Prototypes, Poles, and Topological Tessellations of Conceptual Spaces

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Abstract. The aim of this paper is to present a general method for constructing natural tessellations of conceptual spaces that is based on their topological structure. This method works for a class of spaces that was defined some 80 years ago by the Russian mathematician Pavel Alexandroff. Alexandroff spaces, as they are called today, are distinguished from other topological spaces by the fact that they exhibit a 1-1 correspondence between their specialization orders and their topological structures.

Recently, Ian Rumfitt (apparently not being aware of Alexandroff's work) used a very special case of Alexandroff's method to elucidate the logic of vague concepts in a new way. Elaborating his approach, the color circle's conceptual space can be shown to define an atomistic Boolean algebra of regular open concepts. In a similar way Gärdenfors' geometrical discretization of conceptual spaces by Voronoi tessellations also can be shown to be a kind of geometrical version of Alexandroff's topological construction. More precisely, a discretization à la Gärdenfors is extensionally equivalent to a topological discretization constructed by Alexandroff's method. Rumfitt's and Gärdenfors's constructions turn out to be special cases of an approach that works much more generally, namely, for Alexandroff spaces. For these spaces (X, OX) the

Boolean algebras O*X of regular open sets are still atomistic and yield natural tessellations of X.

1. Introduction. The aim of this paper is to present a general method for constructing natural discretizations of conceptual spaces based on their topological structures. The method can be traced back to an approach that the Russian mathematician Pavel Alexandroff put forward some 80 years ago. Alexandroff's method is based on a functorial 1-1 correspondence between order structures and topological structures that hold for a special class of spaces.

Recently, Ian Rumfitt (apparently not being aware of Alexandroff's work) used a very special case of Alexandroff's method to elucidate the logic of vague concepts in a new way. As a by-product of Rumfitt's account one obtains a topological tessellation of the conceptual space of the colour spectrum. Thereby Rumfitt's construction exhibits some similarity with Gärdenfors' method of discretization of conceptual spaces by Voronoi tessellations. Moreover, the two constructions are special cases of Alexandroff's construction. More precisely, Gärdenfors's discretization of conceptual spaces based on Voronoi diagrams is extensionally equivalent to a topological discretization constructed by Alexandroff's method. Rumfitt's and Gärdenfors's constructions turn out to be very special cases of a general result concerning Alexandroff spaces (X, OX), namely, that the algebra O*X of regular open sets of X is an atomistic Boolean algebra.

A conceptual space may be considered as a continuous realm of possible stimuli or experiences. A task of cognitive science is to understand how this realm is rendered a structured space that can serve as a base for the elaboration of a more or less detailed classification of stimuli or experiences.[1]

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[1] With some good will, Carnap's „attribute spaces" may be considered as forerunners of conceptual spaces in Gärdenfors's sense. For a more detailed comparison of the similarities and differences of the two approaches see Sznajder (2016, section 6). In particular, in contrast to

For this task, Gärdenfors and his collaborators have proposed to employ so called Voronoi tessellations based on an underlying Euclidean structure of the conceptual space to be dealt with:

> A Voronoi tessellation based on a set of prototypes is a simple way of classifying a continuous space of stimuli. The partitioning results in a discretization of the space. The prime cognitive effect is that the discretization speeds up learning. … [A] Voronoi tessellation is a cognitively economical way of representing information about concepts. Furthermore, having a space partitioned into a finite number of classes means that it is possible to give names to the classes. (Gärdenfors (2000, 89))

As I want to show the topological essence of Gärdenfors's (and Rumfitt's) discreti–zations of continuous conceptual spaces is based on a structural correspondence between order structures and topological structures that was discovered by Alexandroff in the 1930s.

A Voronoi tessellation of a conceptual space uniquely determines a topological tessellation that is extensionally equivalent to a regular open tessellation constructed by Alexandroff's method. The constructions of Rumfitt and Gärdenfors boil down to very special cases of Alexandroff's construction. Thus it makes sense to consider the far-reaching generalizations that are suggested by Alexandroff's original construction. This generalization suggests that Alexandroff spaces should be considered as a natural topological habitat of conceptual spaces. They provide the natural framework for conceptual spaces that are interested in those concepts that are empirically meaningful. This claim can be explicated as follows:

(1) An empirically meaningful concept has to be stable in the sense that, if it applies to a situation x, it also applies to small variations x' of x. This stability

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attribute spaces, the regions of conceptual spaces that correspond to concepts are non-homogeneous in the sense that some (generating) points are more prototypical than others.

is accompanied with a certain conceptual vagueness. Stable concepts do not hold with absolute precision. They are unable to single out empirical objects with absolute precision. This should be considered as virtue rather than a vice. Otherwise, concepts would no longer be empirically applicable.

(2) Arbitrary conjunctions of stable concepts should be stable. This requirement is the expression of a reasonable conceptual modesty. Otherwise, we could get rid of the inherent vagueness of empirical concepts by purely logical means, namely, by piling up more and more concepts that eventually result in an absolutely precise conceptualization of reality.

(3) Topologically, the requirements (1) and (2) can be satisfied by the requirement that an empirically relevant conceptual space S is endowed with the structure of an Alexandroff topology (S, OS) such that concepts are characterized as elements of O*S.

In this paper we rely on a topological account of concepts, i.e., concepts are characterized as topologically well-formed regions of a topological space (X, OX), namely, as elements of OX or O*X. This is similar to Gärdenfors's geometrical approach of conceptual spaces according to which concepts are represented by convex regions of an underlying conceptual space that is usually assumed to be an Euclidean space endowed with a metric. In comparison with the geometrical account the topological one to be presented in this paper is more austere insofar as one and the same topological structure may give rise to different geometrical structures.

The outline of this paper is as follows: The preparatory section 2 introduces the basic topological concepts needed in the rest of this paper. Then, in section 3, a special class of conceptual spaces is discussed in detail, namely, "polar spaces" and their topological structures. These spaces were recently defined by Ian Rumfitt in his book *The Boundary Stones of Reason* (Rumfitt 2015). This class of spaces may be considered as an elementary paradigmatic example of the general topological account elaborated in this paper. In section 4 the relation between the topologically defined tessellations of polar spaces and the geometrically defined Voronoi tessellations of Gärdenfors's conceptual spaces is explicated. Section 5 deals with the topology and order structure of an especially subclass of Alexandroff spaces, namely, Artinian Alexandroff spaces. It is shown that this class of spaces may be considered as the most general class of spaces that gives rise to well-behaved classifications and categorizations of objects.

2. Elements of Topology. In this section we recall some topological concepts that are necessary for understanding the rest of this paper. Let us start right-on with the basic definitions:

(2.1) Definition. Let X be a set with power set PX. A topological space is a relational structure (X, OX) with $OX \subseteq PX$ satisfying two requirements that $\emptyset, X \in OX$ and finite intersections and arbitrary unions of elements of OX are elements of OX.The elements of OX are called the open sets of X. The set-theoretical complements of open sets A are called closed sets.[2] If there is no danger of confusion a topological space (X, OX) is simply denoted by X.♦

If X has more than one point several different topological structures OX exist on X. In particular, there are two extreme topological structures (X, O_0X) and (X, O_1X) defined by $O_0X := \{\emptyset, X\}$ and $O_1X := PX$. The topology (X, O_0X) is called the indiscrete topology on X, and the topology (X, O_1X) is called the discrete topology. With respect to set-theoretical inclusion $\subseteq$ all topological structures (X, OX) on X lie between these two (rather uninteresting) topologies, i.e., $O_0X \subseteq OX \subseteq O_1X$.♦

[2] A set may be open and closed. For instance, the sets Ø and X are open and closed for any topological structure (X, OX).

Topological structures can be defined in many equivalent ways. For our purposes particularly useful is the definition in terms of closure operators cl or interior kernel operators int. These operators have to satisfy the so called Kuratowski axioms:

<u>(2.2) Definition.</u> Let X be a (non-empty) set with power set PX. A topological closure operator is an operator PX——cl——>PX satisfying the four requirements (1) – (4) below. Dually, a topological interior kernel operator is a map PX——int——>PX satisfying the following four requirements (1)* - (4)*:

(1)	$cl(A \cup B) = cl(A) \cup cl(B)$.	(1)*	$int(A \cap B) = int(A) \cap int(B)$.
(2)	$cl(cl(A)) = cl(A)$.	(2)*	$int(int(A)) = int(A)$.
(3)	$A \subseteq cl(A)$.	(3)*	$int(A) \subseteq A$.
(4)	$cl(\emptyset) = \emptyset$.	(4)*	$int(X) = X$.

Closure operators cl and interior kernel operators int are interdefinable: cl(A) = **C**int**C**(A), and int(B) = **C**cl**C**(B), **C** being the set-theoretical complement with respect to X.

Every topological closure operator cl uniquely defines a topological structure (X, OX) and viceversa: Given a topological closure operator cl define the class of open sets OX $\subseteq$ PX by OX := {B; B = **C**cl(A); A $\subseteq$ X}. Dually, given a topological interior operator int define OX by OX := {A; A = int(A), A $\subseteq$ X}. For every A $\subseteq$ X the boundary bd(A) of A (in (X, OX) is defined as bd(A) = cl(A) $\cap$ cl(**C**A).♦

<u>(2.3) Proposition.</u> Let (X, OX) be a topological space, cl and int its closure operator and interior kernel operator, respectively. An open subset A $\in$ OX is regular open iff A = int(cl(A)). The set of all regular open subsets of X is denoted by O*X. O*X is a complete Boolean algebra. There is a canonical map OX——j——>O*X defined by j(A) := int(cl(A)) and an inclusion O*X——i——>OX such that $j \bullet i = id_{O^*X}$.♦

The following definition is essential for the type of topological spaces we will deal with in this paper, namely, Alexandroff spaces.

(2.4) Definition (Specialization Order of a Topology). A topological structure (X, OX) defines a partial order (X, ≤) on X by $x \leq y := x \in cl(y)$. This partial order is called the specialization order of (X, OX). The set of maximal elements of (X, ≤) is denoted by M.[3]♦

For traditional topological spaces like Euclidean spaces (E, OE) the specialization order (E, ≤) is trivial, i.e. $x \leq y$ iff $x = y$. In contrast, for the topological structures to be considered in the following, the specialization order is non-trivial and of great importance. Under appropriate conditions, the topological structure (X, OX) of a topological space can be reconstructed from its specialization order (X, ≤).

(2.5) Definition (Topology defined by a partial order (X, ≤)). Let (X, ≤) be partial order. For $A \subseteq X$ define $\uparrow A := \{x; a \leq x \text{ for some } a \in A\}$. The set $\uparrow A$ is called the upper set of A. Analogously, the lower set $\downarrow A$ of A is defined by $\downarrow A := \{y; y \leq a \text{ for some } a \in A\}$. The upper topology (X, OX) corresponding to (X, ≤) is defined by $OX := \{\uparrow A; A \subseteq X\}$.♦

(2.6) Definition. A topological space X is an Alexandroff topological space iff arbitrary intersections of open sets are open. Equivalently (invoking the Kuratowski axioms) a topological space X is an Alexandroff space iff arbitrary unions of closed sets are closed. Clearly, if X is Alexandroff for every $A \subseteq X$ there exist a smallest open set V(A) containing A, namely, the intersection of all open sets that contain A. ♦

In order to distinguish between different classes of topological spaces separation axioms have turned out to be a convenient means. A few important instances of

[3] For the Euclidean topology (**R**, O**R**) one has M = **R**. For the Alexandroff space (N, O**N**) (see (2.8)(iv)) one has M = Ø.

separation axioms relevant for those topological spaces considered in this paper are given in the following definition.

(2.7) Definition (Separation axioms). Let X be a topological space, and x, y different points of X.

(i) X is a T_0-space iff for every $x \in X$ there exists an open set $a \in OX$ such that either $x \in a$ and $y \notin a$, or $x \notin a$ and $y \in a$.

(ii) X is a T_D-space iff for every x there exists an open set a(x) such that $x \in a(x) \in OX$ and $a(x) - \{x\} \in OX$.

(iii) X is a $T_{1/2}$-space iff if every point $x \in X$ is either open or closed.

(iv) X is a T_1-space iff if every point $x \in X$ is closed.

(v) (X, OX) is a T_2-space iff there exists open sets $a \in OX$ and $b \in OX$ containing x and y, such that $x \notin b$ and $y \notin a$.♦

(2.8) Examples of Topological Spaces and their Separation Axioms.

(i) If X has more than one point, the indiscrete topology (X, O_0X) is not T_0. Except the trivial spaces (X, O_0X) all other topological spaces to be considered in this paper are assumed to be T_0–spaces.

(ii) The standard Euclidean topology O**R** of the real line **R** is generated by open intervals (a, b) = {x; a < x < b}. Two distinct points x and y can be separated by small open intervals U(x) and U(y) that are disjoint to each other. Hence (**R**, O**R**) is a T_2–space. All points are closed.

(iii) Let **Q** ⊆ **R** be the subset of rational numbers. The set **Q** as a subset of **R** inherits a topology from **R** by declaring a subset A of **Q** to be open iff A = **Q** ∩ B, for some B ∈ O**R**. The topological space (**Q**, O**Q**) has many clopen (= open and closed) subsets. It is totally disconnected.

(iv) Let (**N**, ≤) be the set of natural numbers endowed with their natural order ≤. A topological space (**N**, O**N)** is defined by stipulating that Ø, **N** and the sets ↑n := {m; n ≤ m} are open for each n ∈ **N**. Then (**N**, O**N)** is an Alexandroff space that satisfies T_0 but not T_1. The points of (**N**, O**N)** are neither open nor closed.

(v) An Alexandroff space (X, OX) satisfies T_1 iff it is discrete.♦

<u>(2.9) Proposition</u>. The topological space (X, OX) is a T_0-Alexandroff space iff its topology OX coincides with the upper topology of the specialization order (X, ≤).

<u>Proof</u>. Check the definitions.♦

The equivalence between the topological structure (X, OX) and the order-theoretical structure of the specialization order (X, ≤) shows that modal logic is a rich source of Alexandroff spaces:

<u>(2.10) Proposition.</u> Let (W, ≤) an S4 Kripke frame such that ≤ is reflexive, transitive, and anti-symmetric. Then the upper topology on W defines a T_0 -Alexandroff space (W, OW).♦

<u>(2.11) Definition</u>. A T_0-Alexandroff space X is an Artinian Alexandroff space iff the specialization order (X, ≤) satisfies the ACC-condition, i.e., (X, ≤) does not have chains $x_1 < x_2 < \ldots < x_n < \ldots$ of infinite length. That is to say, for every x ∈ X after finitely many steps one obtains a maximal element $x \leq m_x$ (The element m_x is, of course, usually not unique).♦

Now we can formulate the main formal result of this paper:

<u>(2.12) Theorem</u>. Let (X, OX) be an Artinian Alexandroff space, M the set of maximal elements of (X, ≤). Then the Boolean lattice O*X of regular open subsets of X is an

atomistic Boolean algebra, i.e., O*X = 2^L, L being the set of atoms of O*X. More precisely, the elements of L are the regular open sets int(cl(m)), m ∈ M. The atomistic Boolean algebra O*X defines a unique regular open tessellation of the set X by X = ∪L ∪ bd(∪L).♦

We will deal with this theorem on two different levels of generality: First, as a particularly simple case, we will consider so-called polar spaces, recently introduced by Rumfitt (2015). These spaces are characterized by specialization orders (X, ≤) of depth 2. Then we consider general Artinian Alexandroff spaces (X, OX) whose specialization orders satisfy the ACC condition, i.e., every properly ascending chain in (X, ≤) has finite length. Their Boolean algebras O*X are still atomistic. The theorem (2.12) is best possible in the sense that Alexandroff spaces (X, OX) that do not satisfy the ACC condition may have Boolean algebras O*X that are not atomistic. An example will be given in (5.8).

3. The Topological Structure of Polar Spaces. In this section we explicate the topological structure of the polar spaces (cf. Rumfitt (2015, chapter (8.4)). In particular, we show that these spaces are very simple Alexandroff spaces.

Rumfitt presents his approach by way of example, discussing the polar topology of the well-known color circle as a conceptual space of color experiences in general terms. He does not, however, calculate its topology in any detail. This will be undertaken in this section.

Let X be a set of coloured objects that is to serve as the underlying set of a conceptual space for color experiences. We are looking for a discretization of X, i. e., a partition of X that allows us to classify color experiences in different categories. Usually, this is done with the aid of certain paradigmatic or prototypical experiences of red, of blue, yellow and so on. (cf. Gärdenfors (2000)). Following Sainsbury (cf. Sainsbury (1996)) Rumfitt argues that the task of classification of colors is best conceptualized as a procedure based on a comparison with certain objects that are to be considered as paradigmatic or prototypical:

> The spectrum enables us to attach senses to colour terms not because it shows boundaries, but because it displays colour paradigms or poles. Sainsbury likens colour paradigms to 'magnetic poles exerting various degrees of influence: some objects cluster firmly to one pole, some to another, and some, though sensitive to the forces, join no cluster'. ... I prefer a simpler analogy which likens paradigms to gravitational poles, that is, massive bodies. If a small body is sufficiently close to a gravitational pole, it will be drawn towards it, rather as we are drawn to classify as red those objects that are sufficiently close in colour to a paradigm, or pole, of red. Rumfitt (2015, 236)

The essential mathematical structure to be extracted from this example is the following: Assume that there is given a set X of objects to be classified, and a subset P of X to be considered as a set of distinguished elements that are "paradigmatic" or "prototypical" objects. In the terminology of Rumfitt (2015) they are called poles. These poles are used to classify the ordinary objects.

More precisely, the set of poles is used to endow X with the structure of a topologically structured conceptual space. Let us assume that for every object $x \in X$ there is a non-empty set $m(x) \subseteq P$ of poles p such that all the $p \in m(x)$ are maximally close to x. It may well be that for some x the set m(x) comprises more than one element, but m(x) is assumed to be always non-empty. It seems plausible to assume that for a pole $p \in P$ one has $m(p) = \{p\}$. This is to be interpreted as the assumption that for a paradigmatic object p the unique maximally close paradigmatic object to it is p itself. The attribution of maximally close poles to each object $x \in X$ can be conceived as a function $X \xrightarrow{m} 2^P$ satifying the two conditions:

(3.1) (i) For all $x \in X$ $m(x) \neq \emptyset$ (ii) For all $p \in P$ $m(p) = \{p\}$.

The function m is called a pole distribution for X and is denoted by (X, m, P). The requirements (3.1) (i) and (ii) guarantee that poles do some classificatory work by classifying the elements of X according to the poles that are maximally close to them: First, poles are distinguished from non-poles as those elements that are, so to speak, "self-classifying", i.e., $m(p) = \{p\}$.[4] Secondly, pole distributions (X, m, P) define a topology on X with the help of the following interior kernel operator 2^X——int——>2^X:

(3.2) Proposition. Let (X, m, P) be a pole distribution, $A \subseteq X$ and define the operator by

$(3.2)_1$ $x \in int(A)$ iff $x \in A$ & $\forall_{p \in P} (p \in m(x) \Rightarrow p \in A)$

is a topological interior kernel operator that defines an Alexandroff topology. Informally formulated, $x \in int(A)$ iff $x \in A$ and moreover all poles that are maximally close to x also belong to A. In other words, the interior of A comprises those elements of A whose maximally close poles also belong to A. Thereby elements of int(A) "have no connection to elements outside A".

Equivalently, the topology corresponding to a pole distribution (X, m, P) can be defined by the closure operator cl

$(3.2)_2$ $x \in cl(A)$ iff $x \in A$ or $\exists_{p \in P}(p \in A$ and $p \in m(x))$.

That is, the closure cl(A) of a set A comprises the members of A together with all objects for which at least one of their maximally close poles is in A. In other words, cl(A) comprises all elements of A that are in A or have at least connection to elements of A. The topological space (X, OX) defined by the operators int or cl is called the polar space of the distribution (X, m, P).♦

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[4] Note that it is not required that m is defined with the aid of a fully-fledged metric on X as Gärdenfors seems to assume.

Proof. The proof that int and cl are topological Alexandroff operators, i.e., define an Alexandroff topology, consists in a routine check that these operators satisfy the Kuratowski axioms (2.2), see Rumfitt (2015, 243 - 246). A closer inspection of the definitions $(3.2)_1$ or $(3.2)_2$ reveals that they even satisfy the stronger Alexandroff condition (2.6).♦

Although Rumfitt introduces the topology (X, OX) given by a pole distribution (X, m, P) he does not describe the topology in any further detail. In particular, he does not mention that (X, OX) is an Alexandroff topology. Further, the fact that the singletons {p} of poles $p \in P$ are open but not regular open is not mentioned. Nor does he show explicitly that O*X is atomistic. All this could have been proven easily with the formal apparatus he has at his disposal. The proofs are mathematically quite elementary, but perhaps not totally obvious, and it may help the reader to get a more concrete idea of what is going on in this area and to motivate the generalization to be proposed in the following.

Endowed with the topology defined by a pole distribution (X, m, P) the color circle (X, OX) is a very special Alexandroff space, namely, (X, OX) is a T_0-space such that the singletons $\{p\} \subseteq P$ are open and all the singletons {x} for $x \in X - P$ are closed.[5] More precisely, the following proposition obtains:

(3.3) Proposition. Let (X, OX) be the topological space defined by a pole distribution (X, m, P). Then X is a T_0–space. For all $x \in X$, the smallest open set that contains x is the set $V(x) := \{x\} \cup m(x)$. For the elements $p \in P$ and $x \in X - P$ one calculates:

$$\text{int}(p) = \{p\} \qquad \text{int}(x) = \emptyset \qquad V(x) = \{x\} \cup m(x)$$

[5] A space having this property is sometimes said to satisfy the $T_{1/2}$-axiom located halfway between T_o and T_1.

cl(x) = {x} cl(p) = {x; p ∈ m(x)}, int(cl(p)) = {x; {p} = m(x)}

Proof. Let us prove V(x) = {x} ∪ m(x). According to the definition of the interior operator int one has

$$y \in int(\{x\} \cup m(x)) \;\Leftrightarrow\; y \in \{x\} \cup m(x) \;\&\; \forall p(p \in m(y) \Rightarrow p \in \{x\} \cup m(x)).$$

Clearly every element in {x} ∪ m(x) satisfies this condition. On the other hand, any smaller set, properly contained {x}∪ m(x), does not satisfy the condition. For two different elements x and y V(x) and V(y) are different. Hence X is a T_0-space. Thus the sets {{x} ∪ m(x), x ∈ X} form a unique minimal base for the topology on X.

Proposition (3.3) provides the topological data that can be used to characterize Rumfitt's polar spaces as a very special class of topological spaces (cf. Bezhanishvili, Esakia, and Gabelaia (2003) and Bezhanishvili, Mines, and Morandi (2003)). In order to understand the discussions there recall the following definitions:

(3.4) Definition. Let (X, OX) be a topological space.

(i) An element x ∈ X is isolated iff {x} ∈ OX. The set of isolated points of X is denoted by ISO(X).

(ii) A subset A ⊆ X is dense in X iff cl(A) = X. A subset A ⊆ X is nowhere dense iff int(cl(A)) = Ø.

(iii) (X, OX) is weakly scattered iff ISO(X) is dense in X, i.e., cl(ISO(X)) = X.

(iv) X is a nodec space iff every nowhere dense set is closed.

(v) (X, OX) is submaximal iff every dense subset of X is open.

(vi) (X, OX) satisfies the McKinsey axiom iff int(cl(A)) ⊆ cl(int(A)) for all A ⊆ X.

(vii) Define the boundary bd(A) of A as bd(A) := cl(A) ∩ cl(CA). Then bd(bd(A)) = bd(A) for all A ⊆ X iff X satisfies the McKinsey axiom.♦

(3.5) Proposition. Let (X, OX) be a polar space defined by a pole distribution (X, m, P). Then X satisfies the following conditions:

(i) X is weakly scattered.

(ii) X is nodec.

(iii) X is submaximal.

(iv) X satisfies the McKinsey axiom.

(v) bd(bd(A)) = bd(A) for all $A \subseteq X$.

Proof. (i): By (3.3) the set ISO(X) of isolated points of X is just the subset $P \subseteq X$. Let (X, OX) be a polar space. Then every x is contained in at least one cl(p) by definition of the polar distribution m. Hence $X = \cup_{p \in P} cl(p) \subseteq cl(ISO(X))$, i.e., X is weakly scattered.

(ii): Let $A \subseteq X$ be nowhere dense, i.e., intcl(A) = Ø. One has to show that A is closed. If $A \subseteq X - P$ then A is closed by (3.3) and the fact that (X, OX) is Alexandroff. If A is not a subset of X – P, there must be a $p \in A$ and $p \in A$. Then A cannot be closed by (3.3).

(iii): The submaximality of X immediately follows from the fact that the points of X are either open ($x \in P$) or closed ($x \in X - P$). From (3.3) it transpires that any dense set must contains the set P of poles. If a dense set D contains any other point x by (3.3) it contains V(x) and is therefore open. Thus a polar space is submaximal.

(iv): Assume $x \in int(cl(A))$. Since (X, OX) is Alexandroff by (3.3) this is equivalent with $\{x\} \cup m(x) \subseteq cl(A)$. Again due to Alexandroff $cl(A) = \cup_{a \in A} cl(a)$. For $p \in m(x)$ this entails that $p \in cl(a)$ for some a. This entails p = a. So $m(x) \subseteq A$. Since {p} for all $p \in m(x)$ is open this is equivalent with $m(x) \subseteq cl(int(A))$. Clearly $cl(m(x)) \subseteq cl(int(A))$ and $x \in cl(m(x))$. Since X is Alexandroff this entails $x \in int(cl(m(x)))$ and therefore $x \in int(cl(m(x)) \subseteq cl(int(A))$. Thus X satisfies the McKinsey axiom.

(v) By definition of the boundary operator bd one has $bd(bd(A)) \subseteq bd(A)$. Hence it is only left to prove $bd(A) \subseteq bd(bd(A))$. By definition one has $bd(bd(A)) = cl(bd(A)) \cap cl(\mathbf{C}bd(A)) = bd(A)) \cap cl(\mathbf{C}bd(A))$. We prove $cl(\mathbf{C}bd(A)) = X$:

$$\mathrm{cl}(\mathbf{C}\mathrm{bd}(A))) = X \Leftrightarrow \mathrm{int}(\mathrm{bd}(A)) = \emptyset \Leftrightarrow \mathrm{int}(\mathrm{cl}(A)) \cap \mathrm{int}(\mathrm{cl}(\mathbf{C}A)) = \emptyset$$

$$\Leftrightarrow \mathrm{int}(\mathrm{cl}(A)) \subseteq \mathbf{C}\mathrm{int}(\mathrm{cl}(\mathbf{C}A)) \Leftrightarrow \mathrm{int}(\mathrm{cl}(A)) \subseteq \mathrm{cl}(\mathrm{int}(A)).$$

In (iv) it was proved that polar spaces satisfy the McKinsey axiom.♦

<u>(3.6) Proposition.</u> Let (X, OX) be the topological space defined by (X, m, P). The minimal open neighborhood V(x) of every $x \in X$ has the form $V(x) = \{x\} \cup m(x)$. The Boolean algebra O*X of regular open sets of (X, OX) is atomistic. The atoms of O*X are the sets intcl(p)) = {x; {p} = m(x)}. Two atoms int(cl(p)) and int(cl(p*)) generate in O*X the set $\{x; \{p, p^*\} \cap m(x) \neq \emptyset\}$.

<u>Proof</u>. Check the definitions and apply (3.3). ♦

In section 5 we will show that the “correct” generalization of polar spaces, for which the Boolean algebra O*X is still atomistic, is the class of Artinian Alexandroff spaces.

<u>4. Topological and Geometrical Tessellations</u>. The aim of this section is to discuss several types of tessellations of conceptual spaces that can be used to set up conceptual classifications based on geometrical and/or topological structure of conceptual spaces. Due to the fact that for polar spaces (X, OX) the Boolean algebra O*X is atomistic it will be shown that the resulting tessellation is particularly simple and essentially unique.

<u>(4.1) Definition</u>. A regular open tessellation T of a topological space (X, OX) is a set of disjoint regular open subsets $A_i \in O^*X$ such that $X = \vee A_i$. The supremum $\vee A_i$ of the A_i taken here in O*X, not in OX. The A_i are called the (regular open) cells or atoms of T. Points of X that are not in the interior of any cell A_i are said to belong to the boundary bd(T) of the tessellation T.♦

(4.2) Examples.

(i) Let (X, OX) be a topological space, A ∈ O*X, A ≠ Ø, X. Denote the Boolean complement of A in O*X by A* (A* = int(**C**A)) Then {A, A*} is a regular open tessellation of X with the two cells A and A*. The boundary bd(T) of T is just the boundary bd(A) (= bd(A*)).

(ii) More generally, let $A_1,\dots, A_n$ be n regular open subsets of X. The set $\{A_1,\dots, A_n\}$ generates a regular open tessellation of X that has 2^m atoms, $m \leq 2^n$.♦

These examples show that regular open tessellations of topological spaces abound. Thus, the problem is not to prove the existence of a regular open tessellation but to single out interesting tessellations that are essentially unique. Polar spaces are an example. They have unique canonical regular open tessellations:

(4.3) Proposition. Let (X, m, P) be a pole distribution for X. Then (X, OX) has a regular open tessellation T by the atoms of the Boolean algebra O*X of X, i.e., T = $\cup_P$ int(cl(p) ∪ bd($\cup_P$ int(cl(p)).

Proof. Let (X, OX) be defined by (X, m, P). As proved in (3.8) the Boolean algebra O*X is isomorphic to the powerset 2^P. Thus, clearly, the atoms of O*X generate a regular open tessellation T of X. The atoms of T are the regular open sets int(cl(p)) = {x;{p} = m(x)}, p ∈ P. Elements of X that have more than one pole maximally close to them are located in the boundary of T, i.e., bd(T) = {x; {p, p'} ⊆ m(x) for some p, p' ∈ P and p ≠ p'.♦

Now, let us consider in some more detail, the most prominent class of tessellations, namely, the so called Voronoi tessellations (cf. Gärdenfors 2000, Decock and Doven 2015, Zenker and Gärdenfors 2015). Voronoi tessellations may be characterized as geometrical tessellations. They are defined by using the underlying geometrical structure of Euclidean conceptual spaces, more precisely, their metrical structure. The general definition is as follows:

(4.4) Definition (Okabe et al. (1992, p. 44)). Given a set P of two or more but a finite number of distinct points in the Euclidean plane, we associate all locations in that space with the closest member(s) of P with respect to the Euclidean distance. The result is a tessellation of the plane into a set of the regions associated with members of P. We call this tessellation the planar ordinary Voronoi diagram generated by P, and the regions constituting the Voronoi diagram ordinary Voronoi polygons.♦

As an illustration of this general definition let us consider the simplest Voronoi tessellation of the Euclidean plane E:

(4.5) Example. Let E be the Euclidean plane endowed with a Cartesian coordinate system (x, y). Choose two points $p_L = (-1, 0)$ and $p_R = (1, 0)$. Then the Voronoi tessellation in the sense of (4.4) taking p_L and p_R as generators is given by the two open cells

$$L: = \{(x,y); d((x,y), p_l) < d((x,y), p_R)\} \qquad R : = \{(x,y); d((x,y), p_R) < d((x,y), p_L)\}$$

The boundary of this tessellation is given by the topological boundary bd(L) = bd(R) of the two half-planes L and R. It is the line

$$\{(x,y); d((x,y), p_L) = d((x,y), p_R\} \quad = \quad \{(x, y); x = 0\}.♦$$

Following the recipe of (4.4) this construction is easily generalized to a partition of the plane into at most 2^n regular open polygons (and their boundaries), n finite number. Let $p_1, ..., p_n$ a finite set of different points. Then a general Voronoi tessellation may be conceived of as the result of the intersection of the (n, 2) = n!/2! (n-2)! pairs of half-planes each defined by the bisectors of pairs (p_i, p_j) points in such a way that the plane is divided into convex open cells together with their boundaries.

Clearly, a geometrically defined Voronoi tessellation of Euclidean space gives rise to a regular open tessellation in the sense of (4.1): By construction all the open Voronoi cells are disjoint to each other, and, due to fact that they are convex (cf. Gärdenfors (2000, 88) they are not only open but even regular open. By the very definition of the Voronoi cells the points not in the interior of a cell are those points that have equal distance to two (or more) paradigmatic points p_i. Hence they are located on the topological boundary of those cells.

A Voronoi tessellation based on the metrical structure of an Euclidean space E not only defines a topological tessellation in the sense of (4.1), it also yields a polar distribution (E, m, P):

<u>(4.6) Proposition</u>. Let T be a Voronoi tessellation of the Euclidean plane E defined by a finite set P of prototypes $p_1, \ldots, p_n$. Then a pole distribution (X, m, P) is defined as follows: Take the Voronoi generators $p_1, \ldots, p_n$ as poles of a pole distribution $X \xrightarrow{m} 2^P$ defined as

$$m(x) = \{p_i;\ x \in cl(\langle p_i \rangle);\ \langle p_i \rangle \text{ the open Voronoi cell defined by the generators } p_i)\}.$$

The resulting topological space (E, OE) defines a regular open tessellation of E by the regular open atoms $int(cl(p_i))$ of O*E such that the $int(cl(p_i))$ are just the Voronoi cells $\langle p_i \rangle$. More precisely, $x \in E$ is contained in the interior of a cell int(cl(p)) iff m(x) = {p}.♦

By this recipe, the cells of the resulting topological tessellation of E coincide with the cells of the Voronoi tessellation of E. Moreover, the geometrically defined boundary of the Voronoi tessellation coincides with the topological boundary. In sum, every geometrically defined Voronoi tessellation based on a finite set of prototypical points gives rise to a uniquely defined topological tessellation defined by (X, m, P) that is extensionally equivalent.

Compared with the geometrical construction of a Voronoi tessellation of the plane a topological tessellation gets along with much less structural presuppositions. This is a conceptual advantage insofar, as certain problems caused by representational artifacts disappear. For example, for Euclidean spaces there are many different metrical structures that define the same underlying topological structure.[6] With respect to these different metrics, one and the same set P of prototypical points may give rise to different Voronoi tessellations. Which one should be chosen as the right one? Another problem that may be attributed to the too specific mathematical apparatus used for the definition of a Voronoi tessellation T of a conceptual space concerns the boundary bd(T) of T. It has been dubbed the "thickness problem" (cf. Douven et al. 2013).

The "thickness problem" may be explicated as follows. By the very construction of Voronoi tessellations, the boundaries of cells are "thin" compared with the interiors of the cells since they are lines consisting of points that have equal distances to two (or more) prototypical points. Douven et al. (2013) rightly point out that this assumption for most conceptual spaces is hardly plausible. For instance, for the conceptual space of the color spectrum the boundary, say, between "red" and "orange" is defined by the points that have exactly the same distance from the prototypical points of "red" and "orange". Empirically, this does not make much sense. What should it mean that a certain shade of color has the same distance from a prototypical "red" and a prototypical "orange"? Moreover, in the general case, there is no reason to assume that boundaries are "thin" compared with the regular open cells of the Voronoi tessellation. Douven et al. (2013) propose to overcome this shortcoming by the introduction of "collated Voronoi diagrams" that arise as the result of projecting similar

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[6] A prominent case is provided by the family of Minkowski-metrics $d_i(x, y)$ for $1 \leq i \leq \infty$. This problem is briefly discussed for i = 1 (taxi cab or Manhattan metric) and i = 2 (ordinary Euclidean metric in Gärdenfors (2000, chapter 3.9). The Euclidean metric d_2 has the structural advantage that the cells of its Voronoi tessellations turn out to be convex.

ordinary Voronoi diagrams onto each other such that their boundaries define a blurred and more or less "thick" area thereby taking into account the vagueness of concepts and their boundaries.

For topological tessellations no "thickness" problem arises, since they do not distinguish between "thick" and "thin" as geometrical tessellations do (in an artificial way). The following example shows that the topological approach easily deals with tessellations with cells whose boundaries are "thicker" than the cells themselves:

(4.6) Example. Let X be the set $\{\alpha, \mathbf{N}, \omega\}$, $\mathbf{N} = \{1, 2, \ldots\}$ the set of natural numbers and α and ω two objects different from all elements of **N** and from each other. Take P = $\{\alpha, \omega\}$ and define a pole distribution (X, m, P) by

$$m(i) = \{\alpha, \omega\} \text{ for } i \in \mathbf{N}, \qquad m(\alpha) = \{\alpha\} \quad , \qquad m(\omega) = \{\omega\}$$

The corresponding topological structure (X, OX) to this pole distribution is given by

$$cl(\alpha) = \{\alpha\} \cup \mathbf{N} \quad , \qquad cl(i) = \{i\} \quad , \qquad cl(\omega) = \{\omega\} \cup \mathbf{N}$$

$$int(\alpha) = \{\alpha\} \quad , \qquad int(\mathbf{N}) = \emptyset \quad , \qquad int(\omega) = \{\omega\}$$

$$intcl(\alpha)) = \{\alpha\} \quad , \qquad int(cl(\omega)) = \{\omega\} \quad , \quad bd(\omega) = bd(\alpha) = \mathbf{N}.$$

The cardinality of the boundary of the regular open cells $\{\alpha\}$ and $\{\omega\}$ is infinite and thus much larger than the cardinality of the regular open cells intcl(α)) = $\{\alpha\}$ and int(cl(ω)) = $\{\omega\}$.♦

The example (4.6) shows that the topological approach has no difficulty to accommodate the "thickness of boundaries". Topological tessellations are flexible enough to accept cells with boundaries that are intuitively much "thicker" than the cells they are boundaries of.

5. Artinian Alexandroff Spaces. Alexandroff topologies (X, OX) are especially apt for bringing to the fore the order-theoretic features of topology by relating the topology OX to the specialization order (X, ≤). The essential feature that distinguishes Alexandroff topological spaces (X, OX) from ordinary topological spaces is the existence of an "open hull" V(A) ∈ OX for every A ∈ PX.

In this section we recall the necessary elements of the theory of Alexandroff topological spaces in order to show that Artinian Alexandroff spaces provide a natural generalization of polar spaces. For quite a long time Alexandroff spaces did not find much attention in mathematics or elsewhere. This changed, when it became clear that topology may have a bearing on theoretical computer science. As it turned out, Alexandroff topological spaces and related structures became important, see Gierz et alii (2003), Goubault-Larrecq (2013).

Recall that an Alexandroff space X is defined as a topological space such that arbitrary intersections of open sets are open (and not only finite ones) (cf. (2.6)). Clearly, every topological space X having only finitely many elements is an Alexandroff space. In other words, Alexandroff topology becomes an interesting concept only for spaces of infinite cardinality. Thus, the Alexandroff topology of the color circle and similar conceptual spaces defined by a prototype distributions (X, m, P) may qualify as an interesting Alexandroff topology (cf. Rumfitt (2015).

Let (X, OX) be a polar space defined by (X, m, P). The specialization order (X, ≤) is defined as x < y iff y is a prototype of x, i.e., y ∈ m(x). The smallest open subset that contains x is the set V(x) = {x, m(x)}. Thus, the polar topology OX is just Alexandroff topology defined by the specialization order (X, ≤). In particular, ↑x := {x, m(x)} = {y; x ≤ y}. Thus, for general Alexandroff spaces (X, OX) with specializiation order (X, ≤) one has:

(5.1) $x \in int(A) := \uparrow x = \{y; x \leq y\} \subseteq A$

Thus, Rumfitt's definition of the interior operator int for polar spaces is just a special case of the standard definition of int for general Alexandroff spaces in that m(x) is replaced by ↑x. This corresponds to the standard definition for general topological spaces. In the case of an Alexandroff the existence of an open neighborhood U(x) contained in int(A) can be expressed more specifically that int(A) must contain ↑x. Thus, Rumfitt's definition of int for polar spaces is just a special case of the general definition of the interior operator in of Alexandroff spaces. Correspondingly, for a general Alexandroff (X, OX) space one obtains for its closure operator:

(5.2) $$cl(A) = \{x; \downarrow x \cap A \neq \emptyset\} \qquad \downarrow x := \{y; y \leq x\}$$

Now we specialize to a particularly well-behaved class of Alexandroff spaces, namely, Artinian Alexandroff spaces. Their specialization order (X, ≤) is of finite depth in the following sense:

<u>(5.3) Definition (cf. Mahdi and El Atrash (2005)</u>.[7] Let (X, OX) be a T_0-Alexandroff space with specialization order (X, ≤). The space (X, OX) is called an Artinian Alexandroff space iff (X, ≤) satisfies the ascending chain condition (ACC-condition) that every ascending sequence

$$a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots$$

eventually stabilizes, i.e., there is an $m \in \mathbf{N}$ such that $a_m = a_{m+1} = a_{m+2} = \ldots$. In other words, every strictly increasing sequence of elements of X terminates after finitely many steps. The space (X, OX) is of depth at most n iff every strictly increasing chain of elements of X at most n elements. Thus, polar space (X, m, P) are Artinian Alexandroff spaces of depth 2.♦

[7] Very useful detailed calculations of the topology of Artinian Alexandroff spaces can be found in Mahdi and El Atrash (2005) and Mahdi et al. (2015).

Let (X, OX) be any topological space, and **2** the canonical Heyting algebra with two elements 0, 1, and 0 ≤ 1. Then every x ∈ X defines a map OX——f_x——>**2** by $f_x(a) = 1$ iff x ∈ a. The maps f_x are frame maps in the sense that they preserve arbitrary joins and finite meets. The set of frame maps of (X, OX) is called the spectrum of (X, OX) and denoted by Sp(X). In sum, for every topological space (X, OX) there is a map X——Sp—>Sp(X). Usually, this map is neither a monomorphism nor an epimorphism. The topological space (X, OX) is called sober iff Sp is an isomorphism. As is easily shown, Hausdorff spaces are sober. The following theorem evidences that Artinian Alexandroff spaces behave topologically quite nicely in that they are sober (even if they are not Hausdorff):

<u>(5.4) Theorem (Picado and Pultr (2015, 6.4.4. Proposition, p. 22).</u> An Alexandroff space (X, OX) is sober iff it is Artinian, i.e. its specialization order (X, ≤) satisfies the ACC-condition.

The simplest class of Artinian Alexandroff spaces are polar spaces defined by pole distributions (X, m, P). More general Artinian Alexandroff spaces share many properties with polar spaces. Thus it seems justified to assert that Artinian Alexandroff spaces may be considered as the "right" generalization of polar spaces. The following proposition enables us to show that ArAlexandroff spaces are very similar to polar spaces:

<u>(5.5) Proposition (Bezhenashivili, Mines, Morandi (2003), Propositions (2.1), (2.8))</u> Let (X, OX) be a topological space, DX the set of all dense subsets. Then DX is a filter (with respect to set-theoretical intersection ∩) generated by the set ISO(X) of isolated points iff the McKinsey axiom int(cl(A)) ⊆ cl(int(A)) holds for all A ⊆ X.♦

The rather technical proof of this proposition is not very illuminating and therefore omitted. Nevertheless proposition (5.5) is quite useful since it guarantees that Artinian Alexandroff spaces satisfy the McKinsey axiom and its consequences:

(5.6) Proposition. Let (X, OX) be an Artinian Alexandroff space, and M the set of maximal points of the specialization order (X, ≤). Then M = ISO(X) and M is dense in X. Moreover, M generates the filter DX of dense sets of X.

Proof. Let n ∈ M be a maximal element of (X, ≤). The singleton {n} is open by the definition of the Alexandroff topology, i.e., ↑n = {n}. Then cl(n) = ↓n = {x; x ≤ n}. Since the ACC-condition for (X, ≤) is satisfied, for all x there is at least one n such that x ∈ cl(n). Hence cl(M) = $\cup_{n \in M}$ cl(n) = X and M is dense in X.

In order to show that M generates DX as a filter one may argue as follows. Let A be a dense subset of X. Then X = cl(A) = $\cup_{a \in A}$cl(a) = $\cup_{a \in A}$↓a. In order that M is contained in cl(A) one must have for all n ∈ M ⊆ A $\cup_{a \in A}$↓a, since n is maximal in (X, ≤). Hence, M is a minimal generator of DX as a filter.♦

(5.7) Corollary. Artinian Alexandroff spaces (X, OX) satisfy the McKinsey axiom, i.e., bd(bd(A)) = bd(A) for all A ⊆ X.♦

Due to (5.7), for Artinian Alexandroff spaces no problem of higher-order vagueness arises, higher-order boundaries coincide with ordinary boundaries, i.e., bd(A) = bd^n(A), for n ≥ 1. As is well-known, the validity of the McKinsey axiom is equivalent to the fact that the logic of Artinian-Alexandroff spaces is S4.1.

On the other hand, Artinian Alexandroff spaces (X, OX) in general are no longer sub–maximal nor nodec as is easily shown by examples (An example will be given in (5.8)). Rather, Alexandroff spaces (X, OX) are submaximal iff the depth of (X, ≤) is ≤ 2. With respect to tessellations, however, general Artinian Alexandroff spaces behave very much like polar spaces: Every Artinian Alexandroff space (X, OX) has a canonical atomistic regular open tessellation defined by the atoms of O*X:

<u>(5.7) Proposition</u>. Let (X, OX) be an Artinian-Alexandroff space with specialization order (X, ≤), and M be the set of maximal elements. Then O*X is an atomistic Boolean algebra with atoms a = int(cl(m)), m ∈ M, as generators. Thereby one obtains a regular open tessellation of X by X = $\cup_{m \in M}$ int(cl(m)) ∪ bd($\cup_{m \in M}$ int(cl(m)).

<u>Proof</u>. Let M ⊆ X be the set of all maximal elements of the specialization order (X, ≤). By definition, for each x ∈ X there is at least one m ∈ M such that x ≤ m. Since the Alexandroff topology is the upper topology of the specialization order (X, ≤) the singletons {m} are open, and the closures cl(m) of {m} are just the down sets ↓m := {x; x ≤ m}.

The sets int(cl(m)) := {x; ↑x ⊆ ↓m} for m ∈ M are atoms of O*X: For different m, m* the sets intcl(m) and intcl(m*) are disjoint and regular open, since int(cl(m)) ∩ int(cl(m*)) = intcl({m} ∩ {m*}) = Ø due to the fact that the singletons {m} and {m*} are open and the operator intcl is a nucleus, i.e., it distributes over finite intersections (cf. Borceux (1994, 29)).

That intcl(m)) is an atom in O*X is seen as follows: Assume x ∈ A = intcl(A) ⊆ intcl(m). Since intcl(m)) is open one has ↑x ⊆ intcl(m)). This entails x ≤ m and therefore m ∈ ↑x ⊆ A. Hence intcl(m) ⊆ intcl(A) = A. Hence A = intcl(m).

Now we prove that any regular open A = $V_{m \in M'}$ intcl(m) for some M' ⊆ M. Let M_A := {m; m ∈ A ∩ M}. Clearly, $V_{m \in AM}$ intcl(m) ⊆ A. If we can show that A ⊆ $V_{m \in AM}$ intcl(m) we are done. Assume x ∈ A and define M_x := {m; x ≤ m and m ∈ M}. Clearly M_x ⊆ A_M. Hence x ∈ cl(A_M). Assume y ∈ ↑x. This is the case iff x ≤ y and this entails M_y ⊆ M_x. So we obtain

$\uparrow x \subseteq cl(A_M)$. This means $x \in intcl(A_M) = intcl(\cup_{m\in AM}\{m\}) = V_{m\in AM}\, intcl(m)$. This is to say A $\subseteq V_{m\in AM}\, intcl(m)$.♦

The result that the Boolean lattice O*X is atomistic for Artinian Alexandroff spaces (X, OX) is is best possible in the sense that Alexandroff spaces that are not Artinian may not have an atomistic algebra O*X. This is shown by the following example:

<u>(5.8) Example</u>. Let X = **N** ∪ **N***,   **N** = {0, 1, 2, 3, ... }, **N*** = {1*, 2*, 3*, ... }. X is rendered a partial ordering (X, ≤) by stipulating n < (n+1), n < (n+1)*, no other non-trivial order relations exist between the elements of X, in particular, it is NOT assumed that n* < (n+1)*. Clearly, (X, ≤) does not satisfy the ACC-condition, since there is the infinite series 0 < 1 < ... that does not stabilize after finitely many steps. Hence the Alexandroff space (X, OX) that corresponds to (X, ≤) is not Artinian. Now consider the sets

$$\uparrow n := \{n, (n+1), (n+2), \dots , (n+1)^*, (n+2)^*, \dots \}$$

By definition these sets are open in the Alexandroff topology (X, OX), since they are upper sets. It is easily calculated that int(cl(↑n)) = ↑n, i.e., the sets ↑n are not only open, but even regular open. The sets ↑n are all different and non-empty, and the sequence of ↑n is converging to Ø. Thus O*X is not atomistic. This shows that the requirement that (X, ≤) satisfies the ACC-condition is necessary to ensure that the Alexandroff space (X, OX) has an atomistic Boolean algebra O*X. As is easily checked, (X, OX) is a T_D–space but not an $T_{½}$-space. Further, the set of isolated points of X is **N*** = {1*, 2*, ... }. Clearly, **N*** is dense in X, i.e., cl(**N***) = X, and N* generates the set DX of open dense sets of X as a filter. According to (Bezhanishvili, Mines, and Morandi (2003, 2.1. and 2.8) this entails that (X, OX) satisfies the McKinsey axiom (3.4)(vi). Hence (5.8) shows that there are Alexandroff spaces that are not sober but nevertheless satisfy the McKinsey axiom.♦

The fact that the topologies of polar spaces and Artinian Alexandroff spaces behave in a rather similar way suggests that the dynamics of conceptual spaces may be described in terms of (partially) structure-preserving maps between these spaces. Let us consider the following example related to the problem of conceptual refinement, in some detail: Let $(X, \leq_1)$ and $(X, \leq_2)$ be two partial orders on a set X. The relation $\leq_2$ is said to be finer than $\leq_1$ iff $\leq_1 \subseteq \leq_2$ as subsets of $X \times X$. The identity map $(X, \leq_1) \xrightarrow{id} (X, \leq_2)$ induces a continuous map of the Alexandroff spaces $(X, O_{\leq 1}X) \xrightarrow{id} (X, O_{\leq 2}X)$ iff the maximal elements of $\leq_1$ are also the maximal elements of $\leq_2$. If this is the case $(X, O_{\leq 1}X) \xrightarrow{id} (X, O_{\leq 2}X)$ may be interpreted as a conceptual refinement of the theory based on the conceptual space $(X, O_{\leq 1}X)$ by the theory based on the conceptual space $(X, O_{\leq 2}X)$. This refinement amounts to a replacement of the concepts defined by $O^*_{\leq 1}X$ by concepts defined by $O^*_{\leq 2}X$. Informally formulated, the simple task of categorizing some experiences to certain (sets of) prototypes is replaced by categorizing these experience by prototypes of different hierarchically types.

Consider the following example: Let $(X, \leq_1)$ be the specialization order of a polar space, and $(X, \leq_2)$ be a partial order such that $\leq_1 \subseteq \leq_2$. Let us further assume that the maximal elements of $\leq_1$ (the "poles") are also the maximal elements of $\leq_2$. Then the partial order $\leq_2$ is a refinement of the partial order $\leq_1$. The identity map of the corresponding Alexandroff spaces $(X, O_{\leq 1}X) \xrightarrow{id} (X, O_{\leq 2}X)$ is easily shown to be continuous (in the usual topological sense). For a maximal element $m \in X$, $cl_{\leq 1}(m) - \{m\}$ is just the set $\{x; x <_1 m\} = \{x; x <_2 m\}$. In contrast to $\leq_1$ the partial order $\leq_2$ induced on $cl_{\leq 1}(m) - \{m\}$ may be non-trivial. If this is the case, $(cl_{\leq 1}(m) - \{m\}, \leq_2)$ is a non-trivial partial order that may be considered as a conceptual refinement of the unstructured set $cl_{\leq 1}(m) - \{m\}$. While $\leq_1$ only allows us to classify the elements of $cl_{\leq 1}(m) - \{m\}$ as belonging to the class of “m-individuals”, the order $\leq_2$ enables us to classify the “m-individuals” further into subclasses, subsubclasses, etc.

More generally, the representation of knowledge systems by conceptual spaces suggests that the dynamics of cognitive achievement may be modelled with the help of (partially structure-preserving) maps between these spaces.

6. Concluding Remarks. Let us take stock. In this paper some general reasons have been put forward for the claim that conceptual spaces should be conceived of as being endowed with the topological structure of Alexandroff spaces. One reason is that Alexandroff topologies are an expedient conceptual device to take care of the important role that prototypes and paradigmatic elements play in human categorization. In the simplest case, this is already evidenced by polar spaces (X, OX) that are defined by pole distributions (X, m, P). Artinian Alexandroff spaces may be considered as a convenient generalization of this case since they preserve most of the useful properties exhibited by polar spaces, and on the other hand allow to get rid of the overly narrow framework that permits only one level of prototypes. Instead, in the case of Artinian Alexandroff spaces a partially ordered hierarchy of prototypes may occur. This enables us to deal with properties of different paradigmatic levels. If paradigms and prototypes play an important role in human categorization, and this seems to be the case, polar spaces, and, more generally, Artinian Alexandroff spaces provide a natural framework.

The basic assumption of the conceptual spaces approach in cognitive science is that concepts can be usefully represented as well-formed subsets of a conceptual space that is structured in one way or other. The basic problem of this approach is to find appropriate structures that allow to single out empirically useful concepts as structurally well-formed. Topological structures are generally recognized as basic spatial structure for all kinds of spaces that show up in all realms of knowledge. Thus, it does not appear unreasonable to expect that topological structures may also play a role in the theory of conceptual spaces. Topological concepts are sufficiently flexible to be adapted to various empirical necessities. It is a matter of empirical research to find out which topological structures for which type of conceptual spaces may be helpful.

Gärdenfors and others have argued that convex structures of Euclidean spaces are appropriate for singling out meaningful concepts. The resulting Voronoi tessellations depend on a rather special geometrical structure. This has the danger of generating structural artifacts that may lack empirical content. For instance, it may be difficult to justify empirically that a given conceptual space should be endowed with a specific metric and not with another one. For instance, the convexity of the Voronoi cells of a certain discretization of a conceptual space depends on a specific metric, the choice of a different metric would yield different cells. As another example of a problem generated by the too specific structure imposed on conceptual spaces is that problem that Douven et al. (2013) have called the problem of "thickness": Assuming a Euclidean structure of conceptual spaces entails a very specific structure of the boundaries of their concepts that is difficult to justify empirically. More precisely, for Euclidean conceptual spaces the boundaries of concepts suffer from "extensional anorexia", so to speak, in that they are overly thin (cf. Douven et al. (2013)). Topological tessellation do not suffer from this illness, since discretizations of conceptual spaces based on their topological structure generate much less representational artifacts than those that are based on a much richer geometrical structure. Topological structures are fundamental spatial structures, arguably even the most fundamental ones. Thus, if we agree with Gärdenfors' thesis (Gärdenfors (2000, 262) that "to understand the structure of our thoughts ... we should aim at unveiling our conceptual spaces", we should aim at understanding the topological structures of our conceptual spaces. The topology of Alexandroff spaces may be a useful tool for this purpose.

REFERENCES

Alexandroff, P., 1937, Diskrete Räume, Rec. Math. [Mat. Sbornik], N.S., 1937, Volume 2(44), Number 3, 501–519.

Bezhanishvili, G., Esakia, L, Gabelaia, D., 2004, Modal Logics of Submaximal and Nodec Spaces, Festschrift for Professor Dick de Jongh, Institute for Logic, Language and Computation, University of Amsterdam.

Bezhanishvili, G. Mines, R., Morandi, P.J., 2003, Scattered, Hausdorff-reducible, and hereditarily irresolvable Spaces, in Topologiy and its Applications 132, 291 – 306.

Borceux, F., 1994 (2008), Handbook of Categorical Algebra 3. Categories of Sheaves, Cambridge, Cambridge University Press.

Decock, L., Douven, I., 2015, Conceptual Spaces as Philosophers' Tools, in Zenker, F, Gärdenfors, P. 2000, Applications of Conceptual Spaces, The Case for Geometric Knowledge Representation (eds.), Springer, 207 – 222.

Douven, I., Decock, L., Dietz, R., Égré, P., 2013, Vagueness: A Conceptual Spaces Approach, Journal of Philosophical Logic 42, 137 – 160.

Goubault-Larrecq, J., 2013, Non-Hausdorff Topology and Domain Theory. Selected Topics in Point-Set Theory, Cambridge, Cambridge University Press.

Gärdenfors, P., 2000, Conceptual Spaces, MIT Press, A Bradford Book.

Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M., Scott, D.S., 2003, Continuous Lattice and Domains, Cambridge, Cambridge University Press.

Johnstone, P., 1982, Stone Spaces, Cambridge, Cambridge University Press.

Keefe, R., Smith, P.,1996, Vagueness: A Reader (edss) Cambridge, MA: MIT Press.

Kuratowski, K., Mostowski, A., 1976, Set Theory. With an Introduction to Descriptive Set Theory, Amsterdam, North-Holland Publishing Company.

Mahdi, H.B., El Atrash M.S., 2005, ON T_0–Alexandroff Spaces, Journal of The Islamic University (S. N Sci. &E) Vol.13, No.2 (2005), 19-46.

Mahdi, H., Nada, S., Muamar, R., 2015, On Alexandroff Shadow Spaces, British Journal of Mathematics & Computer Science 7(6), 429 – 438.

Okabe, A., Boots, B., and Sugihara, K., 1992, Spatial Tessellations: Concepts and Applications of Voronoi Diagrams. New York, NY, John Wiley and Sons.

Picado, J. , Pultr, A., 2012, Frames and Locales, Topology without Points, Basel, Birkhäuser.

Rumfitt, I., 2015, The Boundary Stones of Thought. An Essay in the Philosophy of Logic, Oxford, Oxford University Press.

Sainsbury, R. M., 1996 (1990), Concepts without boundaries, In Keefe and Smith, eds., 1996, 251– 64.

Rose, D., Sizemore, K., and Thurston, B., 2006, Strongly Irresolvable Spaces, International Journal of Mathematics and Mathematical Science, Volume 2006, 1 – 12.

Sznajder, M., 2016, What conceptual spaces can do for Carnap's late inductive logic, Studies in History and Philosophy of Science, Series A, vol. 56, 62 – 71.

Zenker, F, Gärdenfors, P. 2015, Applications of Conceptual Spaces, The Case for Geometric Knowledge Representation (eds.), Springer.