always assumed to be endowed with this canonical similarity relation. The elements of Q are to be interpreted as "quasi-properties" in the sense of Carnap (1923). ♦

Now we are going to show that a quasi-analysis of a similarity structure (S, ~) is nothing but a certain continuous map $(S, \sim) \xrightarrow{f} (PQ^*, \sim)$ of the similarity structure S into a similarity structure PQ* of appropriate quasiproperties Q. This is a quite natural reformulation of Carnap's original project as can be shown by going back to Carnap's original account as presented in Carnap (1923).

In order to make this thesis plausible first let us recall Carnap's orginal formulation of the four basic requirements that every quasi-analysis ("*Quasizerlegung*") of a similarity structure S (*"inhomogene Menge"*) has to satisfy:

(5.5) Axioms for Similarity and Quasiproperties (Carnap (1923, 4 – 5). Let (S, ~) be a similarity structure, Q a set the elements of which are called quasi-properties. Then a quasi-analysis of S by Q is a distribution of quasi-properties which satisfies the following requirements:

(C1) If two elements of S are similar they share at least one quasiproperty $q \in Q$.

(C2) If two elements of S are not similar they do not share any quasiproperty $q \in Q$.

(C3) If two elements a and b are similar to exactly the same elements they have the same quasiproperties.

(C4) There is no proper subset Q' ⊂ Q such that the elements of Q' satisfy (C1) – (C3) for all elements of S.♦

In more formal terms, such a distribution of quasi-properties may be described as a map (S, ~) —f—>(PQ*, ~) that maps the elements s ∈ S onto elements f(s) ∈ PQ* in such a way that the requirements (C1) – (C4) are satisfied.

The following remarks on f are in order. Since the similarity relation ~ on S is reflexive f(a) is not empty for all a ∈ S, the f maps S into P*Q. In terms of the quasi-analytical representation f the requirements (C1) - (C3) can be formulated as

(C1) $a \sim b \quad \Rightarrow \quad f(a) \cap f(b) \neq \emptyset.$

(C2) $f(a) \cap f(b) \neq \emptyset \quad \Rightarrow \quad a \sim b.$

(C3) $co(a) = co(b) \quad \Rightarrow \quad f(a) = f(b).$

Since Ø ∉ PQ* from (C2) one immediately obtains the converse of (C3):

$(C3)^{\#}$ $f(a) = f(b) \quad \Rightarrow \quad co(a) = co(b).$

In other words, the map f is a structure-preserving map between the similarity structures (S, ~) and (PQ*, ~) in a quite strong sense. For our purposes, it is expedient to make a slightly stronger requirement than (C3) that takes into account the order structures involved and helps bring to the fore the topological aspects of the quasi-analytical representation f, namely

(C3)* $co(a) \subseteq co(b) \quad \Rightarrow \quad f(a) \subseteq f(b).$

This renders f also a structure-preserving map with respect to the order structures (S, ≤) and (P*Q, ≤). By (5.3) one even obtains that such a map f even defines a <u>continuous</u> map (S, OS) —f—>(PQ*, OP*Q) between the associated Alexandrov spaces (S, OS) and (P*Q, OP*Q). If one further assumes that (S, ~) satisfies (SNI) this entails that f is a monomorphism, i.e., from f(a) = f(b) one can deduce that f(a) ≠ f(b). This can be proved as follows. Assume f(a) = f(b) and a ≠ b. From a ≠ b one deduces that co(a) ≠ co(b). Then one may assume that there is x ∈ co(a) and x ∉ co(b). That is to say a ~ x and NOT (b ~ b) Therefor f(x) ∩ f(a) ≠ Ø and f(x) ∩ f(b) = Ø. Hence f(a) ≠ f(b). This is a contradiction. Hence f is a monomorphism.

The following remarks on Carnap's requirements (C1) – (C4) are in order. The requirements (C1) and (C2) may be considered as almost analytic for a "reasonable" relation between similarity and properties. Implicitly, they are based on the assumption that all properties considered are "essential" properties in the sense that they determine the similarity between objects. Two objects are similar to each other if and only if they share at least one "essential" property. These two requirements are the only ones that appear in the *Aufbau*. Moreover, they are the only ones that have been discussed in the secondary literature dealing with quasianalysis of similarity structures. Exceptions are Proust (1989), and Mormann (1994, 2009).

The most interesting condition is (C3). Mormann (2009) showed that many (but not all) of the allegedly "devastating" counter-examples for the uniqueness of a quasi-analysis of a similarity structured founder on this requirement.

The requirement (C4) is a kind of Occam's razor. It requires maximal austerity lof the distribution of quasi-properties. As long as the set Q of quasiproperties is finite it is possible at least in principle to construct for a function S——f——>P*Q that satisfies (C1) – (C3)* a function S——f'——>P*Q that also satisfies (C4) simply by brute force. More precisely one provisionally deletes a quasiproperty q_1 and check whether the resulting function "f – {q}" still satisfies (C1) – (C3)*. If this is not the case f already satisfies (C4), if not, one provisionally deletes another quasiproperty q_2 and so on. Since Q is finite this process ends with a quasianalysis f does satisfies (C1) – (C3)*, and (C4). It should be noted that this process in no way produces a unique "minimal" quasianalysis but heavily depends on the order according to which the elements of Q are provisionally deleted. In other words, even for finite similarity structures (S, ~) there may be distinct quasianalyses which satisfy (C1), (C2),(C3)*, and (C4) (see Mormann (2009)).

Goodman and many others argued that this unavoidable non-uniqueness is to be considered as a fatal flaw of the quasianalytical method. Meanwhile, a growing number of scholars for philosophical reasons no longer thinks that Goodman's arguments are really convincing (cf. Proust (1989), Mormann (2005, 2009).

If the set Q of quasi-properties has infinite cardinality, things may become more complicated. Although even in this case a given quasianalytical function f (which satisfies (C1), (C2), and (C3)*) may be improved stepwise by step by deleting superfluous quasiproperties it is not clear, whether this process converges to some definite function or may go on and on for ever without ever reaching an end.

Things become better manageable if one assumes that the starting point S——f——>P*Q is such that for every a $\in$ S the cardinality of f(a) is finite. In this case the quasi-analytical representation f is actually a map S——f——>P_{fin}*Q with P_{fin}*Q the set of non-empty <u>finite</u> subsets of Q. In the terminology of the previous section P_{fin}*Q is characterized as the set of sets of elements that have finite depth. In this case one can prove by induction over the subsets of S that the process of eliminating superfluous quasiproperties of f does terminate with a definite function f* that satisfies all of Carnap's requirements. Of course, this f* need not be uniquely determined.

A (SNI) similarity structure (S, ~) that has a quasianalysis S——f——>P_{fin}*Q may be called locally finite: although S may have infinitely many objects, every s $\in$ S is of finite depth and can be described with the help of finitely many quasiproperties. In this case an optimal quasianalysis, i.e., one with satisfies (C4) can be constructed from any one which satisfies (C1), (C2) and (C3)* but possibly not (C4). Already for finite similarity structures one can show, however, that such an optimal analysis may not be unique. In Mormann (2009, Proposition (3.3)) an existence theorem for quasianalysis that satisfies the first three axioms has been proved. Hence, for similarity structures of finite depth the existence of a minimal quasianalyses satisfying (C4) is guaranteed.

6. Quasi-analysis of Similarity Structures as Sheaves. Conceiving a quasianalysis of a similarity structure (S, ~) as a continuous map f between the topological spaces S and P*Q (or P_{fin}*Q is not the only way of making explicit the topological (and order-theoretical) aspects of the quasianalytical method. Another possibility of connecting the quasianalytical approach with concepts of modern mathematics in a fruitful way is to conceive a quasi-analysis of a similarity structure S as a very special topological bundle, namely, as a sheaf over the Alexandrov space (S, OS) of S. This will be done in the rest of this section.

First, let us succinctly recall the definition of a topological bundle. A topological bundle (E, p, B) is just a continuous map E——p——>B between the topological spaces E and B. The space E is called the total space of the bundle, B its base space, and $p^{-1}(b)$:= {e; p(e) = b} is called the fibre over b.

Topological bundles abound in mathematics, physics, and other sciences. A particularly simple example is provided by trivial bundles having total space E = B x X for some space X and B x X ——p——>B defined as p(b, x) := b. Trivial bundles are also called product bundles. A more interesting type of bundle is represented by the following example. Let **R** denote the real line and $\mathbf{S}^1$ the unit sphere of dimension 1 defined by $\mathbf{S}^1$:= {z; z is a complex number z = a + ib with $|z| = 1$}. Let **R**——p——>$\mathbf{S}^1$ defined by p(r) := exp(2rπ). Then (**R**, p, $\mathbf{S}^1$) is well-known to be a non-trivial topological bundle. This bundle has the property of being locally, but not globally trivial, i.e., every point z $\in$ $\mathbf{S}^1$ has an open neighborhood U(z) such that (p^{-1}(U(z)), p, U(z)) is a trivial bundle.

As we will show now the topological bundles defined by continuous quasianalyses are non-trivial bundles. The first step is to define the sets underlying the total space, the base space, and the fibre of such a bundle:

(6.1) Definition. Let S——f——>P*Q be a continuous quasianalysis of a similarity structure (S, ~). Define the bundle (E(f), p, S) by E(f) := {(s, q); q $\in$ f(s)} and p(s, q) := s. The bundle (E(f), p, S) is called the bundle associated with f.♦

The topology of the base space S is, of course, the Alexandrov topology. The topology on the total space E(f) will be defined in a moment. First some general remarks on the adequacy of an interpretation of Carnap's original conception of quasianalysis. Conceiving a quasianalysis S——f——>P*Q as a bundle (E(f), p, S) immediately leads to an interpretation of the concept of quasianalysis that is very close to Carnap's original intentions. The bundle achieves exactly what a quasianalysis originally should yield, namely, a kind of analysis or resolution of the elements s of S that helps understand the fabric of the various similarity relations in which they stand to each other. More precisely, the fibre p^{-1}(s) := {(s, q); q $\in$ f(s)} over s provides sort of analysis or resolution of the element $s \in S$. The fibre $p^{-1}(s)$ is to be conceived of as not as a proper analysis but as a quasianalysis of $s \in S$ since s "as such" remains intact, it is only represented by its fibre. Thus, moving from the base space S of indecomposable elements to the total space E amounts indeed a "quasianalysis" in a quite literal sense since the „atomic" object s is virtually replaced by a system of elements q that allows for a more fine-grained analysis of the relations of s to other objects of the basic domain S.

Or, to put it in another way, the quasianalytical stalks $p^{-1}(s)$ of the bundle (E(f), p, S) can be used to „explain" the similarity relations in which the base elements $s \in S$ stand to each other. The bundle-theoretic perspective conceiving a quasianalysis of a similarity structure as a topological bundle brings to the fore some novel aspects and possibly fruitful connections of this concept that have gone unnoticed so far by more traditional interpretations. Indeed, the topological bundles that emerge from the quasianalysis of similarity structures are examples of an especially type of bundles, namely, <u>sheaves</u> (cf. Goldblatt 1978, Mac Lane and Moerdijk 1992).

The topological bundle (E(f), p, S) associated to a the quasianalysis S——f——>P*Q of a similarity structure is a very special bundle since the total space E and the base space S are „locally" but not „globally" homeomorphic. More precisely, every element e of the total space E has an open neighborhood $U(e) \in OE(f)$ that is mapped homeomorphically by p onto an open neighborhood V(p(e)) $\in$ OS of p(e). Nevertheless, the topological spaces E(f) and S may be globally non-homeomorphic as is explicitly shown by the bundles that arise from continuous quasianalysis of similarity structures.

Topological bundles (E, p, S) whose projection map p is a local homeomorphism are not mere curiosities. Since the 1950s bundles of this type have been thoroughly investigated and found applications in many mathematical fields and more recently also in computer science. Bundles of this kind are called sheaf spaces, etale spaces, or sometimes simply sheaves (cf. Borceux (1994), Mac Lane and Moerdijk (1991)).

After these preparations, now we can formulate and prove the following theorem that brings together the order-theoretical and topological aspects of the quasi-analysis of similarity structures in a neat way:

<u>(6.2) Theorem</u>. Let S——f——>P*Q be a continuous quasianalysis of a similarity structure (S, ~). Then the topological bundle E(f)——p——>S defined by $E(f) := \{(s, q); q \in f(s)\}$ and $p(s, q) = s$ is a sheaf bundle. Every element $e \in E(f)$ has an open neighborhood U(e) that is mapped homeomorphically by p onto an open neighborhood p(U(p(e))) of $p(e) \in S$. More precisely, the canonical maps q: $f^{-1}(\uparrow q)$——>E(f) defined by $q(s) := (s, q)$ are local homeomorphisms.

<u>Proof.</u> Let $q \in Q$. Assume $q \in f(s) \subseteq Q$ for some $s \in S$. Define $\uparrow q := \{A; q \in A \subseteq Q\}$. The set $\uparrow q$ is upper closed and hence open with respect to the Alexandrov topology of PQ*. Since f is continuous the set $f^{-1}(\uparrow q) \subseteq S$ is open with respect to the order topology OS of S. Now one has $f^{-1}(\uparrow q) = \{t; f(t) \subseteq \uparrow q\} = \{t; q \in f(t)\}$.

If there is no $s \in S$ with $q \in f(s)$ then $f^{-1}(\uparrow q) = \emptyset$. Hence for all $q \in Q$ the set $f^{-1}(\uparrow q)$ is open.

Define $V(q) \subseteq E(f)$ by $V(q) := \{(t, q), q \in f(t)\}$. Then p maps V(q) isomorphically onto the open set $p(V(q)) = \{t; q \in f(t)\} \in OS$. The sets V(q), $q \in Q$, define a topology OE(f)on Ef such that every e = (t, q) $\in$ E(f) has an open neighborhood V(q) that is mapped homeomorphically onto an open neighborhood of p(e).[5]♦

Theorem (6.2) evidences that Carnap's quasi-analys can't be dismissed as an esoteric construction. Rather, it turns out to be an elementary case of a widespread construction that has shown its fruitfulness in many areas of mathematics, logic, and other fields, namely, sheaf theory (cf. Borceux (1994), Goldblatt (1979), Johnstone (1982)). More precisely, a quasi-analysis of a similarity structure S is nothing but the construction of a sheaf bundle (etale space) over S. Thus, a natural generalization of Carnap's account would be to replace the rather simple similarity structures of the base space S by more richer topological structure. Then, a sheaf (E, p, S) yields in a quite literal sense a "quasi-analysis" of the elements of S, namely, $s \in S$ is "quasi-analysed" by its fibre $p^{-1}(s) = \{e; p(e) = s\}$ in the sense the elements of $p^{-1}(s)$ are to be interpreted as "quasi-properties" of s.

7. Concluding Remarks. Although the order-theoretical and topological elucidation of the concepts similarity and quasianalysis that has been carried out in this paper employ quite different conceptual tools than those that Carnap used in the *Aufbau* and other early manuscripts in which he treated issues of similarity and its role for the constitution of the world, does not follow the letter of Carnap's approach in these writings, I contend that it faithfully follows his intentions, namely, to provide a precise explication of some important philosophical notions using the conceptual means of modern logic and mathematics. This becomes evident if one looks at Carnap's original account of the aim and purpose of quasi-analysis revealed as put forward in (Carnap 1923):

> Suppose there is given a set of elements, and for each element the specification to which it is similar. We aim at a description of the set which only uses this information but ascribes to these elements quasicomponents or quasiproperties in such a way that it is possible to deal with each element separately using only the quasiproperties, without reference to other elements. (Carnap 1923, *Quasizerlegung*, p.4)

As should be clear from the considerations of the previous sections Carnap's "inhomogeneous sets" are just similarity structures. The requirements (C1) – (C4) that according to him have to be satisfied by a quasi-analysis are captured in a natural way by the structural requirements for a continuous quasi-analytical representation of a smilarity structure. To formulate it succinctly, a quasi-analysis of a similarity structure (S, ~) is a sheaf bundle (E, f, S) with base space S, S endowed with the Alexandrov

[5] Indeed, the V(q), $q \in Q$, are all disjoint. Hence, they provide in a canonical way a basis for a uniquely defined topology on E.

topology defined by the similarity relation ~. Thus, as a formal theory, Carnap's quasi-analysis of similarity structures fits snugly into the general framework of modern topological bundle theory, more precisely, into the theory of sheaves.

The topological and order-theoretical elucidation of the concept of similarity is faithful to the spirit of Carnap's original approach to a logical analysis of the concept of similarity as put forward in the early manuscript *Quasianalysis - A Method to Order Non-homogeneous Sets by Means of the Theory of Relations* (Carnap 1923) already mentioned in the introductory section of this paper. In the very title of this manuscript, the aim of the quasi-analytical method was explicitly characterized as to "order non-homogeneous sets" using the methods of relational logic. In order to understand the meaning of this thesis one has to know what he meant by "non-homogeneous sets". Roughly, a "non-homogeneous" set is characterized as a set for which it makes sense to distinguish between elements that are "similar" and those that are "not similar" to each other. Thus, the elements of a non-homogeneous set can be classified or ordered in some way or other.

To put it in a nutshell, then, a quasi-analysis is a method for ordering non-homogeneous sets, i.e., sets, for which it makes sense to look for the possibility of an ordering. This is achieved by representing the elements of M as bundles of properties that determine the relations of similarity between them. More precisely, elements are considered as similar if and only if they share a common property; if they do not share a common property, they are said to be not similar. Replacing the space S by a quasi-analytical sheaf bundle (E(f), f, S) with (S, OS) as base space amounts to a "resolution" of the elements $s \in S$ by the fibres $f^{-1}(s) := \{q; q \in f(s)\}$ bundles of quasi-properties q. This resolution provides a kind of reificatory explanation by the otherwise opaque similarity relation ~ defined on S: the elements s and s' are similar to each other if and only if there are elements e and e' in the fibres $p^{-1}(s)$ and $p^{-1}(s')$ of s and s' respectively, that both are contained in the same element V(q) of the distinguished subbasis of the topology of Ef as defined in the proof of theorem (6.2). The fibres may $p^{-1}(s)$ and $p^{-1}(s')$ be may be interpreted as device for "explaining" the similarity (or non-similarity) of their base points s and s' in so far as they provide (by the neighborhood V(q)) a "carrier" of the similarity of s and s', namely, the quasi-property q.

The sheaf structure as well as the more elementary order structure and the topological structure on which it is based all depend on the the concept of similarity neighborhood encapsulated in the operator co. Thus this structure may be considered as the point of departure on the way toward a precise structural analysis of the concept of similarity. Thereby Quine's and Goodman's verdicts according to which this concept is "alien to logic and set theory" can be refuted.

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