Russell's Many Points

Thomas Mormann

Abstract. Bertrand Russell was one of the protagonists of the programme of reducing "disagreeable" concepts to philosophically more respectable ones. Throughout his life he was engaged in eliminating or paraphrasing away a copious variety of allegedly dubious concepts: propositions, definite descriptions, knowing subjects, and points, among others. The critical aim of this paper is to show that Russell's construction of points, which has been considered as a paradigm of a logical construction *überhaupt*, fails for principal mathematical reasons. Russell could have known this, if he had taken into account some pertinent results due to Hausdorff or Tarski. Its constructive aim is to show that one can save Russell's thesis – that points can be defined in terms of events or regions – by using the conceptual resources of modern pointless topology.

1. Points in Russell's Philosophy. Bertrand Russell was one of the protagonists of the programme of reducing "disagreeable" objects to philosophically more respectable ones. Throughout his life he was engaged in eliminating or paraphrasing away a copious variety of suspicious objects: propositions, definite descriptions, knowing subjects, and many others.

For Russell, logical analysis was the method of the new scientific philosophy to which he dedicated his philosophical career after his conversion from British Idealism during the last years of the 19th century. The aim of logical analysis was the elimination of suspicious or otherwise undesired entities from the discourse of scientific philosophy. In *Our Knowledge of the External World as a Field for Scientific Method in Philosophy* (Russell 1914, OKEW) Russell attempted to show

> "by means of examples, the nature, capacity, and limitations of the logical-analytic method in philosophy. …The central problem by which I have sought to illustrate method is the problem of the relation between the crude

> data of sense and the space, time and matter of mathematical physics." (OKEW, 10)

More precisely, Russell attempted to show that the basic mathematical structures of physical spacetime - conceived as structured sets of spatial points and temporal points - could be reconstructed from "crude sense data", later to be characterized as "events". He credited Whitehead with the basic ideas of this approach characterizing his own version as a "rough preliminary account":

> "I owe to Dr. Whitehead the definition of points, the suggestion for the treatment of instants and "things", and the whole conception of the world of physics as a *construction* rather than an *inference*. What is said on these topics here is, in fact, a rough preliminary account of the more precise results which he is giving in the fourth volume of our *Principia Mathematica.*" (OKEW 10, 11)

Actually, points played a role in Russell's thinking already before he had started his collaboration with Whitehead on *Principia Mathematica*. From his youthful *Essay on the Foundations of Geometry* (Russell 1897, EFG)[1] up to *My Philosophical Development* (1959) "points" were a recurrent theme in many of his writings. The most detailed account of spatial points can be found in *The Analysis of Matter* (Russell 1929), the last original work on matters of points (more precisely on instants) was the paper *On Order in Time* (Russell 1936, OT), but still in the retrospective *My Philosophical Development* (Russell 1959) he ascribed to the issue of points an important place in his philosophical development:

> "As regards points, instants, and particles, I was awakened from my "dogmatic slumber by Whitehead. Whitehead invented a method of constructing points, instants, and particles as sets of events, each of finite extent. This made it possible to use Occam's razor in physics in the same sort of way in which we had used it in arithmetic. I was delighted with this fresh application of the methods of mathematical logic." Russell (1959, 77).

[1] In the *Essay on the Foundations of Geometry*, still under the spell of Bradley's idealism, Russell conceived points as "contradictory" objects that could be used to argue for the "dialectical structure" of science according to which the most basic science (= geometry) was not an independent science but pointed to some "higher" science, i.e. physics: "The antinomy of the Point proves the relativity of space, and shows that Geometry must have some reference to matter … ." Russell (1897, 196 – 199).

Taking into account Russell's assertion that "the question of the construction of point-instants … was already very much in my mind in 1911" (Russell 1959, 121) one may say that Russell considered the topic of the logical construction of spatial points and temporal instants as an important philosophical topic for almost 50 years. Nevertheless, he never got it right. Nevertheless he was quite clear about the general idea of how the construction of points from less queer entities such as regions or events should be carried out. In *The Analysis of Matter* (Russell 1927) he compared his reductionist program with the standard approach of point set topology and differential geometry:

> "In *analysis situs* (= topology, T.M.) both points and neighborhoods are given. We, on the other hand, wish to define our points in terms of "events", where "events" will have a one-one correspondence with certain neighborhoods. … We have to assign to our events such properties as will enable us to define the points of a topological space as classes of events, and the neighborhoods of the points as classes of points. (Emphasis mine, T.M.) But we have to remember that we do not want to construct merely a topological space: what we want to construct is the four-dimensional space-time of the general theory of relativity." (Russell 1927, 298)

This constructional programme has a circular structure: we start with a certain set E of events that are assumed to have certain (relational) properties. Then we construct points as certain classes of events and the neighborhoods of these points as certain classes of points of classes of classes of events. Then we may consider the newly constructed neighborhoods of constructed points as events in their own right and consider them as building blocks for the construction of new "second order" points. For these second order points we may construct neighborhoods again, i.e., second order events, and so on. Continuing in this way things tend to be complicated, or so it seems. Therefore it seems advisable to put some constraints on this undesired profusion of higher order points and events. The simplest way to achieve this is to assume that iterating the construction of points and events from already constructed points and events does not yield anything new. That is to say, the classes of points and events of higher order are required to be isomorphic to the classes of points and events of first order. This is a reasonable assumption, since the constructions of points of modern pointless topology do satisfy it. Thus, from now on, this requirement of "stability" will be considered as a condition that every good construction programme has to satisfy.

One of the main negative results of this paper is that Russell's original construction is not stable, rather, it produces a diverging profusion of points and events of ever higher order.
Notwithstanding this flaw I think it is remarkable fact that already in 1927 Russell quite explicitly formulated here the programme of what later was to become "pointless topology", namely, "to assign to events such properties as will enable us to define the points of topological spaces as classes of events, …" (ibidem). Although Russell identified the task of that future discipline with admirable clarity, he did not much to accomplish it. He never offered any elaborated proposal of what the "properties of events" might be that would "enable us to define points in terms of events". Rather, he was content to conceive the class E of events – in modern terms – as a similarity structure (E, ~)[2], i.e. as a set E endowed with a binary relation ~ intuitively to be interpreted as non-trivial overlapping (see section 5). Consequently, Russell never accomplished the task of giving a correct definition of points in terms of events, to say nothing about the envisaged construction of the topological and differential structures of the space-time manifolds. In the following we leave aside the projected higher layers of this construction and concentrate on its most basic level – the construction of points and their neighborhoods.

Cast in terms of some appropriate set theory (instead of the framework of *Principia Mathematica* as Russell did), his plan for the construction of points (spatial, temporal, and spatio-temporal points) may be outlined as follows: One starts with a set E of events, whereby the concept of "event" is assumed to be a primitive term, i.e. there is no explicit definition of what an event is to be, rather, only some intuitive and informal hints are given. The reader may think of events as more or less well-formed regions of the Euclidean plane, or in the special case of purely temporal events, of intervals of the real line **R** endowed with its usual order structure. For the moment, he may take those regions as certain point sets, but it is important to keep in mind that in the end events are not to be thought as point sets – rather, points have to be constructed from events in such a way that events and their relations may be represented by appropriate point sets and set-theoretical relations.

Let pt(E) := {p; p is a point of E} denote the set of points to be constructed from E. For the moment, the only thing we know about points is that they are to be sets of events, i.e., the set pt(E) of points is a subset of PE, PE being the power set of E. In the next sections we are going to determine more precisely, what kind of subsets are points. At first look, this may look a bit involved, nevertheless the basic idea of constructing points as sets of events is intuitively appealing and simple. This is not to say, that the constructions to be carried out are simple. Actually they are not, and for reasons of space it is not possible to present all the concepts and arguments in full detail. The underlying mathematical facts may be succinctly described as follows. The general mathematical framework is provided by topology and the theory of lattices, more precisely by the theory of Heyting algebras. From a modern point of view, Russell's programme of defining points in terms of events is located in a conceptual space that is determined by the following facts:

> FACT1: Let H be a regular continuous Heyting algebra. The elements of H are to be intuitively conceived as "events" in the sense of Russell. Up to isomorphism there is a unique topological space (pt(H), O(pt(H)) such that the Heyting algebra O(pt(H)) of its open sets and H are isomorphic. Hence, without loss of generality one can assume that H is a set-theoretical Heyting algebra, i.e. its elements are subsets of pt(H), and the lattice-theoretical operations of H are just the familiar set-theoretical operations of union and intersection.
>
> FACT2: Russell's construction of ersatz points as maximal co-punctual subsets of H yields a profusion of points of which only a small minority corresponds to real points, i.e. to elements of pt(H). Hence, Russell's construction is not stable under interation.[3]
>
> FACT3. Russell's programme of defining "points in terms of events" can be saved by relying on some more sophisticated topological concepts of modern pointless topology. The most important device of this approach is the notion of the "interior parthood" relation <<. This relation is used to define maximal <u>round</u> filters as ersatz points for which a 1-1-correspondence with the real points of H, i.e. the elements of pt(H), can be established.

Before we go on some explanatory remarks on FACTS may be in order:

<u>Remark 1</u>: The definition of a regular continuous Heyting algebra encapsulates the "appropriate properties of our events" that allow us to "define points in terms of events".

---

[2] Or, in the case of purely temporal events, as an ordered similarity structure (E, ~, <).

[3] In fact, we will not rely on Russell's original construction but on a slightly improved variant. Even the improved construction founders, so Russell's original proposal would score even worse.

The explicit definition of this structure from scratch is rather complicated and would need several pages. Thus some hints to the literature must suffice. Davey and Priestley (1990) provide a useful introduction to general lattice theory. The basics of the theory of complete Heyting algebras may be found in chapter II of Johnstone (1982), for a rather exhaustive treatment of continuous Heyting algebras the reader may consult the authoritative treatise of Gierz et al. (2003).
Important examples of regular continuous Heyting algebras, which will be treated in some detail in the following sections, are the Heyting algebras OX of open sets of "nice" topological spaces (X, OX). Again, the exact definition of "niceness" used here is somewhat involved. Be it sufficient to state that Euclidean spaces belong to the class of nice spaces whose Heyting algebras of open sets are regular continuous Heyting algebras (cf. Gierz el al. 2003). On the other hand, the rational numbers **Q** endowed with their standard order do <u>not</u> form a continuous Heyting algebra.[4] For an informal account of Stone's mathematical achievements, see Piazza (1995).

<u>Remark 2</u>. In section 3 we give an elementary example that shows that Russell's construction yields too many ersatz points, i.e., at least some of the constructed points do not correspond to real points. This and other examples were known to mathematicians and logicians such as Hausdorff and Tarski, probably already to Cantor. Thus, Russell could have known that his method was doomed to fail if he had paid attention to the then contemporary mathematics.

<u>Remark 3</u>: The "new methods" alluded to in FACT3 belong to the realm of mathematics sometimes called "pointless topology". As a forerunner of this discipline one may consider Stone's work on the representation of Boolean algebras B by topologically defined subsets of their Stone (or Boolean) spaces St(B) (Stone (1936, 1938). For a brief but complete account of Stone's representation theorem see Davey and Priestley (1990); for some remarks on the history of the history of pointless topology see Johnstone (1982) and Gierz et al. (2003).

The outline of this paper is as follows: in the next section we discuss the basic properties of Russell's construction of points. For intuitive reasons, this construction may be dubbed the "onion construction"; mathematically, this it is characterized as the construction of maxi-

[4] Continuity as employed here is indeed a generalization of Dedekind completeness.

mal filters for appropriate structures.[5] Applying the axiom of choice allows us to define Russell points and Stone points. In section 3 we consider a very simple set-theoretical model of space and show that it has too many Stone points, and a fortiori, too many Russell points. Thus, Russell's programme is doomed to fail for principal reasons. In section 4 concepts of pointless topology are used to overcome the difficulties into which Russell's original approach gets entangled. In particular it is shown that the replacement of "round" filters for ordinary filters suffices to eliminate the profusion of ersatz points Russell's original construction was plagued with. Section 5 deals with the special case of instants, i.e. temporal points. We conclude with a general assessment of Russell reconstructional programme in section 6.

<u>2. The Onion Construction.</u> Let us take the Euclidean plane P as a typical example of a well-behaved topological space and take as the set E = E(P) of events an appropriate set of well-formed regions of the Euclidean plane. For the moment we need no worry about what precisely is meant by "well-formed region". The reader may think of parts of the plane with "nice" boundaries and without interior crackles and holes, for instances circles, ellipsoides, and similar figures. Intuitively, appropriate families of these regions may be arranged in such a manner that they form an "onion" or a nested system of neighborhoods such that the elements of this system "approximate" a point x of E that lies in the interior of all of them.

The system N(x) of nested neighborhoods of x can be taken as a representative of the point x, since at least for intuitively "nice" spaces such as Euclidean ones two different points x and y give rise to two different "onions" N(x) and N(y), respectively, since we may always find a neighborhood of x that does not contain y, and vice versa.

If we could characterize systems like N(x) independently from and without reference to the points x of which they are neighborhood systems we would have met Russell's challenge of "defining points in terms of events". More precisely, our task is, as Russell put it, "to assign to events such properties as will enable us to define the points of a topological space as classes of events" (ibidem). Russell's own proposal for achieving this task is somewhat complicated. In modern terms, it comes to something like this. The class of events E is considered as a set with a certain structure, to wit, some kind of a mereological or lattice-theoretical structure: given two events (regions) a and b, they may overlap, i.e. there may be a region $(a \wedge b)$ such that $(a \wedge b)$ is a part of the regions a and b, and every region c that

---

[5] Actually, Russell did not use filters but somewhat weaker structures that he called "co-punctual" sets, see sections 2 and 5.

is part of a and b has $(a \wedge b)$ as part. Of course, we have to require that the concept of overlapping is structurally well behaved. For instance, one should require that the overlapping $(a \wedge b)$ of a and b is the same as the overlapping $(b \wedge a)$ of b and a etc. In sum, the class of events should be conceived (at least) as a (semi)lattice $(E, \wedge, 0)$, 0 being the bottom element (cf. Davey and Priestley 1990, Chapter 3, 58). Then we can as usual define "overlapping" in terms of the lattice-theoretical operation by the stipulation that a and b overlap if and only if $(a \wedge b) \neq 0$.[6]

After these preparations we are now ready to approach Russell's proposals of defining ersatz points solely in terms of regions:

(2.1) Definition (Russell (1928, 299). Let E be the class of events. A set $F \subseteq E$ is called co-punctual if and only if every five regions $a_1, \ldots, a_5$ of F overlap:

$$a_1 \wedge \ldots \wedge a_5 \neq 0.$$

A Russell point is a maximal co-punctual subset F of E, i.e. a subset that cannot be enlarged without ceasing to be co-punctual.♦

This definition is the very core of Russell's programme of reducing spatio-temporal points to more respectable entities such as events or regions. Thus, some explanatory remarks may be in order.

The first question that probably comes to mind is whether there is any deeper reason why co-punctional F are defined via the non-trivial overlapping of five regions - why not two, three, or seventeen? Obviously one can define "co-punctuality" with respect to every n, $n \geq 1$. The answer is that Russell wanted to reconstruct the points of the four-dimensional space-time manifold of relativity theory. For some not very clear reasons he believed that for the construction of an n-dimensional space one needed co-punctuality with respect to (n+1) regions. Indeed, he explicitly asserted that for ordinary three-dimensional space one needed co-punctuality with respect to four regions, and for the reconstruction of temporal points (instants) of the one-dimensional time manifold he required co-punctuality with respect to two events (intervals), see Russell (1936) and section 5.

[6] Russell did not traffick with these technicalities but took them for granted.

Secondly, an interesting problem arises with respect to maximal co-punctual subsets of E, i.e. Russell points. How do we know that they exist? Can we construct them in some explicit sense? Russell was well aware that for the construction of maximal co-punctual subsets F of E the axiom of choice or a similar principle was necessary. Points are not for free, rather, one has to rely on a logical (or set-theoretical) principle that is far from trivial. As it turned out, even the modern "correct" construction of points has to use such a principle. On the other hand, the first part of Russell's proposal, to wit, the requirement of "co-punctuality", turned out to have been a less clever idea. As will be shown in the following, the co-punctuality approach is doomed to fail from the outset, regardless of the number of co-punctual regions one requires. As we will show Russell's construction yields too many points.

The details are as follows. Let S be any well behaved space, e.g. Euclidean space. Assume that S has points in the usual sense, i.e., S is as a point set endowed with some further geometrical or topological structure. Defining the regions of S as a set of well behaved point sets we may then attempt to apply Russell's recipe of (re)constructing the points of S as sets of maximal co-punctual regions. The result turns out to be a failure since there are many more Russell points than real points, i.e., elements of S. Describing the outcome as "many more ersatz points than real points" is to put it mildly. Actually, the cardinality of ersatz points is the cardinality of the power set of the power set of "real" points! Thus, Russell's construction is quite off the mark as it is much too prolific in generating ersatz points. The first step to prove this is the following definition:

(2.2) Definition. Let (E, $\wedge$, 0) be a lattice. A Stone point of E is a subset F of E satisfying the following conditions:

(1) $0 \notin F \neq \emptyset$.

(2) If $a, b \in F$, then $a \wedge b \in F$.

(3) If $a \in F$ and $a \leq b$, then $b \in F$.

(4) F is maximal with respect to (1) – (3), i.e. if $x \notin F$, there is an $a \in F$ such that $a \wedge x = 0$.♦

Defining as usual $a \leq b := a \wedge b = a$ one may succinctly characterize a Stone point as a maximal upper subset of E closed under finite intersections. Usually a “Stone point” of E is called maximal filter or ultrafilter of E.[7]

Filters are quite common mathematical objects considered in many realms of logic, lattice theory, topology and elsewhere. The following elementary lemma shows that for our purposes we may replace the Russell points by the more manageable Stone points:

(2.3) Lemma. Let $(E, \wedge, 0)$ be a lattice. Then every Stone point of E is a Russell point of E.

Proof: Let F be a Stone point of E. Let $a_1, \ldots, a_5$ be five regions of F. By the very definition of a Stone point $(a_1 \wedge a_2)$ is a region that belongs to F. Hence point $(a_1 \wedge a_{2)}) \wedge a_3)$ is an element of F. Iterating this argument, one finally obtains that the regions $a_1, \ldots, a_5$ are co-punctual in the sense of Russell. In order to show that F is a maximal set of copunctual elements we proceed by reductio. Assume that F is not a maximal co-punctual subset of E. Then there is a maximal co-punctual set G that properly contains F. Moreover, there is a region x in G that is not a region of F. Since F is a maximal Stone point, there is a y in F such that $x \wedge y = 0$. Hence $x \wedge y \wedge y \wedge y \wedge y = 0$. Hence G is not a maximal co-punctual set in the sense of Russell. This is a contradiction. Thus already F is a maximal co-punctual set, i.e. a Russell point.

In the next few sections we will forget about Russell points and concentrate on the mathematically better behaved Stone points. Before we deal with the technicalities it may be helpful to point out that the neighborhood systems of points of "nice" spaces define Stone points in a natural way. In some more detail, this may be spelt out as follows.[8] Let (X, OX) be a topological space, i.e. a set X endowed with a topological structure OX $\subseteq$ PX. The elements of OX are called the open sets of X. A $\in$ OX with x $\in$ A is called an open neighborhood of x. The system N(x) of all open neighborhoods of x is a filter, i.e. N(x) satisfies the following conditions:

(1) $\varnothing \notin N(x) \neq \varnothing$.

(2) $A \in N(x)$ & $A \subseteq B \Rightarrow B \in N(x)$.

(3) $A, B \in N(x) \Rightarrow A \cap B \in N(x)$.

Invoking the axiom of choice it is easily proved that there exists a maximal filter F(x) containing N(x).[9] Then F(x) satisfies the further maximality condition:

(4) $C \notin F(x) \Rightarrow A \cap C = \varnothing$ for some $A \in F(x)$.

For "nice" topological spaces two different points x and y have neighborhood filters N(x) and N(y) that contain neighborhood A(x) and A(y) of x and y, respectively, such that the intersection A(x) $\cap$ A(y) is empty. Then one can easily prove that x and y define different Stone points F(x) and F(y), i.e. maximal filters that contain N(x) and N(y), respectively, are different. "Nice" spaces in this sense are Euclidean spaces, and more generally, Hausdorff spaces.

In the following section we show that nice spaces have more Stone points than real points. More precisely, we will show that for a space X every real points x defines a of Stone point F(x), but there is a wealth of undesired Stone points that do not correspond to any real points. Actually, Stone points related to real points turn out to be an exception. Since all

Stone space or the Boolean space of E (cf. Davey and Priestley 1990, Chapter 10, 197).

[8] For a succinct presentation of the basic topological concepts used in this paper, the reader may consult the Appendix of Davey and Priestley (1990) or any textbook of topology.

[9] Of course, there is no reason to expect that F(x) is unique or that already N(x) is maximal.

Stone points are Russell points, this implies that Russell's original constructional programme is flawed.

3. A Minimalist Model of Space. In this section we consider a simple set-theoretical model of space and show that it has too many Stone points, and a fortiori, too many Russell points. In later sections it will be shown that this simple model does indeed reflect the essential mechanisms due to which every construction based on Stone or Russell points must fail. Moreover, we show how to modify the construction of points such that the profusion of ersatz points is cut down in such a way that there can be established a 1-1 correspondence between modified Stone points and "real" points.

Let X be a set, and let us regard X as a topological space (X, OX) by endowing it with the discrete topology OX = PX. (X, PX) is called a discrete topological space. With respect to this special topological structure just any $Y \subseteq X$ with $x \in Y$ is an open neighborhood of x. Moreover, the system N(x) of all open neighborhoods of x is the set of all subsets of X containing x. This implies that in this case the system N(x) has the following properties:

(1) $\emptyset \notin N(x) \neq \emptyset$.

(2) $A \in N(x) \ \& \ A \subseteq B \Rightarrow B \in N(x)$.

(3) $A, B \in N(x) \Rightarrow A \cap B \in N(x)$.

(4) $C \notin N(x) \Rightarrow A \cap C = \emptyset$ for some $A \in N(x)$.

Since $\{x\} \in N(x)$ one has $C \in N(x) \Leftrightarrow x \in C$. In other words, for every real point $x \in A$ the neighborhood system N(x) is a Stone point. Moreover two real points x and y are equal iff the neighborhood systems N(x) and N(y) are equal. Hence for (X, PX) the neighborhood systems N(x) are faithful representatives for real points $x \in X$. Since neighborhood systems are maximal filters on PX, and maximal filters on PX can be characterized without explicit reference to points we seem to have made great progress on our way of defining points as systems of events or regions – at least in the special case of X being a discrete topological space (X, PX). There is missing only one piece: Although we know that every "real" point $x \in X$ defines a unique Stone point N(x), it might be that there are still other maximal

filters that do not correspond to any real point x. Russell assumed without argument that this is not the case. As it seems, he took it for granted that all maximal filters on a set X arise as the neighborhood system N(x) of some point x of X. Actually, this assumption is true only for X with finitely many elements. For infinite sets X it is false. For them there are many more abstract neighborhood systems N than real points.
It would take us too long to prove this result in full generality, instead, let us prove that an infinite set X has at least one Stone point that does not correspond to any real point x of X.

(3.1) Proposition . Let X be a set with infinitely many elements, and define F to be the set of subsets Y of X whose complements consist of finitely many elements:

$$F := \{Y;\ Y \subseteq X \text{ and } \mathbf{C}F \text{ finite}\}$$

The there is a Stone point N(F) that contains F but there is no element $x \in X$ such that N(F) = N(x).

Proof. Clearly F is non-empty since $X \in F$ and $\emptyset \notin F$ since X is infinite. Since the finite union of finite sets is finite, F is closed with respect to finite intersections. Moreover F is upward closed. In other words, F is a filter. Invoking the axiom of choice or a similar principle one infers that there is a maximal filter N(F) containing F. But N(F) cannot come from some principal filter N(x). If this were the case, x would be an element of all elements of N(F), a fortiori it would be an element of all elements of F. But clearly $X - \{x\}$ is an element of F that does not contain x. Hence there is at least one abstract neighborhood system that does not correspond to a real point $x \in X$.♦

One might hope that the maximal filter N(F) is somehow an exception and that “usually” a Stone point of X comes from some real point of X. This hope is shattered by the following classical result due to Hausdorff and Tarski:

(3.2) Proposition. Let X be an infinite set with cardinality <X>. Then the set of Stone points St(X) of X has the cardinality of the power set of the power set of X, i.e. <St(X)> = <PPX>. In other words, there are many more Stone points than real points.

Proof. Bell and Slomson (2006, 108, Theorem 1.5).♦

Remark. The proof of (3.2) is somewhat complicated but elementary in the sense that it can be carried out using only the concepts that are already available to us. Its essential ingredient is, of course, the axiom of choice. Without it, or some similar principle, one cannot show the existence of even one maximal filter. This was known already to Russell (cf. Russell (1936), Anderson (1986)). More generally, without the axiom of choice there is no modern set theoretical topology as we know it.

One may object that (3.2) can hardly count as a refutation of a Russellian programme of constructing points from events since the geometrical structure of X, i.e. OX = PX, is just too meager as though it could capture the relevant features of the topological relations between "points" and "neighborhoods" that Russell wanted to put to use. Prima facie, this suspicion is not unreasonable; nevertheless it turns out to be wrong in the end. More precisely it can be shown that for every reasonable space X (in particular for the Euclidean space E) and every reasonable set of events E(X) of X an analogous theorem to (3.2) holds according to which the cardinality of Stone points is much larger than the cardinality of real points of X:

(3.3) Proposition (Theorem of Balcar-Franêk). Let B be an infinite complete Boolean algebra. Then the cardinality of the set of maximal filters on B is the cardinality of the power set PB of B.

Proof. Koppelberg (1989, Theorem 13.6, 197).♦

Now, every maximal filter on O*X gives rise to at least one maximal filter on OX, since the inclusion O*X-----i----->OX is a ∧-monomorphism. This implies that there are at least as many maximal filters on OX as maximal filters on O*X. Hence, Russell's construction yields too many points not just for the trivial discrete topological structure (X, PX) but for every non-finite topological space (X, OX). In other words, our minimalist set theoretical model of space cannot be blamed for the failure of Russell's programme. Rather, as will turn out in the next section, the culprit for the profusion of undesired ersatz points is the too primitive "non-topological" notion of a filter we employed. This shortcoming will be overcome in the next section.

4. Topology to the Rescue. Let us take stock what we have achieved so far, and what remains to be done. Given a topological space (X, OX) with a system of regions OX we did the first step to realize Russell's program of "defining points in terms of events", namely, improving upon Russell's own account of ersatz points as maximal copunctual sets of regions, we defined ersatz points as maximal filters of X. In a sense, this works quite well – for every real point $x \in X$ we can construct an ersatz point. In the case of OX = PX this ersatz point is even unique.[10] The only flaw in this construction is that it yields much too many ersatz points.

In this situation it is natural to attempt to restrict somehow the profusion of ersatz points by singling out a small class of particularly "nice" ersatz points. This is indeed possible, if we put to use the topological structure of the space X. It is here, where we encounter something new that has no counterpart in Russell's original attempt to "define points in terms of events".

In order to use the topological structure of a space as a tool for cutting down the profusion of ersatz points we have to delve somewhat deeper into the details of topology. Recall that given a topological space (X, OX) the closed subsets of X are defined as the set-theoretical complements of the open sets. Every subset Y of X there is the smallest closed subset cl(Y) containing Y, where cl(Y) is defined as the intersection of all closed subsets containing Y. This defines an operator PX---->PX mapping each subset Y to its closed hull cl(Y). Complementarily the open kernel int(Y) of a set Y is defined by int(Y) := Ccl(CY), C being the set-theoretical complement of Y in X. A set Z is called regular open if and only if Z = int(cl(Z)). If Y is any subset of X, an open covering of Y is a family U_i of open subsets of X such that $Y \subseteq \cup U_i$. The covering is finite if and only if it consists of finitely many elements U_i. A subset Y is compact if and only if every open covering contains a finite subcovering. Now we are ready to define the crucial concept of interior parthood:

(4.1) Definition. Let a, b open sets of the topological space (X, OX). The region a is an interior part of the region b, a << b, if and only if the closure cl(a) of a is contained in b, and cl(a) is compact:

$$a << b := \text{cl}(a) \subseteq b \text{ and cl}(a) \text{ is compact.}♦$$

Intuitively, the relation a << b is to convey the meaning that the region a plus its boundary bd(a) is fully contained in b, and moreover, that a is somehow "small". Compactness may indeed be interpreted as a topological version of "smallness" or even "finiteness" as the following topological analogue of the finite/infinite filter (see Proposition (3.1)) shows:

(4.2) Example (A topological version of the Finite/Infinite Filter). Let (X, OX) be a nice non-compact space, for instance the real line **R** or an Euclidean space. Define the filter F as a maximal filter that contains all sets of the form X – K, K ⊆ compact. Then it is easily seen that F cannot contain the open neighborhood filter N(x) of some x ∈ X: if x and y are two different points of X, one can show that x has an open neighborhood U(x) such that cl(U(x)) is compact and does not contain y. This proves that there is a Stone point of (X, OX) that does not correspond to any real point x ∈ X. In other words, Russell's construction yields too many points.♦

Now we engage in the constructive work of showing that the concept of interior parthood may be used to single out a special class of filters as follows:

(4.3) Definition . Let (X, OX) be a topological space with interior parthood relation <<.
(a) A subset F ⊆ OX is a round filter (with respect to <<) if and only if F is a filter and satisfies the further condition that for all b ∈ F there is an a ∈ F with a << b.[11]
(b) A maximal round filter F is called a (round) ersatz point.♦

---

[10] There is no reason to expect that in the general case each real point x gives rise to a unique Stone (or Russell) point. In section 5 we will prove this for the 1-dimensional topological space **R**. As is seems, this idea never occurred to Russell. Even Anderson in his (Anderson 1986) seems to be aware of this possibility.

As is easily shown by Zorn's lemma, round ersatz points exist. Moreover, under some mild conditions on the topology of X the filters N(x) of open neighborhoods of real points x are round filters. Indeed, the class of maximal round filters is the class of ersatz points we were looking for. Let us first deal with the special case (X, PX):

(4.4) Proposition. Let (X, PX) be a discrete topological space. Then the real points of X are in a 1-1-correspondence with the maximal round filters of X.

Proof: First let us prove that for (X, PX) a << b if and only if a is a finite subset of b. Assume that a << b. By definition this means that cl(a) a is a compact subset of b. But cl(a) = a, since the topology of X is trivial; using this fact once again, we observe that every singleton {x} is open. Hence, a set is compact if and only if it has only finitely many elements. By definition a round maximal filter F must contain a finite set. But then it is easily seen that in order to be maximal F must even contain a singleton {x}. Hence it is the principal filter F(x) generated by x. The other direction is trivial.♦

In other words, in the case of a discrete topological space (X, PX) the topologically defined round maximal filters correspond in a 1-1-fashion to the real points of X.
This result not only holds for discrete topological spaces (X, PX), but for a large class of "nice" topological space (X, OX). The task of defining exactly what is meant by "nice", would lead us to far away.[12] Be it sufficient to state that the class of nice spaces comprises the class of metrical spaces, in particular the Euclidean spaces:

(4.5) Proposition. Let (X, OX) be a nice topological space with interior parthood relation <<. Then there is a 1-1-correspondence between the real points x ∈ X and the round ersatz points of X, i.e., the maximal <<-round filters on OX.

Proof (Sketch). The proof of this proposition naturally falls into two parts. First, one has to show that each real point gives rise to a uniquely defined maximal round filter, secondly

[11] For a detailed discussion of "round filters" see Gierz and Keimel (1981) and Gierz et al. (2003).
[12] For the cognoscenti: locally compact regular Hausdorff spaces will do as "nice" spaces.

one has to show that each maximal round filter comes from a uniquely defined real point. Let x be a real point of X. Then we can consider the filter R(x) of regular open neighborhoods of x. Due to niceness, R(x) is easily shown to be a round filter with respect to the interior parthood relation <<. By the axiom of choice there is a maximal round filter N(x) containing x. Again by niceness (Hausdorff property) for two different points x and y maximal round filters N(x) and N(y) must be different. Hence, if X is nice, every real point gives rise to (at least) one ersatz point such that the ersatz points are different for different points.

It remains to show that every maximal round filter corresponds exactly to one real point. Assume that N is a maximal round filter. Consider the intersection $\cap cl(Y_i)$ of the closed hulls $cl(Y_i)$ of all elements of N. Since N is round, we may assume that X is compact without loss of generality. The set $\cap cl(Y_i)$ is non-empty, since otherwise there would be a finite family of sets $cl(Y_i)$ with empty intersection. This is impossible, since N is a filter. Choose a point $x \in \cap cl(Y_i)$. Since for every Y there is a Y' with $cl(Y') \subseteq Y$ we obtain $x \in Y$ for every element of N, i.e. x is an element of every element of the filter N. In other words, N is contained in the neighborhood filter of x. In other words, the neighborhood filter N(x) of x is the only maximal round filter containing x. This establishes a 1-1-correspondence between real points and maximal round filters as ersatz points.♦

This is an important step on the path of realizing Russell's programme of defining "points by events". In should be noted, however, that we yet we have not accomplished our final goal. Up to now, we have assumed that the events we are employ for the construction of ersatz points are point sets, namely, the open sets of a topological structure OX defined on the set X. In other words, in the definition of ersatz points as maximal round filters the real points of X still play an essential role, or so it seems. Actually this is not the case. But it needs some extra effort to show this.

Roughly, it works like this: up to now we started from a topological space (X, OX) endowed with an interior parthood relation << and constructed the ersatz points as maximal round filters. To get rid of the points, one forgets about the set X and retains only the topological structure OX, conceived not as a set of open sets, but as an abstract lattice. As is well known, the lattice OX is a complete Heyting algebra. For topologically nice spaces the lattice OX is even a regular continuous Heyting algebra (cf. Johnstone (1982), Gierz et al. (2003)). The point is that these structures can be defined without reference to points, i.e. the elements of such algebras need not be conceived of as point sets. Moreover, every regular continuous Heyting algebra H comes along with a binary relation << (the way below relation) that may be interpreted as an interior parthood relation. Then one can define the set pt(H) of maximal round filters of H as ersatz points of H. Next one shows that the set pt(H) can be endowed with a topology O(pt(H)) such that (pt(H), O(pt(H)) is a topological space. Finally it is proved that O(pt(H)) and H are naturally isomorphic as Heyting algebras. Thus one has obtained a <u>stable</u> (see section 1) method of defining points in terms of events: starting with the algebra H of events, one constructs the topological space (pt(H), O(pt(H)), taking O(pt(H)) as a system of events in its own right one constructs a set pt(O(pt(H)) of points of second order, for which one can construct neighborhoods O(pt(O(pt(H)) etc. But fortunately these iterations yield nothing new, due to the isomorphism between H and O(pt(H)).

Conceiving H as the class of events this construction may be considered as a mathematically rigorous reconstruction of Russell's programme of "defining points in terms of events" that he envisaged some 80 years ago in *The Analysis of Matter* (Russell 1927) or even earlier in *Our Knowledge of the External World* (Russell 1914). Conceptually, this construction is not too difficult to understand – it amounts to the onion construction discussed in section 2, but technically it is somewhat involved. Hence, the details cannot be given here, see Gierz et al. (2002), Johnstone (1982), Mormann (1997).

Russell never imagined the technical complexity of this endeavor, since he never cared about the details of the relational structure of the set E of events. He simply assumed that the class E of events was endowed with some binary relation of overlapping whose structural properties he never specified. In contrast, the definition of a continuous Heyting algebra, due to Scott (1972) is rather involved.

Finally, it should be noted that our reconstruction only accomplishes a part of Russell's programme: we only defined the topological structure "in terms of events" while Russell planned to reconstruct not only that structure but also the geometrical and differential structure of the spacetime manifold as well.

5. Conclusion. We are left with a mixed assessment of Russell's programme of defining points in terms of regions. On the one hand, it is clear that Russell did not have the technical skills to realize this programme in a mathematically satisfying and rigorous manner. With respect to higher dimensional spaces he never got beyond some informal sketches that might be intuitively appealing but that did not hit upon the conceptual essence of the matter. His attempted construction of instants, i.e. temporal points, scores somewhat better, but this is a special case that depends on the linear order of time, and therefore cannot be applied to the case of general topological spaces. On the other hand, Russell had a surprisingly clear vision of the general task of what today is called "pointless topology".

It was the American mathematician M.H. Stone, who laid the foundations for this discipline in the thirties. Stone's famous representation theorem of Boolean algebras may be considered as the first successful example of constructing points for something (Boolean algebras) that at first view exhibited no spatial features at all. According to the experts Stone's theorem is one of the most important theorems of 20th century mathematics that has influenced an ever-growing variety of mathematical disciplines (cf. Johnstone 1982). Unfortunately, the spaces that Stone constructed as representations for Boolean algebras in this way were quite remote from any intuitive interpretation. In particular, Stone's spaces were quite different from Euclidean and other familiar spaces. Hence, his work, although highly appreciated by mathematicians and logicians (for instance by Tarski), remained virtually unknown to philosophers during the following decades.[13] In particular, Russell never took notice of Stone's work, although this would have helped him a lot in the task of coming to terms with the task of "defining points in terms of events". In philosophy, Russell's topological sketches were received with respect. But there were not many attempts to develop them further. Consequently, up to how, topology - to say nothing of pointless topology - can hardly be said to belong to a philosopher's toolkit even when he is accustomed to use formal methods. If Russell's reconstructional programme, in particular his attempt of developing pointless topology, had been taken up more seriously by his fellow philosophers, topology might have played a major role - to the benefit of philosophical disciplines such as metaphysics, epistemology and ontology.

---

[13] The collection *Topology for Philosophers* (Smith and Zelaniec (eds.) 1996) still provides ample evidence for this claim: Stone's work is not mentioned even once.

In sum, then, I'd contend that Russell has had a point with his early programme of pointless topology, despite its mathematical flaws.

References:

Anderson, C.A., 1989, Russell on Order in Time, Minnesota Studies in the Philosophy of Science XII, Rereading Russell: Essays in Bertrand Russell's Metaphysics and Epistemology, edited by C.W. Savage and C.A. Anderson, 249 – 263.

Bell, J.L., Slomson, A.B., 2006 (1969), Models and Ultraproducts, New York, Dover.

van Bentham, J.F.A.K., 1983, The Logic of Time. A Model-theoretic Investigation into the Varieties of Temporal Ontology and Temporal Discourse, Synthese Library vol. 156, Dordrecht, Reidel.

Demopoulos, W., 2003, Russell's Structuralism, in N. Griffin (ed.), The Cambridge Companion to Bertrand Russell, 393 – 419.

Gierz, G., Keimel, K., 1981, Continuous Ideal Completions and Compactifications, in Lecture Notes in Mathematics 871, Berlin and Heidelberg, Springer, 97 – 124.

Gierz, G., Hofmann, K.H., Keimel, K., Lcs 0 [illegible]

Mormann, T., 1994, A Representational Reconstruction of Carnap's Quasianalysis, PSA 1994, Volume 1, 96-104.

Mormann, T., 1998, Continuous Lattices and Whiteheadian Theory of Space, Logic and Logical Philosophy 6, 35 - 54.

Mormann, T., 2005, Description, Construction, and Representation: From Russell and Carnap to Stone, in G. Imaguire and B. Linsky (eds.), On Denoting 1905-2005, München, Philosophia Verlag, 333-360.

Mac Lane, S., Moerdijk, I., 1992, Sheaves in Geometry and Logic. A First Introduction to Topos Theory, Berlin and New York, Springer.

Mormann, T., 2009, New Work for Carnap's Quasi-analysis, The Journal of Philosophical Logic 48, online publication January 2009.

Piazza, M., 1995, "One must always topologize": Il teorema di Stone, la 'topologia influente' e l'epistemologia matematica, Rivista di storia della scienza (ser. II) 4, 1-24.

Russell, B, 1896 (1996), An Essay on the Foundations of Geometry, London, Routledge.

Russell, B., 1914 (1995), Our Knowledge of the External World as Field for Scientific Method in Philosophy, London, Routledge.

Russell, B., 1927 (2001), The Analysis of Matter, London, Routledge.

Russell, B., 1936 (1994), On Order in Time, in B. Russell, Logic and Knowledge, London, Routledge, 347 – 363.

Russell, B., 1959, My Philosophical Development, London, Routledge.

Scott, D.S., 1972, Continuous Lattices, in Toposes, Algebraic Geometry, and Logic, Lecture Notes in Mathematics 274, Berlin and Heidelberg, Springer, 97- 136.

Smith, B., Zelaniec, W., 1996, Topology of Philosophers (eds.), The Monist 79(1), LaSalle Illinois, The Hegeler Institute.

Stone, M. H., 1936, The theory of representations for Boolean algebras, Transactions of the American Mathematical Society 40, 47 - 111.

Stone, M.H., 1938, The Representation of Boolean Algebras, Bulletin of the American Mathematical Society 44, 807 - 816.

Thomason, S.K., 1984, On Constructing Instants from Events, Journal of Philosophical Logic 13, 85 – 96.

van de Vel, M.L.J., 1993, Theory of Convex Structures, Amsterdam, North Holland.