On the Gettier Problem for

Topological Logic of Knowledge and Belief

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Abstract. Gettier's famous examples intended to show that knowledge cannot always be equated with justified true belief. The Gettier problem can also be considered as a problem for topological epistemic logic: If knowledge and justified belief are conceived as topological operators K and B on topological spaces (to be considered as universes of possible worlds), one may ask whether it happens that there is a proposition A such that $KA \neq A \cap BA$ or not. If this is the case, the epistemological logic defined by the topological operators K and B may be said to be non-traditional since there is for them a "Gettier proposition" where knowledge does not coincide with justified true belief. As far as we know, for the first time, this topological Gettier problem is discussed in Steinvold's PhD dissertation (Steinsvold 2012). In Baltag, Bezhanishvili, Özgün, and Smets (2013), Steinvold's co-derived set account is criticized since it can be easily "Gettierized". Baltag et alii claim (without proof) that their topological account of Stalnaker's logic KB of knowledge and belief does not have this flaw.

In the following we will give some topological conditions that determine whether a topological KB model does or does not cope with the Gettier challenge. First, it will be shown that every KB model defines a (topologically slightly simplified) model that is rather similar to it but a is a model of traditional JTB (knowledge = justified true belief) logic of knowledge and belief. The existence of such doppelgangers of all KB models may be read as a qualified and partial topological rehabilitation of traditional JTB epistemology.

Second, we consider the Gettier problem for a special class of models of Stalnaker's combined logic of knowledge and belief KB, namely, for T_0 Alexandroff spaces. We prove that, that almost all Alexandroff spaces do not face Gettier counterexamples.

Third, we prove that the Stone spaces of Boolean algebras of regular closed subsets of Hausdorff (or T_2) spaces X do not face Gettier counterexamples if the X are not pseu–docompact, and these Stone spaces do face Gettier counterexamples for compact spaces X.

Succinctly formulated, Stalnaker's KB logic of knowledge and belief can avoid Gettier problems for some models, while it falls back to traditional pre-Gettier JTB epistemology for other models. In contrast, co-derived set semantics is doomed to fail always in the sense that its models always fall prey to Gettier counterexamples.

1. Introduction. Gettier's problem for Topological Epistemology. Gettier's famous examples of Gettier intended to show that knowledge cannot always be equated with justified true belief. In the past 60 years – beginning with Gettier's paper (Gettier (1963)) a huge number of examples have been produced that intend to show that under certain circumstances we are inclined to deny that someone knows a proposition A, although he is justified to believe that A and A is true.[1]

[1] For a comprehensive collection of recent examples of Gettier cases see Borges, de Almeida, and Klein (2017).

The Gettier problem can also be considered as a problem for topological epistemic logic in the following way: If knowledge and justified belief are conceived as topological operators K and B on a topological space (X, OX) (to be considered as a universe of possible worlds, $A \in PX$ as a proposition), one may ask whether it happens that

$$KA \neq A \cap BA$$

for some proposition A, i.e., whether there is a true proposition A for which knowledge KA of A does not coincide with the conjunction of the propositions that A is true and that it is justified to believe that BA. In this case, the epistemological logic defined by the operators K and B may be said to be non-traditional since there is a "Gettier proposition" A that exemplifies a Gettier case of knowledge in which knowledge KA does not coincide with justified true belief $A \cap BA$.

This topological Gettier problem for formal topological epistemology is of more recent vintage than the original Gettier problem. For instance, in the Handbook of Spatial Logic (2007) edited by Guram Bezhanishvili, Ian Pratt-Hartmann, and Johan van Benthem there are several chapters (e.g., 5, 6, 8, 9, and 10) that deal with topological logic of knowledge and belief, but the name of Gettier is not mentioned even once.

Meanwhile, this has changed. In Baltag, Bezhanishvili, Özgün, and Smets (2013), the authors criticize the co-derived set account of Steinsvold (2012) since it always falls prey to Gettier's counterexamples in the sense that according to it knowledge <u>always</u> coincides with justified true belief. More precisely, they observe (cf. Baltag et alii (2013, p.12)):

> There is another objection, maybe even more decisive, against the co-derived set semantics, namely that it can be easily "Gettierized". As mentioned above,we have $Int(A) = A \cap t(A)$, which means that in the co-derived set interpretation, knowledge is exactly the same as true (justified) belief. So this semantics is easily vulnerable to all

the well-known Gettier-type counterexamples!

In contrast, they seem to claim (without proof) that their topological account of Stalnaker's combined logic KB of knowledge and belief which conceives belief as the possibility of knowledge (in their terminology B = <K>K := ¬K¬K) does not succumb to this flaw.

To assess their claim, first, it is expedient to observe that both the co-derived set interpretation as well as the topological interpretation of Stalnaker's semantics are defined for a class of models: co-derived set semantics is defined for the class of DSO spaces[2], and the semantics of Baltag et al. (2013) is defined for the class of extremally disconnected (ED) spaces. It is well known that every DSO space is (hereditarily) extremally disconnected space but not every ED space is DSO. Thus, if one accepts Gettier counterexamples as genuine counterexamples for the traditional JTB semantics, Stalnaker's semantics would be clearly better than the co-derived semantics if it is shown that there is an ED space that does not face a Gettier counterexamples. Baltag and his collaborators do not show this. Özgün's simple example of an ED space (X, OX) that is not a DSO space is easily shown not to face Gettier's problem in the sense that there is a proposition $A \in PX$ such that one has

(G) $$\exists_{A \in PX}(KA \neq A \cap BA)$$

with respect to Stalnaker's KB logic (cf. Özgün (2013)). Baltag and his collaborators seem to assume (without actually proving it) that more generally all models of KB do not face Gettier counterexamples. This, however, is not the case. As will be shown in this paper, there are infinitely many models (X, OX) of KB that do face Gettier counterexamples in the sense that all propositions $A \in PX$ satisfy the negation $(G)^C$ of (G):

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[2] A DSO-space is defined as topological space (X, OX) which is dense-in itself, i.e., for all $x \in X$ ($\{x\} \notin OX$), and every derived set d(A), $A \subseteq X$, is open, i.e., $d(A) \in OX$.

$(G)^C$ $$\forall_{A \in PX}(KA = A \cap BA)$$

Moreover, it will be shown that all models (X, OX) of KB can be topologically slightly modified to yield models (a(X), Oa(X)) that satisfy $(G)^C$. These and other results, proved in the following, suggest that, in order to cope with the topological Gettier problem for KB, it is highly desirable to know more of the class of models of KB than just the fact that they have to be ED spaces. To offer some interesting results in this respect is the general aim of this paper. The first step for the realization of this project is the following distinction:

<u>(1.1) Definition</u>. Let (X, OX) be a topological space that is a topological model of Stalnaker's combined logic KB of knowledge and belief, i.e., K = Int and B = Clnt, Int the interior operator and Cl the closure operator of (X, OX).

(i) (X, OX) is a Gettier model of KB iff there is a proposition $A \in PX$ with $KA \neq A \cap BA$. The class of Gettier models of KB is denoted by KB(G).

(ii) (X, OX) is a non-Gettier (or JTB) model of KB iff for all propositions $A \in PX$ one has with $KA = A \cap BA$. The class of non-Gettier models of KB is denoted by $KB(G)^C$.♦

In definition (1.1), (G) and $(G)^C$ are considered as possible axioms for models of Stalnaker's combined logic KB of knowledge and belief. Clearly, every model of KB (X, OX) is either a KB(G) model or a $KB(G)^C$ model.

In the following we will give some topological conditions that determine whether a topological KB model (X, OX) is a KB(G) or a $KB(G)^C$. Moreover, it will be shown that every KB model (X, OX) (be it KB(G) or $KB(B)^C$) defines a (topologically slightly simplified) model (a(X), O(a)X) that is $KB(G)^C$. Since (a(X), Oa(X)) is a topological space that is rather similar to (X, OX): In particular, the underlying sets X and a(X) coincide. (a(X), Oa(X)) is called the JTB doppelganger of (X, OX). The existence of such $KB(G)^C$ doppelgangers for all KB model may be read as a qualified and partial topological rehabilitation of traditional JTB epistemology. In other words, from a topological perspective (traditional) JTB epistemology is a simplified version of post-Gettier epistemology where knowledge and true justified beli9ef are dis–tinguished. Or, to see it from the opposite perspective, post-Gettier epistemology is a more sophisticated, but not essentially different version of traditional epistemology.

(1.2) Theorem. For every KB model (X, OX) (be iit KB(G) or $KB(G)^C$), there is a $(KB(G)^C$ model (a(X), Oa(X)) defined by

$$X = a(X),\ Int_{a(X)}(A) := A \cap ClInt(A),\ Oa(X) = \{Int_{a(X)}(A\};\ A \in PX\}. \blacklozenge$$

This theorem may be considered as a genuine theorem of topological epistemology. It has no direct counterpart in non-topological informal epistemology. Its elementary proof depends on the formal properties of the topological model of knowledge used in standard modal logic S4. To prove the other theorems of this paper a certain amount of topology is necessary. For instance, several variants of the topological concept of compactness will be needed that may not be in the toolkit of all epistemologists. Nevertheless, before we come to present these technical details, let us state the general theorems of this paper in order that the reader can have an idea of what he may expect from this paper.

First, we consider the Gettier problem for a special class of models of Stalnaker's combined logic of knowledge and belief KB, namely, T_0 Alexandroff spaces. It is well known that T_0 Alexandroff spaces are models of KB (cf. Özgün (2013)). We prove that sufficiently complex T_0 Alexandroff spaces do not face the Gettier problem, i.e., they are models of KB(G). What is to understood as "sufficiently complex" will be explicitly defined in (Section 3, Definition (3.6). Only a small class of particularly simple T_0 Alexandroff spaces turns out to be models of $KB(G)^C$. The simplicity of Alexandroff spaces is measured in terms of their specialization order. The proof of this theorem is elementary and uses only the fact that Alexandroff spaces are determined by their specialization order.

The second theorem asserts (roughly) that the Stone space St(X(B)) of the Boolean algebra X(B) of the regular closed subsets of a Hausdorff topological space (X, OX) is a KB(G) model if (X, OX) is not pseudocompact, and St(X(B)) is a $KB(G)^C$ model if (X, OX) is compact. For instance, for (X, OX) the open unit interval (0, 1), St(X(B)) turns out to be a KB(G) model, while for X the closed compact interval [0, 1] St(X(B)) is a model of $KB(G)^C$. More precisely, we will prove the following theorems:

(1.3) Theorem. Let (X, OX) be a T_0 Alexandroff space with specialization order $\leq$.

(a) If (X, OX) contains elements x, y, z with $x < y < z$ then the KB model defined by (X, OX) is a KB(G) model, i.e., it does not face the Gettier problem.

(b) If (X, OX) does not contain a triple $x < y < z$ then there are topologically nontrivial KB models defined by (X, OX) that are $KB(G)^C$ models, that face Gettier counterexamples, and there are other KB(G) models that are models of traditional JTB epistemology. ♦

Informally expressed, (1.3) Theorem asserts that sufficiently complex T_0 Alexandroff spaces (X, OX) do not face Gettier counterexamples. On the other hand, simpler Alexandroff spaces (X, OX) may fall prey to Gettier counterexamples. A similar qualitative pattern with respect to the axioms KB(G) and $KB(G)^C$ holds for general topological spaces. More precisely, we can prove:

(1.4) Theorem. Let X be a compact Hausdorff space, and B(X) the complete Boolean algebra of the regular closed subsets of X. Then the Stone space St(B(X)) of B(X) defines a model of $KB(G)^C$, i.e., St(B(X)) defines a JTB model for Stalnaker's KB logic. ♦

(1.5) Example. Let X = [0, 1] be the closed unit interval of the real numbers $\mathbb{R}$. Then the Stone space St(B(X)) of the Boolean algebra B(X) of regular closed subsets of X defines a model of $KB(G)^C$, i.e., a model of traditional JTB epistemology.♦

(1.6) Theorem. Let X be a locally compact Hausdorff space, and B(X) the complete Boolean algebra of regular closed subsets of X. If X is not pseudocompact, then St(B(X)) is a KB(G) model, i.e., St(B(X)) does not face the Gettier problem.♦

(1.7) Examples. Let X = (0, 1) be the open unit interval of the real numbers $\mathbb{R}$ or any open subset of $\mathbb{R}^n$, for $n \geq 1$. Then St(B(X)) is a model of KB(G) that does not face the Gettier problem.♦

For the proof of (1.4) one needs some knowledge of topological structure of the Stone space St(B(X)) of B(X) and some knowledge about the (Iliadis) absolutes of non-regular topological spaces. For the proof of (1.6), one needs detailed information of the Stone-Cech compactification βX of suitable non-pseudocompact spaces X. For this information we have to rely heavily on the pertinent mathematical literature.

As already said, Baltag et alii (2013, 2019) present Stalnaker's topological logic of knowledge and belief as if this logic (with the ClInt semantics) does not face the Gettier problem. But they do not prove it. They seem to believe that extremally disconnectedness of KB models is enough to ensure that their semantics does not face the Gettier problem. This, however, is not the case. In Mormann (2023) it is shown that for every topological model based on a topological space (X, OX) one can construct a topological model based on a "doppelganger" (a(X), Oa(X)) (with the same underlying set X) for which knowledge is justified true belief. In other words, formal topological epistemology does not get rid of Gettier counterexamples as Williamson seems to suggest (cf. Williamson 2015).

At least one achieves the following: ClInt-semantics can sometimes avoid Gettier problems if appropriate models are chosen, on the other hand, it falls back on traditional JTB epistemology if the models are simplified appropriately. In contrast, co-derived set semantics is doomed to fail always, in the sense that the topological models for co-derived set semantics always fall prey to Gettier counterexamples. This may be expressed as the assertion that co-derived set semantics of belief amounts to traditional pre-Gettier JTB semantics in the sense that in its models necessarily knowledge always coincides with true justified belief. In contrast, for ClInt-semantics one can go beyond traditional JTB semantics by choosing topological models where knowledge und justified true belief can be distinguished. This is a substantial advantage of ClInt semantics of belief over Steinsvold's co-derived set semantics.

The ClInt semantics defines belief as "possibility of knowledge". This definition is plausible for ED spaces, but it does not guarantee that the space X is not nodec, i.e., does not fall prey to the Gettier problem: Mormann's results (2023) are easily strengthened to show that for any ED space X its doppelganger (a(X), Oa(X)) is an ED space as well. The move from (X, OX) to (a(X), Oa(X)) is shown to be topologically "small" in the sense that it amounts only to a slight simplification of the space's topology. Hence, someone, who accepts Gettier examples as compelling arguments against traditional standard epistemology and at the same time takes topological epistemology seriously should look for topological models that are not based on nodec spaces. Thus, for partisans of Stalnaker's KB logic it is highly desirable find ED spaces that are not nodec. Up to now, this task has been rather ignored in the literature.

The simplest example of a Stone space that is not nodec is St(Pℕ), Pℕ the complete Boolean algebra of subsets of ℕ with ℕ a topologically discrete and countable set (that is as usual identified with the natural numbers ℕ).

In order to show that the ClInt semantics of Baltag et alii is "really better" than the co-derivative semantics of Steinsvold et alii one has to show that at least some ED spaces are not nodec spaces.

Stalnaker (2006) did not deal with the Gettier problem for KB logic. Baltag et al. (2013, 2019) seem to assume that the Gettier problem does not concern Stalnaker's combined logic of KB of knowledge and belief, since then models for which the equation "Knowledge = True Justified Belief" holds, had not been found for belief = ClInt and ED. In this paper we show that for many topological models the Gettier equation can easily be derived for some ED models.

The organization of the paper is as follows: In section 2 we prove (1.2) Theorem. We show that all topological models of Stalnaker's logic KB have topological doppelgangers that are models of $KB(G)^C$, i.e., models of traditional JTB logic for which knowledge coincides with true justified belief. In section 3 the Gettier problem for Alexandroff spaces is treated in detail. In sections 4 and 5 the Gettier problem for topological models of KB based on Stone representation spaces is discussed. We conclude with some general remarks on the role of compactness for topological logic in section 6.

2. Every model of KB has a doppelganger that is a model of traditional JTB epistemology.

In this section we prove Theorem (1.2) that every model of KB has a $KB(G)^C$ doppelganger, i.e., a model of KB that is also a model of traditional JTB logic.

First, for the sake of definiteness, let us recall the basics of the syntax and semantics of the modal language to be employed in the following. We consider a bimodal extension L_{KB} of standard propositional logic defined by two modal operators K and B. The formulas of the language L_{KB} are defined on a countable set of propositional letters PROP, Boolean operator $\neg$, $\wedge$, and the modal operators K and B by the following grammar:

$$\varphi := p \mid \neg p \mid \varphi \mid \varphi \wedge \varphi \mid K\varphi \mid B\varphi \mid \quad , \quad p \in PROP.$$

Abbreviations for the connectives $\vee$, $\rightarrow$, and $\leftrightarrow$ are standard. The unimodal fragments of L_{KB} defined by K and B are denoted by L_K and L_B, respectively.

Now, the axioms and the inference rules of Stalnaker's system KB of a combined logic of knowledge and belief can be formulated as follows (cf. Stalnaker (2006), Baltag et al. (2017, 2019):

(2.1) Definition (Axioms and inference rules of Stalnaker's logic of knowledge and belief).

(CL)	All tautologies of classical propositional logic.	
(K)	$K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$	(Knowledge is additive).
(T)	$K\varphi \rightarrow \varphi$	(Knowledge implies truth).
(KK)	$K\varphi \rightarrow KK\varphi$	(Positive introspection of K).
(CB)	$B\varphi \rightarrow \neg B \neg\varphi$	(Consistency of belief).
(PI)	$B\varphi \rightarrow KB\varphi$	(Positive introspection of B).
(NI)	$\neg B\varphi \rightarrow K\neg B\varphi$	(Negative introspection of B).
(KB)	$K\varphi \rightarrow B\varphi$	(Knowledge implies belief).
(FB)	$B\varphi \rightarrow BK\varphi$	(Full belief).

Inference Rules:

(MP)	From φ and $\varphi \rightarrow \psi$, infer ψ.	(Modus Ponens).
(NEC)	From φ, infer $K\varphi$.	(Necessitation).♦

In the topological approach to knowledge and belief, the axiom (NI) plays a special role. It has been shown that (NI) holds only for topological models of a very special kind, namely, models based on extremally disconnected spaces (ED)[3] (cf. Baltag et al (2013, 2019), Stalnaker (2006)). All other axioms and rules of KB are satisfied by all topological spaces. The validity of (NI) guarantees unique definability of the belief operator, i.e., for extremally disconnected spaces, the belief operator is uniquely determined by the knowledge operator as ClInt (cf. Baltag et alii).

Now we going to prove Theorem (1.2). The proof is based on a result of Mormann (2023) and some results of Reilly and Vamanamurthy (1985) and Jankovic (1985). To set the stage, let us start with the following definition:

(2.2) Proposition. Let (X, OX) be an arbitrary topological space. Define

$$a(X) := X \quad , \qquad Int_{a(X)}(A) := A \cap IntClInt(A) \quad , \qquad Oa(X) := \{Int_{a(X)}(A);\ A \in PX\}.$$

Then (a(X), Oa(X)) is a topological space. The topology Oa(X) is at least as fine as the topology OX, i.e., $Oa(X) \subseteq OX$. It may happen that $Int = Int_{a(X)}$. In this case, (X, OX) is called a nodec space.

Proof. Check the definitions.♦

Obviously, the procedure of constructing doppelgangers can be iterated: from X we may move to a(X), a(a(X)), a(a(a(X))), … , etc. This iteration yields nothing new, however. More precisely, one obtains:

[3] Recall that a space (X, OX) is extremally disconnected (ED) iff the closure of every open set is open: ClInt(A) = IntClInt(A) (cf. Willard (2004, 15.G, p. 106)).

(2.3) Lemma. Let (X, OX) be an arbitrary topological space. Then:

(i) $Int_{a(a(X))}(A) = Int_{a(X)}(A)$ and $a(a(X)) = a(X)$.

(ii) $Int_{a(X)}Cl_{a(X)}Int_{a(X)}(A) = IntClInt(A)$.

Proof. This is directly calculated from some well-known properties of the Kuratowski topological operators Int and Cl, see Jankovic and Hamlett (1990), Mormann (2023).♦

(2.4) Corollary. Let (X, OX) be an arbitrary topological space.

(i) Then a(X) satisfies $(G)^C$, i.e., $Int_{a(X)}(A) = A \cap Int_{a(X)}Cl_{a(X)}Int_{a(X)}(A)$.

(ii) The Boolean algebra of regular open subsets of X is isomorphic to the Boolean algebra of regular open subsets of a(X): OX* = Oa(X)*.

Proof. (i) By (2.2) one has $Int_{a(X)}(A) = Int_{a(X)}Int_{a(X)}(A) = Int_{a(X)}(A) \cap Int_{a(X)}Cl_{a(X)}Int_{a(X)}(A))$

$= Int_{a(X)} \cap Int_{a(X)}(Int_{a(X)}Cl_{a(X)}Int_{a(X)}(A)) = Int_{a(X)}(A) \cap (Int_{a(X)}Cl_{a(X)}Int_{a(X)}(A))$

$= Int_{a(X)}(A) = A \cap IntClInt(A) = A \cap (Int_{a(X)}Cl_{a(X)}Int_{a(X)}(A))$ by (2.2)(ii).

(ii) By definition of the set of regular open subsets O*X and (2.2)(i)we have

$$O^*X = IntClInt(PX) = Int_{a(X)}Cl_{a(X)}Int_{a(X)}(PX) = Oa(X)^*.$$ ♦

Informally expressed, (2.3) asserts that any topological space (X, OX) can be refined in such a way that the refined topological space (X, Oa(X)) satisfies the axiom $(G)^C$. If (X, OX) already satisfies $(G)^C$ then X = a(X)).

It is well known (see Bezhanishvili, Esakia, Gabelaia (2007)) that the modal logic corres–ponding to nodec spaces is the extension of the standard unitary modal logic S4 by the axiom

(Zem) $\Box p = p \wedge \Box\Diamond\Box p$

This logic is to be denoted by S4.Zem (for Zeman). The axiom (Zem) was introduced for the first time by J.J. Zeman (1969) as an axiom of modal logic without any reference to topology and epistemology.[4]

The modal logic S4 may have been the first word of modal logic on the epistemological concept of knowledge. It has certainly not been the last one. Most epistemologists consider it as too weak. Instead of S4, they prefer some normal extension of S4. For example, according to Lenzen (1979), the "correct" logic of knowledge is isomorphic to a modal system at least as strong as S.4.2 and at most as strong as S4.4.

If one follows this line of thought, one observes that for models of KB, i.e., ED topological spaces (X, OX), (Zem) coincides with $(G)^C$:

(2.5) Lemma. Let (X, OX) be a topological model of of Stalnaker's combined logic KB, i.e., an ED space. Then

$$\mathrm{IntClInt}(A) \Leftrightarrow \mathrm{ClInt}(A) \text{ for all } A \in PX. \blacklozenge$$

For extremally disconnected spaces (X, OX) the operator B := ClInt can be intuitively plausible interpreted as a formal model of justified belief (Stalnaker (2006)). More precisely, it is well known that the tandem of operators Int and ClInt satisfies for ED spaces the axioms of Stalnaker's combined logic KB of knowledge and belief, in particular, the axiom of negative introspection (NI) (Baltag et alii (2019), Stalnaker (2006)).

On the other hand, if one subscribes to a traditional (JTB) epistemology for which knowledge coincides with justified true belief and observes (2.2) one may wish to take into account also

[4] Another name for S4.Zem is S4.04, see Segerberg (1971, Ch.II, 7, p. 153). Further results on S4.Zem can be found in the papers Sobocinsky (1970) and Georgacarakos (1976).

(Zem) (or, equivalently $KB(G)^C$)). That means, a traditional epistemologist may be inclined to favor S4.2.Zem as a good candidate for the office of the "correct" topological epistemology. Surprisingly, this is a path that has not been taken up to now. In order to tackle Lenzen's problem of finding the "correct" logic of knowledge seriously, we have, of course, to consider the question: Do non-trivial models of S4.2.Zem exist? The answer to this question is Yes. More precisely, in Mormann (2023) it has been shown that for any space X a space a(X) exists, with a topology Oa(X) at least as fine as OX, that satisfies Zem. The space a(X) has very similar topological properties as the space (X, OX). For instance, the sets of regular open subsets of X and a(X) coincide. Therefore, (a(X) may be called a doppelganger of X. It will be shown that the doppelganger a(X) of an ED space X is also an ED space and, moreover, a $KB(G)^C$ space. This fact entails that also the Clnt semantics is not immune against the Gettier problem: Some topological models of Stalnaker's combined logic KB of knowledge and belief do face the Gettier problem in the sense that for them knowledge coincides with justified true belief. Thus, the natural question arises: Are there any models of KB that do NOT have this weakness and do satisfy the "Gettier axiom" KB(G) instead of the "anti-Gettier axiom" $KB(G)^C$ (= Zem)? Fortunately for partisans of ClInt semantics, who subscribe to a post-Gettier, non-traditional epistemology of knowledge, this is the case. In contrast to Steinvold's co-derived set semantics of belief, the ClInt semantics does <u>not always</u> face the Gettier problem but only sometimes. To prove this is, however, not totally obvious, but requires some work.

Now, we will prove (1.2) Theorem according to which for any ED space (X, OX) its doppelganger (X, Oa(X)) is an ED space with respect to its interior operator

$$Int_{a(X)}(A) = A \cap IntClInt(A)$$

It is well known that ED spaces and nodec spaces can both be characterized in topological terms:

<u>(2.6) Definition (Jankovic (1985, Section 2, p. 83)</u>. Let (X, OX) be a topological space, $A \subseteq X$.

(i) The set A is said to be semi-open iff $A \subseteq ClInt(A)$. The class of semi-open sets of (X, OX) will be denoted by SOX.

(ii) The set $A \subseteq X$ is said to be pre-open iff $A \subseteq IntCl(A)$. The class of pre-open sets of (X, OX) will be denoted by POX.

Clearly, open sets are semi-open <u>and</u> pre-open. The opposite implication does not hold in general. For instance, consider the half-open interval A := [0, 1) of the Euclidean real line. Then one has $[0, 1) = A \subseteq ClInt(A) = [0, 1]$ and $IntCl(A) = (0, 1) \subseteq A = [0, 1)$, i.e., A is semi-open and pre-open. A is not, however, open. Rather, we get:

<u>(2.7) Proposition (Jankovic (1985, Proposition 2.1, Corollary 2.5), Reilly and Vamanamurthy (1985, Theorem 3))</u>.

(i) $SO(X, O_{a(X)}X) = SO(X, OX)$.

(ii) $PO(X, O_{a(X)}X) = PO(X, OX)$.

(iii) $Oa(X) = SO(X) \cap PO(X)$. ♦

<u>(2.8) Proposition ((Jankovic (1985, Proposition 4.1)</u>. A space (X, OX) is ED iff $SO(X, OX) \subseteq PO(X, OX)$. ♦

Putting (2.6) and (2.7) together we eventually obtain:

<u>(2.9) Proposition</u>. A space (X, OX) is ED iff $(X, O_{a(X)}X)$ is ED. ♦

(2.3) immediately implies that O*X and O*a(X) coincide. This and (2.8)) may be considered as compelling arguments that (a(X), Oa(X)) is called a (topological) doppelganger of (X, OX).

Thus, many topological models of KB logic do face the Gettier problem: Just take any ED space (X, OX) as a model of KB and consider its doppelganger a(X). Then a(X) is also a KB model that is also a model of $(G)^C$! In a(X) knowledge $Int_{a(X)}(A)$ of any A <u>always</u> coincides with true justified belief $A \cap ClInt(A) = A \cap Cl_{a(X)}Int_{a(X)}(A)$.

In other words, (1.2) Theorem holds and many models of Stalnaker's logic KB satisfy the axiom $KB(G)^C$ (= Zem)! Stalnaker's logic KB turns out not to be per se immune against Gettier counterexamples. In other words, it is neutral with respect to the problem whether traditional JTB logic or post-Gettier logic should be considered as the "correct" logic of knowledge and belief. The best of what we show we can is that there are some topological models of KB that are models of $KB(G)^C$. This is indeed the case as we will show in the next section in which most Alexandroff spaces will be shown to have this property. In the subsequent sections we will prove analogous results for another class of ED spaces that may be preferable for certain reasons (to be explained in due time) over Alexandroff spaces,

<u>3. The Gettier problem for Alexandroff spaces</u>. In this and the subsequent sections we will prove theorems (1.3) – (1.7)). To set the stage let us first recall the exact definition of Alexandroff spaces:

<u>(3.1) Definition</u>. A topological space (X, OX) is an Alexandroff space iff the interior operator Int distributes over <u>arbitrary</u> intersections, i.e., $Int(\cap A_\lambda) = \cap Int(A_\lambda)$. ♦

<u>(3.2) Definition</u>. Every T_0 topological space (X, OX) defines a partial order $\leq$ on X defined by $x \leq y := x \in cl(y)$. This (partial) order is called the specialization order of the topological space (X, OX). For $A \subseteq X$ define

$\uparrow A := \{y; \exists x (x \in A \text{ and } x \leq y\}$ and $\downarrow A := \{y; \exists x (x \in A \text{ and } x \geq y\}$.♦

It is well known that for a T_0 Alexandroff space (X, OX) its open sets OX are just the upper sets of the specialization order ≤, i.e.:

(3.3) $$A \in OX \Leftrightarrow A = \uparrow A.$$

In Özgün (2013) the following proposition is proved that is very useful for KB logic:

(3.4) Proposition. Let (X, OX) be a T_0 Alexandroff space. Then (X, OX) is an ED space.

Proof. Özgün (2013, (Proposition 3, p. 22))♦

Thus, by a well-known result of Baltag et al. ((2019, (Theorem 5, p. 223)) T_0 Alexandroff spaces provide a rich arsenal for KB models.

In order to show that ClInt semantics is more general than the DSO semantics of Steinsvold et alii, let us reconsider an example of a small T_0 Alexandroff space (X, OX) given by Özgün to show that not all ED spaces are DSO spaces. Her example can be easily shown to cope also with the challenge of Gettier counterexamples:

(3.5) Proposition. Let (X, OX) be defined by X = {1, 2, 3} and OX = {Ø, X, {2}, {1, 2}}. Then (X, OX) is a $KB(G)^C$ model, i.e., (X, OX) does not face Gettier counterexamples.♦

Proof. Consider A = {2, 3}. Then Int(A) = {2} and ClInt(A) = Cl({2}) = {1, 2, 3} = X. Therefore $A \cap Cl(Int(A)) = \{2, 3\} \neq \{2\} = Int(A)$, i.e., (X, OX) does not satify S4.Zem = (G).♦

This toy example shows that there are ED spaces that are not nodec. It may be considered, however, as not fully convincing to show that the ClInt semantics of KB does not always fall prey to Gettier counterexamples, since its small underlying space (X, OX) of three elements is certainly unrealistic for a topological universe of possible worlds. However, Özgün's result can be easily generalized for many T_0 Alexandroff spaces:

(3.6) Definition. Let (X, OX) be a T_0 Alexandroff space with specialization order $\leq$. The space (X, OX) satisfies the xyz-condition iff there are elements x, y, z $\in$ X such that $x < y < z$.(X

(3.7) Proposition. Let (X, OX) be a T_0 Alexandroff space with specialization order $\leq$. Assume (X, OX) satisfies the xyz-condition. Then (X, OX) is not a nodec space.

That is, A is not nodec.

Proof. Let x, y, z $\in$X and $x < y < z$, and $\uparrow z := \{w; z \leq w\}$. Then we calculate:

$$\mathrm{Int}(\{x\} \cup \uparrow z)) = \uparrow z, \text{ and } (\{x\} \cup \uparrow z) \cap \mathrm{ClInt}(\{x\} \cup \uparrow z)$$

$$= (\{x\} \cup \uparrow z) \cap \mathrm{Cl}(\uparrow z) = (\{x\} \cup \uparrow z) \cap \downarrow\uparrow z) = (\{x\} \cup \uparrow z).$$

Hence, for A := $\mathrm{Int}(\{x\} \cup \uparrow z)$ one has $\mathrm{Int}(A) \neq A \cap \mathrm{ClInt}((A)$, i.e., (X, OX) is not a nodec space.♦

Clearly, there are many T_0 Alexandroff spaces that satisfy the premise of (3.). Hence, we can be certain that many T_0 Alexandroff spaces can cope with Gettier's challenge. But it may be further observed that (3.7) is not a necessary condition for coping with Gettier. Even some Alexandroff spaces with a simpler specialization order turn out to be not nodec. An important example is the so-called "digital line" or "Khalimsky line". The Khalimsky line may be considered as a starting point for digital topology (cf. Kong et al. (1991), Kopperman (1994), Melin (2008)). It is a polar space (Mormann (2021)) that can be defined as follows:

(3.8) Proposition. Let $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ be the set of integers. Denote the set of odd numbers by $2\mathbb{Z} + 1 = \{...-3, -1, 1, 3, ...\}$. Define the map $m: \mathbb{Z} \rightarrow 2\mathbb{Z} + 1$ by

$$m(2n) := \{2n-1, 2n+1\}, \qquad m(2n+1) := \{2n+1\}.$$

For $A \subseteq \mathbb{Z}$ define

$$x \in Int(A) := x \in A \text{ and } \forall p(p \in m(x) \Rightarrow p \in A)$$

Then Int defines a topology on $\mathbb{Z}$ (cf. Rumfitt (2015, p. 243), Mormann (2021, Proposition 2.2)). The corresponding topological space ($\mathbb{Z}$, O$\mathbb{Z}$) is called the "digital line" or the "Khalimsky line". ($\mathbb{Z}$, O$\mathbb{Z}$) is not nodec.

<u>Proof</u>. To prove that ($\mathbb{Z}$, O$\mathbb{Z}$) is not nodec, consider the following example. By definition, the singletons of the points 2n+1are open, while the singletons of the points 2n are closed. Now consider the subset $A \subseteq \mathbb{Z}$ defined by A := {2, 3, 4, 5, 6}. One calculates:

$$Int(A) = \{3, 4, 5\} \qquad , \qquad IntCl\{3, 4, 5\} = \{2, 3, 4, 5, 6\}.$$

Hence one obtains

$$Int(A) = \{3, 4, 5\} \neq A \cap IntCl(A) = \{2, 3, 4, 5, 6\} \cap \{2, 3, 4, 5, 6\}.$$

That is, Int(A) ≠ A ∩ IntCl(A), i.e., the Khalimsky line ($\mathbb{Z}$, O$\mathbb{Z}$) is not a nodec space.♦

Some simple but non-trivial Alexandroff spaces, however, are nodec spaces. Consider the following example: Denote by $\mathbb{Z}$ again the set of integers. Define a topology on $\mathbb{Z}$ as follows. Let $3\mathbb{Z}$, $3\mathbb{Z}+1$, $3\mathbb{Z}+2$ the equivalence classes of integers mod 3 of rest 0, 1, and 2, respectively. Define the numbers 0, 1, and 2 as paradigmatic representants of $3\mathbb{Z}$, $3\mathbb{Z}+1$, $3\mathbb{Z}+2$, respectively. Subsets of $3\mathbb{Z}$, $3\mathbb{Z}+1$, $3\mathbb{Z}+2$ are defined to be open iff they contain 0, 1, or 2, respectively. Then we obtain:

<u>(3.9) Proposition</u>. The space ($\mathbb{Z}$, O$\mathbb{Z}$) is a polar space that is nodec.♦

(3.6) ensures that Stalnaker's KB logic possesses a class of models that do not face the Gettier problem. This argument may perhaps be considered as not fully satisfying for coping with

Gettier's challenge, since being Alexandroff is a very strong assumption and for (X, OX) to be interpreted as a universe of possible worlds. It seems difficult to render plausible. Moreover, as Steinsvold has argued compellingly, the epistemic capacity of an agent that is based on an Alexandroff space, may be assessed as unrealistically high ("immodest").

Indeed, if an epistemic agent is represented by a T_0 Alexandroff topology this agent's epistemic capacities are assumed to be very powerful. That this agent's topology is closed under infinite intersections of open sets suggests that she is able to put infinitely many pieces of evidence into a single one single piece. Compared with the closure of non-Alexandroff topology that is limited to deal only with finitely many pieces of evidence we may say that the Alexandroff agent's epistemic capacities are rather unlimited and "immodest". Thus, following Steinsvold (2020) we may conclude that we should not use Alexandroff topologies if we want to represent an epistemic agent with (more or less) realistic and modest epistemic capacities. Or, to put it differently, ClInt semantics scores better than co-derivative semantics with respect to the challenge posed by Gettier situations, the price to pay for this advantage is considerably high. It would be highly desirable to have a class of KB models that do not exhibit the weakness of immodesty and are able to cope with Gettier's counterexamples as well. These KB(G) models could be considered as really better than those based on DSO spaces.

In the following two sections we will tackle this problem and eventually identify a class of KB models that are not Alexandroff but nevertheless can cope with Gettier's challenge.

4. The Gettier Problem for Stone Spaces I. In this section we begin with studying the Gettier problem for another class of models of KB logic, namely, models of KB based on Stone representation spaces of certain complete Boolean algebras. In this section, we will determine a class of Stone spaces which do face Gettier counterexamples.

The simplest example of a completely regular, compact Hausdorff space is probably the closed unit interval X = [0, 1]. Applying (1.6) to [0, 1] we get that the Stone space St(B[0, 1]) is a $KB(G)^C$ model that faces the Gettier problem for Stalnaker's KB logic. For this model, traditional epistemology is valid, i.e., one has always that $KA = A \cap BA$. This proves (1.7).

(4.1) Definition (Porter and Woods 1988, Ch. 6.5 (a), p. 452). Let X and Y be topological spaces and let f be a (not necessarily continuous) surjection from X onto Y. f is called irreducible iff for A a proper closed subset of X, always $f(A) \neq Y$.♦

(4.2) Definition (Mioduszewski and Rudolf (1969, I, §6, 7). A (not necessarily continuous) map $f:X \to Y$ from (X, OX) to (Y, OY) is called an HJ-map (Henriksen-Jerison map) iff

$$Int_X f^{-1}(Cl_Y(V)) \subseteq Cl_Y(f^{-1}(V)) \text{ for each regular open subset } V \in O^*(Y).$$♦

(4.3) Lemma. The map $id_a:X \to a(X)$ is a HJ-map.

Proof. One has to prove $Int_X(Cl_{a(X)}(V)) \subseteq Cl_{a(X)}(V))$. That is obviously true.♦

(4.4) Definition (Mioduszewski and Rudolf (1969, I, §5). A (not necessarily continuous) map $f:X \to Y$ from the topological X into the topological space Y is called θ-continuous iff for each $x \in X$ and each open neighborhoood V of f(x) there exists an open neighborhood U of x such that $f(Cl_X(U)) \subseteq Cl_Y(V)$.♦

(4.5) Lemma. The map $id_a:X \to a(X)$ is θ-continuous for all topological spaces (X, OX).

Proof. Assume $x \in Int_{a(X)}(V) = V \cap IntClInt(V)$. Define U:= IntClInt(V). Then clearly $x \in U$. and $Cl(U) = ClIntClInt(V) = Cl_{a(X)}Int_{a(X)}(V) \subseteq Cl_{a(X)}(V)$. Hence, id is θ-continuous.♦

Clearly, a continuous map $f:X\rightarrow Y$ is θ-continuous. The converse does not hold in general. In particular, the map $id_a:X\rightarrow a(X)$ is not continuous but only is θ-continuous. Rather, we get: If Y is a regular space, then a θ-continuous map is continuous.

<u>(4.6) Definition</u>. An ED-resolution of a Hausdorff space (Z, OZ) is an irreducible θ-continuous map $p:(E(Z), OE(Z))\rightarrow(Z, O(Z))$, such that (E(Z), OE(Z)) is ED and Hausdorff.♦

The existence of an ED-resolution for each Hausdorff space was given by Iliadis (1963). A detailed presentation of Iliadis resolution can be found in Iliadis and Fomin (1966) and Csaszar (1991). Iliadis' construction can be briefly described as follows. Let (Z, OZ) be a Hausdorff space. Denote by E(Z) the set of all convergent open ultrafilters of Z. The sets $O_U := \{\xi \in E(Z), U \in \xi\}$ form, by definition, a base of open sets in E(Z).This defines a topological space ((E(Z), OR(Z))). Since Z is Hausdorff, there is a well-defined natural map $p: E(Z)\rightarrow Z$ defined by

$$p(\xi\}) := \cap\{Cl(U); U \in\xi\}.$$

Moreover, it can be proved that p is irreducible and θ-continuous and E(Z) is ED. The space E(Z) is compact iff if Z is almost compact (Csaszar (1991a, Theorem 2.6)).

Mioduzewski and Rudolf (1969) constructed a modified Iliadis resolution $p:E^{♣}(Z)\rightarrow Z$ by improving the topology E(O(Z)) of the original resolution (E(Z), O(E(Z)) in such a way that the natural map p is rendered continuous. This improved resolution will be used to prove Theorem (1.5).

To obtain the topology $O^{♣}(E(Z))$ of the improved Iliadis resolution $E^{♣}(Z):= (E(Z), O^{♣}(E(Z))$ of Z, Mioduszewski and Rudolf enlarged O(E(Z)) by the sets $p^{-1}(U)$, $U \in OZ$. Then they proved in detail that $E^{♣}(Z) = (E(Z), O^{♣}(E(Z))$ is still an ED-resolution of Z (see Mioduszewski and Rudolf (1969), (3.1) – (3.3)), i.e., $p: (E^{♣}(Z), O^{♣}(E(Z))\rightarrow(Z, O(Z))$ is irreducible and continuous (and not only θ-continuous).

This improved resolution has a universal property for certain ED spaces E. This property we will exploit for ED spaces of the form E = a(E(X)), with X being a compact Hausdorff space. Explicitly, it can be proved:

(4.7) Theorem (Mioduszewski and Rudolf (1969)). Let $p:E^{\clubsuit}(a(E(X)) \to a(E(X))$ be the conti– nuous improved Iliadis resolution of the ED Hausdorff space a(E(X))), and let $id_a:E(X) \to a(E(X))$ be the (θ-continuous) HJ-map defined in (4.2). Then there exists a unique continuous map $h^{\clubsuit}: E(X) \to E^{\clubsuit}(a(E(X))$ such that

$$id_a = h^{\clubsuit} \bullet p \qquad \begin{array}{ccc} & & E^{\clubsuit}(a(E(X)) \\ & h^{\clubsuit} & \downarrow p \\ E(X) & \xrightarrow{\;id_a\;} & a(E(X)) \end{array}$$

Thus, the map $id:E(X) \to a(E(X))$ is not only a θ-continuous, but even a continuous map.♦

Since $id_a^{-1}:a(E(X)) \to E(X)$ clearly is a continuous 1-1 inverse of $id_a:E(X) \to a(E(X))$ we get the following corollary which quenches the proof of (1.5):

(4.8) Corollary. The map $id:E(X) \to a(E(X))$ is a homeomorphism. Thus, E(X) is nodec space, i.e., $Int(A)= Int_{a(X)}(A) = A \cap IntClInt(A)$ for all $A \subseteq E(X)$. ♦

In a sense, Thorem (1.5) confirms the general idea that compact topological spaces are less complicated and better behaved than non-compact ones. Consequently, their logic of knowledge and belief should be simpler than that of non-compact spaces. In the next section we will prove the complementary part of the assertion, this conjecture will be – grosso modo – further confirmed by the theorems proven in the next section, where we prove that the epistemological logic corresponding to non-compact spaces is a non-traditional post-Gettier logic that does not face Gettier counterexamples.

5. The Gettier Problem for Stone Spaces II. In this section we will consider some non-compact Hausdorff spaces X for which their Boolean algebras B(X) of regular closed subsets of X have Stone spaces St(B(X))) that are not nodec, i.e., their models of Stalnaker's combined logic of knowledge and belief are post-Gettier logic. Thereby, finally, we will have shown that an important class of dense-in-themselves topological universes of possible worlds do not face the Gettier problem. For technical reasons we assume that X fulfils, besides of not being pseudocompact, certain further conditions. Examples of such spaces include the countable topologically trivial space $\mathbb{N}$ of natural numbers, the Euclidean spaces $\mathbb{R}^n$ with their standard Euclidean topology, and many more.

More precisely, in this section it is assumed throughout that X is a locally compact and completely regular, but not compact space.[5] These assumptions ensure that the Stone-Cech compactification βX of X is well behaved, and the following holds:

Let RC(X) the complete Boolean algebra of regular closed subsets of X. Then it is well-known that B(X) is isomorphic to the Boolean algebra of regular closed sets RC(βX) of the Stone-Cech compactification (βX) of X. Let E(X) be the projective cover (Iliadis absolute) of X. Let St(X) be the Stone space of RC(βX). According to Wheeler (1979, p. 568) one has

(5.1) $$St(X) = \beta E(X) = E(\beta X))$$

By Theorem 1(e) and (1)(i) of Wheeler (1979) we have E(X) is locally compact and not pseudocompact, since X is assumed to be locally compact and not pseudocompact. As discussed in the previous section for each completely regular Hausdorff space X there is an ED

[5] A paradigmatic example of such a space is the open unit interval (0, 1) of the real numbers $\mathbb{R}$.

space E(X), called the (Iliadis) absolute of X and a perfect irreducible map E(X)→X. Moreover, E(X) is completely regular (Iliadis and Fomin 1966, Note 3, p.44).

(5.2) Definition. A set $A \subseteq X$ is C-embedded in (X, OX) iff for every continuous function $f:A \rightarrow \mathbb{R}$ there is a continuous extension $f:X \rightarrow \mathbb{R}$.♦

(5.3) Proposition (Theorem 1.1.3, (7), p. 2, Angoa-Amador et al. 2018). X is pseudo-compact iff X does not have C-embedded copies of $\mathbb{N}$.♦

(5.4) Proposition (Theorem 1.3.8), p.16, Angoa-Amador et al. 2018, Encyclopedia d-16, p. 28). Let c be the cardinal of $\mathbb{R}$. If X is not pseudocompact, then the remainder $X^* = \beta X - X$ has at least 2^c elements.♦

After these preparations we can prove theorem (1.6) as follows. First, we observe that not all points of X* are isolated, since by (5.4) this would contradict the compactness of X*. Hence, there is a non-isolated point $a \in X^*$.[6] The open neighborhoods of a point $a \in \beta X - X$ have the form $U(a) \cap (\beta X - X)$ with $U(a) \in O(\beta X)$. The point $a \in \beta X - X$ is non-isolated in $\beta X - X$ iff for all open neighborhoods U(a) of a there is a point $a^* \in U(a)$ with $a^* \in \beta X - X$ and $a \neq a^*$. Then $a \notin Int(X \cup \{a\})$ since $= X \cup \{a\}$ does not contain a*. Hence $Int(X \cup \{a\}) = X$. Since X is open and dense in βX (Gillman and Jerison (1960, 6.9(d), p. 90) one obtains:

$$X = Int(X \cup \{a\}) \neq (X \cup \{a\}) \cap ClInt(X \cup \{a\}) = (X \cup \{a\}) \cap ClInt(X)$$

$$= (X \cup \{a\}) \cap Cl(X) = (X \cup \{a\}) \cap \beta X = X \cup \{a\}$$

[6] For specific cases of X much stronger assertions hold. For instance, for $\mathbb{N}$ we have that $\mathbb{N}^* - \mathbb{N}$ has no isolated points at all.

In other words, there is subset $X \cup \{a\} \subseteq \beta X$ with $Int(X \cup \{a\}) = X$ such that

$$Int(X \cup \{a\}) \neq (X \cup \{a\}) \cap ClInt(X \cup \{a\}) = X \cup \{a\}$$

That is, βX is not a nodec space. In other words, βX is a KB(G) model of Stalnaker's combined logic KB of knowledge and belief. By Wheeler (General remarks (3), p. 567, (1979)) we have

(5.5) $$E(\beta X) = \beta E(X) = St(X(B))).$$

By Theorem 1 (Wheeler (1979, p.569) X is locally compact, pseudocompact etc. iff E(X) is locally compact, pseudocompact etc.

Hence, we can rehearse the same argument just presented for X for the absolute E(X). Thereby we get that $E(X) \cup \{a\}$ is not open for some $a \in E(\beta X) - E(X)$. Thus, St(X(B)) is not nodec. This proves Theorem (1.6). It should be noticed that this proof essentially depends on the fact that for non-pseudocompact spaces the Stone-Cech compactification $\beta(X)$ is extensionally much larger than X. It is well known that for metrical spaces compactness is equivalent to pseudocompactness (Angoa-Amador et al. ((2017, Proposition 1.1.13, p.8). Thus, we have proved (1.7).

6. Concluding Remarks. In order to determine whether the Stone space of a Boolean algebra is nodec or not, several variants of compactness have been essential. For metrical spaces compactness and pseudo-compactness are equivalent. (Angoa-Amador et al. ((2017, Proposition 1.1.13, p.8)

Compactness is a sort of topological finiteness. Compact spaces are topologically more easily dealt with than non-compact spaces. This fact should also have logical consequences. One is that (roughly speaking) that non-compact spaces X do not face the Gettier problem for St(B(X)).

Gettier situations described by propositions A for which knowing that A does not coincide with the conjunction of believing that A and the truth that A are exceptional and rare. Topologically, the exceptional character of such a Gettier proposition is shown by the fact that the extension of A is nowhere dense in the universe of possible worlds.

Gettier counterexamples to traditional JTB epistemology may be said to be a result of to a kind of inexhaustibility of the universe of possibilities.

References.

Aiello, M., van Benthem, J., Bezhanishvili, G., 2003, Reasoning about space: the modal way, Journal of Logic and Computational 13(6), 889 – 920.

Angoa-Amador, J., Contreras-Carreto, A., Ibarra-Contreras, M., 2018, Basic and Classic Results on Pseudocompact Spaces, in M. Hrusek et al. (eds.), 2018, Pseudocompact Topological Spaces, 1 – 39.

Baltag, A., Bezhanishvili, N., Özgün, A., Smets, S., 2013, The Topology of Belief, Belief Revision and Defeasible Knowledge, , in Logic, rationality, and interaction, 4th International workshop in logic, Hangzou, China, October 9 -12, 2013.

Baltag, A., Bezhanishvili, N., Özgün, A., Smets, S., 2019, A topological approach to full belief, Journal of Philosophical Logic 48, 219 – 244.

Bezhanishvili, G., Esakia, L., Gabelaia, D., 2004, Modal Logics of Submaximal and Nodec Spaces, Festschrift for D. de Jongh, Dedicated on his 65th Birthday, University of Amsterdam.

Blasczczyk, A., 1974, Extremally disconnected resolutions of T_0-spaces, Colloquium Mathematicum XXXII (1), 57 – 68.

Borges, R., de Almeida, C., Klein, P.D., 2017, New Essays on the Gettier Problem (Eds.), Oxford University Press, Oxford.

Csaszar, A., 1991a, Iliadis Absolutes for Arbitrary Spaces, Acta Mathematica Hungarica 57(1-2), 111 – 120.

Csaszar, A., 1991b, General Absolutes of Topological Spaces, Acta Mathematica Hungarica 58(1-2), 141 – 151.

Davey, B.A., Priestley, A., 1990, Introduction to Lattices and Order, Cambridge, Cambridge University Press.

Fine, N.J., Gillman, L., 1960, Extension of Continuous Functions in βN, Bulletin of the American Mathematical Society 66, 376 – 381.

Frolik, Z., Non-homogeneity of βP – P, Commentarii Math. Univ. Carolin. 8, 705 -709.

Georgacarakos, G.N. 1976, Semantics for S4.04, S4.4, and S4.3.2, Notre Dame Journal of Formal Logic, 17(2), 297 – 302.

Gettier, E., 1963, Is justified true belief knowledge? Analysis 23, 121 – 123.

Gleason, A.M., 1958, Projective topological spaces, Illinois Journal of Mathematics, 482- 489.

Gillman, L., Jerison, M., 1960, Rings of Continuous Functions, Springer, New York, Heidelberg.

Hart, K.P., Nagata, J., Vaughan, J.E., 2004, Encyclopedia of General Topology (Eds.), Elsevier, Amsterdam.

Hrusak, M. et al., 2018, Pseudocompact Topological Spaces (Eds.), Springer Nature.

Iliadis, S.D., 1963, The absolutes of Hausdorff spaces, Dokl. Akad. Nauk SSSR 149, 22 -25. In Russian.

Iliadis, S.D., Fomin, S.V., 1966, The method of centered systems in the theory of topological spaces, Russ. Mathematical Surveys 31(4), 37 -62.

Jankovic, D.S., 1985, A note on mappings of extremally disconnected spaces, Acta Mathe–matica Hungarica 46(1-2), 83 – 92.

Jankovic, D., Hamlett, T.R., 1990, New Topologies from Old via Ideals, The American Monthly 97(4), 295 – 310.

Johnstone, P.T., 1982, Stone spaces, Cambridge University Press, Cambridge.

Lenzen, W., 1979, Epistemologische Betrachtungen zu [S4, S5], Erkenntnis 14, 33 – 56.

Kong, T. Y., Kopperman, R., & Meyer, P. R. (1991). A topological approach to digital topo–logy. *The American Monthly, 98*(10), 901–917.

Kopperman, R. (1994). The Khalimsky line as a foundation of digital topology. In Y.-L. O. A. Toet, D. Foster, H. J. A. M. Heijmans, & P. Meer (Eds.), *Shape in picture,* 3–20. Berlin: Springer.

Melin, E. (2008). *Digital geometry and Khalimsky spaces, Uppsala dissertations in mathe–matics 54*.

Mioduszewski, J., Rudolf, L., 1969, H-closed and extremally disconnected Hausdorff spaces, Dissertationes Mathematicae LXVI, Warszawa, Panstwowe Wydawictwo Naukowe.

Mormann, T., 2023, JTB epistemology and the Gettier problem in the framework of topological epistemic logic, Review of Analytic Philosophy 3(1), 1 – 43.

Njastad, O., 1965, On some Classes of Nearly Open Sets, Pacific Journal of Mathematics 15 (3), 961 – 979.

Özgün, A., 2017, Evidence in Epistemic Logic. A topological Perspective. Thesis.

Parikh, R., Moss, L.S., Steinsvold, C., 2007, Topology and Epistemic Logic, in Handbook of Spatial Logics, edited by Aiello M, Pratt-Hartmann, I., van Benthem, J., Springer, 299 – 341.

Ponomarev, V. I., Shapiro, L. B., 1976, Absolutes of Topological Spaces and their Continuous Maps, Russian Mathematical Surveys 31(5), 138 – 154.

Reilly, I.L., Vamanamurthy, M.K., 1985, On α-continuity in topological spaces, Acta Mathematica Hungarica 45, 27 – 42.

Rumfitt, I., 2015, The Boundary Stones of Thought. An Essay in the Philosophy of Logic, Oxford, Oxford University Press.

Sarsak, M., Ganster, M., Steiner, M., 2006, On S_i-metacompact spaces, Tatra Mountains Mathematical Publication 34, 1- 7.

Segerberg, K., 1971, An Essay in Classical Modal Logic, University of Uppsala.

Sobocinski, B. 1970, Certain extensions of modal system S4, Notre Dame Journal of formal logic 11(3), 347 – 367.

Stalnaker, R.C., 2006, On Logics of Knowledge and Belief, Philosophical Studies 128(1), 169 – 199.

Steinsvold, C., 2012, Topological Models of Belief Logics, PhD thesis 2007, City University of New York, New York, USA, Akademiker Verlag, Saarbrücken/Germany.

Steinsvold, C., 2020, Some Formal Semantics for Epistemic Modesty, Logic and Logical Philosophy 29, 381 – 413.

Wheeler, R.F., 1979, Topological Measure Theory for Completely Regular Spaces and their Projective Covers, Pacific Journal of Mathematics 82(2), 565 – 584.

Willard, S., 2004, General Topology, Dover Publications, Mineola, New York.

Williamson, T., 2015, A note on Gettier cases in epistemic logic, Philosophical Studies 172, 129 - 140.

Woods, G.R., 1969, Certain Properties of $\beta X - X$ for σ-compact spaces X, Dissertation.

Woods, G.R., 1972, Ideals of pseudocompact sets and absolutes of Hewitt realcom–pactifications, General Topology and its Applications 2, 315 – 331.

Zeman, J. J., 1969, Modal systems in which necessity is “factorable”, Notre Dame Journal of Formal Logic 10(3), 247 – 256.

APPENDIX:

The Topology of Knowledge and Belief. To set the stage, in this section we recall the necessary basics of elementary set-theoretical topology that are needed for the formulation of the interior semantics for epistemic logic of knowledge and belief as presented by Baltag, Bezhanishvili, Özgün, and Smets (cf. Baltag et al. (2013, 2015, 2016, 2019)). This semantics is used throughout this paper. First of all, recall the definition of a topological space:

(A.1) Definition. Let X be a set with power set PX. A topological space is an ordered pair (X, OX) with OX $\subseteq$ PX that satisfies the following conditions:

(i) Ø, X $\in$ OX.

(ii) OX is closed under finite set-theoretical intersections $\cap$ and arbitrary set-theoretical unions $\cup$.

(iii) (X, OX) is an Alexandroff space iff OX is closed under arbitrary intersections. ♦

The elements of OX are called the open sets of the topological space (X, OX). As usual, a topological space (X, OX) is denoted by X, if no misunderstanding concerning the topology OX is possible. The set-theoretical complement A^C of an open set A $\subseteq$ X is called a closed set. The set of closed subsets of (X, OX) is denoted by CX. The interior kernel operator Int and the closure operator Cl of (X, OX) are defined as usual: The interior kernel Int(A) of a set A $\in$ PX is the largest open set that is contained in A; the closure Cl(A) of A is the smallest closed set containing A.[7]

(A.2) (Separation axioms). Let (X, OX) be a topological space, $x, y \in X$, $x \neq y$.

(i) T_0 axiom: There exist an open set $U \in OX$ such that either $x \in U$ and $y \notin U$, or $y \in U$ and $x \notin U$.

(ii) T_1 axiom: There exists open sets U, $V \in OX$ containing x and y respectively, such $y \notin U$, and $x \notin V$.

(iii) T_2 axiom: There exists disjoint open sets U, $V \in OX$ containing x and y respectively.♦

Each of these axioms is independent of the axioms for a topological space, T_2 implies T_1, and T_1 implies T_0, the reverse does not hold.

Topologies on a set X can be partially ordered set-theoretically:

(A.3) Definition. Let (X, OX) and (X, O'X) two topologies on the same set X. Then OX is said to be coarser than O'X iff OX is a subset of O'X, i.e., $OX \subseteq O'X$. If OX is coarser than O'X this is also expressed by saying that O'X is finer than OX.♦

Clearly, the coarsest topology on X is $O_0X = \{\emptyset, X\}$ and the finest topology is $O_1X = PX$. For all topologies OX one has

[7] For details, see Willard (2004), Steen and Seebach Jr. (1982), or any other textbook on set-theoretical topology.

(A.4) $$O_0X \subseteq OX \subseteq O_1X.$$ ♦

The topological operators Int and Cl are well-known to satisfy the Kuratowski axioms (cf. Kuratowski and Mostowski (1976)):

(A.5) Proposition (Kuratowski Axioms). Let (X, OX) be a topological space, A, D $\in$ PX. Define the interior kernel operator Int of (X, OX) by Int(A) := $\cup\{U ; U \in OX \text{ and } U \subseteq A\}$. Dually, the closure operator Cl is defined by Cl(A) := $\cap\{K; K \in CX \text{ and } A \subseteq K\}$. The operators Int and Cl satisfy the following axioms:

(i) $\mathrm{Int}(A \cap D) = \mathrm{Int}(A) \cap \mathrm{Int}(D)$. $\mathrm{Cl}(A \cup D) = \mathrm{Cl}(A) \cup \mathrm{Cl}(D)$.

(ii) $\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A)$. $\mathrm{Cl}(\mathrm{Cl}(A)) = \mathrm{Cl}(A)$.

(iii) $\mathrm{Int}(A) \subseteq A$. $A \subseteq \mathrm{Cl}(A)$.

(iv) $\mathrm{Int}(X) = X$. $\varnothing = \mathrm{Cl}(\varnothing)$. ♦

In the following the Kuratowski axioms are used without explicit mention. Moreover, we will use freely the fact that the operators Int and Cl are inter-definable:

(A.6) $$\mathrm{Int}(A) = \mathrm{Cl}(A^C)^C \text{ and } \mathrm{Cl}(A) = \mathrm{Int}(A^C)^C.$$

Often, it is expedient to conceive the operators Int and Cl as operators Int:PX→PX and Cl:PX→PX defined on PX in the obvious way. Hence, the concatenation of these operators makes perfect sense.

(A.7) Definition. Let (X, OX) and (Y, OY) be two topological spaces. A map f:X→Y is a continuous map from (X, OX) to (Y, OY) iff for all A $\in$ OY $f^{-1}(A) \in$ OX. ♦

(A.8) Definition. An open cover of a space ((X, OX) is a collection A of open subsets $X_i \in OX$ whose union $\cup X_i$ is all of X. A subcover of a cover of X is a subcollection of a cover of X.

(i) A space X is compact iff each open cover of X has a finite subcover.

(ii) A space is X is locally compact iff each point of X has a neighborhood base consisting of compact sets.

(iii) A space is almost compact iff every open cover contains a finite set F such that the closures of F cover X.

(iv) A space X is pseudocompact iff every continuous real-valued function $f:X\to\mathbb{R}$ is bounded.♦

(A.9) Definition. A compactification of a space (X, OX) is we a compact space (Y, OY) in which X is dense.

(A.10) Definition. A topological space (X, OX) is extremally disconnected (ED) iff the closure of every open is open, i.e., Int(Cl(Int(A))) = Cl(Int(A)) for all subsets A of X.♦

(A.11) Definition. A topological space (X, OX) is a nodec space iff its operators Int and Cl satisfy for all $A \subseteq X$ the equation Int(A) = A $\cap$ IntClInt(A).♦

(A.12) Definition. (i) A topological space (X, OX) is regular iff whenever A is a closed set in X and $x \notin A$ then there are disjoint open sets U and V with $x \in U$ and $A \subseteq V$.

(ii) A topological space (X, OX) is completely regular iff whenever A is a closed set in X and $x \notin A$ there is a continuous function $f:X\to[0, 1]$ such that f(x) = 0 and f(A) = 1, I = [0,1] the unit interval.♦

(A.13) Proposition. Let (X, OX) be a topological space with interior kernel operator Int and closure

operator Cl, and A, D $\in$ PX. Then

(i) IntClIntCl(A) = IntCl(A) and ClIntClInt(A) = ClInt(A).

(ii) IntCl(Int(A) $\cap$ D) = IntClInt(A) $\cap$ IntCl(D).

Proof. The identities (i) are well known, (ii) is also well known for A and D $\in$ OX. The proof of (A.12)(ii) can be found in Kuratowski and Mostowski (1976, Ch.I, §8).♦

(A.14) Definition. Let (X, OX) a topological space. For A $\subseteq$ X the set of limit (or accumulation) points of A is called the derivative of A and denoted by dA.♦

(A.15) Definition. A topological space (X, OX) is a DSO space iff it is dense-in-itself and for all A $\subseteq$ X the derivative dA of A is open.♦