Structural Mereology: A Formal Elucidation and Some Metaphysical Applications

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Abstract. The aim of this paper is to provide a formal elucidation of structural mereology and to show that it has a bearing on a variety of metaphysical problems, among them the structure of complex states of affairs and the appropriate definitions of wholes. The account of structural mereology presented in this paper takes its inspiration from the concept of structure as it has been developed in modern mathematics. The point of departure is David Lewis's account of the mereology of sets. It is shown that the mereology of sets naturally generalizes to a more general structural mereology of structured sets which takes into account the structure of mereological wholes for structure-specific concepts of parthood and composition. Usually, the resulting systems of structural parts are not Boolean. This enables us to shed new light on a variety of metaphysical concepts such as the structure of complex states of affairs, structural universals, wholes, and quantities.

Key words: Classical Mereology, Structural Mereology, Structured Universals, Conceptual Completion, Quantities, Similarity Structures, Wholes.

1. Introduction. The basic idea of structural mereology is that mereological concepts such as parthood and composition should be conceived as structure-related concepts that take into account the structure of the objects that mereologists are studying in their investigations. This vague description may be interpreted in many different ways. Conceived in this broad sense, one may assert that structural mereology may be even traced back to the mereological considerations of Plato and Aristotle who attempted to take into account the concepts of structure or form in their mereological considerations in one way or other. The pertinent texts of this classical dispute on matters of structural mereology are *Socrates' Dream* in the *Theaetetus* and Aristotle's *Metaphysics Z 17*, respectively (cf. Scaltsas 1994, chapter 4, 60ff). Plato held that the whole is identical with its parts. Accordingly, he claimed that the syllable "BA" is identical to the letters "B" and "A".

In contrast to Plato, Aristotle maintained that the whole and its parts are different.[1] The syllable "BA", said Aristotle, is not just the letters "B" and "A" but something else, too, since when the syllable is "dissolved, the whole, i.e. the syllable, no longer exists, but the elements

[1] Perhaps the positions of Plato and Aristotle may not be separated in such a neat and clear-cut manner, see Harte (2002). But this is of no concern for the present paper.

of the syllable exist." Hence, he concluded, the syllable consists of the elements plus a further item, which is of a completely different type than the elements, namely, its substance.

The dispute between Plato and Aristotle on structural mereology has found a certain rehearsal in the dispute between Armstrong, Lewis and other contemporary philosophers on the possibility of non-mereological composition of structural universals (cf. Armstrong 1980, Lewis 1984). Moreover, this contemporary discussion enables us to tap another source of inspiration for the elaboration of a modern account of structural mereology, namely, the theory of structures in the sense of modern mathematics.

In this paper, we only need some rudiments of the elementary theory of structures, for a more sophisticated account the reader may consult Awodey (1996). According to this theory, a structured set is a system (X, R), X being a set and R a set of relations defined on some sets derived from X by some familiar operations such as taking subsets, constructing Cartesian products or power sets. Structured sets abound in mathematics: order structures, topological spaces, differentiable manifolds, groups, vectorspaces, and virtually all objects studied in modern mathematics may be conceived as structured sets in this sense. By conceiving mathematical entities as structured sets the theory of sets obtains a prominent role in mathematics as a foundational theory of all of mathematics.

As David Lewis pointed out that sets provide natural examples for Boolean mereological systems in that the (non-empty) subsets of a set X may be considered as the parts of X in the sense of mereology. In this way, every set X defines in a canonical way a classical Boolean mereological system PX – {Ø}, PX the powerset of X, and Ø the empty set. Having at our disposal a mereological system for every set X, one may expect that also structured sets (X, R) may possess a reasonable theory of parthood and composition, i.e. a mereology. More precisely, if the parts of a set X are its subsets, one may expect that the parts of a structured set (X, R) be its structured subsets – the latter concept to be defined in an appropriate way. As will be shown, this is indeed the case. Thereby structural mereology may be described as the study of mereological concepts such as parthood and composition for <u>structured</u> sets (X, R).

For the moment it may not be clear how this structural mereology, based on the concept of structure in the sense of modern mathematics, is related to the structural mereology favored by Plato, Aristotle, or Husserl, to mention the classical protagonists of this approach. Although the main aim of this paper is not to make a direct contribution to the "metaphysics of structure" (cf. Harte 2002) in the following I hope to show that the modern mathematical concept of structure helps elucidate the classical approach by bringing to the fore hitherto unnoticed conceptual possibilities and variations.

The outline of this paper is as follows. As a first step toward a general structural mereology in the next section we consider the mereology of inhomogeneous sets. Inhomogeneous sets have appeared in the literature under many names, tolerance spaces, similarity structures, or simple undirected graphs, to name just three. The term „inhomogeneous sets“ was coined by Carnap in an unpublished manuscript of the 1920s that may be considered as the origin of his famous method of „quasianalysis“. Informally speaking, an inhomogeneous set is a set endowed with a reflexive and symmetric relation to be interpreted as a binary similarity relation. Ordinary „homogeneous“ sets may be conceived as special inhomogeneous sets whose similarity is trivial, i.e. the identity.

Following David Lewis, the mereological systems of ordinary („homogeneous“) sets are Boolean algebras. More precisely, the parts of a set are its subsets. Generalizing Lewis's recipe in a natural way, it turns out that the mereological systems of inhomogeneous sets are non-Boolean Heyting algebras. As an application of the mereology of inhomogeneous sets it is shown that the mereological structure of many complex states of affairs in the sense of Armstrong's factual ontology (cf. Armstrong 1986, 1992, Mormann 2010) can be described in terms Heyting algebras. Thereby the usage of the problematic notion of structured universals can be avoided.

Structurally the relation between the mereologies of homogeneous and inhomogeneous sets is analogous to that between classical (Boolean) and intuitive (Heyting) propositional logic. Thus one may be inclined to assess their differences as not too impressive.[2] In order to show that mereological systems of structured sets may indeed considerably differ from classical systems in the next section we deal with some mereological systems that naturally arise from Euclidean geometry. It is shown that the mereology of Cartesian rectangles of the Euclidean plane defines a complete non-distributive and non-orthocomplemented lattice. Mereological systems of this kind exhibit metaphysically interesting phenomena of „creative fusion“ according to which the mereological fusion of two (or more) objects may be considerably larger than the collection of the fused objects taken separately. Turning upside down the traditional classification mereological systems with „creative fusions“ become the generic case, while those with non-creative Boolean fusions are an exception. In the final section it is shown that the approach of generalized mereology developed so far does not depend on the theory of

[2] A second look reveals that the mereological systems of inhomogeneous sets differ from that of ordinary sets in that they do not satisfy the so-called supplementation principles that some authors consider as essential for „real“ mereological systems.

structured sets or relational systems but can be set up in a more general setting that does not depend on set theory at all. Rather, the appropriate framework for mereology as a general theory of the parthood relation is category theory in the sense of Mac Lane and Eilenberg (cf. Lawvere and Rosebrugh 2001).

My point is not so much to argue for or against Aristotelian metaphysics, or to put forward a more or less novel interpretation of Aristotle's doctrines of substance and form etc.[3] Rather, I'd like to offer some formal elucidations, i.e., the task of this paper is not to give a formal reconstruction of Aristotelian mereology as such. Rather, I'd like to pursue some motifs of Aristotelian mereology in the framework of a structural mereology inspired by the concept of structure as used in modern mathematics. It may be that this kind of structural mereology may help elucidate some issues of Aristotelian mereology proper. But the interpretation of classical metaphysical concepts such as „form" and „substance" is not the task of this paper.

2. Homogeneous and Inhomogeneous Sets. From a set-theoretical point of view sets exhibit a totally homogeneous structure in that all elements of a set are on an equal footing; for any two elements x and y of a set S, there is nothing to say about them than that either they are equal or that they are different. *Tertium non datur*. In contrast, an inhomogeneous set S may be thought as a set whose elements are not all on an equal footing in that they may be compared with each other in a way that goes beyond the binary distinction that they differ or do not differ, namely that they are, or they are not, neighbors of each other in some sense. The idea of inhomogenity may be rendered precise in various ways. Perhaps the simplest and most natural way is via similarity structures.

A similarity structure (S, ~) is defined as a set S endowed with a reflexive and symmetric binary relation $\sim \subseteq S \times S$. Two elements are said to be similar iff $(a, b) \in \sim$. As usual this is denoted by $a \sim b$. Similarity structures may be conceived as undirected simple graphs: the vertices of the graph are the elements of S and two different vertices are similar iff they are the vertices of the same edge. For instance, for S = {1, 2, 3, 4, 5} the graph

(2.1)

```
      3
      |
1 ——— 2 ——— 5
      |
      4
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[3] For a detailed discussion of the more subtle features of a(n) (Neo)Aristotelian mereology see Koslicki (2006, 2008) and and the many contributions of Kit Fine to this field, for instance Fine (1999).

represents the similarity structure (S, ~) according to which each of the elements 1, 2, 3, 4, 5 is similar to itself, 1 and 2 are similar, 2 and 3 are similar, 2 and 5 are similar, 2 and 4 are similar, and no other similarities between the elements of S occur. For instance, 1 and 4 are not similar. In other words, the graph structure is to be interpreted in such a way that it gives complete information concerning the similarity and dissimilarity of its elements. If we deal with several similarity structures and have to distinguish their similarity relations, they are denoted by (S, $\sim_S$) and (T, $\sim_T$). More formally, a similarity structure (S, $\sim_S$) may be characterized as a structured set or a relational systems. Then structural parts of inhomogeneous sets alias similarity structures can be defined as follows:

(2.2) Definition. A structural part of a similarity structure (S, $\sim_S$) is a similarity structure (T, $\sim_T$) such that T ⊆ S and $\sim_T \subseteq \sim_S$. The set of structural parts of (S, $\sim_S$) is denoted by PART(S, ~). A partial order ≤ on PART(S, ~) is defined by

$$(U, \sim_U) \leq (V, \sim_V) := U \subseteq V \text{ and } \sim_U \subseteq \sim_V \qquad (U, V \subseteq S)$$

(U, $\sim_U$) is a part of (V, $\sim_V$) iff (U, $\sim_U$) ≤ (V, $\sim_V$). The top element of the partial order PART(S, ~), ≤) is (S, $\sim_S$) and the bottom element is the empty similarity structure (Ø, $\sim_Ø$).♦

Definition (2.2) is a natural generalization of Lewis's definition of parts of sets (cf. Lewis 1991, 3ff). It may look innocent enough, nevertheless, it ushers us outside the realm of classical Boolean mereology, since the mereological systems PART(S, ~) are non-Boolean Heyting algebras (cf. Mormann (2010)). In other words, Boolean mereology is too narrow a framework to capture all kinds of reasonable mereological systems, in particular, the Boolean frame does not include the mereological systems arising from similarity structures, i.e., inhomogeneous sets.

Moving from Boolean to Heyting algebras should not be considered, however, simply as a weakening of the formal requirements imposed on mereological systems. In the following section it will be shown that the Heyting framework enables us to make metaphysically important distinctions that disappear for Boolean systems. For instance, for Heyting algebras there is natural distinction between connected and non-connected objects that vanishes for Boolean algebras. In other words, Boolean algebras turn out not to be the most appropriate framework for certain metaphysical applications.

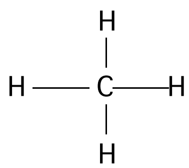
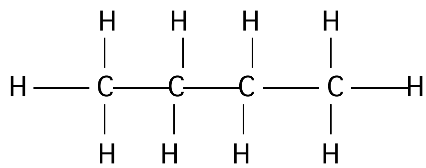
3. Complex States of Affairs and Structural Wholes. Now let us apply the apparatus of the mereology of similarity structure for the elucidation of the structure of complex states of

affairs (cf. Armstrong 1997, Mormann 2010). Looking at structural formulas of elementary chemistry such as H—O—H, H—C—H, O—O, Na—O—H and at graphs of similarity structures such as (2.1) it is more or less obvious that similarity structures and structural formulas of chemistry are closely related to each other. One further definition is necessary to make this relation explicit:

<u>(3.1) Definition</u>. Let $S_1, S_2, ..., S_n$ be non-empty, mutually disjoint sets and $S := S_1 \cup S_2 \cup ... \cup S_n$, n = 1, 2, 3, ... Then a similarity structure (S, ~) is called an n-sorted similarity structure. The S_i are called the sorts of (S, ~). ♦

A similarity structure as defined in section 2 is just a 1-sorted similarity structure. In the following we will only deal with 2-sorted similarity structures but all arguments go through for n-sorted structures with n ≥ 3.

A 2-sorted similarity structure ($S_1 \cup S_2$, ~) may be graphically represented as a "labeled" graph in such a way that the vertices of S_1 are labeled, say, by the letter H, and the vertices of S_2 are labeled by the letter C, respectively:

(3.2) (i)  (ii) 

Obviously, the 2-sorted graphs of (3.2)(i) and (3.2)(ii) correspond to the structure formulas of methane and butane, respectively. More complex molecules involving more than two sorts of atoms may be described analogously with the help of n-sorted graphs, n ≥ 3.

What has been said about the mereological structure of 1-sorted similarity structures (S, ~) in section 2 directly applies to n-sorted similarity structures since the labeling does not affect the mereological structure of a graph. In particular, n-sorted similarity structures S give rise to mereological systems PART(S, ~) of structural parts in the same way as 1-sorted structures do. In particular, the mereological systems PARTS(S, ~) in general are non-Boolean Heyting systems (cf. Mormann (2010), Theorem (2.10)).

As an example of how the formal apparatus of non-classical mereology developed so far may be put to work we offer the following elucidation of the structure of complex states of affairs such as "being a methane molecule" and similar ones. For extensive discussion of these and similar issues see (Armstrong 1997, Lewis 1986, Mormann). For the following we only need some rudiments of Armstrong's factual ontology. According to this approach, the world is a world of states of affairs. States of affairs are constituted by (thin) particulars a, b, ..., and (relational) universals H, C, N, etc. Then the states of affairs that the particulars a, and b, are hydrogen atoms, and c and d are oxygene atoms that form an oxygene molecule are denoted by Ha, Hb, and cOd, respectively. The more complex state of affairs of being a methane molecule can be analyzed as follows:

<u>(3.3) Mereological Analysis of Methane</u>. Assume a, b, c, d, e to be particulars, H the universal hydrogen and C the universal carbon. Define S_1 = {Ha, Hb, Hc, Hd}, S_2 = {Ce} and S := $S_1 \cup S_2$. Denote by — the binary bonding universal, and define a similarity relation ~ on S by

~ := {Ha—Ce, Ce—Ha, Hb—Ce, Ce—Hb, Hc—Ce, Ce—He, Hd—Ce, Ce—Hd}

Then the similarity structure (S, ~) is the state of affairs that the particulars a, b, c, d, e instantiate a methane molecule such that a, b, c, and d instantiate the universal H, e instantiates the universal C, and the binary bonding universal — is instantiated by the set {Ha—Ce, Ce—Ha, Hb—Ce, Ce—Hb, Hc—Ce, Ce—He, Hd—Ce, Ce—Hd}.♦

Since bonding is symmetric the complex state of affairs of being methane may be displayed in the apparently familiar way as

(3.4)

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      Hb
       |
 Ha—Ce—Hc
       |
      Hd
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Denoting the similarity structure (3.4) by (M, ~) the lattice PART(M, ~) of its structural parts can be calculated by (2.2). Before we give an explicit calculation one should note that the graphs (3.2) and (3.4) are conceptually quite different. Actually, according to Armstrong's factual ontology graphs like (3.2) are not ontologically well-defined. According to the very definition of universals it does not make sense to assume that a universal such as H appears twice or more times in any context. Thus, even if one assumes that there is a kind of second-order relational bonding universal — that bonds universals such as H and C, symbols such as H—C—H in which the universal H appears twice do not make sense – if we want to keep the original sense of what universals are. On the other hand, symbols like (3.4) are not plagued by such difficulties. States of affairs such as Ha, ..., Hd, and Ce are particulars, more precisely, they are thick particulars (cf. Armstrong 1997). The only thing to note is that the relational universal — is to be conceived as defined for thick particulars such as Ha and Ce. But there is no particular conceptual difficulty to make such an assumption, at least not for Armstrong's factual ontology. In sum, while similarity structures such as (3.4) are meaningful in the framework of an Armstrongian factual ontology, symbols such as (3.2) strictly are not.

After these general remarks let us engage in the task of calculating the mereological structure of "being methane" explicitly. First, one observes that PART(M, ~) has five atoms, to wit, Ha, Hb, Hc, Hd, Ce. It should be noted that the states of affairs Ha—Ce, ..., Hd—Ce are neither atoms nor are generated by atoms in that they are suprema of atoms. Nevertheless they are mereologically not trivial in that they have non-trivial parts. Using (2.2) one calculates

(3.5) $\quad$ Hj, Ce < Hj & Ce < Hj—Ce < (Hj—Ce) & Hk < Hj—Ce—Hk $\quad$ (j, k ∈ {a, b, c, d}, j ≠ k)

The important point of (3.5) is that parts such as Hj & Ce and Hj—Ce are different, but there are no atoms that make a difference between these parts.[4] More precisely, Hj—Ce is neither an atom nor is generated by atoms. Analogously one shows that other parts of M such as Hj—Ce—Hk (j ≠ k) are neither atoms nor are generated by atoms in the sense that they are fusions of atoms. This evidences that PART(M, ~) is not a Boolean lattice. As has been shown in Mormann (2010) this fact can be used to yield the "correct" mereological analyses of methane, butane and iso-butane without using the problematic notion of structural universals. Rather, structural universals turn out as a convenient *façon de parler* but they lack any ontological weight. The state of affairs "x is a methane molecule" is analysed as a complex state of affairs whose parts are other states of affairs whose ultimate parts are atomic states of affairs that are constituted by (thin) particulars and simple universals. Through this reductive analysis alleged structural universals such as H_2O or CH_4 that in an ontologically superficial analysis of water or methane seem to play an analogous role to that of basic universals such as H or C play in the analysis of hydrogen atoms or carbon atoms Ha or Ce, are to be considered as a convenient shorthand of the non-Boolean mereological structures that actually do the real work.

The structural mereology of inhomogeneous sets is not only useful for elucidating the structure of complex states of affairs. It may be used also to shed some new light on a fundamental, albeit neglected notion of mereology, to wit, the notion of whole. According to the standard lexicographic definition, mereology is the theory of parts and wholes. Nevertheless, although much has been written about parts, parthood relations and its relatives,

[4] Evidently, the difference between Hd—Ce and Hd & Ce is analogous to the difference between the syllable „BA“ and the separated letters „B“ and „A“. Hence, in the debate between Plato and Aristotle on the relation between parts and wholes we side with the latter.

much less can be found on the notion of wholes. Standard mereology has not much to say about the fundamental question of how to distinguish a scattered object from a whole, or so it seems. For similarity structures (S, ~) the following definition singles out in a naturally way those parts (T, ~) of (S, ~) that may be conceived as (relative) wholes (with respect to the universal whole (S, ~)):

(3.6) Definition. A part $(T, \sim_T)$ of a similarity structure $(S, \sim_S)$ is a whole part of $(S, \sim_S)$ iff for $a, b \in T$ $a \sim_T b$ iff $a \sim_S b$.♦

Obviously, $(S, \sim_S)$ is a whole part of itself. Moreover, the the singleton subgraphfs {a} defined by elements $a \in S$ are whole parts of $(S, \sim_S)$. For the similarity structure of methane (3.4) the subgraphs Hj—Ce are whole parts, j = a, b, c, or d, while the subgraphs Hj & Ce are not whole parts. For reason of vividness, they may be called scattered parts. Generally, arbitrary intersections of whole parts are again whole parts. Hence, since (S, ~) is a whole part for every part $(T, \sim_T)$ there is a smallest whole part $(WH(T), \sim_{W(T)})$ that contains $(T, \sim_T)$ defined by

(3.7) $(WH(T), \sim) := \cap\{(K, \sim); (T, \sim) \leq (K, \sim) \text{ and } (K, \sim) \text{ is a whole part of } (S, \sim)\}$

(WH(T), ~) may be called the whole hull of (T, ~). It is the smallest whole that contains (T, ~). For instance, one easily calculated WH(Hj & Ce) = Hj—Ce and WH(Hj & Ce & Hk) = Hj—Ce—Hk etc. With the help of the operator WH one may single out among the parts of (S, ~) a special class of parts WH(T) that may be characterized as wholes in so far as they exhibit maximal coherence in that two elements $a, b \in WH(T)$ are similar relative WH(T) iff they are similar relative to the similarity relation valid in S: $a \sim_{W(T)} b \Leftrightarrow a \sim_S b$.

For $T \in PART(S, \sim)$ the part WH(T) may be conceived as sort of completion of T that exhibit the highest possible degree of coherence or cohesiveness that the elements of T can possess in the context of (S, ~). This completion may be conceived of in a natural way as an operator PART(S, ~)—WH—>PART(S, ~) that maps a part (T, ~) to its completion $(WH(T), \sim_{WH(T)})$:

(3.8) Lemma. The operator PART(S, ~)—WH—>PART(S, ~) has the following properties:

(WH1) $T \leq WH(T)$.

(WH2) $T \leq T' \Rightarrow WH(T) \leq WH(T')$.

(WH3) $WH(T) = WH(WH(T))$.

(WH4) $WH(T \cap T') = WH(T) \cap WH(T')$.

An operator satisfying (WH1) – (WH3) is called a closure operator. A closure operator that also satisfies (WH4) is called a nucleus (cf. Johnstone 1982, Borceux 1994).♦

For our purposes it is not necessary to delve into the details of the theory of nuclei. Let us be content to state the following easily proved proposition:

(3.9) Proposition. Let (S, ~) be a similarity structure. Then the set WHPART(S, ~)) of whole parts of (S, ~) is a (complete) Boolean algebra. The whole parts are exactly those parts that are invariant under the closure operator WH:

$$T \in WHPART(S, \sim)) \Leftrightarrow T = WH(T)$$

For a trivial similarity structure (S, ~), i.e. a set S, one has PS = PART(S, ~) = WHPART(S, ~).♦

Proof. Use the properties (WH1) - (WH4) of the completion operator WH and the definitions (2.2) and (2.7) to show that the natural order relation ≤ defined on WHPART(S, ~)) defines a Boolean structure (cf. Davey and Priestley 1990).♦

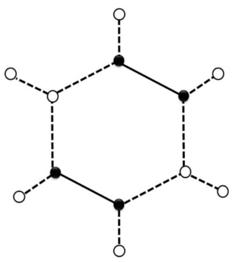
The distinction between whole parts and scattered parts is possible not only for parts of similarity structures (S, ~). Rather, for many a structure one may define an approporiate closure operator W that can be used to single out a class of whole parts from the larger class of parts of that structure. Moreover, if (S, $\sim_S$) is a trivial similarity structure ($\sim_S$ = id_S), then any subset T ⊆ S turns out to be a whole part, i.e. T = WH(T), and the distinction between whole parts and non-whole parts breaks down and we obtain PART(S, ~) = WH(PARTS(S, ~)) = PS. In other words, exactly the non-Booleaness of the mereological systems of most similarity structures enables us to define the distinction between scattered and non-scattered parts.
Definition (3.6) is not the only option to distinguish scattered parts from non-scattered ones. Another plausible option is based on the intuition that wholes should be connected in some sense. Since PART(S, ~) uses to be a non-Boolean Heyting algebra the following definition – mimicking the standard definition of connectedness in topology - offers a reasonable characterization of connectedness:

(3.10) Definition . Let (S, ~) be a similarity structure. A structural part (T, $\sim_T$) of (S, ~) is connected if and only if it is not the union of two (non-trivial) disjoint structural parts.♦

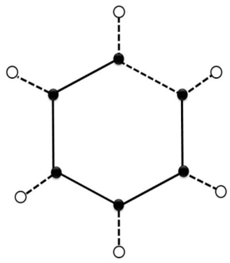
(3.11) Example. Consider the similarity structure (M, ~) of (3.4) characterizing a methane molecule. Then the parts Hb and Ha—Ce—Hc of M are connected, while Ha & Hd is not connected, since it is the union of the non-trivial disjoint parts Ha and Hb.♦

It seems plausible then to assume that "real wholes" should be connected wholes in the sense of (3.2) and (3.10). One should note, however, that the intersection of "real wholes" in this sense need be no longer a "real whole", i.e. the algebraic structure of "real wholes" is less comfortable than that of not necessarily connected wholes. Moreover, the ring-shaped molecule of benzene (C_6H_6) shows that there is no logical relation between wholeness and connectedness:

(3.12)



whole and non-connected



non-whole and connected

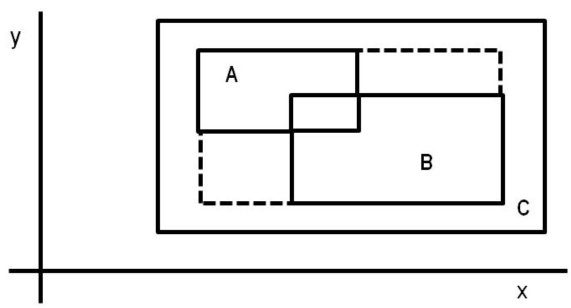
In sum, moving from homogeneous to inhomogeneous sets enables us to distinguish between whole parts and scattered parts, and between connected and non-connected parts in a natural way. The basic ingredient for this distinction is provided by the structural component of a similarity relation. Traditional mereology of (homogeneous) sets turns out to be a very special case of the mereology of inhomogeneous sets, namely, in which these structural distinctions become trivial. Or, to put it differently, Aristotelian Non-Boolean structural mereology turns out to be a refinement of the non-structural traditional Boolean mereology. In the next section we will discuss two further examples of structural mereologies. The first arises from the geometrical structure of Euclidean space, the second may be considered as a prototypical algebraic structure. For both structures the resulting mereological systems are highly non-Boolean (not even Heyting), both support interesting non-trivial notions of connectedness and wholeness analogous to the case of similarity structures.

4. The Mereologies of Rectangles. Already the mereological systems of inhomogeneous sets indicate that the framework of Boolean mereology is too narrow to describe all kinds of mereological phenomena. Nevertheless, it may be expedient to explore in detail the mereo-

logical systems of some other kinds of structured sets in order to strengthen evidence that their mereologies may wildly differ from the familiar Boolean mereological systems of sets. For this purpose in this section the mereological system Cartesian rectangles of the Euclidean plane are studied in some detail.

Let E(x, y) be the Euclidean plane E endowed with a distinguished (orthogonal) Cartesian coordinate system (x, y), i.e. x and y are two orthogonal lines of E. Let A, B, ... be rectangles of E whose sides are parallel to x and y, respectively. These rectangles, together with the sets E and Ø, are called Cartesian rectangles. I propose to take Cartesian rectangles as the structural parts of the structured set E(x, y). This class of structural parts of E(x, y) is denoted by PART(E(x, y)).[5] Evidently, there is a well-defined and natural notion of parthood between Cartesian rectangles. If the the Cartesian rectangle A is a part of the Cartesian rectangle B this is denoted by A ≤ B. The plane E itself may be considered as an „infinite" Cartesian rectangle that comprises all other Cartesian rectangles as its parts. For technical reasons we also consider the empty set Ø as an element of CART. Ø is called the trivial rectangle and is contained in all elements of PART(E(x, y)). Clearly, the relation ≤ thereby defined on PART(E(x, y)) is a partial order relation, i.e. ≤ is reflexive, antisymmetric, and transitive. Moreover, the bottom element of PART(E(x, y)) is Ø and its top element 1 is the whole Euclidean plane E. Hence PART(E(x, y)) is a plausible candidate for a structural mereological system. It not only depends on the set-theoretical structure of the Euclidean plane E but takes into account also some aspects of its rich geometrical structure, to wit, the Cartesian coordinates x and y.

(4.1)



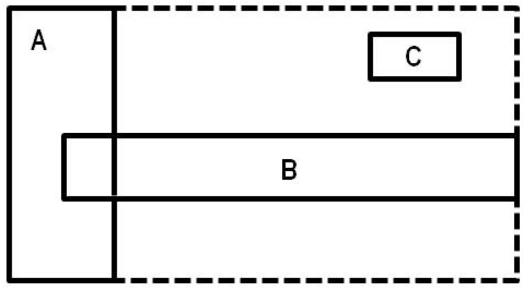
As can be seen immediately from this diagram, for any two Cartesian rectangles A and B, there exist a unique smallest Cartesian rectangle A ∨ B that contains them both. As is easily seen, A ∨ B is just the intersection of all Cartesian rectangles C that contain A and B, i.e. A ∨ B is the

[5] In the following it is assumed throughout that all Cartesian rectangles are non-degenerate in that their sides both have positive lengths. No other subsets of E are considered as structural parts of E(x, y). In particular, points or line segments are not considered as structural parts of E.

supremum of A and B. The same is true for infinitely many Cartesian rectangles A_i, $i \in I$. Hence for any set of Cartesian rectangles A_i there is a well-defined supremum $\vee A_i$. In other words, (PART(E(x, y)), ≤) is complete sup-semilattice.

It should be noted that in (PART(E(x, y)), ≤) the supremum $A \vee B$ of A and B behaves differently than the set-theoretical union $X \cup Y$ of subsets X and Y. The supremum $A \vee B$ may contain parts $C \leq A \vee B$ that do not overlap neither with A or B. In contrast, every non-empty subset W of $X \cup Y$ either overlaps non-trivially with X or Y (or with both). The following diagram gives an example:

(4.2)



This fact distinguishes systems like (PART(E(x, y)), ≤) from Boolean mereological systems such as (PX, ⊆). A subtler difference between the two kinds of mereological systems comes to the fore when we consider the relation of overlapping in PART(E(x, y)). Given two Cartesian rectangles A and B clearly there is either a uniquely determined largest Cartesian rectangle C that is part of both A and B, or there is no proper Cartesian rectangle contained in A and B. Hence any pair of Cartesian rectangles A and B has an infimum (A, B), denoted, as usual, by $A \wedge B$. Note, that the Cartesian rectangle $A \wedge B$ need not always be the set-theoretical intersection of the point sets that represent A and B.[6]

Summing up these pedestrian and informal considerations one concludes that (PART(E(x, y)), ≤) is a lattice (cf. Davey and Priestley 1990). Characterizing (PART(E(x, y)), ≤) as a lattice is not the whole story to be told about this structure. It is natural to ask what type of lattice (PART(E(x, y)), ≤) is. Perhaps the most important general question concerning a given lattice is whether it is distributive or not. Recall that a lattice L is distributive if for all A, B, C ∈ L

[6] Rather, not only set-theoretical operations but also the topological structure of gets involved in the calculation of the infimum $A \wedge B$ in PART(E(x, y)). Indeed, $A \wedge B = cl(int(A \cap B))$, cl and int being the closure operator and the interior kernel operator of the topological structure.

(4.3) $$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C).$$

As (4.1) shows, PART(E(x, y)) is not distributive. Moreover, all proper Cartesian rectangles A lack complements, i.e. there is no Cartesian rectangle A* such that $A \wedge A^* = Ø$, $A \vee A^* = E$, and $A^{**} = A$. This shows that in a certain sense mereological systems like PART(E(x, y)) are less similar to classical Boolean systems than Heyting systems PART(S, ~). On the other hand, PART(E(x, y)) satisfies some requirements that locate it closer to traditional mereological systems than Heyting algebras:

(4.4) Definition. (Axiom of Strong Supplementation (SSP), (Simons 1987, 29). A mereological system (M, ≤) satisfies (SSP) if for all x, y ∈ M one has: If x is not a part of y, then there is a part z of x such that z and y are disjoint:

$$(x)(y)(\text{NOT}(x \leq y) \Rightarrow \exists z(z \leq x \text{ and } z \wedge y = Ø).♦$$

(4.5) Theorem.

(1) The mereological system (PART(E(x, y), ≤) satisfies (SSP).

(2) Mereological systems (PART(S, ~), ≤) in general do not satisfy (SSP).

Proof. (1) Direct inspection of the Euclidean plane, (2) already the similarity structure x——y shows that PART(S, ~) need not satisfy (SSP).♦

Structural mereologies like PART(S, ~) satisfy only the weaker so called proper part axiom that may be conceived as a sort of extensionality principle. It asserts that if the proper parts of an object are all proper parts of a second object then it is a part of the second object:

(4.6) Definition (Proper Part Principle (PPP), (Simons 1987, 28). A mereological system (M, ≤) satisfies the proper part principle (PPP) if for all x, y ∈ M one has:

$$\text{IF } (\exists z(z < x \text{ AND } (u)(u < x \Rightarrow u < y)) \text{ THEN } x \leq y.♦$$

By definition Heyting systems satisfy (PPP). Moreover, according to (Simons 1987, 29) the axiom (SSP) entails (PPP), but, of course, the reverse implication does not hold.

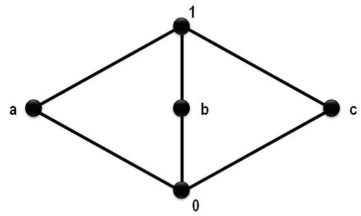
In order to structurally distinguish lattices such as (PART(E(x, y)), ≤) from other types of structural mereological systems the following definition will be useful:

(4.7) Definition. A lattice (L, ≤) is modular iff for all A, B, C ∈ L with C ≤ B entails that (C ∨ A) ∧ B) = C ∨ (A ∧ B).♦

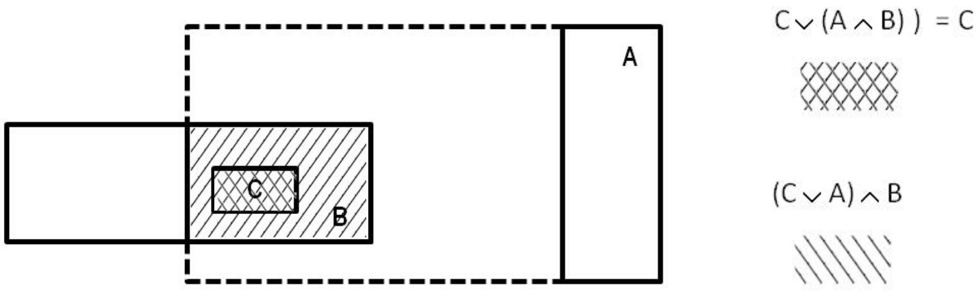
The smallest non-modular lattice is the so-called pentagon:

1
b a
c
0

One can show that the pentagon is typical for non-modular lattices since a lattice L is modular if and only if it does not contain a pentagon as a sublattice. For our purposes the important thing to note is that distributivity entails modularity. This is seen as follows: Assuming C ≤ B the application of the distributive law yields (C ∨ A) ∧ B) = (C ∧ B) ∨ (A ∧ B) = C ∨ (A ∧ B). Hence every distributive lattice is modular. The opposite implication does not hold as the



classical counter-example of the „diamond“ shows:
As is easily calculated, the diamond is modular, but not distributive. Now let us show that the Cartesian rectangles also usher us in the realm of non-modular systems. For this purpose consider the following constellation of elements of PART(E(x, y)) characterized by the relations C ≤ B, and A and B disjoint:

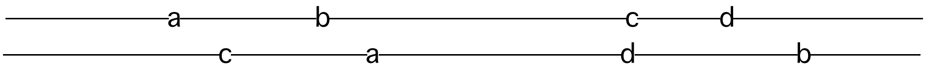


(4.8) Theorem. The lattice (PART(E(x, y)), ≤) is NOT modular.

Proof. Direct inspection yields that the triple A, B, and C violate the law of modularity.♦

To round up our presentation of structural mereological systems such as PART(E(x, y) let us observe that PART(E(x, y) is only one of a family of more or less similar systems. Evidently PART(E(x, y)) has n-dimensional analogues PART($E^n(x_1, ..., x_n)$) for n-dimensional Euclidean spaces E^n for all natural numbers $n \geq 1$. Particularly simple is the 1-dimensional analogue PART($\mathbf{R}^1$) whose elements are just the closed intervals of the real line **R**. The join of two intervals $[a, b], [c, d] \subseteq \mathbf{R}$ is the interval [min(a, c), max(b, d)], or, in terms of convex geometry, its convex hull. Two typical cases are the following ones:

(4.9)



In the same vein as PART(E(x, y)), all these mereological systems turn out to be non-distributive and being without complements.

Bounded variants of mereological systems of Cartesian parts are obtained if one considers instead of the entire Euclidean plane E a Cartesian rectangle $F \in$ PART(E(x, y)) and its Cartesian parts:

(4.10) $\qquad$ PART(F(x, y)) := {A; $A \in$ PART(E(x, y)) and $A \subseteq F$}

An other variant of the structural mereological system PART(E(x, y)) is obtained by admitting only Cartesian rectangles $[n, n+k] \times [m, m+l]$ as structural parts, $n, m, l, k \in \mathbf{Z}$. This mereological system of discrete structural parts may be denoted by $PART_{DIS}$(E(x,y)). Again, $PART_{DIS}$(E(x,y)) is non-modular and lacks complements. As a structural bonus, however, the lattice $PART_{DIS}$(E(x,y)) is atomistic, i.e. all element are the suprema of finitely many atoms $[n, n+1] \times [m, m+1]$.

Finally, as a far-reaching generalization, let us mention the mereological system of the Euclidean plane E that takes all closed convex subsets of E as structural parts. Recall that a set X is convex iff with two points $a, b \in X$ one always has $[a, b] \subseteq X$. The plane E endowed with the canonical convex structure may be conceived as a structured set (E, []) for which a structural mereological system (PART(E, []), $\leq$) can be defined whose structural parts are the convex subsets of E.

<u>5. A Mereological Analysis of Quantities.</u> Some twenty years ago David Armstrong proposed to

conceive quantities as structural universals (cf. Armstrong 1988). Since then, his proposal has beed widely discussed. Recently, it came under heavy attack from various sides (cf. Eddon 2007, Gibb 2007, Morganti 2010 among others). The aim of this section is to use the insights that we have gained from the study of non-classical mereological systems such as a special case of mereological systems such as (PART(E(x, y)), ≤) to defend a version of Armstrong's account of quantities as universals. It follows the lines of the reductive account of structural universals given in section 3. There, the state of affairs „x is a methane molecule" that apparently involves the structural universal „being methane" is analysed in the framework of Armstrong's factual ontology as follows: „x is a methane molecule" is a thick particular whose parts are other states of affairs as described in (3.3) and (3.4). The smallest parts of „x is a methane molecule" are atomic states of affairs that are constituted from thin particulars and simple universals.

The aim of this section is to give a similar reductive account of structural universals as they allegedly occur in Armstrong's analysis of quantities. For this purpose first we consider a canonical model of real-valued quantities that emerges in a natural way from the system of „1-dimensional rectangles" PART(E^1(x)). The Euclidean line E^1 endowed with a Cartesian coordinate may be canonically identified with the real numbers **R**. Then we may define the following structural mereological systems:

<u>(5.1) Lemma</u>. Denote by **R*** := {a; 0 ≤ a ∈ **R**} the set of non-negative real numbers.

(i) For a ∈ **R*** define PART*([0, a]) := {[0, b]; b ≤ a}. Then (PART*([0, a]), ≤) is a (linear) Heyting algebra with bottom element [0, 0] and top element [0, a]. (PART*([0, a]) is order isomorphic to the interval of real numbers [0, a] by the canonical isomorphism that for 0 ≤ b ≤ a maps the subinterval [0, b] of [0, b] onto its endpoint b.♦

(ii) Define PART*(**R***) := $\cup_{a \in \mathbf{R}^*}$PART*([0, a])= {[0, a]; a ∈ **R***}. The set-theoretical inclusion of intervals defines a partial order ≤ on PART*(**R***) with bottom element [0,0].

Obviously, PART*([0, a], ≤) does not satisfy (SSP) but, as every Heyting algebra does, it satisfies (PPP). ...

Let A*, B*, C*, ... be a family of length universals and a, b, c particulars such that states of affairs A*a, A*b, B*c, ... obtain in case the particular a instantiates the length universal A*, the particular b instantiates the length universal B*, etc. As Armstrong pointed out states of affairs, in particular such states as A*a, B*b, ... are particulars. More precisely they may be characterized as „thick particulars." In Armstrong's factual ontology (Armstrong 1997) there may exist monadic or relational universals V, W, ... that can be instantiated by thick particulars. Thereby, complex states of affairs are constituted, e.g. VA*a, WB*aC*c, ... to be interpreted as the states of affairs that V is instantiated by A*a, and that the state of affairs B*a and C*c (as particulars) are related to each other by the binary universal W etc. Hence, states of affairs such as A*a, B*b, etc. may occur as structural parts in more complex states of affairs in the same way as the states of affairs such as Ha (a is a hydrogen atom), Ce (e is a carbon atom), ... occur in the complex state of affairs „being methane" as described in section 3.

The details are as follows. Assume that for „length universals" such as A*, B*, ... there is a binary universal „K" such that the complex state of affairs „A*a K B*b" to be interpreted as „The length A* of a is smaller than the length B* of b". In order that K be a reasonable length comparison „smaller than" it should satisfy the following adequacy conditions:

<u>(5.2) Assumption (Continuity of Length Universals)</u>.[7] The particulars a, b, ..., the length universals A*, B*, ... and the binary universal K defined for atomic states of affairs A*a, B*b, ... are assumed to satisfy the following requirements:

(1) IF A*a AND B < A THEN B*b for some b < a.

(2) IF (A*a AND B*a) THEN (A = B).

(3) IF (A*a K B*b) THEN (A < B).♦

The first condition guarantees a „continuous" instantiation: if an object a instantiates a length A* it has parts that instantiate all smaller lengths. If length were quantized, this would not hold, of course. It is a matter of empirical research, so to speak, whether one should assume (5.2)(1) or not. Thus, subscribing to (5.2)(1) corresponds to the classical case of a continuous length in a continuous Newtonian world (cf. Davies 2001).

Condition (5.2)(2) ensures that the length of a particular a is well defined, i.e. no particular a can instantiate different length universals A* and B*; finally (5.2)(3) ensures that the comparison between length states of affairs by the binary universal K covaries with the arithmetical relation <. Thereby a strict correlation between the binary universal K and the arithmetical relation < is guaranteed.

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[7] In some sense it is an empirical question, whether length or any other quantity is continuous or not. If it would turn out that length is quantized, instead of (5.2) different stipulations had to be made. This would, of course, influence the mereological structure of complex states of affairs A*a. I hope to deal with this issue in detail in another paper. For now, be it sufficient to say that instead of R* the natural numbers **N** would determine the relations between length states of affairs.

Now we are prepared to give the following mereological analysis of length state of affairs analogous to the mereological analysis of methane:

<u>(5.3) Mereological Analysis of quantitative states of affairs.</u> Let a be a particular with a mereological system PART(a, ≤). Then the instantiation A*a of the simple length universal A* by the particular a is a complex state of affairs that gives rise to the mereological system PART(A*a, ≤) defined as follows:

(1) PART(A*a) := {B*b; B ≤ A and b ≤ a}

(2) B'*b' ≤ B''*b'' := B' ≤ B'' and b' ≤ b'' (B'*b', B''*b'' ∈ PART(A*a))

The bottom element 0 of PART(A*a, ≤)) is 0*Ø and the top element 1 is A*a.♦

One should note that the universal K instantiating relations between state of affairs A*a, B*b is a first-order universal for thick particulars in the same way as the bonding universal — instantiated by chemical states of affairs Hc, Ce, ... If one is prepared to admit second-order universals that are instantiated by first-order universals, one may assume K to be an asymmetric and transitive binary universal for length universals A*, B*, ... such that if A*KB* obtains then A < B. This may be more elegant than the procedure described in (5.2) but depends on the contingent fact, so to speak, that K – as a strict order relation – is necessarily asymmetric. Hence no universal shows up more than once, as would be the case for alleged complex „chemical universals“ such as H_2O. Compared with the second-order account the one of (5.2) is ontologically more austere.

<u>6. Modular Mereological Systems.</u> Let us come back to the topic of modular versus non-modular mereological systems discussed in section 4. Only announcing that PART(E(x, y)) does not satisfy the axiom of modularity, may not be considered as a sufficient reason for introducing this concept. The reader may rightly expect some positive application of modularity, i.e. an example of some nice mereological systems that do satisfy the axiom of modularity. Here it comes.

Recall that a group in the sense of mathematics is a set G endowed with a multiplication •: G x G ⇒ G that is associative, (for all a, b, c one has (a•(b•c) = (a•b)•c) and each element a has an inverse a^{-1} such that a • a^{-1} = e, e being a distinguished element called the neutral element of G. A group G is commutative or abelian if for all a, b one has a • b = b • a.

Groups abound in mathematics, physics, and elsewhere. Perhaps the best known example is the group (**Z**, +) of integers ..., -1, 0, 1, 2, ... with ordinary addition + as multiplication and 0 as neutral element.

A natural choice for the structural parts of groups are subgroups: a subgroup H of a group G is a subset H of G that contains the unit e, is closed under multiplication, and for every element h of H the inverse h^{-1} also belongs to H. Thus for every group G there is a structural mereological system SUB((G, •) ≤) of subgroups, the parthood relation ≤ is just ordinary set-theoretical inclusion ⊆. As is easily seen all subgroups of the group of integers **Z** have the form

(6.1) $(n\mathbf{Z}, +) := \{..., -n, 0, n, 2n, ...\} \qquad n \in \mathbf{N}$

Then an elementary calculation yields that SUB((**Z**, +), ≤)) is a modular mereological system. More generally, one can show that for every abelian group G the mereological system SUB((G, •) ≤) is modular. For general groups this is no longer valid. To get a neat and palatable result for general groups one has to require something more for a structural part than just being a subgroup. Rather, structural parts of general groups should be normal subgroups, or, at least, this choice leads to a neat and elementary result.

For any subgroup H of G and any g ∈ G define the set $gHg^{-1} := \{ghg^{-1}; h \in H\}$ is again a subgroup, usually different from H. If $H = gHg^{-1}$ for all g ∈ G, then H is called a normal subgroup. Clearly, if G is abelian all of ist subgroups are normal. Technically speaking, normal subgroups are stable under the inner automorphisms of G. Every subgroup H can be embedded canonically in a normal subgroups, its normalisator N(H).[8] Hence we may consider the structural mereological system SUB_N((G, •) ≤) of normal subgroups of G. For this system one has the following classical result:

(6.2) Theorem (Schmidt 1994, 2.1.4 Theorem, p. 43). The partial order (PART(G, •), ≤) of normal subgroups of a group G is a modular lattice. The bottom element 0 of PART(G, •), is the unit group {e} of G and the top element 1 is G itself.♦

Groups abound in mathematics, physics, and elsewhere. Elementary examples show that PART(G, •), is not always distributive (cf. Mormann 2010). Thus, in some sense, Theorem (6.2) is a best possible result. This theorem may be conceived as an important theorem of structural mereology, although, of course, mathematicians did not conceive it as such. It gives

[8] In an analogous way as the completion WH of subgraphs in (3.8) „normalization" of subgroups may be conceived as a closure operator N.

us a profusion of structural mereological systems that essentially differ from the familiar Boolean systems. Modular lattices have a rich and interesting structural theory (REF, REF). So it may be conceived as an important result of structural mereology. The investigation of the structure of the lattices of subgroups is still a living subject of contemporary mathematics (cf. Schmidt 1994). This evidences that the investigations of structural mereology need not be trivial, even for those apparently simple structures such as groups.

7. Structural Mereology and The Problem of False Axioms. Some time ago Mark Johnston blamed classical mereology as a „telling case of misplaced generality“ (Johnston 2002) qualifying most of its axioms as simply „false.“ He proposed to reinterpret classical mereology in a radical way to save it reserving to it at least a small place in the more comprehensive field of ontology by reinterpreting it as a theory of mereological sums and not as a theory of ontological constitution (ibid., 130).

In this paper we have pursued another way to overcome the „misplaced generality“ of traditional mereology, namely, to conceive traditional mereology as a special case of the general discipline of structural mereology that deals with a plurality of structural mereological systems in an analogous way as Euclidean geometry may be conceived as a special limiting case of the much more comprehensive field of Riemannian geometry.

In a sense, the approach of this paper is a generalization of the one that David Lewis pursued some twenty years ago when he pointed out that sets have a natural mereological structure (Lewis 1991). While Lewis tapped set theory as a source for mereological systems, in this paper I propose to use geometry, topology or, even more generally, relational systems, as rich and variegated sources for structured mereological systems. Depending on the specifics of the structured systems under consideration the mereologies of these systems more or less resemble classical Boolean mereologies. More precisely, Boolean mereologies come out as limiting cases of structured mereologies analogously as sets may be conceived as limiting cases of structured sets.

Arguably, geometry and topology are more useful to feed our mereological intuitions concerning real world systems than set theory. After all, we live in a spatio-temporally structured universe. Hence it is only natural that geometrical and topological structures play a role for mereological considerations. The primacy of geometry and topology over set theory can be formulated as the maxim that set theory is to be conceived as an abstraction or a limiting case in which the geometrical and the topological structure becomes trivial. This can be rendered precise in many different ways. For instance, from a topological point of view, a set X may be conceived as a special topological space endowed with the trivial topology, to wit, either with the discrete or the indiscrete topology (cf. Willard 1970, chapter 2). As has been explained elsewhere, one may take the (regular) open sets OX (O*X) of the topological space X as its structural parts; another option would be to the (regular) closed subsets CX (C*X) of X as structural parts. For the the special case X endowed with the discrete topology all these options boil down to the same. This means, the topological perspective gives us a richer spectrum of what may be conceived as a part of X than the set-theoretical one.

The variety of structural mereologies such as PART(S, ~), PART(E(x, y)), PART(**R***, ≤), and PART(G, •), which could be continued *ad libitum*, hopefully suffices to render plausible the thesis that the mereological concepts of part, whole and their relatives are inexorably plurivalent in that they depend to a large extent on the structural context where they occur. Searching for „the" axioms of mereological systems is a futile enterprise. There is no need to determine the boundaries of the variety of admissible mereological systems once and for all by elaborating a closed list of types that contains all of them. Rather, the concept of a (structural) mereological system is bound to become a variable in the same way as the concepts of „number" or „space" have acquired a variable meaning.[9]

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[9] In this paper we have only studied some rather elementary examples. In a more sophisticated manner category theory may be used to explore the variety of structural mereological systems (cf. Lawvere and Rosebrugh 2001, Mormann 2009).

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