Similarity and Continuous Quality Distributions

In the philosophy of the analytical tradition, set theory and formal logic are familiar formal tools. I think there is no deep reason why the philosopher's tool kit should be restricted to just these theories. It might well be the case—to generalize a dictum of Suppes concerning philosophy of science—that the appropriate formal device for doing philosophy is mathematics in general; it may be set theory, algebra, topology, or any other realm of mathematics. In this paper I want to employ elementary topological considerations to shed new light on the intricate problem of the relation of qualities and similarity.[1] Thereby I want to make plausible the general thesis that topology might be a useful device for matters epistemological.

1. Introduction

The idea of defining qualities by means of a similarity relation between particulars lies at the heart of Carnap's quasianalysis. Goodman launched an apparently devastating attack against Carnap's approach.[2] In this paper I want to show that the sweeping statement that any attempt to define qualities via similarity is doomed to fail has been pronounced too hastily. For this purpose I will rely on concepts and ideas borrowed from topology.

The outline of the paper is as follows: in section 2 we'll briefly recall the basic definitions of topology necessary for what follows. In section 3 we'll show that the relations between similarity and qualities may be conceived of as maps (to be called quality distributions) between certain sets. In section 4 these sets are endowed with natural topologies. This enables us to drastically reduce the apparent redundance of possible mapping relations between similarity relations and qualities. Sometimes we even achieve uniqueness. This amounts to what may be called a topological definition of qualities via similarity.[3]

2. Closure operators and continuity

To fix notation and terminology we recall the definitions of the topological concepts we will need in the following.[4]

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(2.1). Definition. Let X be a set, $P(X)$ its power set. A topological closure operator on X is an operator cl: $P(X) \rightarrow P(X)$ with the following properties ($Y, Y' \subseteq X$):

(CL1)	$Y \subseteq cl(Y)$	(Reflexivity)
(CL2)	$cl(cl(Y)) = cl(Y)$	(Transitivity)
(CL3)	If $Y \subseteq Y'$ then $cl(Y) \subseteq cl(Y')$	(Monotony)
(CL4)	$cl(Y \cup Y') = cl(Y) \cup cl(Y')$	

A set $Y \subseteq X$ is called closed (with respect to cl) iff it is invariant with respect to cl, i.e., iff $cl(Y) = Y$. A set $Y \subseteq X$ is called open (with respect to cl) iff it is the set theoretical complement of a closed set.[5]

Topologizing a set X allows us to classify the subsets of X in a variety of ways. We may distinguish between "natural" and "non-natural" ones. The former are topologically well-behaved or "nice" sets, while the latter are "topologically wild" subsets of X.[6] However, the definition of topological structures and closure operators on a set X can hardly be considered as a goal in itself. Their *raison d'être* is that they allow a precise definition of the concept of continuous maps. Among the various equivalent definitions of continuity we choose the following:

(2.2) Definition. Let X and Y be topological spaces. A map f: $X \rightarrow Y$ is continuous if and only if the induced map f^{-1}: $P(Y) \rightarrow P(X)$ maps closed subsets of Y onto closed subsets of X.

It is the concept of continuity that will be crucial for the approach to be developed in this paper. The following lemma is well-known:

(2.3) Lemma. Let X, Y, and Z be topological spaces and f: $X \rightarrow Y$, g: $Y \rightarrow Z$ continuous maps. Then the concatenation $g \bullet f$: $X \rightarrow Z$ is continuous.

3. Qualities and similarity

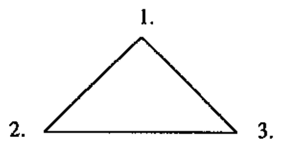
In this section I want to show how the topology may be used to shed new light on the intricate relation between the concepts of similarity and qualities. As a point of departure (and as a target) I take Goodman's negative claim that "[S]imilarity between particulars does not suffice to define qualities."[7] The argument Goodman is attacking here runs as follows: at first sight one might think that several particulars all similar to each other, have to share some common quality or other. If this were the case qualities could be identified with the most comprehensive classes of particulars which are all similar to each other. The following example shows that this argument is not sound:

> Suppose .... we have three discs, the first one half red and half blue, the second one half blue and half yellow, and the third one half yellow and half red:
>
> | rb | by | yr |
> |---|---|---|
> | 1 | 2 | 3 |

> Each two of the three discs have a color in common, but there is no color common to all of them. Dyadic likeness between particulars will not serve to define those classes of particulars that have a common quality throughout.[8]

As far as it goes this argument is correct. I do not think, however, that it suffices to dismiss once and for all the possibility that there might be a kind of defining relation between likeness and qualities.

To argue against Goodman's negative assessment, first we have to introduce some formal machinery. We assume throughout that the set S of particulars we are dealing with is finite. Hence S may be taken to be the set of natural numbers $\{1, 2, ..., n\}$. A similarity relation is a reflexive and symmetric, but not necessarily transitive, relation $\sim \subseteq S \times S$. Then a similarity structure, denoted by $(S,\sim)$, is a set S endowed with a similarity relation $\sim$. A similarity structure $(S,\sim)$ can be conveniently represented by a (numbered) graph such that two different elements of S form an edge iff they are similar. In this way, Goodman's example discussed above corresponds to the following triangle:



In order to topologize the problem of how similarity and qualities are related, I intend to conceive the attribution of qualities to particulars, as a map whose domain is the set of particulars and whose codomain is a set of sets of qualities. Of course, not just any map will do. The following definition intends to capture what may be considered as a minimal requirement:

(3.2) Definition. Let $(S,\sim)$ be a similarity structure, and Q be a set whose elements are to be interpreted as qualities. A quality distribution is a map $f: S \rightarrow P(Q)$ that satisfies the requirement $s \sim s' \Leftrightarrow f(s) \cap f(s') \neq \emptyset$.

(3.2) renders Goodman's informal requirement that "a relation of likeness obtains between two particulars if and only if they share at least one among certain qualities"[9] a structural constraint on the map f. It can be cast in an even more convenient form if we observe that the set $P(Q)$ is a (rather special) similarity structure itself. Define a similarity relation on $P(Q)$ in the following way: the empty set is similar only to itself, two non-empty sets P, P' are similar iff they have a non-empty intersection, i.e., $f(s) \cap f(s') \neq \emptyset$.

(3.3) Corollary. Let $(S,\sim)$ be a similarity structure, and Q be a set of qualities. A quality distribution in the sense of (3.2) is a structure-preserving map $f: S \rightarrow P(Q)$ between similarity structures, i.e., for all $s,s' \in S$ we have $s \sim s' \Leftrightarrow f(s) \sim f(s')$.

If we could define qualities by similarity alone this would mean that, given the similarity structure $(S,\sim)$, there is only one structure-preserving map $f: S \rightarrow P(Q)$ in the sense of (3.3). As Goodman's bicolored disks show, this is not the case. For one and the same similarity structure $(S,\sim)$ there might exist several quality distributions. Does this mean, as Goodman wants to make us believe, that there is no general relation between similarity and qualities? I don't think so. It might well be the case that among the different quality distributions that fit a given similarity structure we might be able to establish a ranking according to which some distributions are better than others. This is the line of argument I want to pursue next. For this task we need the following definitions that can be traced as far back as 1923—to Carnap's first account of quasi-analysis.

(3.4) Definition. Let $(S,\sim)$ be a similarity structure. A subset T of S is called a similarity circle if and only if it satisfies the following two conditions:

(i) for all x,y $(x,y \in T \Rightarrow x \sim y)$
(ii) for all x $(x \notin T \Rightarrow \exists y(y \in T$ and x and y are not similar)).

The set of similarity circles of $(S,\sim)$ is denoted by $SC(S)$.

A quality distribution all whose qualities are similarity circles is defined as follows:

(3.5) Definition and lemma. Let $(S,\sim)$ be a similarity structure, and $SC(S)$ its set of similarity circles. Define $f_{SC}: S \rightarrow P(SC(S))$ by $f_{SC}(s) := \{T: s \in T\}$. Then f is a quality distribution in the sense of (3.2) all of whose qualities are maximal. It is called the standard distribution.

Goodman's example of bicolored discs shows that there are quality distributions other than f_{SC}. His example can be generalized as follows:

(3.6) Goodman's example generalized. Let S be a similarity structure. Let the set of qualities Q_G defined by $Q_G := \{\{x,y\}: x,y \in S$ and $x \sim y, x \neq y\}$. Define $f_G: S \rightarrow P(Q^G)$ by $f_G(x) := \{\{x,y\}: x \sim y, x \neq y\}$. Then f_G is a quality distribution.

As is easily seen, for $(S,\sim)$ as in (3.1), f_G is just the bicolored discs quality distribution, and the standard distribution f_{SC} is the distribution that attributes one and the same quality to each element of S. Thus, most similarity structures $(S,\sim)$ have at least two different quality distributions: on the one hand we have the standard distribution f_{SC}, on the other hand we have the Goodmanian distribution f_G. For most similarity structures it's quite easy to construct many other quality distributions (some examples will be discussed in the following). If the cardinality of S is large the number of quality distributions rapidly increases. This fact, as we might reformulate Goodman's criticism against the feasibility of defining qualities by similarity, undermines any reasonable and controlled relation between similarity and qualities. At least this is the case if we stick to a set theoretical account. It seems we don't have any criteria for distinguishing "good" from "bad" quality distributions. Here topology comes to the rescue. In the next section we'll embark on the task of distinguishing between "good", i.e., continuous quality distributions and "bad", i.e., discontinuous ones. Moreover, we will show that in many cases exactly one good quality distribution can be selected whose qualities are thereby definable by similarity alone.

4. Continuous quality distributions

Conceptualizing the attribution of qualities to particulars as a map f: $S \rightarrow P(Q)$ is the essential presupposition for applying topological considerations. The next thing we have to do is to look for appropriate topological structures on S and $P(Q)$. Then we have to find out whether f is continuous or not with respect to these topologies. In this way Goodman's negative assessment of the feasibility of defining similarity by qualities can be defused: we only admit continuous quality distributions. This will yield the pleasing result that in many cases there is only one continuous distribution fitting a given similarity structure. Hence we might consider the qualities of this distribution as being defined by similarity alone. Goodman's criticism against the feasibility of such an approach is based on the fact that he allows for all kinds of quality distributions and does not impose any continuity constraint on them.

It is a rather amusing fact that the topological structures we use to get rid of Goodmanian quality distributions can be traced back to Carnap, one of the "good philosophers" Goodman targeted in the argument quoted above. The ammunition for a topological counterattack against Goodman's criticisms is not to be found in the well-known acount of quasianalysis of the *Aufbau*, however, but in the unpublished *Quasizerlegung*. There Carnap developes a more sophisticated version of quality distribution than in the *Aufbau*. In contrast to the *Aufbau* account, in *Quasizerlegung* quality distributions have to satisfy two further constraints that turn out to be the source for the topological concepts to be employed in the following. For their definitions we need some preparatory definitions which are topological reformulations of concepts Carnap introduced some 70 years ago:

(4.1) Definition. Let $(S,\sim)$ be a similarity structure and $x \in S$. The similarity neighborhood of x is the set $co(x)$ of all elements similar to x: $co(x) := \{y: x \sim y\}$. The relation between similarity circles and co is as follows:

$$T \in SC(S) \Leftrightarrow T = \bigcap \{co(x): x \in T\}.$$

Proof: (i). Let T be a similarity circle, and $x \in T$. Because all elements of T are similar to each other we have $T \subseteq co(x)$. Hence we get $T \subseteq \bigcap \{co(x): x \in T\}$. On the other hand, let $y \in co(x)$ for all $x \in T$. Then y is similar to all x of T. Then by definition of a similarity circle y already belongs to T.

(ii) Let T be a subset of S such that $T = \bigcap \{co(x): x \in T\}$. Any $y \in T$ belongs to $co(x)$ for all $x \in T$, hence it is similar to all elements of T. Thus T satisfies the condition (3.4)(i). Assume y does not belong to T. Then there is an $x \in T$ with $y \notin co(x)$, i.e., y is not similar to x. That means, T satisfies (3.4)(ii). Hence T is a similarity circle.

Already in *Quasizerlegung* Carnap used the concept of similarity neighborhood to impose the following structural constraint on good quality distributions:

$$(4.2) \quad co(x) = co(y) \Leftrightarrow f(x) = f(y)$$

In the following we shall use a slightly stronger condition than (4.2). It has the advantage of being a genuine topological condition, to wit, it is the condition of continuity. First we have to define topological structures on similarity structures:

(4.3) Definition and Lemma. Let $(S,\sim)$ be a similarity structure. Define an operator cl on S by $cl(R) := \{y$: there is an $x \in R$ with $co(x) \subseteq co(y)\}$. cl is a topological closure operator on S.

The proof that cl is a topological closure operator is well known and depends on the fact that the inclusion of similarity neighborhoods renders S a partial order. The topology defined by (4.3) is also called the order topology. Thus, the similarity structures S and $P(Q)$ present themselves with ready made topological structures. From now on it is assumed throughout that the sets S and $P(Q)$ are endowed with these topological structures. With respect to these topologies we get the following pleasing result:

(4.4) Proposition. Let $(S,\sim)$ be a similarity structure. A quality distribution f: $S \rightarrow P(Q)$ is continuous iff it preserves the order structures of S and $P(Q)$, i.e., iff $co(x) \subseteq co(y) \Rightarrow f(x) \subseteq f(y)$.

The proof of (4.4) only depends on the fact that co renders S and $P(Q)$ partially ordered sets. It is a well-known fact that order-preserving maps are continuous with respect to the order topologies. In this paper, continuity of quality distributions $f: S \rightarrow P(Q)$ is always to be understood with respect to these topologies. According to the Topological Maxim to be formulated below, continuous functions are to be preferred, ceteris paribus, to non-continuous ones. Since the condition in (4.4) is only slightly stronger than Carnap's original requirement (4.2), by hindsight we might say that Carnap got it almost right back in 1923.[10] Applying (4.4) we can now tackle the main task of this paper, namely to distinguish between "good" and "bad" quality distributions. The result is almost optimal:

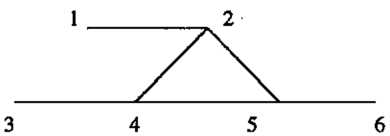
(4.5) Corollary. (i) For all similarity structures $(S,\sim)$ the standard quality distributions $f_{SC}: S \rightarrow P(SC(S))$ are always continuous. (ii) The Goodmanian quality distributions $f_G: \rightarrow P(Q_G)$ are not always continuous.

Proof: (i) Let $(S,\sim)$ be a similarity structure, and $f_{SC}: S \rightarrow P(SC(S))$ the standard quality distribution. According to (4.4) we have to show that f_{SC} is order-preserving. Assume $co(x) \subseteq co(y)$ and let T be a similarity circle with $T \in f_{SC}(x)$. By (4.1) we have $T \subseteq co(x)$. Hence $T \subseteq co(y)$, i.e., $T \in f_{SC}(y)$.

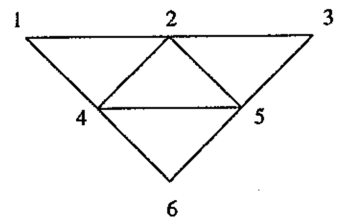
(ii) It is sufficient to give an example of a similarity structure $(S,\sim)$ whose Goodmanian quality distribution f_G is not continuous. The simplest example is provided by Goodman's original bicolored disc example. Moreover, as is easily seen, Goodmanian quality distributions of most similarity structures are not continuous. The content of (4.5) may be formulated in the following slogan: The SC-method of assigning quality distributions is reliably topologically well-behaved, the G-method is not.

This does not exclude that some very Goodmanian quality distributions turn out to be continuous as is shown by the following example:

(4.6)



The quality distribution f_G of (4.6) is continuous and actually different from the standard distribution f_{SC}. Thus we find that some similarity structures have essentially different continuous quality distributions. For them, the criterion of continuity does not suffice to reach uniqueness. (4.5) is not to be interpreted as the claim that the SC-method yields always an optimal distribution, however. The standard distributions f_{SC} are less than optimal in certain respects which are not covered by the distinction between continuous and discontinuous distributions. Consider the following example:



As is easily proved, the standard distribution $f_{SC}: S \rightarrow P(SC(S))$ needs exactly four qualities, whose extensions are {1,2,4}, {2,4,5}, {2,3,5}, and {4,5,6}. Actually we can do without q={2,4,5}. This means, the distribution $f_{SC}: S \rightarrow P(SC(S))$ can be reduced to a continuous quality distribution $f': S \rightarrow P(SC(S)\text{-}\{q\})$. Taking the plausible stance that quality distributions should be as parsimonious as possible, we are led to the conclusion that distributions of type f_{SC} are not always the best ones. This deliberation we already found in Carnap's paper of 1923. There he defined an economy condition that can be formulated in the following form:[11]

(4.7) Definition. Let Q be a set of qualities, $Q' \subseteq Q$ a subset. Q' induces a continuous map $r': P(Q) \rightarrow P(Q')$ defined by $r'(P) := Q' \cap P$. Let $f: S \rightarrow P(Q)$ be a continuous quality distribution. A reduction of f to Q' is a quality distribution $f': S \rightarrow P(Q')$ such that $r' \bullet f = f'$. The map f is said to cover f'; f is said to be irreducible if and only if there is no reduction of f to Q' except for $Q = Q'$. If f' is a reduction of the the standard distribution f_{SC} it is said to be of the first kind.

The constraints of irreducibility and continuity exclude quite different kinds of distributions:

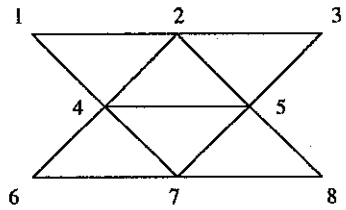
(4.8) Lemma. (i) The Goodmanian quality distributions $f_G: S \rightarrow P(Q_G)$ are irreducible but not always continuous. (ii) The standard quality distributions f_{SC}: $S \rightarrow P(SC(S))$ are continuous but not always irreducible.

Hence, in general, neither standard nor Goodmanian quality distributions can be considered optimal. To get optimal quality distributions we have to invest more work. Since the G-method is topologically unreliable, i.e., does not generally produce continuous distributions, a natural strategy is to see whether the SC-method of standard distributions might be improved in such a way that we end up with a continuous and irreducible distribution. This indeed can be done. Starting with the standard distribution f_{SC} we can always obtain an irreducible continuous distribution simply by removing superfluous qualities:

(4.9) Lemma. Let $f': S \rightarrow P(Q')$ be a quality distribution covered by a continuous quality distribution $f: S \rightarrow P(Q)$. Then f' is a continuous quality distribution. In particular, quality distributions of the first kind are continuous.

Proof: Let f': $S \to P(Q)'$ be a quality distribution covered by the continuous distribution f: $S \to P(Q)$, i.e., $f' = r' \bullet f$, r': $P(Q) \to P(Q')$ induced by the inclusion $Q' \subseteq Q$. Obviously the map r' is continuous. Then f' is continuous by (2.3).

However, the following problem remains. We cannot be sure that this process of reduction leads to a uniquely determined irreducible continuous distribution. The following example shows that there are similarity structures that possess more than one continuous, irreducible quality distribution:



The standard distribution f_{SC}: $S \to P(SC(S))$ can here be reduced to two essentially different irreducible continuous distributions: either we may remove the similarity circle $q = \{2,4,5\}$, or we may remove the similarity circle $q' = \{4,5,7\}$. We have no reason to prefer one of these distributions to the other. Thus, in general we have to give up hope of reaching a unique optimal quality distribution. It may be the case, however, that there are unique, continuous and irreducible quality distribution for similarity structures of a special kind. This is indeed the case. For similarity structures that are topologically not too complicated unique irreducible continuous distributions with maximal qualities exist. Such a theorem was proved by Brockhaus some 30 years ago (without any reference to topology).[12] Its content may be informally stated as follows: the standard quality distribution f_{SC} has a unique minimal reduction iff there is a subset $SC(S,2) \subseteq SC(S)$ which covers S, and all elements of $SC(S,2)$ are extensionally generated by at most two elements. The precise definition is as follows:

(4.10) Theorem. A similarity structure $(S,\sim)$ has a unique continuous irreducible quality distribution f: $S \to P(Q)$ of the first kind iff S can be covered by a class $SC(S,2) \subseteq SC(S)$ defined as follows:

(i) $T_i \in SC(S,2)$ iff there are $x_{i1}, x_{i2} \in T_i$ such that $co(x_{i1}) \cap co(x_{i2}) = T_i$.
(ii) $\bigcup SC(S,2) = S$.

The proof of (4.10) is somewhat lengthy, hence it is deferred to the appendix. To get a feeling for the condition involved in (4.10) let us make the following observations:

(1) If $(S,\sim)$ is a transitive similarity structure, i.e., a set endowed with an equivalence relation, then its similarity circles T_i, i.e., its equivalence classes, are all generated by one x_i. Hence the unique irreducible continuous quality distribution of $(S,\sim)$ is the distribution that maps each element x of S to its equivalence class $[x] = \{y: y \sim x\}$.

(2) All examples of similarity structures dealt with so far by philosphers satisfy the condition of (4.10). Hence their standard quality distributions have unique irreducible reductions. In the counterexample of figure 5 one of the qualities q or q' belongs to any quality distribution. q and q' have exactly three generators, $q = co(2) \cap co(4) \cap co(5)$, $q' = co(4) \cap co(5) \cap co(7)$. Hence the condition of (4.10) does not apply to this similarity structure.

(4.8) gives the motivation for characterizing similarity structures according to how many generators are needed for their necessary similarity circles:

(4.11) Definition. A similarity structure $(S,\sim)$ is of the n-th order ($n > 1$) iff S can be covered by a class $SC(S,n) \subseteq SC(S)$ of similarity circles that are generated by at most n elements. This means the following:

(i) $T \in SC(S,n)$ iff there are $x_1,\ldots,x_n \in T$ so that $co(x_1) \cap \ldots \cap co(x_n) = T$.
(ii) $\bigcup SC(S,n) = S$.

Now we are able to express succinctly the main result (4.10) of this section as follows:

(4.12) Theorem. A similarity structure $(S,\sim)$ has a unique continuous irreducible quality distribution of the first kind if and only if it is of the first or second order.

Let us take stock of what we have achieved so far: according to (4.10), for similarity structures of the first and second order qualities can be defined by similarity alone. At least, this holds if we are prepared to impose topological constraints on the distributions admitted. These topological constraints are quite natural ones: they flow directly from the topological structures similarity structures are endowed with by nature. This means, these structures are not invented ad hoc, they are already there as soon as we talk about similarity. Hence, if one is not blind to the topological aspects of reality for a large class of similarity structures, there is one and only one optimal system of corresponding qualities. That means, contra Goodman, that in this case qualities can be defined by similarity. Thus, at least for similarity structures of the first or second order, Carnap's quasianalytical approach is completely vindicated.

The general format of the problem we have dealt with is the following: Let X and Y be sets. The task is to classify, somehow or other, maps f: $X \to Y$. More precisely, we'd like to keep the "good" maps and discard the "bad" ones.

Set-theoretically, this is a hopeless task, since all maps $f: X \to Y$ are on an equal footing. There is a profusion of alien maps with undesired properties which we do not want to count as reasonable functorial relations between X and Y. Not so in topology. Here is a great divide between "good", i.e., continuous maps, and "bad" ones that do not enjoy this property. That means, if we conceptualize X and Y as topological spaces we can impose the topological constraint of continuity on maps between X and Y. Thereby the undesired profusion of maps might be drastically reduced.

The introduction of topological considerations may be expressed in the following "Topological Maxim":

> *Topological Maxim*: Whenever you meet a set theoretical map $f: X \to Y$ try to achieve the following two tasks:
> (1) Try to conceptualize X and Y as topological spaces in a natural way.
> (2) Find out whether f is continuous or not.

If f turns out not to be continuous probably the functional relation between X and Y is ill-defined. In any case, ceteris paribus, continuous maps should be preferred to non-continuous ones.

In this paper the Topological Maxim has been applied to similarity structures $(S,\sim)$ and $(P(Q),\sim)$ whose natural topologies are the order topologies. In this case, "good" quality distributions are continuous maps between the topological spaces S and $P(Q)$.

5. Appendix: Proof of Theorem (4.10)

By presupposition all quality distributions we consider are of the first kind. Hence we may assume that they are maps $f: S \to P(Q)$ with $Q \subseteq SC(S)$. Let us call a quality $q \in SC(S)$ necessary iff it occurs in all quality distributions. Necessary qualities can be characterized as follows:

(5.1) Lemma. Let $(S,\sim)$ be a similarity structure. $q \in SC(S)$ is necessary iff the following holds: There are $x_q, y_q \in q$ such that $q = co(x_q) \cap co(y_q)$.

Proof: (a) Let $q \in SC(S)$ and assume that $x_q, y_q \in q$ such that $q = co(x_q) \cap co(y_q)$. We show that q is necessary. Since q is a similarity circle x_q and y_q are similar to each other. Let $f: S \to P(Q)$ be a quality distribution of the first kind. There must be a similarity circle q' with $q' \in f(x_q) \cap f(y_q)$. Due to $q' = \bigcap\{z: z \in q'\}$ we have $q' \subseteq co(x_q) \cap co(y_q)$, i.e., $q' \subseteq q$. Similarity circles do not properly include each other, hence we get $q' = q$, and q is necessary.

(*b*) Let us assume that for q there are no $x_q, y_q \in q$ such that $q = co(x_q) \cap co(y_q)$. We show that q is not necessary, i.e., there is a quality distribution where q does not occur. Let $f: S \to P(Q)$ be a quality distribution with $q \in Q$. Let $x_q, y_q \in q$. According to the premiss there is an element $z \in S$ with $z \in co(x_q) \cap co(y_q)$ but $z \notin q$.

The elements x_q, y_q, and z are similar to each other. Hence there must be a $q_{xyz} \in SC(S)$ with $x,y,z \in q_{xyz}$, since f is of the first kind. After these preparatory remarks we define a quality distribution $f': S \to P(SC(S)-\{q\})$ as follows:

$$f'(v) := \begin{cases} f(v), & q \notin f(v) \\ f(v) - \{q\} \cup \{q_{xyz}: x,y,z \in q,\ z \in co(x) \cap co(y),\ v \in q_{xyz}\}, & q \in f(v). \end{cases}$$

Obviously, q does not occur in f' anymore. To show that f' is indeed a continuous quality distribution let us make the following deliberations. Let $v,w \in S$. We have to show that $v \sim w \Leftrightarrow f'(v) \sim f'(w)$. First assume $v \sim w$. We have to distinguish several cases:

(i) $q \notin f(v)$, $q \notin f(w)$. In this case f' and f coincide for v and w, and there is nothing to show.

(ii) $q \in f(v)$ and $q \notin f(w)$. Since f is a quality distribution there is a $q' \in f(v) \cap f(w)$. According to the definition of f' q' belongs to the intersection of $f'(v)$ and $f'(w)$ as well. An analogous argument applies to the case $q \notin f(v)$ and $q \in f(w)$. Hence $f'(v) \sim f'(w)$.

(iii) $q \in f(v) \cap f(w)$. In this case there is a $z \in co(v) \cap co(w)$ with $z \notin q$. Hence there must be a $q_{vwz} \in SC(S)$ with $v,w,z \in q_{uvz}$ and $q_{vwz} \in f'(v) \cap f'(w)$. Hence $f'(v) \sim f'(w)$. Now assume $f'(v) \sim f'w)$. The distribution f' is covered by the standard distribution f_{SC}. Hence $v \sim w$. This completes the proof of (5.1).

Now, theorem (4.10) immediately follows: if the similarity structure $(S,\sim)$ is of the second order it follows from (5.1) that it is covered by a set of necessary similarity circles. These necessary similarity circles can be used to construct a unique minimal quality distribution. On the other hand, if there is a unique minimal quality distribution, all of its qualities are obviously necessary. Hence by (5.1) they are generated by at most two of their elements. This means, $(S,\sim)$ is of first or second order.

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NOTES

1. Cf. Carnap [1928], Goodman [1951, 1972], Quine [1961].
2. Goodman [1951], ch. V, and [1972: 442–43].
3. The main ideas of this paper are topological reformulations of concepts Carnap developed in a still unpublished manuscript *Die Qasizerlegung – Ein Verfahren zur*

Ordnung nichthomogener Mengen mit den Mitteln der Beziehungslehre of 1923, RC-081-04-01, University of Pittsburgh.

4. As a general reference for topology the reader may consult Dugundji [1966] or any other of the standard textbooks.

5. As is well known, the definition of a topological closure operator on a set X is equivalent to the definition of a topological structure on X by specifying a family of open subsets of X satisfying the well-known axioms, see Dugundji [1966], Theorem 8.3.

6. Applications of these kinds are dealt with in Mormann [1993: 219–40].

7. Cf. Goodman [1972: 442–43].

8. Goodman [1972: 442].

9. Goodman [1972: 441–42].

10. In the *Aufbau*, Carnap dropped (4.2), probably for reasons of pedagogical simplification. Later authors such as Goodman and others who dealt with formal aspects of quasianalysis never took into consideration a condition like (4.2) or (4.4).

11. Cf. *Quasizerlegung*, p. 5.

12. Cf. Brockhaus [1963].

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