Measurement Scales and Welfarist Social Choice

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Abstract. The social welfare functional approach to social choice theory fails to distinguish between a genuine change in individual well-beings from a merely representational change due to the use of different measurement scales. A generalization of the concept of a social welfare functional is introduced that explicitly takes account of the scales that are used to measure well-beings so as to distinguish between these two kinds of changes. This generalization of the standard theoretical framework results in a more satisfactory formulation of welfarism, the doctrine that social alternatives are evaluated and socially ranked solely in terms of the well-beings of the relevant individuals. This scale-dependent form of welfarism is axiomatized using this framework. The implications of this approach for characterizing classes of social welfare orderings are also considered.

Keywords. grading; measurement scales; social welfare functionals; utility aggregation; welfarism

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1. Introduction

The social welfare functional approach to social choice theory fails to distinguish between a genuine change in individual well-beings from a merely representational change due to the use of different measurement scales. It is possible for a change in well-beings to be exactly compensated by a change in the scales with which they are measured. When this is the case, the social ordering of the alternatives remains unchanged. In effect, the real change in well-beings is rendered invisible by the compensating change in measurement scales. We introduce a generalization of the concept of a social welfare functional that explicitly takes account of the scales that are used to measure well-beings so as to distinguish between these two kinds of changes. This generalization of the standard theoretical framework permits a wide range of responses to the compensating changes described above; invariance of the social ordering is no longer required.

Welfarism is the doctrine that social alternatives are evaluated and socially ranked solely in terms of the well-beings of the relevant individuals. We argue that a satisfactory formulation of this doctrine also requires a specification of the scales used to measure well-beings (i.e., welfarism is *scale dependent*), otherwise the numbers with which well-beings are described cannot be interpreted as being measures of well-being.[1] Similarly, to say that the length of an object is the number l requires a unit of measurement for this statement to be meaningful. We axiomatize scale-dependent welfarism using our new framework. The implications of our approach for characterizing classes of social welfare orderings are also considered.

The Arrovian approach to social choice theory (Arrow, 1951) relies exclusively on information about how individual preferences order the social alternatives when determining how to socially order them. Arrow precludes from the outset taking account of any information about the individual well-beings when making this social choice other than what is embodied in the individual orderings. To overcome this limitation of the Arrovian approach, Sen (1970, 1974) has introduced the concept of a social welfare functional. A *social welfare functional* assigns a social ordering of the alternatives to each profile of individual utility functions, where a *profile* is an ordered list of entities (here utility functions), one for each person.[2] These utility func-

[1] A similar point has been made by Roemer (1996, Sec. 2.5) about axiomatic models of bargaining and by Marchant (2008) about models for adjudicating claims in a bankruptcy.

[2] A formally equivalent aggregation procedure was introduced by Luce and Raiffa (1957,

tions provide the information about the individual well-beings that is used to determine the social ordering. Sen's framework can accommodate different assumptions about what kinds of utility comparisons are possible both intrapersonally and interpersonally, including the Arrovian special case in which utilities are ordinally measurable and interpersonally noncomparable.

Sen's framework provides a formal way of considering the merits of social objectives that are only meaningful when some kinds of interpersonal utility comparisons are possible. In particular, social welfare functionals have been used to axiomatize a wide range of methods for aggregating individual utilities for the purpose of socially comparing different alternatives. For example, there are axiomatizations of the utilitarian sum, maximin utility, and the constant elasticity of substitution functions that constitute the Atkinson (1970) class, all of which presuppose that some kind of interpersonal utility comparisons are possible—gains and losses in the case of utilitarianism, levels in the case of maximin, and ratios in the case of the Atkinson class. Social welfare functionals have also been used to address questions in population ethics (see Blackorby, Bossert, and Donaldson, 2005b).[3]

Welfare economics is built on welfarist foundations. In the standard formulation of welfarism, well-beings are identified with utilities, *however measured*. Social alternatives are collectively ordered using a social welfare ordering (or a social welfare function that represents it) on the vectors of individual utilities that are obtained with them (see Jehle and Reny, 2011, Chap. 6). No other information other than these utility values is taken account of when determining the social ordering of the alternatives. One of the most important implications of the conventional understanding of welfarism is that all of the information embodied in a social welfare functional can be equivalently described by a single social welfare ordering of the attainable vectors of individual utilities. Specifically, with n individuals, for any profile of utility functions $U = (U_1, \ldots, U_n)$ and any pair of alternatives x and y, how x and y are socially ranked when the profile is U is the same as how the vectors of individual utilities $U(x) = (U_1(x), \ldots, U_n(x))$ and $U(y) = (U_1(y), \ldots, U_n(y))$ are ranked by the social welfare ordering.

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p. 343). In their formalism, a social ordering is assigned to matrices whose columns are the individuals, whose rows are the alternatives, and whose entry in column i and row j is the utility that individual i obtains with alternative j.

[3]Surveys of the literature that employs social welfare functionals may be found in Sen (1977), d'Aspremont (1985), d'Aspremont and Gevers (2002), and Bossert and Weymark (2004).

When the domain of a social welfare functional is unrestricted (Unrestricted Domain), the existence of such a social welfare ordering is equivalent to requiring the social welfare functional to satisfy Pareto Indifference and Independence of Irrelevant Alternatives. Pareto Indifference requires two alternatives to be socially indifferent if each individual is equally well off with them. Independence requires the social preference for a pair of alternatives to be the same for two different profiles if these profiles coincide on this pair. This independence condition is the social welfare functional analogue of Arrow's independence axiom. Henceforth, we refer to these Pareto and independence conditions as the *welfarism axioms*. The two welfarism axioms are also jointly equivalent to Strong Neutrality, which requires that the only features of alternatives that are used to socially rank them are their utilities—the alternatives' physical descriptions and the profile of utility functions that generate these utilities are irrelevant.

With a social welfare functional, different assumptions about the measurability and interpersonal comparability of utilities are formalized by requiring the social ordering of the alternatives to be invariant to any transformation of the profile of utility functions that preserves the meaningful utility comparisons, a property of a social welfare functional called *information invariance*. For example, if only levels of utility are interpersonally comparable, then the social ordering of the alternatives must be invariant to a common increasing transform of each person's utility function. Similarly, if both levels and differences in utilities are meaningful (what is known as *cardinal full comparability*), then invariance is only required if each individual's utility function is subjected to a common increasing affine transform. The modeling of utility comparisons in terms of invariance transforms was developed by Sen (1970, 1974, 1977), d'Aspremont and Gevers (1977), and Roberts (1980), among others.[4] This approach builds on the way that measurement scales have been formalized and classified into scale types in the representational theory of measurement. Pioneering contributions to our understanding of these concepts include Stevens (1946) and Luce (1959). See Krantz, Luce, Suppes, and Tversky (1971) and Roberts (1979) for classic presentations of measurement theory and Luce, Krantz, Suppes, and Tversky (1990, Chap. 20) and Narens (2002) for systematic discussions of scales and scale types.

In this article, a *utility scale* consists of the possible utility values linearly

[4]Invariance transforms were used by Luce and Raiffa (1957, pp. 343–346), but not in any systematic way.

ordered from best to worst and an interpretation of what these utility values mean (e.g., a specification of the units in which they are measured). This way of modeling a scale is closely related to the concept of a language in the theory of grading proposed by Balinski and Laraki (2007, 2010).

Sen (1977, p. 1542) has drawn attention to the importance of distinguishing between real and representational changes in well-beings in the following insightful passage:

> with cardinal full comparability (indeed even with ratio-scale full comparability), the invariance property is unable to distinguish between (i) everyone having more welfare (better off) in some real sense and (ii) a reduction in the unit of measurement of personal welfares. Invariance in all the cases considered here, including cardinal full comparability (or ratio-scale full comparability), requires that the social ordering should not change at all if everyone's welfare function is, say, doubled. But while this requirement is reasonable enough if interpreted as a halving of the unit of measurement, it is quite a restriction if the interpretation is that of a general increase in personal welfare, since the social welfare ordering need not be accepted to be a mean-independent function of individual welfares. But in all cases of measurability-comparability frameworks discussed here (and in other works), the invariance requirement covers both interpretations since there is no natural "unit" of measurement of personal welfare.

Sen's point can be illustrated with a stylized example.[5] Consider a two-person society. Suppose that when individual well-beings are measured using a common scale S, the utility vectors obtained with x and y when the profile of utility functions is U are $(40, 70)$ and $(50, 50)$, respectively. Now suppose that, for whatever reason, all welfares decrease by half using the same scale S resulting in the profile of utility functions U'. The utility values for x and y are then $(20, 35)$ and $(25, 25)$. If the unit of measurement is now halved, we have new scale S'. With this new scale, all measured values are restored to their initial values and, hence, the profile of utility functions is again U even though individual welfares have not changed. The utilities associated with x and y with the three combinations of a profile of utility functions and a measurement scale are summarized in Table 1.

[5]We discuss this example more fully in Section 5.

	x	y
(U,S)	$(40,70)$	$(50,50)$
(U',S)	$(20,35)$	$(25,25)$
(U,S')	$(40,70)$	$(50,50)$

Table 1: Independence of Irrelevant Alternatives conflates real and scale changes in utilities.

Suppose that x is socially preferred to y in the first case because it has a higher sum of utilities, but that the social preference is reversed in the second case because priority is given to the worst off whenever anybody's utility falls below some fixed threshold that is scale-dependent. If well-being is ratio-scale measurable, then this preference must be invariant to a revaluation of utilities due to a change in the unit of measurement. Hence, y should also be socially preferred to x in the third case. But the utility values for x and y in this case are the same as in the first case, so this social preference is inconsistent with Independence of Irrelevant Alternatives. The problem is that this independence axiom takes no account of the scales used and so when scale changes exactly compensate for real changes in utilities, it is as if nothing has happened. This axiom rules out a perfectly appropriate social response to real changes in personal welfares when these are countervailed by changes in the scale with which they are measured.

We agree with Sen that information invariance is a reasonable requirement when all that has changed is the measurement scale. The failure of a social welfare functional to distinguish between a real change in individual well-beings from one that is obtained by using different scales to measure them is due to the combination of using the traditional formulation of welfarism (either directly or by assuming Unrestricted Domain, Pareto Indifference, and Independence of Irrelevant Alternatives) and the scale invariance property. Having pointed out the failure of a social welfare functional to properly distinguish between real and representational changes in well-beings, Sen does not propose a way of doing so. That is what we do here.

A fundamental problem with the social welfare functional approach is that it is not possible to interpret what the utility values measure because the scales used to interpret them have not been specified. Thus, when somebody's utility for an alternative changes from, say, 1 to 2, it is not known whether this change involves a change in the scale in which these numbers are expressed or not. It is for this reason that a social welfare functional can fail to correctly identify when a welfare change is real. As a consequence, the formalization of welfarism using social welfare functionals is incomplete because an individual's well-being is underspecified if the scale with which it is being measured is not known.

In order to deal with this problem, we generalize Sen's definition of a social welfare functional by including the scales that are used to measure individual utilities in its domain in addition to the profile of utility functions.[6] We allow for the possibility that different individuals use different scales of measurement, so the profile of utility functions is supplemented with a profile of scales. To distinguish our functionals from those of Sen, ours are called *scale-dependent social welfare functionals*. The specification of a scale is an essential component of the procedure by which numbers are assigned to alternatives and interpreted as being utility values.[7]

By taking account of the scales of measurement, not just the profile of utility functions, when making collective decisions, we can (i) restrict the application of the Pareto, independence, and neutrality axioms to comparisons that employ the same measurement scale and (ii) restrict the application of the information invariance axioms to comparisons in which only changes in scales are involved. Thus, we are able to distinguish real welfare changes from those that are due to changes in the measurement scales, something which is not always possible using Sen's framework.

Our framework allows us to provide a more satisfactory formulation of what constitutes welfarism, what we call *scale-dependent welfarism* so as to distinguish it from the traditional formulation of welfarism. According to scale-dependent welfarism, the only information that is used to evaluate and socially rank alternatives are the utilities of the relevant individuals together with the scales with which these utilities are measured. We believe that this formulation of what welfarism requires is more satisfactory than the standard one because it includes an interpretation of what the utility values mean. Moreover, scale-dependent welfarism permits social rankings to reflect real welfare changes even when accompanied by a change in the measurement

[6]Blackorby, Bossert, and Donaldson (2005a) have generalized Sen's definition of a social welfare functional so as to take account of non-welfare information. We discuss the relationship between our approach and theirs in Section 4.

[7]Similarly, when assigning lengths to objects, it is necessary to know what units of length are being used in order to interpret these numbers as being lengths.

scales that would otherwise obscure the fact that there has been any real change in well-beings at all.

We characterize scale-dependent welfarism using the restricted forms of the Pareto, independence, and neutrality axioms described above. There is no longer a single social welfare ordering of utility vectors. Instead, there is one for each profile of measurement scales. Nevertheless, given the ordering for one profile of scales, the ordering for any other profile of scales is uniquely determined once an information invariance condition has been specified. The axiomatic characterizations of various social welfare orderings found in the literature make essential use of the traditional formulation of welfarism that is implied by the standard Pareto and independence axioms for social welfare functionals when the domain is suitably unrestricted. By limiting these axioms so that they only apply for fixed measurement scales, we show that there is much greater freedom in choosing a social welfare ordering that is compatible with the other axioms that one might wish to adopt. Thus, welfarism is a less restrictive doctrine when measurement scales are properly accounted for than when they are not.

The rest of this article is organized as follows. Section 2 sets out our formal framework. In Section 3, we review how welfarism is formalized and axiomatically characterized using Sen's social welfare functionals. Then, in Section 4, we reformulate the basic welfarism axioms in scale-dependent terms and provide a scale-dependent analogue of the welfarism theorem established using the traditional approach. In Section 5, we illustrate our theorem with two examples of social aggregation procedures that are not welfarist in the traditional sense, but do satisfy our scale-dependent form of welfarism. In Section 6, we describe how information invariance restricts how social welfare orderings for different scales are related and discuss the implications of our approach for characterizing classes of social welfare orderings. Finally, we provide some concluding remarks in Section 7.

2. The Formal Framework

We are concerned with the problem of determining a social preference ordering for a set of alternatives as a function of the utility functions that a group of individuals have for them. We restrict attention to a fixed set of individuals but our analysis straightforwardly extends to variable population comparisons. The set of alternatives can be anything for which a social preference might be wanted: feasible allocations in an economy, candidates in an election, etc. Formally, the set of alternatives is X, with X assumed to contain at least three alternatives so that all of our assumptions are non-vacuous. The set of individuals is $N = \{1, \ldots, n\}$, with $n \geq 2$. Each individual has a utility function defined on X and these functions are aggregated to determine the collective ordering.

Utilities can be thought of as grades measured using some scale. Balinski and Laraki (2007, 2010) have developed a general theory of collective evaluation in which the individual inputs take the form of grades assigned to the alternatives. Examples of such problems include the evaluation of research projects by an expert panel, prioritizing job candidates by a hiring committee, and the evaluation of evidence for different medical interventions by a scientific panel. Our concept of a scale is based on Balinski and Laraki's way of modeling grades. Grading scales may be person specific. Formally, person i's *grading scale* is a triple $S_i = (\mathcal{G}_i, \succ_i, \mathcal{I}_i)$, where $\mathcal{G}_i$ is the set of available grades, $\succ_i$ linearly orders these grades from best to worst ($g \succ_i g'$ indicates that g is a better grade than g'), and $\mathcal{I}_i$ is the interpretation procedure that fixes the meaning of the grades in $\mathcal{G}_i$. A *grading function* for individual i is a function $G_i \colon X \to \mathcal{G}_i$; that is, it assigns a grade to each alternative.[8]

Balinski and Laraki (2010, Chaps. 7–8) provide a variety of examples of grading scales that are used in a wide range of applications. Although they consider interpretations in some detail, interpretations do not appear explicitly in their formal analysis.[9] Commonly used scales include the grades {*excellent, very good, good, fair, poor*} used in a variety of applications or the set of letter grades {*A, B, C, D, F*} often used in academic evaluation, ordered in the usual way with concrete interpretations of their meanings. The interpretation specifies the characteristics of an alternative that merit a

[8]A grading function corresponds to the homomorphism that is used in the representational theory of measurement to map the objects in an empirical relational system into the numbers in a numerical relational system. See Roberts (1979, Chap. 2) for a succinct presentation of this theory. Narens (2002, p. 50) defines a scale to be a set of real-valued grading functions.

[9]Balinski and Laraki (2010, p. 176) assume that everybody uses the same grading scale $(\mathcal{G}, \succ)$, which they call a *language*. They claim that individuals over time will tend to converge on a common understanding of their grading language (Balinski and Laraki, 2010, pp. 166–169). Intuitively, this claim seems quite plausible. However, because their framework contains no *formal* treatment of interpretation procedures, it is not obvious how to spell out the content of this claim in precise terms. Our own approach lends itself better to studying questions about interpretations of grades and their consequences for collective decision-making.

particular grade. In the case of course letter grades, the interpretation could fix the meaning of an A by saying that the mastery of the course material is outstanding and exhibits independence, a B by saying that this mastery is outstanding but exhibits only moderate independence, and so on.[10]

Grades can be a set of real numbers, in which case $\mathcal{G}_i \subseteq \mathbb{R}$ and the relation $\succ_i$ is simply the inequality $>$ for the real line. In many cases, non-numerical grades can be accommodated by encoding them as numbers. For example, self reports of health status (SRHS), which are often used in making decisions about the allocation of resources for health care, typically use the grades {*excellent, very good, good, fair, poor*}. It is common to convert these verbal grades to the numerical grades {5, 4, 3, 2, 1} for statistical analysis. However, the grades only have ordinal significance, so it is equally valid to use a nonlinear set of grades such as {10, 4, 3, 2, 1} in which only the value assigned to an excellent outcome has been changed (see Allison and Foster, 2004). Similarly, the letter grades {*A, B, C, D, F*} can be described by the numbers {5, 4, 3, 2, 1} or, equivalently, by {4, 3, 2, 1, 0} as in done in the United States when computing grade point averages. Two scales may use the same set of grades but attach meanings to these grades using a different interpretation procedure. For example, self reports of pain and the Mohs hardness scale both use the numbers from 1 to 10 as grades, but differ in their interpretations of these grades.

In assigning grades to alternatives, the question arises whether the grade of x can be determined by examining the properties of x by itself or whether grades are necessarily comparative. Our framework is flexible enough to accommodate either possibility. For example, the hardness of a mineral is determined by its crystalline structure, and so is non-comparative. While hardness is "absolute" in this sense, as an empirical matter, comparative methods may aid in determining a mineral's hardness, as when one rock is assigned a larger hardness value than a second because the former can scratch the latter. Similarly, mass is non-comparative. In contrast, weight is not. In our framework, the absoluteness or otherwise of the grades is settled by the interpretation.[11]

We are concerned with the use of scales to measure utilities in welfarist

[10]A more refined example is that of the Danish course grading system discussed by Balinski and Laraki (2010, p. 133).

[11]Whether a property is intrinsic or extrinsic and how this distinction relates to whether a property is comparative are complicated issues on which there is no consensus. See Weatherstone and Marshall (2014).

social choice, rather than in grading more generally. Utilities are numbers and therefore a utility scale is a numerical grading scale. We employ notation for a utility scale that reflects this fact. Formally, for each individual $i \in N$, *i's utility scale* is a triple $S_i = (\mathcal{U}_i, >, \mathcal{I}_i)$, where $\mathcal{U}_i \subseteq \mathbb{R}$ is the set of *permissible utility values*, $>$ is the standard "larger than" relation for the real line, and $\mathcal{I}_i$ is the *interpretation procedure* that fixes the meaning of the utilities in $\mathcal{U}_i$. We assume that the cardinality of $\mathcal{U}_i$ is at least three. Because $>$ is a common feature of all utility scales, we suppress further mention of this component of a scale.

For example, when the social alternatives are lotteries, as in the von Neumann and Morgenstern (1944) approach to decision-making under uncertainty, and there is a best and worst alternative, utility values are often normalized to lie in the unit interval $[0, 1]$, with 1 interpreted as being the best possible level of well-being. That is one scale. By applying an increasing affine transform that results in the utility values lying in $[1, 3]$, 1 is then interpreted as being the worst possible level. Disambiguation of the meaning of the numbers used to express utility values is made possible by the specification of the scale.

In contrast to the examples of grading scales that we have considered, little attention has been devoted to the question of how to interpret the utilities that are used to measure an individual's well-being. In part, this is because there is no consensus about the nature of well-being. Well-being is variously thought of in terms of mental states (e..g., pleasure), preference satisfaction, and objective lists of goods.[12] Different interpretation procedures are appropriate for different conceptions of well-being. For our purposes, we do not need to settle these interpretive questions; we only need to suppose that some interpretation has been adopted.

The set of *admissible utility scales for individual i* is $\mathcal{S}_i$. We allow for the possibility that different individuals use different scales and that they have different sets of admissible scales. Our analysis also applies if the $\mathcal{S}_i$ are the same for everybody. A *profile of utility scales* is an n-tuple $S = (S_1, \ldots, S_n)$. The set of *admissible profiles of utility scales* is $\mathcal{S} \subseteq \Pi_{i \in N} \mathcal{S}_i$. We assume that $\mathcal{S}$ contains at least two profiles. If there is only one profile, then our framework is formally equivalent to the one used by Sen (1970). In principle, $\mathcal{S}$ could be the set of all logically possible profiles of utility scales. Depending on the particular aggregation problem being considered, it might be natural

[12]See the essays in Part II of Adler and Fleurbaey (2016).

to restrict $\mathcal{S}$ in various ways.

For each individual $i \in N$ and utility scale $S_i = (\mathcal{U}_i, \mathcal{I}_i) \in \mathcal{S}_i$, i's *utility function* is a function $U_i\colon X \to \mathcal{U}_i$. We can think of U_i as a report that lists, for each $x \in X$, which utility value i has assigned to x. A *profile of utility functions* is an n-tuple $U = (U_1, \ldots, U_n)$. The set of all logically possible profiles of utility functions for the profile of utility scales S is $\mathfrak{U}^S$. For all $x \in X$ and $U \in \mathfrak{U}^S$, $U(x) = (U_1(x), \ldots, U_n(x))$ is the vector of utility values obtained with x. A *profile* is a pair $T = (U, S)$ such that $U \in \mathfrak{U}^S$ where $S \in \mathcal{S}$. Let $\mathcal{T}$ be the set of all profiles that can obtained with $\mathcal{S}$.

Let $\mathcal{R}$ denote the set of all orderings of X.[13] A *scale-dependent social welfare functional* $F\colon \mathcal{D} \to \mathcal{R}$ determines a social preference ordering $F(T)$ ("socially weakly preferred to") of the alternatives in X for each profile T in the set of *admissible profiles* $\mathcal{D} \subseteq \mathcal{T}$. This definition of a social welfare functional generalizes that of Sen (1970) by pairing each profile of utility functions with a profile of measurement scales in the domain. For notational simplicity, let $R_T = F(T)$ denote the social preference ordering that results if the profile is T. The strict preference and indifference relations corresponding to R_T are denoted by P_T and I_T, respectively.[14]

The analogue of a scale-dependent social welfare functional for a grading scale is a *grading functional* that aggregates a profile of grading functions and a profile of grading scales into a collective ranking of the alternatives. Our formal results and analysis also apply to grading functionals when the grades are numerical.

3. Scale-Independent Welfarism

In this section, we review how welfarism is formalized and axiomatically characterized using the social welfare functionals introduced by Sen (1970, 1974). Sen does not include scales in his definition of a social welfare functional.[15] For him, a utility function for individual i is a function $U_i\colon X \to \mathbb{R}$ and a profile is an n-tuple U of such functions. A (Sen) *social welfare functional*

[13] A binary relation R on X is (i) *reflexive* if for all $x \in X$, xRx, (ii) *complete* if for all distinct $x, y \in X$, $xRy \vee yRx$, and (iii) *transitive* if for all $x, y, z \in X$, $xRy \wedge yRz \to xRz$. An *ordering* is a reflexive, complete, and transitive binary relation. A *linear ordering* is an ordering for which $xRy \to \neg(yRx)$ for all distinct $x, y \in X$.

[14] For all $x, y \in X$, (i) $xP_Ty \leftrightarrow [xR_Ty \wedge \neg(yR_Tx)]$ and (ii) $xI_Ty \leftrightarrow [xR_Ty \wedge yR_Tx]$.

[15] Henceforth, we restrict attention to the aggregation of utilities and, for simplicity, generally omit the qualifier "utility" when referring to a utility scale.

is a mapping $F\colon \mathcal{D} \to \mathcal{R}$, where $\mathcal{D}$ is now the domain of profiles of utility functions that are to be aggregated. In this section, a profile is simply a profile of utility functions U, so $F(U)$ will be denoted by R_U.

We now formally define the unrestricted domain condition and the two welfarism axioms.

Unrestricted Domain. The domain $\mathcal{D}$ consists of all possible profiles of utility functions on X.

Unrestricted Domain is appropriate when there are no *a priori* restrictions on the profiles that are to be aggregated.

Pareto Indifference. For all $x, y \in X$ and all $U \in \mathcal{D}$, if $U(x) = U(y)$, then $xI_U y$.

Pareto Indifference requires two alternatives to be socially indifferent when each individual obtains the same utility from them. This axiom ensures that the social preference bears some relationship to the individual utilities, but only in the weak form of respecting universal indifference. It precludes a social preference from being imposed by tradition or by an outside party.

Independence of Irrelevant Alternatives. For all $x, y \in X$ and all $U, U' \in \mathcal{D}$, if $U(x) = U'(x)$ and $U(y) = U'(y)$, then $xR_U y \leftrightarrow xR_{U'} y$.[16]

Independence of Irrelevant Alternatives requires the social preference for a pair of alternatives to be the same in two profiles if the individual utilities obtained with them in the two profiles coincide on this pair. This axiom implies that the social preference for any pair of alternatives is independent of the utilities obtained with any other alternative.

The welfarist requirement that alternatives be socially evaluated solely in terms of the well-beings (here, utilities) of the individuals is captured by the following neutrality axiom.

Strong Neutrality. For all $w, x, y, z \in X$ and all $U, U' \in \mathcal{D}$, if $U(w) = U'(y)$ and $U(x) = U'(z)$, then $wR_U x \leftrightarrow yR_{U'} z$.

Strong Neutrality precludes taking account of any non-utility information about the alternatives when determining a social preference. In particular,

[16]In this and similar expressions, the biconditional that is symmetric to the one in the consequence (here, $yR_U x \leftrightarrow yR_{U'} x$) is redundant and so is omitted.

the physical descriptions and names of the alternatives and the profile of utility functions that generate the utilities have no relevance except as mediated through the utilities they generate.

Consider a subset Ω of $\mathbb{R}^n$ whose elements are vectors of utility numbers. A social ordering R^* of Ω is called a *social welfare ordering*. If this ordering can be represented by a function $W\colon \Omega \to \mathbb{R}$, the representation W is called a *social welfare function*. For the profile of utility functions U, the set of *attainable utility vectors* is

$$\Omega_U = \{u \in \mathbb{R}^n \mid u = U(x) \text{ for some } x \in X\}.$$

For the domain $\mathcal{D}$ of profiles of utility functions, let

$$\Omega_{\mathcal{D}} = \cup_{U\in\mathcal{D}}\, \Omega_U.$$

When the domain is unrestricted, $\Omega_{\mathcal{D}}$ is all of $\mathbb{R}^n$.

A social welfare ordering R^* of $\Omega_{\mathcal{D}}$ is *homologous* to the social welfare functional $F\colon \mathcal{D} \to \mathcal{R}$ if for all $x, y \in X$ and all $U \in \mathcal{D}$,

$$xR_U y \leftrightarrow U(x)R^*U(y). \tag{1}$$

When the social welfare functional F satisfies (1), for any profile of utility functions U in its domain, the social ranking of any pair of alternatives x and y can be inferred from how the social welfare ordering R^* ranks the corresponding utility vectors $U(x)$ and $U(y)$. In effect, all of the information embodied in F is summarized by the single ordering R^*. As Theorem 1 demonstrates, this informational parsimony is a direct consequence of the traditional formulation of welfarism.[17]

Theorem 1 (Sen (1977); Hammond (1979))**.** *For a social social welfare functional $F\colon \mathcal{D} \to \mathcal{R}$ that satisfies Unrestricted Domain, the following conditions are equivalent:*

(i) *Pareto Indifference and Independence of Irrelevant Alternatives,*

[17] The equivalence of the combination of Pareto Indifference and Independence of Irrelevant Alternatives with (i) Strong Neutrality and (ii) the existence of a social welfare ordering that is homologous to the social welfare functional when Unrestricted Domain is assumed are due to Sen (1977) and Hammond (1979), respectively. Sen's Theorem is a Pareto Indifference version of a theorem established by d'Aspremont and Gevers (1977) using a stronger Pareto principle.

(ii) *Strong Neutrality, and*

(iii) *there exists a social welfare ordering R^* of $\mathbb{R}^n$ that is homologous to F.*

The ordering R^* in this theorem is unique. If utilities were restricted to be nonnegative (resp. positive), then the homologous social welfare ordering would only be defined on the nonnegative (resp. positive) orthant of $\mathbb{R}^n$.

Pareto Indifference by itself implies a more limited form of welfarism that is profile dependent. With *profile-dependent welfarism*, the only information that is used to socially rank alternatives is the individual utilities associated with them and the profile of utility functions that generate these utilities. The following neutrality axiom formalizes this property.

Profile-Dependent Neutrality. For all $w, x, y, z \in X$ and all $U \in \mathcal{D}$, if $U(w) = U(y)$ and $U(x) = U(z)$, then $wR_U x \leftrightarrow yR_U z$.

As is the case with Strong Neutrality, Profile-Dependent Neutrality takes no account of the physical descriptions or names of the alternatives when socially ranking them.

For the profile of utility functions U, the social welfare ordering R_U^* of Ω_U is *homologous* to the social preference ordering R_U of X if for all $x, y \in X$,

$$xR_U y \leftrightarrow U(x)R_U^* U(y). \tag{2}$$

In contrast to (1), the social welfare orderings in (2) are profile dependent.

The analogue of Theorem 1 for profile-dependent welfarism is Theorem 2.[18]

Theorem 2 (Blackorby, Donaldson, and Weymark (1990))**.** *For a social social welfare functional $F \colon \mathcal{D} \to \mathcal{R}$, the following conditions are equivalent:*

(i) *Pareto Indifference,*

(ii) *Profile-Dependent Neutrality, and*

(iii) *for each $U \in \mathcal{D}$, there exists a social welfare ordering R_U^* of Ω_U that is homologous to R_U.*

[18]Proofs of Theorems 1 and 2 may be found in Bossert and Weymark (2004) and Blackorby, Donaldson, and Weymark (1990), respectively.

Theorem 2 holds for any domain. Neither Pareto Indifference nor Profile-Dependent Neutrality place cross-profile restrictions, so the equivalences in this theorem apply profile by profile. Independence of Irrelevant Alternatives does place cross-profile restrictions and, as we see from Theorem 1, it implies that each social welfare ordering R^*_U in Theorem 2 is the restriction of a *single* social welfare ordering R^* of $\mathbb{R}^n$ to the set of attainable utility vectors Ω_U for the profile U when Unrestricted Domain is satisfied.

4. Scale-Dependent Welfarism

We have argued in Section 1 that a satisfactory formulation of welfarism requires that measurement scales be explicitly considered. We provide a scale-dependent formulation of welfarism in this section and axiomatically characterize it using scale-dependent versions of the axioms presented in Section 3. With *scale-dependent welfarism*, the only information that is used to socially rank alternatives is the individual utilities associated with them and the scales with which these utilities are measured. With scale-dependent welfarism, it is possible to distinguish between real changes in utility from changes due to the use of different measurement scales. The Pareto, independence, and neutrality axioms considered in Section 3 fail to make this distinction. We thus need to restrict the scope of these axioms to comparisons that employ the same scales. This is what we now do using the framework introduced in Section 2. Recall that in this framework, a scale-dependent social welfare functional is a mapping $F\colon \mathcal{D} \to \mathcal{R}$, where an element of $\mathcal{D}$ is a profile $T = (U, S)$ consisting of an n-tuple of utility functions U and an n-tuple of scales S.

Scale-Dependent Pareto Indifference. For all $x, y \in X$ and all $T = (U, S) \in \mathcal{D}$, if $U(x) = U(y)$, then xI_Ty.

Scale-Dependent Independence of Irrelevant Alternatives. For all $x, y \in X$ and all $T = (U, S)$, $T' = (U', S) \in \mathcal{D}$, if $U(x) = U'(x)$ and $U(y) = U'(y)$, then $xR_Ty \leftrightarrow xR_{T'}y$.

Scale-Dependent Strong Neutrality. For all $w, x, y, z \in X$ and all $T = (U, S)$, $T' = (U', S) \in \mathcal{D}$, if $U(w) = U'(y)$ and $U(x) = U'(z)$, then $wR_Tx \leftrightarrow yR_{T'}z$.

These axioms differ from their counterparts in Section 3 by explicitly taking account of the measurement scales that give utility values their meanings.

Moreover, they hold measurement scales fixed. Consider, for example, Scale-Dependent Independence. For this axiom to require that the social ordering of a pair of alternatives be the same for two profiles, it is not enough for the individual utility *values* to be the same in both profiles. In addition, each individual's utility values must be measured using the same scale so that, in fact, he derives the same *utility* from w and y and from x and z.

Our scale-dependent welfarism theorem employs a scale-dependent analogue of Unrestricted Domain.

Scale-Dependent Unrestricted Domain. The domain $\mathcal{D}$ is $\mathcal{T}$.

With this domain assumption, for any fixed profile of scales S in the set of admissible profiles $\mathcal{S}$, any profile of utility functions U that is in the set of logically possible utility profiles $\mathfrak{U}^S$ for these scales is included in $\mathcal{T}$. For example, if S is admissible and everybody uses the same scale in which the permissible utility values are $\{0, 1, 2\}$, then *any* profile U for which for each person i and each alternative x, $U_i(x)$ is one of these three numbers is in the domain. This axiom is a natural generalization of Unrestricted Domain when profiles of utility functions are paired with the measurement scales that give utility values their meanings.

Which utility vectors are attainable now depends on both the profile of utility functions and the profile of scales used. Formally, for the profile $T = (U, S)$, the set of *attainable utility vectors* is

$$\Omega_T = \{u \in \mathbb{R}^n \mid u = U(x) \text{ for some } x \in X\}.$$

Let

$$\mathcal{D}_S = \{T \in \mathcal{D} \mid \alpha(T) = S\}$$

be the set of all profiles in the domain for which the profile of scales is S, where $\alpha(T)$ is the second component of T (T's profile of scales). The set of *attainable utility vectors for the profile of scales S* is

$$\Omega_S = \cup_{T \in \mathcal{D}_S} \Omega_T.$$

Scale-Dependent Unrestricted Domain implies that $\Omega_S = \Pi_{i \in N} \mathcal{U}_i$ for all $S \in \mathcal{S}$, where $\mathcal{U}_i$ is individual i's set of possible utility values with the scale S_i.

Let $F|\mathcal{D}_S$ be the restriction of the scale-dependent social welfare functional $F \colon \mathcal{D} \to \mathcal{R}$ to the subdomain $\mathcal{D}_S$. A social welfare ordering R^*_S of Ω_S is *homologous* to $F|\mathcal{D}_S$ if for all $x, y \in X$ and all $T = (U, S) \in \mathcal{D}_S$,

$$xR_T y \leftrightarrow U(x) R^*_S U(y). \tag{3}$$

The social welfare orderings in (3) are scale-dependent.

Theorem 3 is our scale-dependent welfarism theorem. The proof of this theorem may be found in the Appendix.

Theorem 3. *For a scale-dependent social social welfare functional $F\colon \mathcal{D} \to \mathcal{R}$ that satisfies Scale-Dependent Unrestricted Domain, the following conditions are equivalent:*

(i) *Scale-Dependent Pareto Indifference and Scale-Dependent Independence of Irrelevant Alternatives,*

(ii) *Scale-Dependent Neutrality, and*

(iii) *for each $S \in \mathcal{S}$, there exists a social welfare ordering R^*_S of $\Pi_{i \in N} \mathcal{U}_i$ that is homologous to $F|\mathcal{D}_S$.*[19]

Theorem 3 applies for any set $\mathcal{S}$ of admissible profiles of scales. In particular, it applies if in each profile of admissible scales S, everybody uses the same scale. None of the axioms in this theorem place any restrictions on how the social preferences for profiles with different scales are related, which is why the social welfare orderings in this theorem are scale dependent. Similarly, in Theorem 2, there are no cross-profile restrictions on the social preferences. Consequently, the social welfare orderings are profile-dependent in that theorem. The scale-dependent axioms in Theorem 3 do place cross-profile restrictions on the social preferences but only for fixed scales, and that is why the social welfare orderings are scale-dependent but not profile-dependent.

It is straightforward to define scale-dependent versions of the various other axioms that have been considered for social welfare functionals and to determine their implications for the structure of a social welfare ordering, so we shall only do so informally. We illustrate what is involved with two Pareto and one anonymity axioms. *Scale-Dependent Weak Pareto* regards one alternative to be socially strictly preferred to another if everybody is better off with the first alternative than with the second using a fixed profile of scales. *Scale-Dependent Strong Pareto* regards one alternative to be at least as good as another if everybody is weakly better off with the first alternative than with the second using a fixed profile of scales and, furthermore,

[19]Formally, Theorem 1 can be thought of as being a special case of Theorem 3 in which the set of admissible scales $\mathcal{S}$ contains only a single profile of scales.

there is strict social preference if, in addition, at least one person is strictly better off. These Pareto principles imply that each social welfare ordering R_S^* is weakly monotonic and strictly monotonic, respectively. With *Scale-Dependent Anonymity*, the social preference is invariant to a permutation of the individual utility functions and measurement scales. If everybody uses the same scale in S, this axiom implies that R_S^* is symmetric.

Blackorby, Bossert, and Donaldson (2005a) have proposed a generalization of a social welfare functional in which non-welfare information may be used in addition to the individual utility functions in order to determine the social preference. They model non-welfare information in an abstract way that can encompass a wide variety of interpretations. Moreover, their framework is flexible enough to distinguish between social and individual non-welfare information. Examples of non-welfare information that they consider include the presence or absence of democratic institutions and an individual's length of life or propensity to work hard. Formally, a profile of scales can be interpreted as being non-welfare information in their sense.[20] As a consequence, a scale-dependent social welfare functional is a particular instantiation of their social aggregation procedure.

Blackorby, Bossert, and Donaldson (2005a) define analogues of the axioms considered here, but reformulated so as to apply to their formal framework. While it is in principle possible using their framework to take account of non-welfare information when forming a social preference, they show that their axioms imply that, in fact, non-welfare information is ignored, and so the social aggregation procedure is welfarist in the traditional sense; that is, non-welfare information is irrelevant. Their independence condition only applies when both the utilities and non-welfare information for the alternatives being compared are the same, echoing our own proposal that Independence ought to apply only when utilities are measured using the same scales. However, their Pareto indifference axiom applies when the utilities are the same even if the non-welfare information is not held fixed, which is too strong when the non-welfare information is about measurement scales. If their Pareto

[20] At a formal level, this is true. However, utility scales are different in kind from the examples of non-welfare information considered by Blackorby, Bossert, and Donaldson (2005a) in that scales specify information about well-beings more fully, rather than supplement well-being information with further considerations. Analogously, having said that the length of something is l, to add that this measurement is in meters is not to provide further "non-length" information, but rather to more fully specify how l should be interpreted.

principle is instead formulated so that it only applies to comparisons in which the non-welfare information is held fixed, a more limited form of welfarism is characterized, one that is conditional on the non-welfare information, in precisely the same way that our formulation of welfarism is conditional on the scales being used to measure utilities. Thus, the more limited scope of our Pareto axiom is what prevents our scale-dependent account of welfarism from collapsing to the traditional form of welfarism.

5. Two Scale-Dependent Welfarist Examples

In this section, we present two examples of scale-dependent social welfare functionals that are welfarist in our sense. The first is based on the numerical example used in Section 1. The second shows that the method of majority judgment advocated by Balinski and Laraki (2007, 2010) when applied to utility aggregation is a scale-dependent social welfare functional.

Example 1. This example formalizes distributional objectives that are quite plausible, but which cannot be captured by the conventional formulation of welfarism, thereby illustrating the added flexibility provided by our approach.

We suppose that everybody uses the same scale and that there is a scale-dependent utility level that is minimally acceptable socially. We take this fact into account when determining a social preference. We give priority to equity considerations when there are some individuals who have not achieved this threshold, but not if the threshold has been met by everybody. Specifically, our hybrid social aggregation procedure is utilitarian for comparisons in which everybody reaches the minimally acceptable level and it employs the maximin utility criterion otherwise. This functional is welfarist in the scale-dependent sense, but not in the traditional one, because which number represents the minimally acceptable utility depends on the scale used to measure utilities.

Formally, for all $S \in \mathcal{S}$, we assume that $S_i = S_j$ for all $i, j \in N$. Let μ_S be the *minimally acceptable utility* as expressed in the common scale. For any profile of utility functions $U \in \mathfrak{U}^S$ and any alternative $x \in X$, let

$$\mu U(x) = \min\{U_1(x), \dots, U_n(x)\}$$

be the *minimum utility* achieved in the profile U with the alternative x. The *hybrid* scale-dependent social welfare functional $F_H\colon \mathcal{T} \to \mathcal{R}$ is defined by setting, for each profile $T = (U, S) \in \mathcal{T}$ and each pair of alternatives $x, y \in X$,

$$xR_T y \leftrightarrow \begin{cases} \Sigma_i U_i(x) \geq \Sigma_i U_i(y), & [\mu U(x) \geq \mu_S] \wedge [\mu U(y) \geq \mu_S] \quad (4a) \\ \mu U(x) \geq \mu U(y), & [\mu U(x) < \mu_S] \vee [\mu U(y) < \mu_S]. \quad (4b) \end{cases}$$

First, we show that F_H is a social welfare functional. That is, for any profile T in its domain, the corresponding social preference R_T is an ordering of X. Reflexivity and completeness follow directly from the corresponding properties of the relation $\geq$ on $\mathbb{R}$. To show transitivity, let $xR_T y$ and $yR_T z$. There are two cases to consider. In the first case, $\mu U(y) < \mu_S$. In this case, (4b) applies to both of the comparisons with y, and so $\mu U(x) \geq \mu U(y) \geq \mu U(z)$. Hence, we also have $\mu U(z) < \mu_S$, and therefore (4b) also applies to the comparison of x and z. Because $\mu U(x) \geq \mu U(z)$, it follows that indeed $xR_T z$. In the second case, $\mu U(y) \geq \mu_S$. Note that $xR_T y$ and $\mu U(y) \geq \mu_S$ jointly imply that $\mu U(x) \geq \mu_S$ as well.[21] If it is also the case that $\mu U(z) \geq \mu_S$, then (4a) applies to the comparisons of both x and z with y, and so $\Sigma_i U_i(x) \geq \Sigma_i U_i(y) \geq \Sigma_i U_i(z)$. Because (4a) also applies to the comparison of x and z and $\Sigma_i U_i(x) \geq \Sigma_i U_i(z)$, it then follows that $xR_T z$. If, instead, $\mu U(z) < \mu_S$, then (4b) applies to the comparison of x and z, and we again have $xR_T z$ because $\mu U(x) \geq \mu_S > \mu U(z)$, which completes the demonstration that R_T is transitive.

By construction, F_H satisfies Scale-Dependent Unrestricted Domain. We now show that it also satisfies the two scale-dependent welfarism axioms. Consider any profile $T = (U, S) \in \mathcal{T}$ for which $U(x) = U(y)$. Because both $\Sigma_i U_i(x) = \Sigma_i U_i(y)$ and $\mu U(x) = \mu U(y)$, we have $xI_T y$ regardless of which of (4a) or (4b) applies. Hence, F_H satisfies Scale-Dependent Pareto Indifference. Now consider any two profiles $T = (U, S), T' = (U', S) \in \mathcal{T}$ for which $U(x) = U'(x)$ and $U(y) = U'(y)$. Then, $\mu U(x)$ and $\mu U(y)$ both meet the threshold μ_S if and only if both $\mu U'(x)$ and $\mu U'(y)$ do as well. Thus, for both profiles, the social ranking of x and y is determined by the same case in the definition of F_H. That is, both rankings are determined by (4a) or they are both determined by (4b). Because both of these cases only consider the two alternatives being compared and the individual utilities achieved with

[21] In general, no social improvement lets anyone slip below the minimum when previously everybody had achieved this threshold. In other words, a social improvement preserves acceptability.

them, it follows that R_T and $R_{T'}$ coincide on $\{x, y\}$. Hence, F_H also satisfies Scale-Dependent Independence of Irrelevant Alternatives.

We have thus shown that F_H is a scale-dependent social welfare functional that satisfies the scale-dependent unrestricted domain, Pareto, and independence axioms used in Theorem 3. This does not establish that F_H is not welfarist in the traditional sense because we have not shown that there is not an alternative way of describing this functional in scale-independent terms. We now show that there is not. For this purpose, it is sufficient to show that there exist two profiles $T = (U, S)$, $T' = (U, S') \in \mathcal{T}$ for which $F(T) \neq F(T')$, for then there can be no social welfare ordering R^* that is homologous to F_H. To do this, we more fully specifiy the example considered in Section 1.

As before, there are just two individuals, 1 and 2. Suppose that with the profiles of scales S and S', the common set of permissible grades is $\mathbb{R}_+$ for both profiles, but their interpretations differ. Specifically, with the interpretation used in S', the unit in which utility is measured is half of the one used in S. For S, let $\mu_S = 25$. It is then natural to set $\mu_{S'}$ equal to 50, reflecting the fact that the unit of measurement has been halved.[22] For the alternatives x and y, suppose that the utility values are $U_1(x) = 40$, $U_1(y) = 50$, $U_2(x) = 70$, and $U_2(y) = 50$. Interpreting these numbers using the common scale in S, both individuals achieve the minimally acceptable utility of 25 with both x and y. That is, $\mu U(x) \geq \mu_S$ and $\mu U(y) \geq \mu_S$. Thus, (4a) in the definition of F_H applies. Because $\Sigma_i U_i(x) = 40 + 70 > 50 + 50 = \Sigma_i U_i(y)$, we have $x P_T y$. On the other hand, with S', individual 1's utility of 40 falls below the minimally acceptable utility of 50. Hence, $\mu U(x) < \mu_{S'}$ and therefore (4b) in the definition of F_H applies. Because $\mu U(y) = 50 > 40 = \mu U(x)$, it then follows that $y P_{T'} x$. Hence, $F(T) \neq F(T')$, as was to be shown.

Example 2. Balinski and Laraki (2007, 2010) recommend using their method of majority judgment to rank entities of various kinds on the basis of the grades that they receive. They have applied this method to such collective evaluative tasks as the judging of wines and sporting events. We show that majority judgment is a scale-dependent social welfare functional and that this functional is welfarist in the scale-dependent sense.

As in Example 1, we assume that for all $S \in \mathcal{S}$, $S_i = S_j$ for all $i, j \in N$.

[22] As an application of this formalism, suppose that utility is given by life expectancy at birth and that in S this life expectancy is measured in years, whereas in S' it is measured in half years.

This assumption corresponds to Balinski and Laraki's stipulation that there is a common grading language. For any profile of utility functions $U \in \mathfrak{U}^S$ and any alternative $x \in X$, for $k = 1, \ldots, n$, the *kth median* $\sigma_k U(x)$ in the profile U with the alternative x is defined recursively by setting

$$\sigma_1 U(x) = \text{smedian}\{U_1(x), \ldots, U_n(x)\}$$

and

$$\sigma_k U(x) = \text{smedian}(\{U_1(x), \ldots, U_n(x)\} \setminus \cup_{j=1}^{k-1}\{\sigma_j U(x)\}), \quad k = 2, \ldots, n,$$

where *smedian* is the smallest median. For example, the smallest median of $\{1, 2, 3, 4\}$ is 2.[23] Thus, $\sigma_k U(x)$ is the smallest median utility after the first $k - 1$ smallest medians have been removed. Let

$$\sigma U(x) = (\sigma_1 U(x), \ldots, \sigma_n U(x)).$$

The *majority judgment* scale-dependent social welfare functional $F_{MJ} \colon \mathcal{T} \to \mathcal{R}$ is defined by setting, for each profile $T = (U, S) \in \mathcal{T}$ and each pair of alternatives $x, y \in X$,

$$xR_T y \leftrightarrow \sigma U(x) \geq_{\text{lex}} U(y), \tag{5}$$

where $\geq_{\text{lex}}$ denotes the lexicographic ordering in which for any $a, b \in \mathbb{R}^n$, (i) $a >_{\text{lex}} b$ if for some $j \leq n$, $a_j > b_j$ and $a_i = b_i$ for all $i < j$ and (ii) $a =_{\text{lex}} b$ if $a = b$. That is, the vectors are ranked by their first components if they differ. If they do not, the second components are used instead, and so on.

By construction, F_{MJ} satisfies Scale-Dependent Unrestricted Domain. The relation $\geq_{\text{lex}}$ is binary and orders $\mathbb{R}^n$. Hence, by (5), F_{MJ} satisfies scale-dependent welfarism. By Theorem 3, it also satisfies the scale-dependent versions of the Pareto indifference, independence, and neutrality conditions.

6. Information Invariance

In Sen's approach to determining social preferences, information invariance conditions play an important role in the axiomatic characterizations of particular classes of social welfare functionals (or of specific social welfare functionals) and their corresponding social welfare orderings (see, e.g., Bossert

[23] We follow Balinski and Laraki (2010, pp. 4–6) in using braces to denote multisets with possibly repeated instances of the same element. So, for instance, $\{1, 2, 2\}$ and $\{1, 2\}$ are different multisets that have different smallest medians (2 and 1, respectively).

and Weymark, 2004). As we have explained, the combination of welfarism in its traditional formulation with an information invariance condition cannot properly distinguish real changes in utilities from mere changes in scale. In this section, we describe how information invariance conditions are modeled in our framework and how they constrain how social welfare orderings for different scales are related. We also discuss the implications of using our approach for characterizing classes of social welfare orderings.

We begin by reviewing how information invariance for a social welfare functional is typically modeled using invariance transforms.[24] An *invariance transform* is an n-tuple of functions $\phi = (\phi_1, \ldots, \phi_n)$, where $\phi_i \colon \mathbb{R} \to \mathbb{R}$ for all $i \in N$, that is applied to a profile of utility functions U in order to obtain an informationally-equivalent profile. The transform ϕ operates component-wise by means of function composition, mapping U into $\phi \circ U = (\phi_1 \circ U_1, \ldots, \phi_n \circ U_n)$, where $\circ$ denotes the function composition operator. Different assumptions concerning the measurability and comparability of utility are captured by specifying a class Φ of invariance transforms and requiring a social preference to be invariant to the application of a transform from this class to any profile of utility functions in the domain of the social welfare functional F.[25]

Information Invariance for Φ. For all $U, U' \in \mathcal{D}$, if $U' = \phi \circ U$ for some $\phi \in \Phi$, then $R_U = R_{U'}$.

For example, Φ could be the class of all n-tuples of increasing affine transforms ϕ for which the unit scaling term is the same for each of the ϕ_i.[26] Transforms in this class preserve comparisons of utility differences both intrapersonally and interpersonally, and so utility is *cardinally measurable and unit comparable*. As a second example, Φ could be the class of all n-tuples of increasing transforms, in which case utility is *ordinally measurable and noncomparable*. In this case, social preferences only depend on the individual preferences, as in Arrow (1951). With the scale-dependent social welfare functional in Example 1, interpersonal comparisons of both utility

[24]For more details, see Bossert and Weymark (2004). This is not the only way that information invariance is modeled, but it is the most common. See Bossert and Weymark (2004, Section 5) for an alternative approach.

[25]In order for Φ to partition the domain of the functional into cells of informationally-equivalent profiles, $(\Phi, \circ)$ must be an algebraic group. See Roberts (1980).

[26]Formally, $\phi \in \Phi$ if and only if for all $i \in N$, $\phi_i(t) = \alpha_i + \beta t$ for all $t \in \mathbb{R}$, where α_i and β are scalars with $\beta > 0$.

levels and differences are needed, so Φ must only include common increasing affine transforms. However, the functional in Example 2 only makes use of interpersonal comparisons of utility levels, so in this case, Φ can be any set of common increasing transforms. The main classes of invariance transforms that have been considered may be found in Bossert and Weymark (2004).

When Theorem 1 applies (i.e., when the social welfare functional is welfarist in the traditional sense), there is an equivalent invariance condition to Information Invariance for Φ for the social welfare ordering (SWO) R^*.

SWO Information Invariance for Φ. For all $u, v, u', v' \in \Omega_{\mathcal{D}}$, if there exists a $\phi \in \Phi$ such that $u' = \phi(u)$ and $v' = \phi(v)$, then $uR^*v \leftrightarrow u'R^*v'$.

An important implication of this condition is that any utility transform ϕ maps an indifference contour of the social welfare ordering R^* into another indifference contour of this relation (see Blackorby, Donaldson, and Weymark, 1984). It is this property that provides much of the power of an information invariance condition, but it is also what accounts for the failure to distinguish between real and scale changes in utilities when the social aggregation procedure is welfarist in the traditional sense. For example, in the case in which utilities are ordinally noncomparable, one can always find an admissible transform ϕ that maps any utility vector u back to itself and maps any utility vector v that is socially indifferent to it into any other vector v' that has $v'_i \geq u_i \leftrightarrow v_i \geq u_i$ for all $i \in N$. Because R^* is transitive, this mapping only preserves social indifference if the social welfare ordering R^* is dictatorial or anti-dictatorial (i.e., there is some individual for whom any decrease in his utility is regarded as being a social improvement).[27] If R^* is weakly Paretian, we are left with an Arrovian dictator (Blackorby, Donaldson, and Weymark, 1984). If, however, utilities are distinguished by the scales in which they are measured, the pre- and post-transform indifference contours need not be the same, which opens up further possibilities.

Consider any two profiles of scales $S, S' \in \mathcal{S}$, where $S_i = (\mathcal{U}_i, \mathcal{I}_i)$ and $S'_i = (\mathcal{U}'_i, \mathcal{I}'_i)$ for all $i \in N$. Let ϕ be any invariance transform. The profile S' is a *ϕ-transform* of S if for all $i \in N$, $\mathcal{U}'_i = \phi_i(\mathcal{U}_i)$ and the interpretation of $\phi_i(u_i)$ in the scale S'_i is the same as the interpretation of u_i in the scale S_i for all $u_i \in \mathcal{U}_i$. Informally, individual i uses the transform ϕ_i to translate each

[27] This inference implicitly assumes that R^* is continuous. If it is not, then the same conclusion follows by considering how the upper and lower contours sets of R^* are transformed by the admissible transforms.

utility value measured using his original scale into a new value measured using the transformed scale with the utility value $\phi_i(u_i)$ in the new scale having the same meaning as u_i in the original scale. For example, if ϕ_i doubles each utility value, then a utility value of 2 in the transformed scale has the same meaning as a utility value of 1 prior to its application. Note that with a ϕ-transform, the cardinality of $\mathcal{U}'_i$ must be that same as that of $\mathcal{U}_i$ for all i.

The two information invariance conditions considered above can be reformulated using our framework in order to take account of the dependence on the scales used to measure utilities.

Scale-Dependent Information Invariance for Φ. For all $T = (U, S)$, $T' = (U', S') \in \mathcal{D}$, if there exists a $\phi \in \Phi$ such that (i) $U' = \phi \circ U$ and (ii) S' is a ϕ-transform of S, then $R_T = R_{T'}$.

Scale-Dependent SWO Information Invariance for Φ. For all $u, v \in \Omega_S$ and $u', v' \in \Omega_{S'}$, if there exists a $\phi \in \Phi$ such that (i) $u' = \phi(u)$ and $v' = \phi(v)$ and (ii) S' is a ϕ-transform of S, then $u R^*_S v \leftrightarrow u' R^*_{S'} v'$.

Thus, for a scale-dependent social welfare functional, invariance of the social preference for the alternatives in X is conditional on the invariance transform being applied to both the profile of utility functions and to the profile of scales, not just to the profile of utility functions. For a social welfare ordering, the invariance condition requires the social ranking of a pair of utility vectors obtained with the pre-transform social welfare ordering R^*_S to coincide with the social ranking of the corresponding transformed utility vectors obtained with the post-transform social welfare ordering $R^*_{S'}$. This condition implies that an indifference contour (resp. upper contour set, lower contour set) of R^*_S is mapped by an invariance transform $\phi \in \Phi$ into an indifference contour (resp. upper contour set, lower contour set) of $R^*_{S'}$, where S' is the profile of scales obtained by applying ϕ to S. Moreover, nothing more can be inferred from this condition by itself or in combination with any of the standard axioms.

In contrast, with the standard formulation of welfarism, the pre- and post transformed social welfare orderings must be identical. As we have noted, it is for this reason that information invariance conditions play such an important role in characterizing classes of social welfare orderings that exhibit considerable structure when the traditional framework is employed. For example, d'Aspremont and Gevers (1977) show that the utilitarian sum-of-utilities ordering is the only social welfare ordering that is anonymous and strongly Paretian when utilities are cardinally measurable and unit comparable. In our framework, these two axioms imply that a social welfare ordering R_S for the scale S is symmetric and strictly monotonic when everybody uses the same measurement scale. Adding the information invariance condition merely links the social welfare orderings for different profiles of scales in the way described above.

In general, with scale-dependent welfarism, there is complete freedom to choose any social welfare ordering that satisfies all but the informational invariance condition for one profile of scales S. Once this is done, the information invariance condition completely determines the social welfare ordering for any other profile of scales. In particular, in the case of ordinally measurable and noncomparable utilities, any social welfare ordering that is both weakly monotonic and sensitive to the utilities of at least two individuals can be used for the profile S. This claim may appear to contradict Arrow's Theorem, but it does not. Just as with the use of grades by Balinski and Laraki (2007, 2010), the interpretation of the utilities, as provided by the scales in which they are measured, conveys useful information over and above what is provided by the individual preferences, and this is what allows us to escape from the nihilism of Arrow's Theorem when utilities are ordinally measurable and noncomparable.

7. Conclusion

We have introduced a generalization of the concept of a social welfare functional that explicitly takes account of the measurement scales that are used. This generalization makes it possible to distinguish between a real change in individual well-beings from one that is merely representational due to the use of different scales to measure them. Using this framework, we have described a scale-dependent form of welfarism and shown how it can be characterized using scale-dependent versions of the standard welfarism axioms. The implications of our approach for characterizing classes of social welfare orderings have also been considered. Allowing social preferences to depend on utility scales properly limits the role that information invariance conditions play, thereby substantially enlarging the range of social aggregation procedures that are normatively appealing compared to what is possible with the traditional formulation of welfarism.

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Appendix

Proof of Theorem 3. Suppose that F satisfies Scale-Dependent Unrestricted Domain. Consider any $S \in \mathcal{S}$. This profile of scales is fixed throughout this proof.

(i) $\Rightarrow$ (ii). Suppose that F satisfies Scale-Dependent Pareto Indifference and Scale-Dependent Independence of Irrelevant Alternatives. Consider any $w, x, y, z \in X$ and any $T^1 = (U^1, S)$, $T^2 = (U^2, S) \in \mathcal{D}_S$ such that $U^1(w) = U^2(y)$ and $U^1(x) = U^2(z)$. Let $U^1(x) = U^2(z) = u$ and $U^1(y) = U^2(w) = \bar{u}$. By Scale-Dependent Unrestricted Domain, there exists an alternative $t \in X$ and profiles $T^3 = (U^3, S)$, $T^4 = (U^4, S)$, $T^5 = (U^5, S) \in \mathcal{D}_S$ such that (a) $U^3(x) = U^3(t) = u$ and $U^3(y) = \bar{u}$, (b) $U^4(z) = U^4(t) = u$ and $U^4(w) = \bar{u}$, and (c) $U^5(t) = u$ and $U^5(y) = U^5(w) = \bar{u}$.[28] By Scale-Dependent Independence of Irrelevant Alternatives, $xR_{T^1}y \leftrightarrow xR_{T^3}y$. Scale-Dependent Pareto Indifference and the transitivity of R_{T^3} imply that $xR_{T^3}y \leftrightarrow tR_{T^3}y$. Similarly, $tR_{T^3}y \leftrightarrow tR_{T^5}y \leftrightarrow tR_{T^5}w$. Applying the same argument once again, $tR_{T^5}w \leftrightarrow tR_{T^4}w \leftrightarrow zR_{T^4}w$. Scale-Dependent Independence of Irrelevant Alternatives then implies that $zR_{T^4}w \leftrightarrow zR_{T^2}w$. We have thus shown that $xR_{T^1}y \leftrightarrow zR_{T^2}w$. Hence, F satisfies Scale-Dependent Strong Neutrality.

(ii) $\Rightarrow$ (i). Suppose that F satisfies Scale-Dependent Strong Neutrality. By setting $w = y$ and $x = z$ in the definition of this neutrality axiom, Scale-Dependent Independence of Irrelevant Alternatives follows. By setting $U = U'$ and $x = y = z$, Scale-Dependent Strong Neutrality implies that $wR_T x \leftrightarrow xR_U x$ when $U(w) = U(x)$. Reflexivity of R_T then implies that $wI_T x$ and, hence, that Pareto indifference is satisfied.

(ii) $\Rightarrow$ (iii). Suppose that F satisfies Scale-Dependent Strong Neutrality. For any $u, \bar{u} \in \Pi_{i \in N} \mathcal{U}_i$, by Scale-Dependent Unrestricted Domain, there exists a profile $T = (U, S) \in \mathcal{D}_S$ and a pair of alternatives $x, y \in X$ such

[28] Except for the case in which $w = x$ and $y = z$, t can be chosen from the set $\{w, x, y, z\}$.

that $U(x) = u$ and $U(y) = \bar{u}$. Define R^*_S on $\{u, \bar{u}\}$ by setting $uR^*_S\bar{u} \leftrightarrow xR_Ty$ and $\bar{u}R^*u \leftrightarrow yR_Tx$. By Scale-Dependent Strong Neutrality, this ranking of u and $\bar{u}$ does not depend on the profile in $\mathcal{D}_S$ or on the alternatives in X used to obtain u and $\bar{u}$. To show that R^*_S is transitive, consider any any $u, \bar{u}, \hat{u} \in \Pi_{i\in N}\mathcal{U}_i$ with $uR^*_S\bar{u}$ and $\bar{u}R^*_S\hat{u}$. By Scale-Dependent Unrestricted Domain, there exists a profile $T = (U, S) \in \mathcal{D}_S$ and alternatives $x, y, z \in X$ such that $U(x) = u$, $U(y) = \bar{u}$, and $U(z) = \hat{u}$. Because $U(x)R^*_SU(y)$ and $U(y)R^*_SU(z)$, by the construction of R^*_S it follows that xR_Ty and yR_Tz. Transitivity of R_T then implies that xR_Tz. Hence, $U(y)R^*_SU(z)$ or, equivalently, $uR^*_S\hat{u}$. Thus, R^*_S is transitive.

(iii) ⇒ (ii). This implication follows immediately from the definitions.[29]

□

References

Adler, M. D., Fleurbaey, M. (Eds.), 2016. The Oxford Handbook of Well-Being and Public Policy. Oxford University Press, New York.

Allison, R. A., Foster, J. E., 2004. Measuring health inequality using qualitative data. Journal of Health Economics 23, 505–524.

Arrow, K. J., 1951. Social Choice and Individual Values. Wiley, New York, second edition 1963.

Atkinson, A. B., 1970. On the measurement of inequality. Journal of Economic Theory 2, 244–263.

Balinski, M., Laraki, R., 2007. A theory of measuring, electing and ranking. Proceedings of the National Academy of Sciences USA 104, 8720–8725.

Balinski, M., Laraki, R., 2010. Majority Judgment: Measuring, Ranking, and Electing. MIT Press, Cambridge, MA.

Blackorby, C., Bossert, W., Donaldson, D., 2005a. Multi-profile welfarism: A generalization. Social Choice and Welfare 24, 253–267.

Blackorby, C., Bossert, W., Donaldson, D., 2005b. Population Issues in Social Choice Theory, Welfare Economics, and Ethics. Cambridge University Press, Cambridge.

Blackorby, C., Donaldson, D., Weymark, J. A., 1984. Social choice with inter-

[29] Our proof follows the same basic proof strategy that is used by Bossert and Weymark (2004) to prove Theorem 1. Their proof in turn draws on the proof of Theorem 1 in d'Aspremont (1985).

personal utility comparisons: A diagrammatic introduction. International Economic Review 25, 327–356.

Blackorby, C., Donaldson, D., Weymark, J. A., 1990. A welfarist proof of Arrow's theorem. Recherches Economiques de Louvain 56, 259–286.

Bossert, W., Weymark, J. A., 2004. Utility in social choice. In: Barberà, S., Hammond, P. J., Seidl, C. (Eds.), Handbook of Utility Theory. Volume 2: Extensions. Kluwer Academic Publishers, Boston, pp. 1099–1177.

d'Aspremont, C., 1985. Axioms for social welfare orderings. In: Hurwicz, L., Schmeidler, D., Sonnenschein, H. (Eds.), Social Goals and Social Organizations: Essays in Memory of Elisha S. Pazner. Cambridge University Press, Cambridge, pp. 19–76.

d'Aspremont, C., Gevers, L., 1977. Equity and the informational basis of collective social choice. Review of Economic Studies 44, 199–209.

d'Aspremont, C., Gevers, L., 2002. Social welfare functionals and interpersonal comparability. In: Arrow, K. J., Sen, A. K., Suzumura, K. (Eds.), Handbook of Social Choice and Welfare. Vol. 1. North-Holland, Amsterdam, pp. 459–541.

Hammond, P. J., 1979. Equity in two person situations: Some consequences. Econometrica 47, 1127–1135.

Jehle, G. A., Reny, P. J., 2011. Advanced Microeconomic Theory, 3rd Edition. Financial Times Prentice Hall, Harlow, England.

Krantz, D. H., Luce, R. D., Suppes, P., Tversky, A., 1971. Foundations of Measurement, Volume I: Additive and Polynomial Representations. Academic Press, New York.

Luce, R. D., 1959. On the possible psychophysical laws. Psychological Review 66, 81–95.

Luce, R. D., Krantz, D. H., Suppes, P., Tversky, A., 1990. Foundations of Measurement, Volume III: Representation, Axiomatization, and Invariance. Academic Press, New York.

Luce, R. D., Raiffa, H., 1957. Games and Decisions: Introduction and Critical Survey. Wiley, New York.

Marchant, T., 2008. Scale invariance and similar invariance conditions for bankruptcy problems. Social Choice and Welfare 31, 693–707.

Narens, L., 2002. Theories of Meaningfulness. Lawrence Erlbaum Associates, Mahwah, NJ.

Roberts, F. S., 1979. Measurement Theory with Applications to Decision-making, Utility, and the Social Sciences. Addison-Wesley, Reading, MA.

Roberts, K. W. S., 1980. Interpersonal comparability and social choice theory.

Review of Economic Studies 47, 421–439.
Roemer, J. E., 1996. Theories of Distributive Justice. Harvard University Press, Cambridge. MA.
Sen, A. K., 1970. Collective Choice and Social Welfare. Holden-Day, San Francisco.
Sen, A. K., 1974. Informational bases of alternative welfare approaches: Aggregation and income distribution. Journal of Public Economics 3, 387–403.
Sen, A. K., 1977. On weights and measures: Informational constraints in social welfare analysis. Econometrica 45, 1539–1572.
Stevens, S. S., 1946. On the theory of scales of measurement. Science 103, 677–680.
von Neumann, J., Morgenstern, O., 1944. Theory of Games and Economic Behavior. Princeton University Press, Princeton, NJ.
Weatherstone, B., Marshall, D., 2014. Intrinsic vs. extrinsic properties. In: Zalta, E. N. (Ed.), The Stanford Encyclopedia of Philosophy, Fall 2014 Edition. URL: http://plato.stanford.edu/archives/fall2014/entries/intrinsic-extrinsic/.